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Part II. Economic Optimization in Agricultural and Resource Economics

On Economic Optimization:
A Nontechnical Survey

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For at least two centuries economic principles have involved three fundamental concepts. First, individual or group behavior can be explained—at least in part—as the result of pursuing one's advantage. Second, a given system of individuals or nations may possess a kind of harmony or equilibrium when each individual or nation pursues its own advantage. Third, if the environment is properly structured, the working of an economy may bring about individual optima and group equilibria. As early as Cournot [1838], these ideas began to receive an explicit mathematical treatment. It was Cournot who first used calculus to analyze the three classical notions of optimum, equilibrium, and process in markets. The methods that he initiated dominated analytical economics for over a century. The more or less definitive form of this neoclassical, marginalist economics was established by Jevons [1871], Marshall [1890], and Walras [1874] and culminated with Hicks's *Value and Capital* [1939] and Samuelson's *Foundations of Economic Analysis* [1948].

From the vantage point of our generation it is clear that something substantial was lost in the neoclassical mathematization of classical economic thought. The issue involves alternative assumptions about the underlying structure of choices and its role in bringing about compatibility between individual optimization and group equilibrium. It is now clear, thanks to Samuelson [1949, 1959a, 1959b] and others, that some of the classical ideas—for example, the theory of rent (Malthus, West) and the theory of trade (Ricardo, Mill)—are most naturally expressed by means of linear programming

models, a postneoclassical development. But this linear programming struc-
ture could not be accommodated in the "smooth" neoclassical world. With
the advent of Arrow and Debreu's analysis [1954] of general equilibrium this
problem was overcome. Classical linearities and inequalities could be incorpo-
rated into the general economic optimization framework. Thus, modern opti-
mization theory not only helps to mathematize and illustrate classical ideas, it
makes it possible to identify the fundamental unity in two centuries of eco-
nomic thought.

The transition to the modern period began to occur even before the clos-
ing of the neoclassical system by Hicks and Samuelson. The catalysts for this
include Leontief, Von Neumann, and Wald. Leontief's input-output or inter-
industry model [1928, 1936] and Von Neumann's growth model [1937,
1945] captured essential features of classical thought and through the use of
algebra forced a shift away from the calculus of the neoclassical school. The
game theory of Von Neumann [1928] and Von Neumann and Morgenstern
[1944] made possible a profound new formalization of the multiperson joint
optimization problem inherent in economics and introduced the axiomatic
method and topology. Wald [1936, 1951] contributed the first rigorous
proof of the existence of general equilibrium among economic optimizing in-
dividuals. The full impact of this reorientation came at mid-century when the
duality of constrained optimization and economic valuation was established
by Gale, Kuhn, and Tucker [1951] and Kuhn and Tucker [1951] and when
efficient optimization algorithms were discovered. Especially because of
Dantzig's simplex method [1949] for linear programming, optimization be-
came a tool for planners as well as a theory for economists. Further back-
ground material relating optimization concepts to the history of economic
thought will be found in Samuelson [1948], Koopmans [1951], Dorfman,
Samuelson, and Solow [1958], and Leontief [1960].

During the past two decades modern optimization theory and methods
have continued to develop, and at the same time their effective application
has spread to a growing variety of important applied problems. The literature
is indeed by now so vast as to preclude a comprehensive survey. In this over-
view, therefore, we shall present a nontechnical summary of the most impor-
tant concepts involved in these developments for economic theory and ap-
plied analysis. The applications of optimization theory to problems in agricul-
tural and resource economics are reviewed elsewhere in this volume (see "Op-
timization Models in Agricultural and Resource Economics" by Richard H.
Day and Edward Sparling).

Optimization Models

For a very long time the mathematical development of theoretical and ap-

plied economics was severely circumscribed by the limited class of optimization model types with which it could cope. Though many fundamental barriers remain, the breakthroughs just recalled resulted in a spectrum of operational optimizing models. This spectrum can be broken down according to the components of an optimization model which we will outline. Various important examples are then illustrated. Next, the concept of infinite programming is used to show how the classical and neoclassical optimization approaches are related. Remarks on the distinction between problems and models conclude the chapter.

The following more or less standard definitions will facilitate our discussion: "Optimizing" is finding a best choice among possible or feasible alternative choices. An "optimization model" is a specific formalization of a problem in terms of its comparable alternatives, the criterion for comparing alternatives, and the feasible alternatives. A "mathematical optimization model" consists of a "choice space," which is the set of comparable alternatives, an "objective function," which describes how alternatives are to be compared, and a "feasible region," which is a subset of the choice space and contains those alternatives that are eligible for choice. The feasible region is usually — though not always — defined by equations or inequality constraints.

The choice space. Virtually all economic optimization models involve real linear spaces in which each comparable choice may be represented by a vector of real variables. If the dimension of the choice space is finite, so that the number of choice variables is finite, we have a finite-dimensional optimization model. Otherwise the model is called infinite-dimensional. If each variable or component in the choice space can take on any real value, we have "continuous" or "real" optimization; if each variable may take on only discrete values, we have "discrete" or "integer" optimization. If some variables are discrete while others are continuous, we have "mixed integer" optimization. According to the type of choice space, then, we may distinguish six types of optimization models as summarized in the following outline.

Optimization Models by Type of Choice Space

1. Finite-dimensional optimization
    1.1 Continuous or real variables
    1.2 Discrete or integer variables (integer programming)
    1.3 Continuous *and* discrete variables (mixed integer
          programming)
2. Infinite-dimensional optimization
    2.1 Continuous variables
    2.2 Discrete variables
    2.3 Continuous *and* discrete variables

The linear and quadratic programming problems and the neoclassical optimizing fall in category 1.1. The transportation problem is an example of category 1.2. Models including increasing returns to scale utilize category 1.3. Optimal control and dynamic programming models are often defined for an infinite future and provide examples of category 2.1. Categories 2.2 or 2.3 would arise if "lumpy" or discrete capital goods were incorporated into the infinite optimization, though to date this does not appear to have been done.

The objective (criterion, utility, payoff) function. In mathematical optimization alternatives are compared by means of their real value as given by some real valued function. This function defines a preference ordering on the alternatives in the choice space. Objective functions may be classified according to their mathematical properties: smoothness or continuity properties, concavity or convexity properties, separability or interdependence properties, and special forms. Thus we have the following outline which gives some of the relevant distinctions.

Function Characteristics

1. Continuity properties
   1.1 Semicontinuous, upper or lower (allows for step functions)
   1.2 Continuous functions
   1.3 Differentiable functions
   1.4 Twice-differentiable functions
   Etc.
2. Concavity properties
   2.1 Concave (convex)
   2.2 Strictly concave (convex)
   2.3 Pseudo concave (convex)
   2.4 Quasi concave (convex)
   2.5 Strictly quasi concave (convex)
   Etc.
3. Separability properties
   3.1 Partially separable
   3.2 Completely separable
4. Special functional forms
   4.1 Linear
   4.2 Quadratic
   4.3 Power
   Etc.
5. Monotonicity properties
   5.1 Nondecreasing (increasing)
   5.2 Strictly increasing (decreasing)

In discrete or mixed optimization problems the objective function is usually defined on the continuous space within which the choice space is imbedded. The preference ordering is then defined for all continuous choices even though only discrete ones are allowed.

The feasible region. If the choice is unrestricted in the choice space, the optimization model is called "unconstrained." In this case the feasible region is the entire choice space. Otherwise, when the feasible region is a proper subset of the choice space, it is called "constrained." Feasible regions are classified according to various criteria: closedness (containing limit points), bound-

edness, or more generally by their compactness or noncompactness; convexity properties; and special functional forms. If the feasible region is defined by an equation, or a set of equations, then we have an "equality-constrained optimization model." If it is defined by inequalities, it is called an "inequality-constrained problem." Because an equation can be expressed by two inequalities, the latter contains the former as a special case. Nonetheless, because mathematical techniques employed in each differ markedly, equality and inequality cases should be regarded as separate categories. In either event it is necessary to define constraint functions. For each type of constraint function we get a specific type of optimization problem. Thus the classes of functions enumerated above are relevant from this point of view too. The following outline summarizes the most important criteria for determining types of feasible regions.

