Application of Copulas to Analysis of Index Insurance Contracts

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Index Insurance: General Idea

- Losses associated with realizations of one random variable (*primary risk*) are hedged with a contract written on realizations of another random variable (*index*)
- Index is typically
  - well-defined
  - objectively measured
  - "related" to the primary risk variable

Index Insurance: Overview

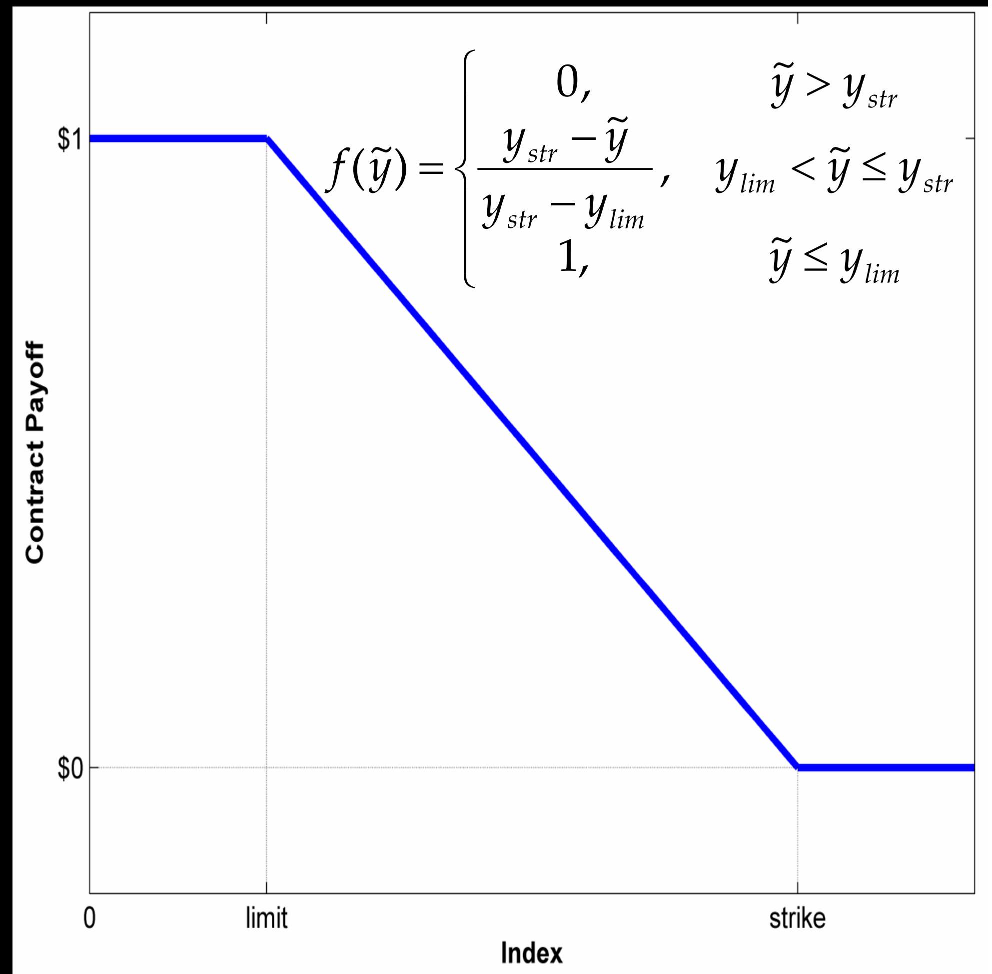
- Area-yield contracts
  - insurance contracts on county yields are used to protect against losses in farm-level yields
- Weather derivatives
  - insurance contracts on realizations of weather variables are used to protect against losses caused by weather
    - yield shortfall
    - energy cost
    - lost revenue (concerts/sporting events/flight cancellations)

Index Insurance: Literature

- Design and pricing
  - Area Yield
    - Miranda (1991); Skees, Black and Barnett (1997)
  - Weather Derivatives
    - Turvey (2001), Mahul (2001)
- Risk-reducing effectiveness
  - Martin, Barnett, and Cobble (2001)
  - Vedenov and Barnett (2004)
  - Chen, Roberts, and Thraen (2006)
  - Deng, Barnett, and Vedenov (2007)

Analysis of Index Contracts: Typical Approach

- An option-type (“standard”) contract is usually considered
- Parameters of the contract are determined by optimizing a measure of risk
- Mean-variance or partial moment analysis is typically applied



Analysis of Index Contracts: Can We Do Better?