Feasible Region Characteristics

1. Unconstrained optimization: The feasible region is the entire space.
2. Equality-constrained optimization: The feasible region is defined by equations. See the table of function characteristics above.
3. Inequality-constrained optimization
   3.1 Compactness or noncompactness
   3.2 Geometric properties
       (1) nonconvexity
       (2) convexity
       (3) strictly convexity
       (4) polyhedral form (as in linear programming)
   3.3 Constraint function types. See the table of function characteristics given earlier.

The basic questions of optimization theory must be posed for each optimization model or class of models: (1) Do solutions exist? (2) How many are there? (3) How can solutions be characterized? (4) How can solutions be found? The answers and the methods used to obtain them depend of course on the characteristics of each model type. The neoclassical economists rarely concerned themselves with the possibility of multiple optima, indeed, they used the calculus of smooth functions to structure models which possessed unique solutions although they were characteristically vague about, or even ignored, the exact model characteristics to justify their results. The implications of these mathematical issues for economic theory are of greater importance than is usually recognized. Indeed, with the appreciation of function characteristics in optimization theory (especially convexity properties) has come the significant realization that what was once assumed to be true of all

private or public ownership economies could in fact be proven true only for economies with very special and not too realistic constraint and preference structures.

Modern optimization theory addresses itself very often to situations in which many "best" solutions exist such as in linear programming. In this example the set of solutions forms a simplex, a "polyhedral face" generated by its extreme points. The location of such extreme points was found by Dantzig to involve sequences of straightforward algebraic calculations. This shows how theoretical characteristics yield insights leading to answers to the question: How can solutions be found? Another example, brilliantly expounded in Samuelson's classic "Market Mechanisms and Maximization" [1949] shows how the duality properties of constrained optimization can be used to guide a sequence of relatively simple adjustments to the constrained optimum thus, in effect, mimicking the market process. We shall return to these issues later. However, at this point we illustrate a few of the most important optimization models in a way that brings out some of their distinctive features.

In figure 1 the isoquants of an objective function are illustrated by more or less concentric, somewhat irregular curves. The arrows normal to these isoquants indicate the direction of locally steepest ascent of the objective function. Point A is the optimizer of the unconstrained problem. The curve in the
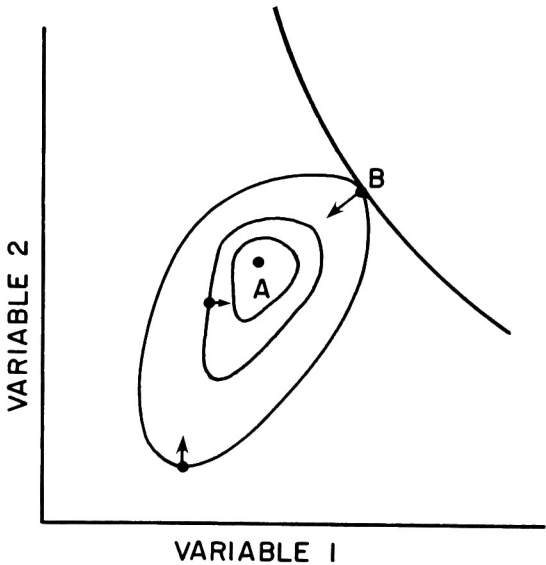
Figure 1. Unconstrained and equation-constrained optima

upper right part of the diagram illustrates an equation constraint to which the choice would be confined for an equality-constrained problem, in which case the optimizer is the point B, as point A is no longer feasible. Figure 2 shows how the mathematical programming model varies in its structure according to changes in the choice space and in the type of objective and constraint func-
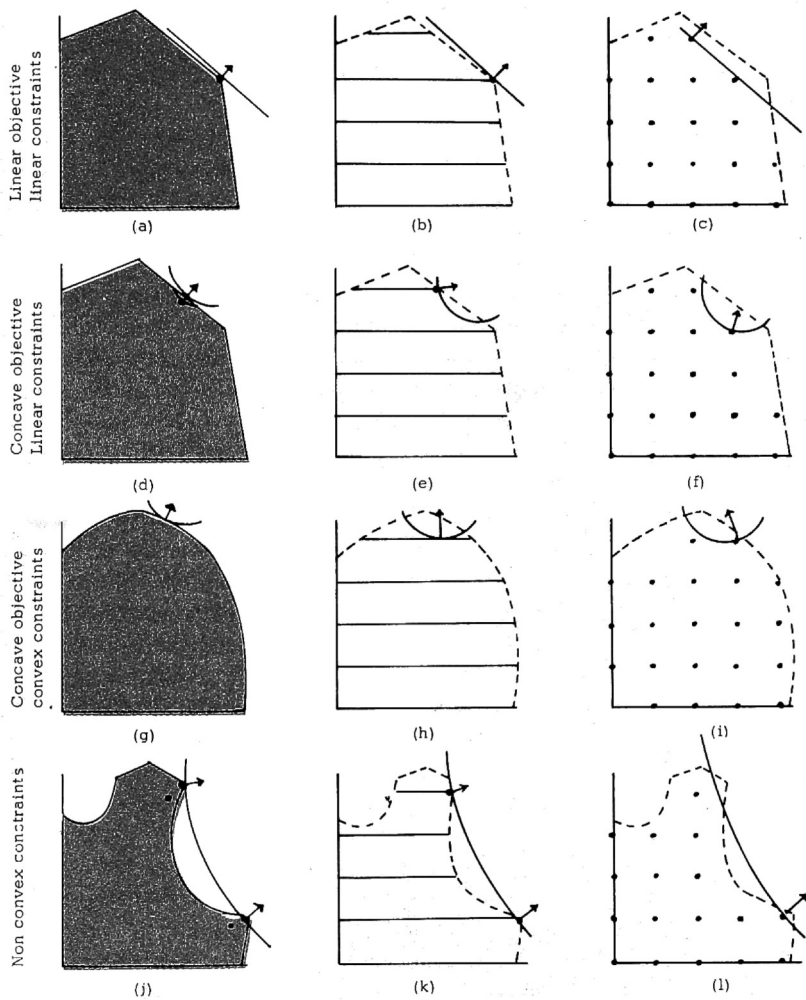
Figure 2. Some types of mathematical programming models

tion. Diagram (a) is the linear programming model, (b) the mixed integer, and (c) the integer linear programming model. In the first column the choice variables can vary continuously; the feasible region is the shaded area on the diagram. In the middle column variable 2 must be an integer while variable 1 can be continuous; the feasible region consists of the parallel lines. In the last column only integer variables are allowed; the feasible region is represented by the dots. The rows show how problems in these three categories change as objective functions and/or constraint functions change from linear to concave or convex or to nonconcave or nonconvex functions. Diagram (l), for example, illustrates integer programming with quasi concave objectives and nonconvex constraints.

Representing choices as integer variables introduces mathematical difficulties of a most formidable nature. There is some intellectual irony in this fact for in the pure integer case the number of feasible alternatives, if the feasible region is bounded, is finite; an exhaustive search is possible. In the continuous case the number of contenders for choice is nondenumerably infinite, even in a problem with only one dimension, and exhaustive search is impossible. Yet it is usually easier to solve continuous models at least approximately than it is to solve discrete ones. In the linear programming model where objective and constraint functions are linear, an unfortunate consequence of this fact is
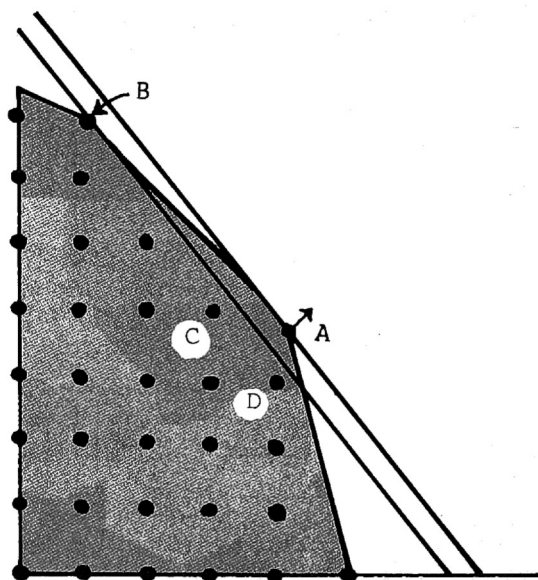

Figure 3. Continuous (A) and integer (B) solutions; suboptimal integer solutions (C, D)

easily illustrated. Figure 3 shows the true feasible region consisting of integer valued variables (dots) inside the shaded convex feasible region where the variables are assumed to be continuous. The continuous variable optimum is A, a point very far removed from the ("true") integer optimum B. If we round the continuous solution A to its nearest integer value, we get C or D, points also far removed from the optimum.