- Shortcomings
  - Linear payoff structure may not necessarily be the best
  - Mean-variance analysis is not always consistent with expected utility maximization
- Alternative approach
  - Search for an optimal structure of contract payoff under expected utility framework

Optimal Insurance Contracts: Conventional Insurance

- General problem (Raviv, 1979)

$$\underset{I(\cdot)}{\mathrm{MaxE}}\, U(w - x + I(x) - P)$$

$$\text{s.t. } EV(W_0 + P - I(x) - c(I(x)) \geq V(W_0)$$

- x = loss, $I(x)$ = indemnity, P = premium
- EU = expected utility of insured
- EV = expected utility of insurer
- $c(I(x))$ = cost of insurance

Optimal Insurance Contracts: Conventional Insurance

- Under typical assumptions (Arrow, 1971)
  - $c(I(x)) = lx$ (proportional cost of insurance)
  - $V(\bullet)$ is linear (insurer is risk-neutral)

the optimal contract is co-insurance with a deductible

$$I(x) = \begin{cases} 0, & x \le \bar{x} \\ x - \bar{x}, & x > \bar{x} \end{cases}$$

Optimal Insurance Contracts: Index Insurance

- Mahul (2000), Mahul and Wright (2003)
- Assumptions
  - *c*(*I*(*x*)) = *lx* (proportional cost of insurance)
  - *V*(•) is linear (insurer is risk-neutral)
  - risk variable is a linear function of the index

$$\tilde{x} = \alpha + \beta\tilde{y} + \varepsilon, \quad \tilde{y}, \varepsilon \ \text{independent}$$

  - quadratic utility function

Optimal Insurance Contracts: Index Insurance

- Mahul (2000), Mahul and Wright (2003)
- Results similar to Arrow
  - payoff is zero below or above certain trigger, positive otherwise
- Problem
  - correlation is not a good measure of co-dependence
    - *Example:* $z \sim N(0,1)$ and z^2 are perfectly related, but not correlated

Optimal Insurance Contracts: Index Insurance

- General formulation

$$\underset{I(\cdot)}{\mathrm{Max}}\,\mathrm{E}U = \iint\limits_{\substack{\mathrm{supp}\,x\\ \mathrm{supp}\,y}} u(w + x + I(y) - P)h(x,y)\,dx\,dy$$

$$\text{s.t. } EV(W_0 + P - I(y) - c(I(y)) \geq V(W_0)$$

- *x* = realization of risk variable
- *I*(*y*) = indemnity conditional on realization of the index, $I(y) \geq 0$
- *c*(*I*(*y*)) = cost of insurance
- *h(x,y)* = joint pdf of *x* and *y*

Optimal Insurance Contracts: Index Insurance

- Assuming
  - $c(I(y)) = (1+\gamma)y$ (proportional cost of insurance)
  - $V(\bullet)$ is linear (insurer is risk-neutral)

the problem reduces to

$$\underset{I(\cdot)}{\mathrm{Max}}\,\mathrm{E}U = \int_{y_L}^{y_H} dy \int_{x_L}^{x_H} u(w + x + I(y) - P)h(x,y)dx$$

$$\text{s.t. } P = (1+\gamma)\,\mathrm{E}_y\, I = (1+\gamma)\int_{y_L}^{y_H} I(y)h_y(y)dy$$

where $h_y(y)$ is the marginal pdf of the index

Optimal Insurance Contracts: Index Insurance

- Euler-Lagrange equation reduces to a set of conditions that have to be satisfied at all *y*

$$\begin{cases} I(y) > 0, & \int_{x_L}^{x_H} u'(w + x + I(y) - P)h(x,y)dx - \lambda h_y(y) = 0 \\ I(y) = 0, & \int_{x_L}^{x_H} u'(w + x - P)h(x,y)dx - \lambda h_y(y) < 0 \end{cases}$$

$$P = (1+\gamma)\,\mathrm{E}_y\, I = (1+\gamma)\int_{y_L}^{y_H} I(y)h_y(y)dy$$

$$\lambda = \frac{(1+\gamma)}{P}\int_{y_L}^{y_H} I(y)dy \int_{x_L}^{x_H} u'(w + x + I(y) - P)h(x,y)dx$$