Infinite-dimensional programming. Optimizing over an infinite horizon arises in economic theories of capital and growth. When formalized, these theories lead to programming models in which the choice space is infinite-dimensional. (We shall take up this class of models later.) What is scarcely appreciated by economists, though fundamental in mathematics, is the extremely close relationship between infinite-dimensional and finite, continuous problems. We touch on this point next because it affords an opportunity to show how concepts from mathematical optimization theory can be exploited to reveal the underlying unity of various schools of economic thought to which we referred in the first section of this chapter.

Let us consider the purely competitive optizimizing problem of the price-taking firm that produces an output in amount y, using an input in amount x according to a production function illustrated by the smooth curve in figure 4. If the profit isoquants are parallel to the straight line marked $\pi = P_y - Q_x$, then the optimum is point E. If instead of the smooth neoclassical curve f(x) we used the linear approximation OBD, we would obtain an approximation to the neoclassical problem which we could represent using linear programming. The solution of this problem is point B in figure 4. By choosing a better linear approximation of the production function—say, OABCD—point A, which is closer to the neoclassical solution, is chosen. By making finer and finer linear approximations we could in this way come as close to the smooth optimum solution E as we pleased, just as a circle can be approximated as closely as we like by a polygon. As we do so, the dimension of the approximating linear programming problem increases, going in the limit to infinity where the approximation is perfect. In this way we see a type of duality between infinite-dimensional linear programming and finite-dimensional non-linear programming.

The mathematical duality just illustrated is analogously reflected in the history of economic thought. It is well known (we recall our earlier references to Samuelson) that the classical theory of production, most clearly expounded by Ricardo in his exegesis of the Malthus-West theory of rent, involves a linear programming problem like that illustrated by the piece-wise linear production function in which the input variable is interpreted to be the amount of land with different qualities and in which the yield declines as more land is brought into cultivation. By increasing the number of qualities of land we see

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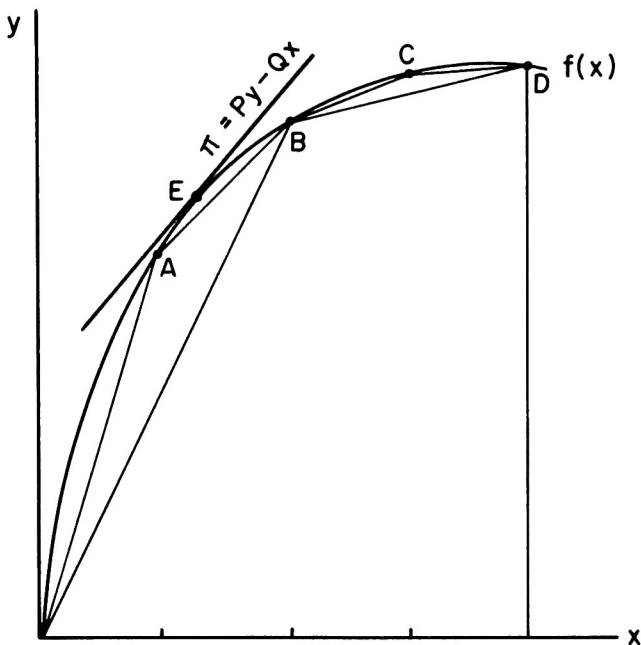
Figure 4. Linear programming as an approximation
of neoclassical optimizing

the convergences of the classical linear programming to the neoclassical smooth production function point of view developed many decades later.

The logical duality of the classical-neoclassical points of view should now alert us to a need for care in how we interpret the term "approximation." Whether we regard the classical linear programming model as an approximation of the neoclassical smooth optimization model or vice versa is a matter not of logic but of relevance, convenience, or interpretation in a particular application. Either may be used as an approximation to some real optimization problem, and one may be preferred to the other on empirical or computational grounds, depending on the nature of the problem at hand. A few economists still seem to think that neoclassical economics is *economics* whereas other forms of optimization theory are methods of operations research of no intrinsic economic interest and useful only in computation settings. Nothing could be further from the truth, as the above exercise demonstrates. Indeed, the neoclassical framework is of no more or less interest or relevance than its classical predecessor, and the very much more general formulation of modern optimization theory encompasses both and establishes their underlying unity.

It is also important to distinguish between model and problem optima. For example, we often use models involving continuous variables when, clearly, many economic variables are discrete in nature (machines, factories, farm buildings). Actual choice situations therefore must often distinguish among "lumpy" alternatives. The continuous optimization *model* must then be thought of as an approximation to an underlying discrete optimization *problem*. That the approximation may not be close is a possibility we have already illustrated in figure 3.

The interested reader should become acquainted with the following texts, which among them cover all of the major optimization model types. We list them in (roughly) ascending order of difficulty: Heady and Candler [1958], Hadley [1962], and Gale [1960] cover linear programming; Hu [1969] is concerned with integer and mixed-integer models; Hadley [1964], Intriligator [1971], Mangasarian [1969], and Karlin [1959] among them cover nonlinear programming, dynamic programming, and optimal control; Canon, Cullum, and Polak [1970] and Leuenberger [1969] give a unified treatment of programming, programming in infinite spaces, and optimal control. Aubin [n.d.] provides an advanced synthesis of optimization and game theory emphasizing duality relationships. There are also several excellent expository pieces by Dorfman [1953] on linear programming, Dorfman [1969] on optimal control, and Baumol [1958].

Parametric Programming and Comparative Static Analysis

In both theoretical and applied economics the study of how optima change in response to changes in the situation of the decision maker is of extreme interest. In optimization theory this study is called parametric programming or perturbation analysis. In economic theory it is called comparative statics. By means of it economists have constructed special theories of consumer demand, of producer supply, and of derived producer demand. Moreover, the careful mathematical study of optimizing behavior plays a central role in modern general equilibrium theory. This is because the theoretical analysis of the existence and properties of general equilibria depend on how well-behaved or smooth optimal sets are in their response to market situations.

In neoclassical models in which unconstrained or equality-constrained optimizations are specified, functions are assumed to be sufficiently smooth to make possible application of ordinary calculus. Equations are defined by setting the gradient of the objective function, or of the Lagrangian (in the equality-constrained case), equal to zero. Any optimum must satisfy these equations. These so-called first-order conditions are then interpreted as implicit functions which can be solved to give the decision variables as functions of

the parameters and exogenous variables of the problem. Even if this explicit functional dependence cannot be derived practically, it is often possible to infer its qualitative character such as "an increase in price will cause a fall in demand" and so forth. Econometricians are especially interested in those models for which the equations can be solved, for then the parameters of the optimization model may be estimated in reduced form.

As we have already noted, the classical models did not have sufficient regularity to make possible the application of calculus, and no doubt largely for that reason interest in them waned until the modern era, when the tools for inequality-constrained optimizations were perfected. Efficient algorithms for parametric programming made possible a reconsideration of step supply and demand functions and the kinked total cost functions of the classical production theory. They also made it possible to conduct traditional comparative static analysis for a vastly expanded range of economic problems.

The achievement was not without cost, however, for the neoclassical equilibrium and welfare theory completed by Hicks, Lange, and Samuelson did not cover the more general optimizations used in practical decision making. The methods for studying the modern optimization models in the general equilibrium setting, however, were not long in coming. Arrow and Debreu [1954] and McKenzie [1955] showed how topological methods and convex analysis could be used to extend the results on existence and efficiency of competitive equilibria to an economy made of modern (and classical) mathematical programmers.

The supply and demand functions that emerge from the modern point of view include, in addition to the traditional smooth neoclassical variety of Marshall's principles as shown in figure 5 (b), the classical step functions and the modern multivalued mappings or correspondences of the kind illustrated in figure 5 (a), (b), and (c). In (a) we find several prices at which the underlying optimizing behavior can take on any one of several possible supplies or demands. In (c) this indeterminance is continuous. In (d) optimizing behavior of a consumer also becomes indeterminant after some *income* level is reached. Any quantity within a given range inside the shaded area might be picked. The incorporation of the integer and mixed integer cases into the main stream of economic theory was given an impressive beginning by Charles Frank [1969]. But a complete comparative static treatment of it as needed for general equilibrium theory remains a task for future contributors. The kinky step function and correspondences that derive from modern parametric programming often have more complex qualitative appearances than their neoclassical counterparts and can indeed seldom be expressed in mathematically closed form. Instead they must be derived computationally and except for

(a) "Classical" or linear
    programming supply
    and demand functions

(b) "Neoclassical" smooth
    supply and demand
    functions

(c) "Modern" supply
    and demand
    correspondences

(d) A "modern" Engel
    curve correspondence
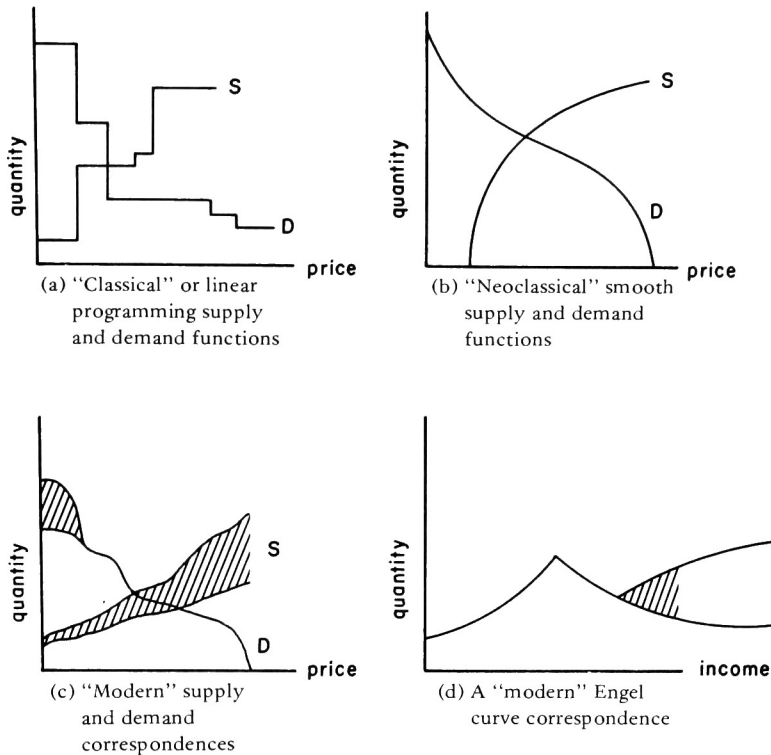
Figure 5. Comparative statics

the smallest problems computers must be used. The trouble and expense of these computations and the corresponding lack of hard, general results in such situations no doubt explain in part the continued vitality of the simpler, better behaved, and less realistic neoclassical models.

The classical work on comparative statics is Samuelson's *Foundations of Economic Analysis* [1948]. He gave the definitive form of the neoclassical parametric optimization and, through the copious exploration of discrete (not infinitesimal) changes and inequalities, anticipated much of the qualitative character of the modern economic structures. Early treatments of parametric linear programming are presented by Simon [1951], Hildreth [1957], and Manne [1956]. Much less has been done in comparative statics for nonlinear and infinite programming, although the very recent work of Araujo, Chichilnisky, and Kalman [1973] promises to provide a breakthrough in this area.

Duality

Our classical predecessors emphasized the fact that use-value was a necessary but not sufficient property for a thing to possess value in exchange. It had also to be scarce or costly to acquire. The neoclassical economists began to unravel the logical mysteries connected with this simple insight. But they failed to unravel them all. It was not until the Kuhn-Tucker theorem for nonlinear programming and the duality theory of linear programming appeared that the essential classical insights on value theory were fully mathematized. The full duality of optimization became evident: as values determine choice, so choice imputes values. Moreover, a resource has economic value only when more of it would allow preferred choices to be made, or when less of it would force acceptance of less preferred alternatives.

In the latter form we see an application of perturbation or comparative static analysis, for one way in which to formalize the duality concepts of value is to study how the value of the best choices varies when one resource at a time is varied slightly. One arrives in this way at the generalized marginal values, shadow prices, Lagrangian or dual variables of general optimization theory, and various versions of the Kuhn-Tucker theorem. Figure 6 (a) illustrates this comparative static view. When constraint one (denoted C1) is perturbed so that the feasible region expands, the best choice shifts from A to B with an increase in the value of the program. Hence, C1 has an imputed value which

(a) Constraint   (b) Tangent and
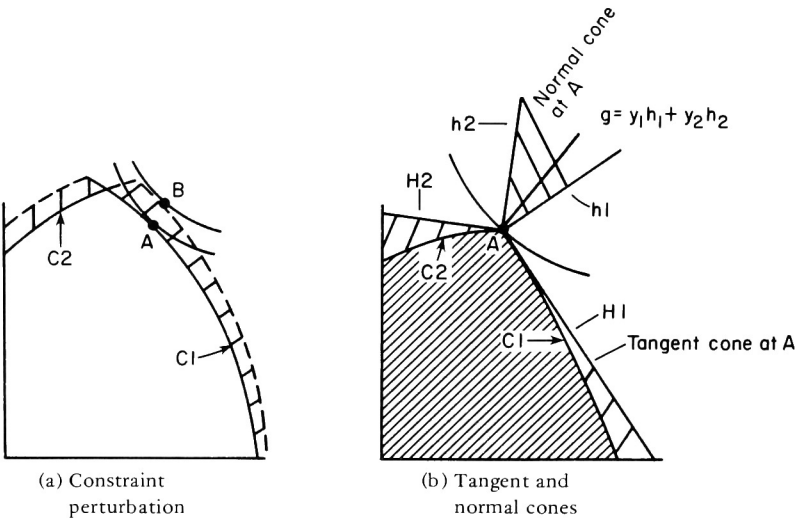   perturbation   normal cones

Figure 6. Two views of duality

is roughly the increment of value divided by the increment by which the constraint is augmented. In contrast, when C2 is shifted, the optimum does not change. Hence, no value is imputed to constraint C2. These imputed values, either positive or zero as the case may be, are, roughly speaking, the partial derivative of the optimal value of the paragram with respect to changes in the limitations or "right-hand-side" coefficients.

An alternative, essentially geometric view of duality is illustrated in figure 6 (b). Here we show an optimum A at which *both* constraints are binding. At this point each constraint possesses a plane of support which is the tangent plane at the optimum point A. The two planes of support are denoted H1 and H2 and are determined by the normal vectors h_1 and h_2. These in turn determine a supporting cone called the tangent cone. It is the intersection of all the half spaces determined by the planes of support containing the feasible region. The gradient g of the objective function, which points in the direction of steepest ascent, can be expressed as a linear combination of the normal vectors that define the supporting cone with weights, say, y_1 and y_2. That is $g = y_1 h_1 + y_2 h_2$. These y's are the dual variables or economic imputations implied by the optimum choice at A.

This geometric point of view brings out in stark relief the relationship between imputed values and the convex shape of the constraints and objective functions of the optimizing problem. Indeed imputed value is difficult to determine or even to interpret in some of the less regular optimization models. In these latter cases little can be said about the possibility or efficacy of decentralized market mechanisms. On the other hand the computation of optima is likewise difficult so that central planning may still be difficult or impossible to carry out in such cases. Procedures more or less the same as trial and error must be invoked.

The references given at the end of the section on optimization models all have good discussions of duality. Much of the contemporary work in duality theory stems from Rockafellar's *Convex Analysis* [1970]. Balinsky and Baumol [1968] supply an elaborate economic exegesis of duality in nonlinear programming, and Leuenberger [1969] gives a good advanced treatment. Aubin [n.d.] provides an extremely general abstract development.

Algorithms

It is often said that modern optimization concepts were given their great impetus by the electronic computer and George Dantzig's simplex method, for it is one thing to know that an optimum exists and quite another to know how to find one economically. The simplex method for linear programming was extended to various quadratic programming models, to mixed integer

programming, and to other examples. Very quickly thereafter various gradient methods appeared for nonlinear programming when the functions were convex or concave. Gradient methods (or methods of steepest ascent) of various kinds were suggested, some of which were built directly on the Kuhn-Tucker theorem and some of which were geometrically motivated.