Optimal Insurance Contracts: Index Insurance

- Euler-Lagrange equation does not have a closed-form solution, but can be solved numerically for any $u(\cdot)$
- Key issue
  - joint distribution of the index and risk variable is not known
  - a limited sample of pairwise observations is typically available instead

Copulas to the Rescue

- A two-dimensional copula is a function C: $[0,1]\times[0,1] \to [0,1]$ such that
  - $C(u,0) = C(0,v) = 0$ (grounded)
  - $C(u,1) = u$ and $C(1,v) = v$
  - $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) > 0$ (2-increasing)

Copulas and Distributions

- Sklar's Theorem
  - If $H_x(x)$ and $H_y(y)$ are cdfs of x and y, then for any copula function $C(u,v)$,

    $$H(x,y) = C(H_x(x), H_y(y))$$

    is a joint cdf of (x,y) with margins $H_x(x)$ and $H_y(y)$
  - For any joint cdf $H(x,y)$ with margins $H_x(x)$ and $H_y(y)$, there exists a copula function $C(\cdot,\cdot)$ such that

    $$H(x,y) = C(H_x(x), H_y(y))$$

Copulas and Distributions

- Copula density can be defined as

$$c(u,v)=\frac{\partial^2 C(u,v)}{\partial u \partial v}$$

- Copula density connects marginal and joint pdfs

$$h(x,y)=c(H_x(x),H_y(y))\cdot h_x(x)\cdot h_y(y)$$

Implications for Index Contracts

- Copulas allow one to recover joint distribution of primary risk variable and index from empirical data
- If $\{X_i, Y_i\}_{i = 1,\ldots,n}$ are pairs of observations
  - use $\{X_i\}$ to derive marginal pdf $h_x(x)$ of x
  - use $\{Y_i\}$ to derive marginal pdf $h_y(y)$ of y
  - use marginal pdfs to calculate marginal cdfs
  - use a copula density $c(\cdot,\cdot)$ to combine marginal pdfs into the joint pdf $h(x,y)$

Which Copula to Use?

- A variety of copula functions are available
  - Parametric copulas
    - Gaussian, Student's *t*, Frèchet family, Archimedian Copulas
  - Nonparametric copulas
    - empirical copula, kernel copula
- The "best" parametric copula can be chosen based on MLE or other methods
- Nonparametric copulas provide more flexibility, does not impose parametric assumptions

Kernel Copula

- Estimate joint pdf of observations $\{X_i, Y_i\}_{i = 1,\ldots,n}$

$$\hat{h}(x,y) = \frac{1}{n\tau_x\tau_y}\sum_{i=1}^{n} K\left(\frac{x-X_i}{\tau_x}\right)K\left(\frac{y-Y_i}{\tau_y}\right),$$

- Estimate joint cdf

$$\hat{H}(x,y) = \int_{x_L}^{x} d\xi \int_{y_L}^{y} \hat{h}(\xi,\eta)\,d\eta$$

- Estimate copula

$$\hat{C}(u,v) = \hat{H}(\hat{H}_x^{-1}(u), \hat{H}_y^{-1}(v))$$

Kernel Copula

- If marginal pdfs are also kernel estimates

$$\hat{H}_x(x) = \int_{x_L}^{x} d\xi \frac{1}{T\tau_x} \sum_{i=1}^{n} K\left(\frac{\xi - X_i}{\tau_x}\right), \quad \hat{H}_y(y) = \int_{y_L}^{y} d\eta \frac{1}{T\tau_y} \sum_{i=1}^{n} K\left(\frac{\eta - Y_i}{\tau_y}\right)$$

then the estimated joint kernel density

$$\hat{h}(x, y) = \frac{1}{n\tau_x\tau_y} \sum_{i=1}^{n} K\left(\frac{x - X_i}{\tau_x}\right) K\left(\frac{y - Y_i}{\tau_y}\right),$$