Implicit in every optimization model whose numerical solution is sought, is the question, "What is the optimum way to find the optimum?" That is, "How can the cost of using a given optimizing model be minimized?" One of the very early discoveries connected with the new simplex algorithm was its astonishing efficiency for general classes of problems. Yet, no one has ever shown it to be the best algorithm for general linear programming problems. Indeed, new modifications and improvements continue to appear, and better ways of finding optima for special types of linear programming models are found in a seemingly unending progression.

The technical issues involved can be illustrated by a smooth, unconstrained minimization model that has a geometric analog, the finding of a lowest point, A, in a valley. Now imagine that the diagrams in figure 7 are the contour maps of this valley. A ball could be released at point 0. If it were propelled solely by gravity, it would presumably follow the path of most rapid descent, a smooth curve as shown emanating from the initial point and minimizing its elapsed time of arrival to the optimum point A. This would be an optimum way of finding the minimum if we evaluate cost as time elapsed. But this path involves a continuous adjustment to the local gradient as the latter varies continuously. And it assumes away inertia. Because of the latter the ball would wander off the optimum path, then veer back and forth across it as shown by the dotted line. The ball would not in fact follow the path of steepest descent but a more or less suboptimal one. Practical numerical methods are somewhat similar to the latter kind of path. Indeed, computation algorithms must be blind to the situation as a whole. They proceed for a time in a given direction generally downward, mistakenly move up, then correct the error and determine a new locally best (but globally suboptimal) direction of descent. The path for such an algorithm is illustrated in figure 7 (b).

It can be stated categorically that optimal algorithms are rare and experience must be used to infer how good a given procedure is and under what conditions a given algorithm works well. This is partly because algorithms for digital computers always involve sequences of relatively simple computations based on purely local information. They begin at some initial, perhaps arbitrary starting point, compute some purely local information that indicates a direction in which a new guess may be chosen to improve on the initial guess, and calculate how far to go in that direction. A new guess is chosen and new local information about neighboring alternatives is computed and the

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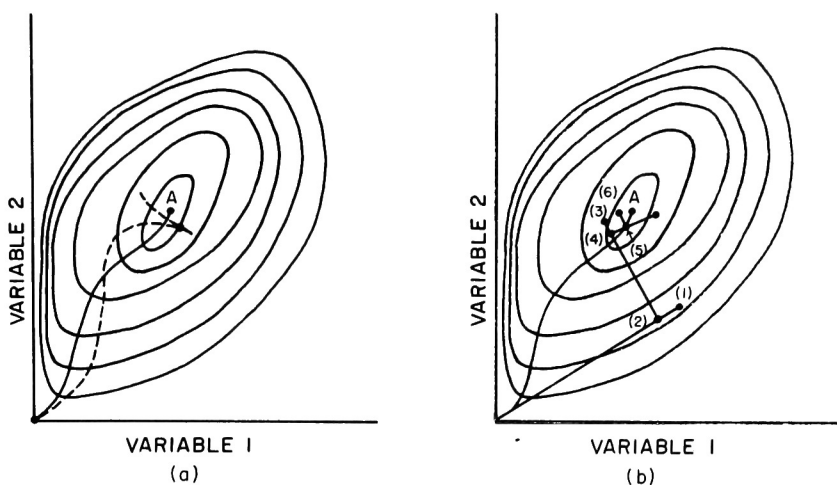(a)   (b)

Figure 7. (a) The "optimal" path to A in the absence of friction and inertia is indicated by the solid line, with inertia it is indicated by the dashed line; (b) the path of a typical computer or learning algorithm

process continues. In this way a sequence of suboptimizations is generated which under favorable conditions converges to a final best solution. The reader familiar with the behavioral economics of Simon [1957] and Cyert and March [1963] should note here the striking similarity between optimizing algorithms and behavioral economics.

Early computational experience with Dantzig's simplex method [1949] is discussed in an interesting manner by Orchard-Hays [1956]. The concept of an optimum algorithm and many examples involving unconstrained problems with one or only a few variables will be found in Wilde [1968]. The sequences of suboptimizations involved in most algorithms would appear to be analogous to the behavior of decision makers in complex organizations and in market economies. This suggests that the study of such algorithms should have considerable interest for economists. The formal mathematical study of algorithms was initiated by Zangwill [1969]. A recent contribution is by Fiacco [1974]. The relationship between otpimizing algorithms and behavioral economics was pointed out in Day [1964] and developed in the context of the theory of the firm in Day and Tinney [1968]. Related articles were prepared by Baumol and Quandt [1964] and Alchian [1950]. The reader interested in computational algorithms for various of the optimization models should find the following references of interest. A complete exegesis of the simplex method is given in Orchard-Hays [1961]. Important early nonlinear programming

algorithms are those of Frank and Wolfe [1956] for the quadratic program-
ming case, the "methods of feasible directions" in Zoutendijk [1960], and
Rosen's gradient projection methods [1960, 1961] for convex (concave) pro-
gramming. Algorithms based on differential equations that converge to the
Kuhn-Tucker conditions and mimic the market process stem from Samuelson
[1949] and include Arrow and Hurwicz [1960] and articles in Arrow, Hur-
wicz, and Uzawa [1958].

Efficiency and Games

The classical notion that many agents simultaneously pursue their several in-
dividual advantages in an economy and that the outcome for each depends on
the actions of all possessed formidable analytical difficulties that were not
fully resolved until Von Neumann's theory of games was developed into a
fundamental working tool for economists by Von Neumann and Morgenstern
[1944] in their famous book and applied by Debreu [1952] in his paper.

In this theory not just one but many utility or objective functions guide
choices so that the optimizing theory as we have reviewed it so far is inade-
quate. Indeed, the notion of "optimum" must be expanded. This has been
done in various ways, but the one central to most work in economics rests on
Pareto's concept of an "efficient" or "Pareto optimal" set of actions in which
no one agent can choose a preferred action without forcing another player in
the game to choose a less preferred alternative.

The theory of games made possible a deeper understanding of many forms
of market competition, as developed, for example, in Shubik's *Strategy and
Market Competition* [1959]. It also became a basic tool in studying the theo-
ry of risky decisions. Games against nature were constructed to formalize the
problem facing a single agent when he could only guess what state his envi-
ronment might take. The application to statistical inference, the scientific
counterpart of this theory, was developed very early by Wald [1945].

But in spite of the extension of optimizing concepts involved in the theo-
ry of games, the close relationship to conventional optimizing theory became
increasingly evident. For example, it was seen that every two-person, zero-
sum game was equivalent to a linear programming model. Kuhn and Tucker
[1951] showed that Pareto optimal solutions to a class of multiobjective op-
timization problems could be characterized by the optimum of a linear com-
bination of those objectives. This quite general duality is at the heart of wel-
fare economics which shows the efficiency properties of competitive equilib-
ria.

The basic idea in this relationship between Pareto optima and conventional
optimization is captured in the diagram shown in figure 8 (a) already familiar

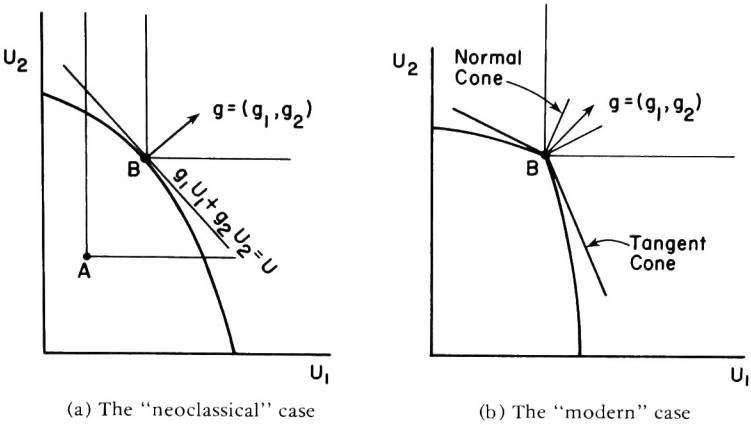(a) The "neoclassical" case   (b) The "modern" case

Figure 8. Efficient choices

to generations of economists. If more of U_1 or U_2 is better than less, then all vectors in the cone emanating from the point A are "Pareto better" or more efficient than the point A itself. But no point in the cone emanating from the point B is feasible except B. Hence B is a Pareto optimum or efficient point with respect to the variables U_1 and U_2.