is the joint pdf of *x* and *y* implied by kernel copula

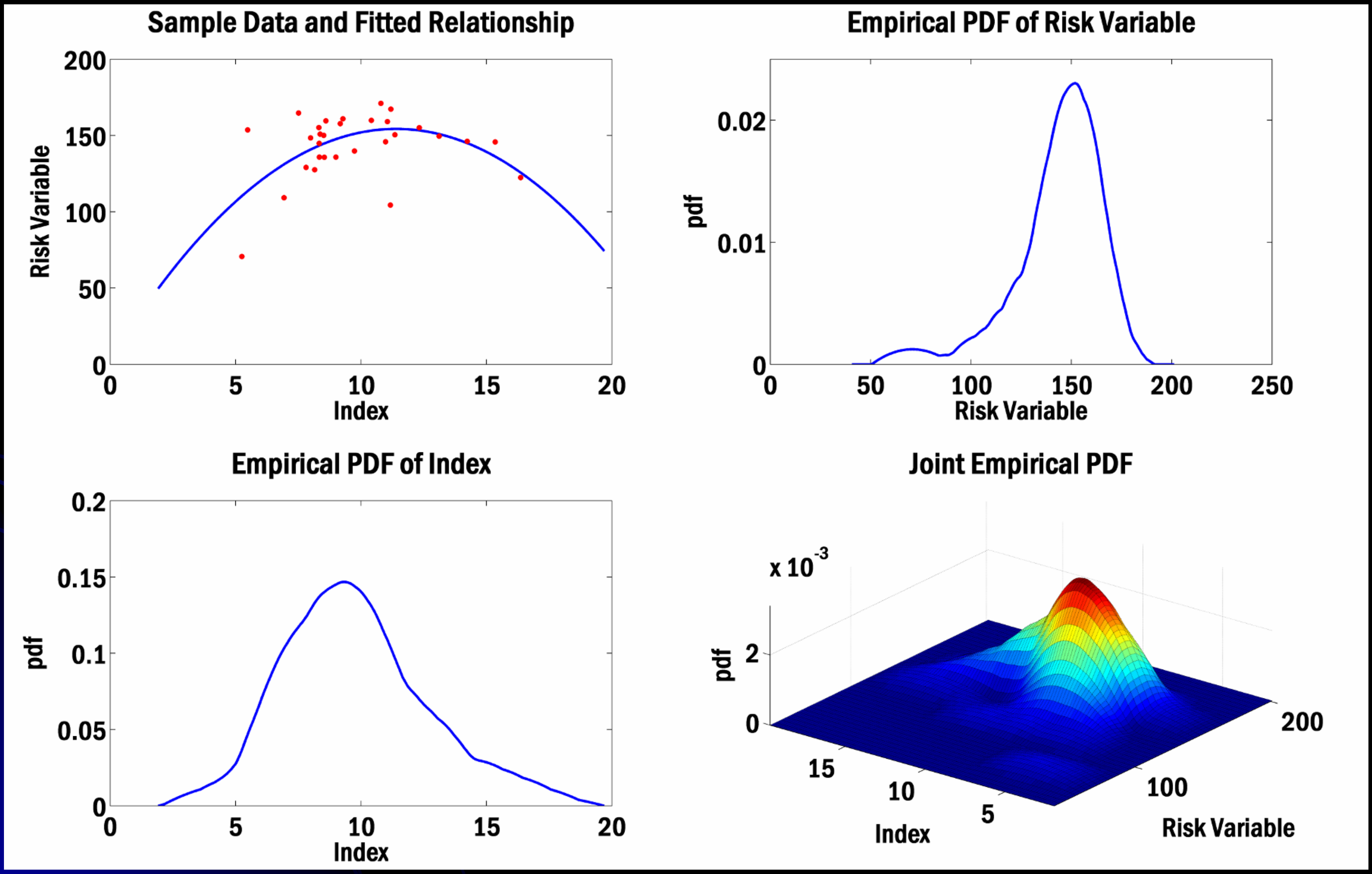
Application

- District-level yields vs. weather variables at nearby weather stations
- Two data series (1972-2001)
  - Illinois corn/June rainfall (CRN1710)
  - Georgia cotton/July temperature (CTN1380)
- CRRA utility function
- Empirical pdf implied by kernel copula

Application

- Three scenarios
  - use index itself as a trigger variable of the "standard" contract
  - regress yield on index and use fitted values as a trigger variable of the "standard" contract
  - calculate optimal contract payoff schedule

Illinois Corn



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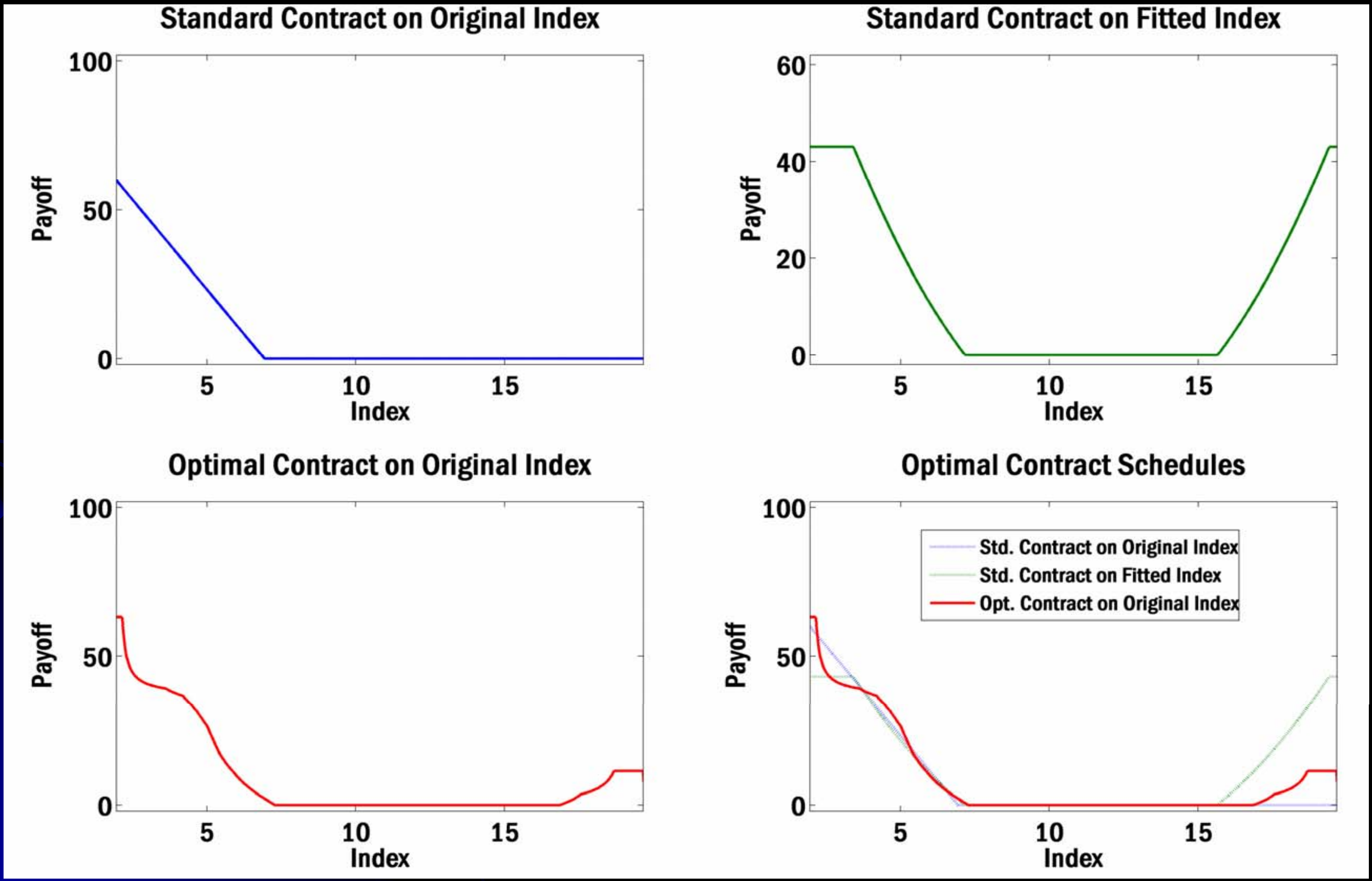