The set of attainable utility combinations is supported at B by the plane H represented by the vector $g = (g_1, g_2)$. Hence B optimizes a linear combination $g_1 U_1 + g_2 U_2$. Now if U_1 and U_2 are considered to be the satisfaction levels for agents one and two, respectively, then we see how the Pareto optimum B is represented or "supported" by an ordinary optimum. One may also interpret g as the gradient of a social welfare function $\varphi(U_1, U_2)$, which is optimized at point B.

In figure 8 (b) the situation is shown where the set of attainable (U_1, U_2) combinations is convex as in (a) but not smooth. This "modern" case is analogous with the duality diagram for nonlinear programming shown in figure 6 (b). Here the normal cone at point B gives a set of weights (g_1, g_2) so that maximizing $g_1 U_1 + g_2 U_2$ will give back point B as a solution to the implied convex programming problem.

A good discussion of modern optimization and welfare economics is offered in Dorfman, Samuelson, and Solow [1958, chapter 14]. They bring out the point suggested in figure 8 that the efficient or Pareto optimal solutions to the "game" or multioptimization problem can be obtained by means of parametric programming. An early application of this technique is Manne's study [1956] of the United States petroleum refining industry. The concept

of efficient production used by Manne, so closely related to that of the Pareto optimal solutions of a game, was developed by Koopmans [1951]. Standard works on the theory of games in addition to Von Neumann and Morgenstern's classic are the studies by Blackwell and Girshick [1954], Savage [1954], and Karlin [1959]. Elementary expositions are contained in Hurwicz [1945] and Marschak [1946].

Decomposition and Coordination

In the theory of the market economy the relationship between group and individual optima is brought out by showing that a price system exists in such a way that when all agents optimize independently with respect to it then the resulting actions are Pareto optimal and compatible with those prices, i.e., the markets are cleared and all firms and households survive. Koopmans [1957, part I] provides a classic modern nontechnical discussion. The problem of finding the "best" social choice might then be viewed as finding a price adjustment process that will guide a sequence of suboptimizations to an efficient or Pareto optimal point. Such adjustment processes are called tâtonnement processes and represent one general means by which the problem of decomposition of social choice and coordination of individual choices is studied. To be compatible with the requirement of leading to Pareto better solutions, such a process would have to lead to a choice lying in the cone emanating from the initial starting point. Arrow and Hahn [1971] discuss tâtonnement-type models of market processes and Arrow and Hahn [1971] cite earlier work in bibliographical notes.

A second setting in which the relationship between group and individual optima is studied is illustrated by Robinson Crusoe, subject of the most famous parable of the centrally planned economy. A Robinson (or a socialist state) is decomposable into Robinson the consumer and Robinson the producer by means of a price system so that Robinson the consumer can achieve the highest feasible utility at minimum cost and Robinson the producer can maximize his profit of production. This analogy, fully developed by Koopmans [1957], shows that with sufficiently convex technology and preference both the competitive market economy and the socialist economy share the same social equilibria. From this point of view the market is seen to be a device for decomposing the economy's overwhelmingly complex problem of resource allocation into a host of relatively simple, individual suboptimizations which are coordinated by the price system to achieve allocations that are efficient. The idea that marketlike computational procedures could be developed for carrying out central planning was developed by Arrow and Hurwicz [1960].

With this background it is hardly surprising that some of the computer al-

gorithms for solving large-scale complex optimization problems have characteristics similar to those of market tâtonnement or socialist planning processes. In these algorithms the master problem is decomposed into a set of much simpler optimization submodels, one of which plays the role of the coordinator, helmsman, planning bureau, or whatever. Each is solved and the solutions are passed back and forth between them. On the basis of the new information the submodels are reoptimized, and so a sequence of suboptimizations with feedback is generated which, when well conceived, will converge to the solution of the master problem. An early example of such a decomposition procedure was proposed by Dantzig and Wolfe [1961] and applied by Kornai and Liptak [1965]. Kornai [1967] discussed it thoroughly in the national planning setting. Malinvaud [1967] prepared an excellent general discussion of several alternative planning procedures. Lasdon [1970] developed a quite comprehensive text from the computational point of view.

Two fundamental problems complicate the theory and impede progress in its development. One is the formalization of data processing, decision making, and administrative costs and the determination thereby of the optimal level of decentralization. The second is the problem of incentives. Decentralized procedures must be coordinated by an appropriate system of incentives and/or constraints to bring about a compatibility between decentralized optima and the central optimum. Some progress in the former has been made by the developers of team theory, J. Marschak and Radner [1972], who exploit concepts from decision and game theory to formalize the problem of determining optimal decisions and information networks in organizations. This work stems from Marschak's early concern with developing an economic organization theory. Attention to some of the dynamic aspects of such theory is found in T. Marschak [1959, 1968]. The incentive problem has been tackled by Groves [1973].

Multiple Goals

Increasing attention is being paid to the decision problem in which many goals or objectives are pursued. Formally, the problem is much like the n-person game theory in which many objective functions are simultaneously optimized. Not unexpectedly, then, one way of approaching the problem is by means of the efficiency or Pareto optimality concept with which we have already been concerned in several different settings. In particular, the Kuhn-Tucker efficiency theorem mentioned in the section on efficiency and games serves as the basis for an interactive planning procedure involving a planner who has several measurable goals. In this procedure, developed by Geoffrion, Dyer, and Feinberg [1972], a decision maker is asked to specify an initial set

of "weights" g_1 and g_2. An efficient point is found if his goal functions are concave. He is then asked to choose new weights and a new efficient point is found. If the decision maker has a sufficiently regular utility function combining the separate goals or objective functions into a single overall goal, then this iterative sequence will lead to the optimal fulfillment of all the goals. If not, he is left with a number of efficient or Pareto optimal possibilities.

Another approach, suggested by Georgescu-Roegen [1954], is that of lexicographic orderings in which the several goals or objective functions are arranged in a hierarchy. Each is maximized or satiated one after the other until no further scope remains for choice. The relationship of such a procedure to rational choice axioms was investigated by Chipman [1960]. Encarnacion proposed various applications (for example, see Encarnacion [1964]). Day and Robinson [1973] established sufficient conditions for such choice models to be compatible with the requirements of general equilibrium theory. A comprehensive collection involving these and other approaches was assembled by Cochrane and Zeleny [1973].

Risk and Uncertainty

Although the formal study of risk began during the classical era of economics with Bernoulli and Laplace, and though its importance was recognized and accounted for in the neoclassical period by Marshall and Walras and later by Knight [1921] and Hart [1942], its formal treatment by means of optimization theory is of modern origin, at least so far as economic theory is concerned. Early attempts to study the problem mathematically in an economic setting were made by Makower and J. Marschak [1938] and by Tintner [1941]. But it is in Von Neumann and Morgenstern's seminal game theory book and in Savage's fundamental work on decision theory [1954] that the decision-theoretic foundations were definitively established. The Von Neumann-Morgenstern approach is that of *expected utility* and makes it possible to study the "best choice" which accounts for risk using conventional optimization theory.

In brief, probabilities (assumed usually to be subjective in nature) are assigned to states of the world. The utility is then conceived to be a random variable whose expected value is to be maximized by choosing an appropriate act or decision, subject to the constraints of the problem. Risk-averting, risk-preferring, and risk-neutral individuals can be represented in this way and the propensity to hedge, to carry portfolios, or to gamble can be explained. Application in economics are by now widespread and, depending on the specific form of the underlying spaces, the risky decision problem is converted into a linear, quadratic, or more general nonlinear optimization problem. Arrow

[1951] prepared a very comprehensive early survey, and Van Moeseke [1965] published an excellent discussion using modern nonlinear programming theory. Markowitz [1958] took as his subject the analysis of "portfolio" type behavior using quadratic programming. Dillon [1971] wrote a comprehensive review of various approaches.