Illinois Corn

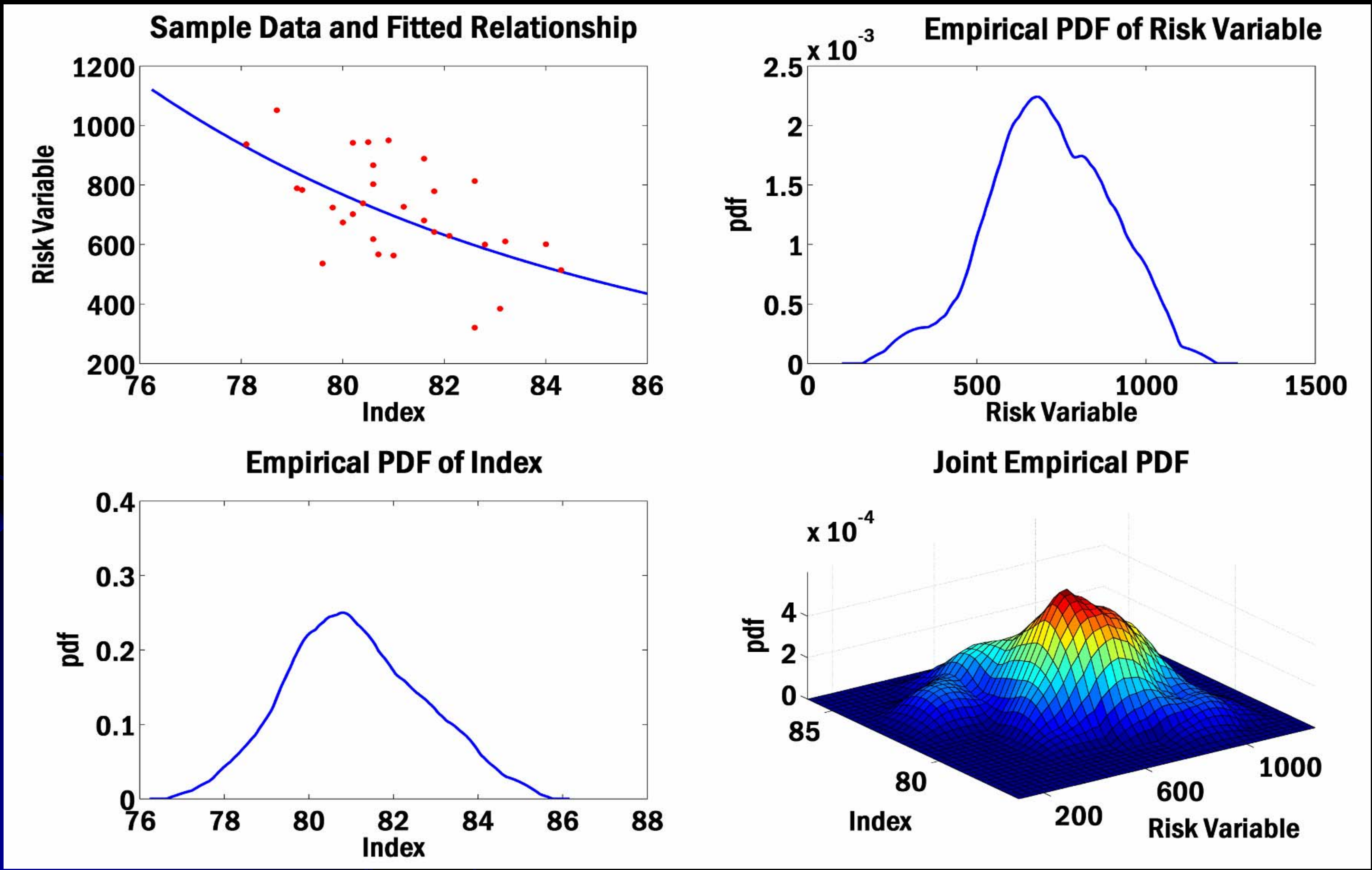
Optimal Parameters of “Standard” Contracts

	Liability	Limit	Strike
Original Index	79.61	0.31	6.93
Fitted Index (quadratic)	43.07	80.02	133.37