Bayesian decision theory represents an extension of the Von Neumann-Morgenstern approach to the dynamic setting in which the decision maker faces a sequence of choices. At each stage he may modify his subjective probabilities on the basis of current information. An optimal choice can then be made. This approach represents a true formalization of Knight's concept of uncertainty as opposed to risk, for it explicitly treats the probabilities as unknown. The dynamic nature of the Bayesian point of view is brought out lucidly in J. Marschak's "On Adaptive Programming" [1963]. Various applications to problems in econometrics are developed in Zellner [1971]. Cyert and De Groot [1975] examine its application to the theory of the household.

So far the approaches summarized involve properly accounting for the possibility of doing better or worse than one expects by incorporating into the objective function terms that account for risk and uncertainty. A different component that deserves equal attention is the feasible region, for one outcome of a risky or uncertain decision is the impossibility of carrying out the desired choice. At worst this situation spells disaster, at best it forces a new choice. Again, the specific ways in which this problem has been formalized are numerous, but one must mention all of the "safety-first" and risk-programming procedures—for example, those developed by Charnes and Cooper [1959] and Roy [1952] and reviewed by Sengupta [1969]. Day, Aigner, and Smith [1971] provided an exposition of this approach in the setting of the elementary theory of the firm; in their paper three variants of the approach are discussed—one which minimizes the probability of disaster or maximizes the safety margin (safety), one which maximizes expected utility given a fixed probability of disaster (safety-fixed), and one which maximizes expected utility given that a minimum level of safety (probability of survival) has been reached (safety first). The last approach leads to a lexicographic ordering of survival probabilities and expected utility.

Another approach to the study of decision making under uncertainty is that of game theory where the agent is characterized by a game against nature or where two or more agents, represented by two or more persons in the game, account for the most damaging strategy against them. This approach, founded by Von Neumann [1928], was developed by Von Neumann and Morgenstern [1944]. More recent texts include those by Blackwell and Girshick [1954], Savage [1954], and Karlin [1959]. Elementary expositions are provided by Hurwicz [1945] and J. Marschak [1946].

In all of the above approaches probabilities are explicitly involved. A general principle that need not make explicit use of probability is the "principle of cautious optimizing" outlined elsewhere by Day [1970]. In this approach the decision maker being modeled optimizes in the usual way except that he limits his choices to alternatives "close enough" to a safety zone. The region of "safe enough" solutions can be based directly on a safety metric or "danger distance" instead of a probability of disaster. It therefore generalizes and places on a behavioral footing the idea of safety-first decision making. An alternative but closely related way of modifying the feasible region to account for uncertainty is Shackle's idea of focus loss [1949]. In the form developed by Boussard and Petit [1967] for a firm with a linear programming choice structure, the agent has a focus on loss or disaster level associated with each activity. In addition, the firm has an allowable level of loss usually associated with some minimal survival income. Each activity has an allowable proportion of the total allowable loss, and each activity adds to the allowable loss by increasing the total expected income.

Dynamic Optimization

The role of foresight in decision making can be illustrated by means of a diagram which incorporates several of the fundamental ingredients of dynamic optimization theory. This is done in figure 9. "States of the world" are represented by axis s and acts or decisions (we do not distinguish between the two here) are represented by axis a. Associated with each state is a feasible interval of choices. The interval is determined by a correspondence (see figure 5) in the upper right quadrant. For example, the set of feasible choices associated with the initial state s_0 is that part of the vertical line through the point s_0 that lies in the shaded graph representing the feasibility correspondence. To each act on the upper vertical axis is associated a payoff or outcome as determined by the concave payoff function in the upper left quadrant. For example, π_0 is the payoff associated with the act a_0. The environmental transition is represented in the lower left quadrant and shows how the state changes in response to each act. Thus, if the agent chooses a_0, the succeeding state will be s_1. By measuring the act on the same scale as the payoff we can project the act chosen onto the left horizontal axis, in this way generating a dynamic process.

Now suppose we begin with a rational but myopic decision maker. He knows the feasible region given the state s, and he knows the payoff function. But suppose that he does not know or try to estimate the environmental transition function. Given that he does the best he can in the given situation beginning at s_0, he chooses a_0, which leads to s_1, at which point he picks a_1.
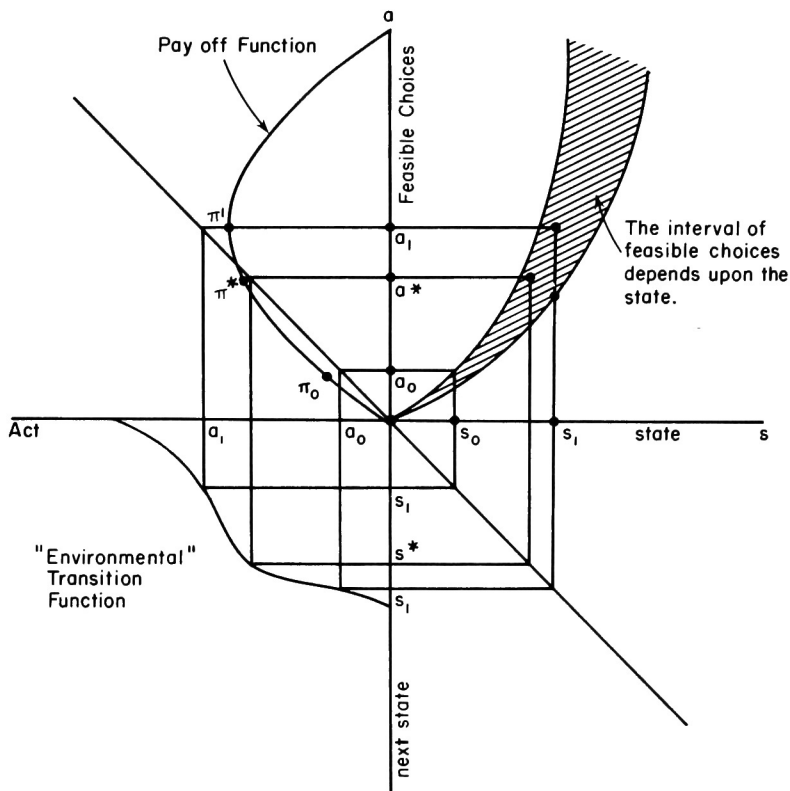
Figure 9. "Short-sighted" and "far-sighted" optimizing

This leads (coincidentally!) back to s_0 and to choice a_0 again. Subsequently an oscillation between a_0 and a_1 occurs. The sequence of payoffs is $[\pi_0, \pi_1, \pi_0, \pi_1, \pi_0, \ldots]$.

Now consider a not-so-myopic individual who knows and takes account of the environmental impact of his actions. Beginning at s_0 he chooses a_0 as before but instead of a_1 at s_1 he picks a^*, realizing that if he picks a_1 as his myopic counterpart did, he will be prevented, because of the environmental feedback, from achieving such a good gain in the next period. His "far-sighted" choice yields a payoff level π^* that can be maintained in perpetuity. Because π^* lies above the chord connecting π_0 and π_1 the average payoff yielded by this far-sighted strategy is better than the myopic strategy.

Of course intertemporal optimization theory encompasses much more

complicated situations than the simple one illustrated, but the basic idea is the same: by taking account of the future consequences of present acts one is led to make choices which, though possibly sacrificing some present payoff, will lead to a preferred sequence of events. To complete the basic ingredients of the theory, a utility function or preference ordering must be specified which will rank feasible alternative time paths such as the two alternative paths just illustrated and determine how much present payoff should be sacrificed for the sake of future enjoyment.

The importance of foresight in economic decisions was noted by our classical predecessors, but widespread application of it by the common man was a possibility about which Smith, Malthus, and Ricardo were hardly sanguine. Later economists, however, realized that the concept of foresight was essential for obtaining a deeper understanding of capital accumulation than they had inherited in classical doctrine. An early breakthrough was Böhm-Bawerk's analysis of time preference [1884], a concept formalized by Fisher [1906] using the newly forged neoclassical theory of preference and utility. The redevelopment of these concepts using modern control theory—for example, as exposited by Arrow [1968] —has led to a huge literature on optimal economic growth, a product of the last two decades. At its basis is an intertemporal utility function of a very restricted nature, the existence of which has been investigated by Koopmans [1960] and Koopmans, Diamond, and Williamson [1964].

When this point of view is extended to the multioptimization problem inherent in general equilibrium theory, one must look for the existence of intertemporally efficient (Pareto optimal) choices and the existence of prices that would permit individual intertemporal optimizations, using these prices to achieve the efficient solutions. The role of such intertemporal efficiency prices has been investigated by Malinvaud [1953]. The neoclassical version of general equilibrium theory from the point of view of intertemporally optimizing firms and households was worked out by Hicks [1939], but his focus was that of temporary rather than dynamic equilibrium. A contemporary line of development that extends the optimal control point of view to the game situation is the work on differential games. A recent collection of studies of this kind (Kuhn and Szegö [1971]) includes papers by Berkovitz, Blaquiere, Friedman, Rockafellar, and Varaiya. An example is by Simaan and Takayama [1974].

The utility aspects of decision making are omitted altogether in on important line of optimal growth theory, namely, the line emanating from Von Neumann's general equilibrium model [1945]. In this theory only the technology of the economy is specified. No time preferences enter the argument. Instead, a technologically maximal rate of growth is defined and its existence

is determined. The existence of "prices" that would support such a rate of growth by profit-maximizing individuals is also established so that a behavioral analog exists in part. Indeed, the possibility that a real economy could follow such an optimal balanced growth path has been investigated by Tsukui [1968]. Dorfman, Samuelson, and Solow [1958] and Koopmans [1964] provide an excellent exegesis of this theory.

Much contemporary work has concentrated on generalizing the individual optimization model to account for information costs, uncertainty and the joint estimation, and control or dual control problems. This line of work had led to a fusion of Bayesian statistical decision theory as developed, for example, by Zellner [1971] and stochastic control theory as described by Aoki [1967]. An influential control theoretic study dealing with the dual control problem of simultaneously deciding and obtaining improved information is Fel'dbaum [1965]. The Bayesian approach involves dynamic programming techniques as developed by Karlin [1955], Bellman [1957], Bellman and Dreyfus [1962], and Blackwell [1967]. A good review of control theory, dynamic programming, and the closely related calculus of variations is given by Intriligator [1971].

Recursive Optimization

The existence of optimal intertemporal strategies and the implications on individuals or economies whose behavior satisfies the conditions of intertemporal optimality have been the focus of most dynamic optimization theory applied to economics in recent years. The question of whether or not individuals of less than heroic stature could in their daily enterprise discover such behavior has only recently begun to receive attention. One way to approach the problem is to break the complex intertemporal optimization problem down into a sequence of much simpler, possibly myopic or relatively short-sighted suboptimizations with feedback. The decision maker does not know the environmental transition equations but merely approximates them, or more simply he forecasts relevant information on the basis of past observations without trying to estimate the structure of the system as a whole. Then, protecting himself from blunders of short-run overcommitment by rules of caution or uncertainty or risky decision making, he optimizes the current situation. When new observations are available, he reestimates and forecasts the relevant information variables and optimizes anew. Thus a sequence of optimizations with feedback is generated which explains actual behavior and which, if the true environment is well behaved, may converge to a path that is intertemporally optimal in some sense, just as a sequence of tâtonnementlike adjustments may lead to a general equilibrium that is efficient or Pareto optimal.

On the other hand, such a convergence may not occur, as in the example with which we introduced the idea of dynamic optimization in the preceding section. It is probably not hard to convince oneself that such convergence does not always occur and perhaps only rarely occurs in the real world.

Sequences of optimizations with feedback are called recursive programs. We have seen that such systems arise not only in the attempt to develop a formal theory of adaptive behavior as just outlined but also in a variety of seemingly quite different settings. We have, for example, observed that mathematical programming algorithms have this structure. We have observed that tâtonnement and decentralized decision processes have this character also. The explicit mathematical representation of economic behavior using recursive optimization originated with Cournot, who used it to investigate the behavior of competing duopolies. Later variants of duopoly theory that preceded the theory of games and Chamberlin's monopolistic competition theory [1948] used an essentially similar type of model. A growth model based on such a principle was stated by Leontief [1958], and a general class of recursive programs was developed by Day and Kennedy [1970] and Day [1970]. Various applications to quantitative modeling of industrial sectors and agricultural regions have been undertaken by Day and others, some of which have been collected in Takayama and Judge [1973] and a number of others in Day and Cigno [in preparation]. Applications to general equilibrium theory are supplied by Cigno [in preparation] and Allingham [1974]. These applications are based on the premise that economic agents' decisions are best characterized by local suboptimizations of partial models of the economy as a whole which are updated and re-solved period after period in response to new information about what other agents have done and what the economy as a whole has done. Like their counterparts in the field of optimization algorithms the recursive optimization models usually cannot be shown to be the best way for the agents to suboptimize. However, some progress has been made in showing that recursive programs based on plausible behavioral hypotheses may converge to the results obtained from the optimal control point of view.

A special and very limited class of recursive programs arises when an optimal strategy can be derived from the dynamic programming point of view which shows how, on the basis of current information and past choices, the next decision can be decided on in the best way. A special group of models falling in this category that have been widely applied in econometrics is the linear decision rules of Holt, Modigliani, Muth, and Simon [1960] and Theil [1964].

On the Normative Content of Optimizing

In reviewing the theory and application of optimization concepts one is struck by the contrasting interpretations given to mathematical programming

and game theory models. On the one hand, an optimization model is formulated to find the "best" solution of some problem. On the other hand, it may be used to characterize the behavior of a real world decision maker. The fallibility of the latter, however, is all too evident to each of us. Moreover, what is "best" clearly rests on a subjective basis—namely, what the agent thinks he likes and wants and what he thinks he can do at the time. These subjective constituents of optimal choice may change whenever something new is learned. How to learn in the optimal way is a problem shrouded in mystery, despite progress in decision theory. What is true of the agents in an economy in general is also true of the model builder and theorist in particular: a solution to an optimizing model is contingent on the structure of the model, something that ultimately rests on the subjective perceptions of mind and on current scientific theories and models which must always be approximations to the real world itself. We thus come to the conclusion that optimality is essentially a logical property of model solutions. Any normative content attributed to optimal solutions must be subjective in character.

Indeed, if one takes into account one's mortal existence and the problem of accommodating the unknown and unknowable preferences of generations yet unborn, one wonders what meaning, if any, the notion of intertemporal optimality has. As the problems associated with global economic development become better known, and as the very-long-run implications of present industrial activity receive increasing attention, this question too is bound to receive more and more attention.

Concluding Remarks

If we were to model our world system microeconomically using optimization theory, we should have to conceive of a game with some three or four billion players, a number growing at a rate of some five thousand per hour. Moreover, the collection of players and its organization into groups of various kinds such as families and firms are variables determined in a complex way by the evolution of the process as a whole. We know also that nature—man's environment—consisting of our nonhuman neighbors, who likewise are evolving in complex living systems, and the physical world must be considered as a player in this game. But what a player! With an uncountable number of strategies at his disposal.

The fact is that none of us takes into account the actions of very many players in this ultimate game. Even if we should like not to do so, we ignore, because we must, all but a few chosen friends and enemies and acquaintances about whom we care or become aware. We account for tiny facets of the universe in our specialized thoughts. And when we turn inward to explore our preferences, we find opening up before us a mystery as infinitely varied and

as unfathomable as interstellar space. We make simplified models of ourselves, solve them, and act. But we throw these models away and start over, or we modify them over and over again. If we fail to do so, our humanity withers and we become like automata in our consistency.

Thus, optimization theories can never yield a complete theory of being or becoming. The man who finds his only poetry in mathematical programs, games, and marginal calculations, who fails to listen to his hunches and feel his senses to the full, who ignores the pleadings of the spirit as it cries out from the works of poets and prophets and painters and from dreams, is less a man.

Still optimization models help us in our battle to create order. The simple clarity of their insights is poetry! Possibly even, their concepts belong to the a priori properties of mind by which, according to Kant, thought's content is defined, by which thought's possibilities and limits are demarcated like fiery poles fixing emblazoned zones in the night. In this case it would be futile for some new doctrine of economy to try to get along without them, just as it would be futile for man to try to get along without science and art in general.

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