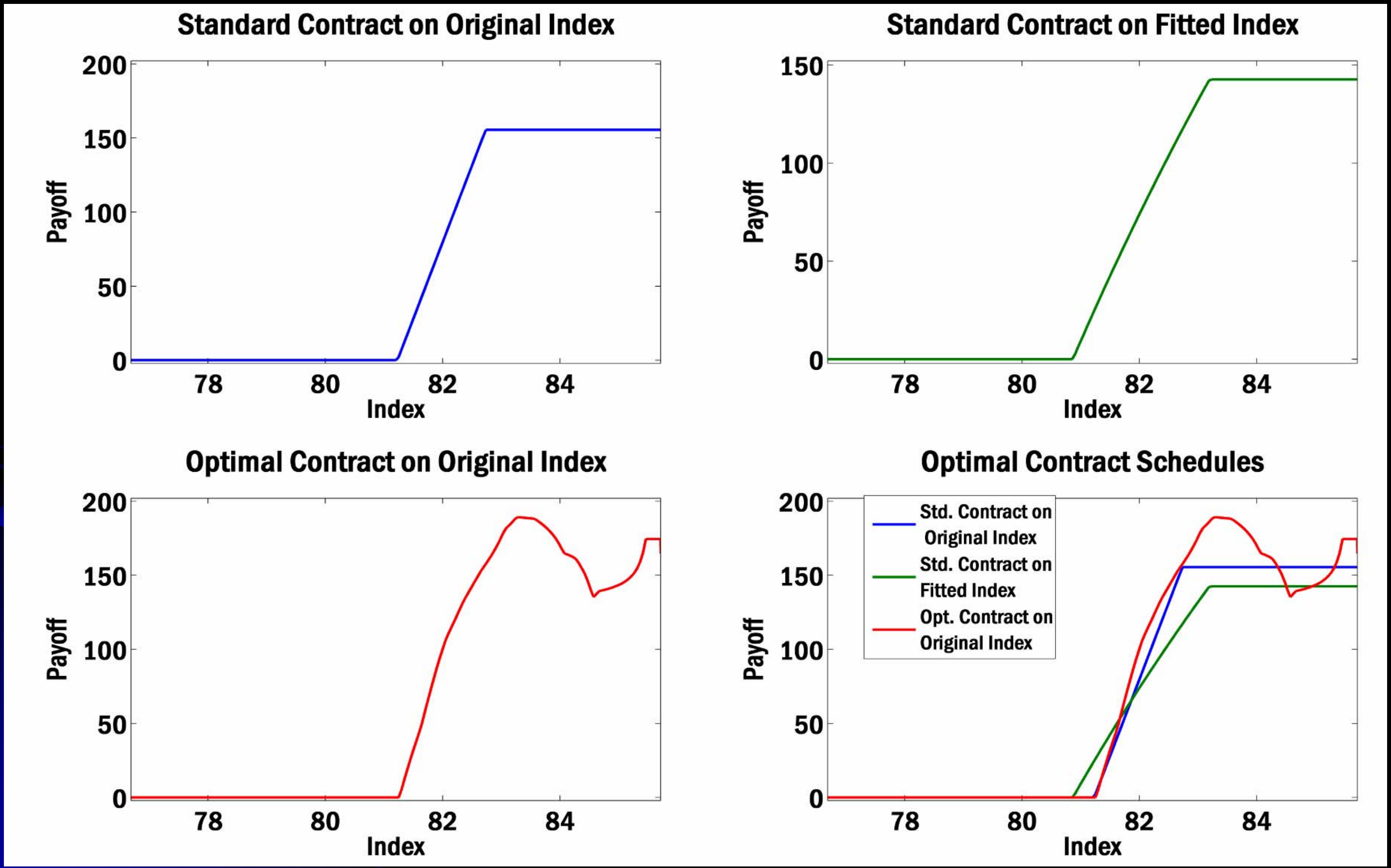
Contract Summary

Contract Type	Expected Utility	Premium
“Standard” on Original Index	–6.6792	2.615
“Standard” on Fitted Index	–6.6812	3.156
Optimal on Original Index	–6.6786	2.742

Georgia Cotton



Georgia Cotton



Georgia Cotton

Optimal Parameters of "Standard" Contracts

	Liability	Limit	Strike
Original Index	155.50	82.73	81.23
Fitted Index (log-log)	142.54	705.4	564.3

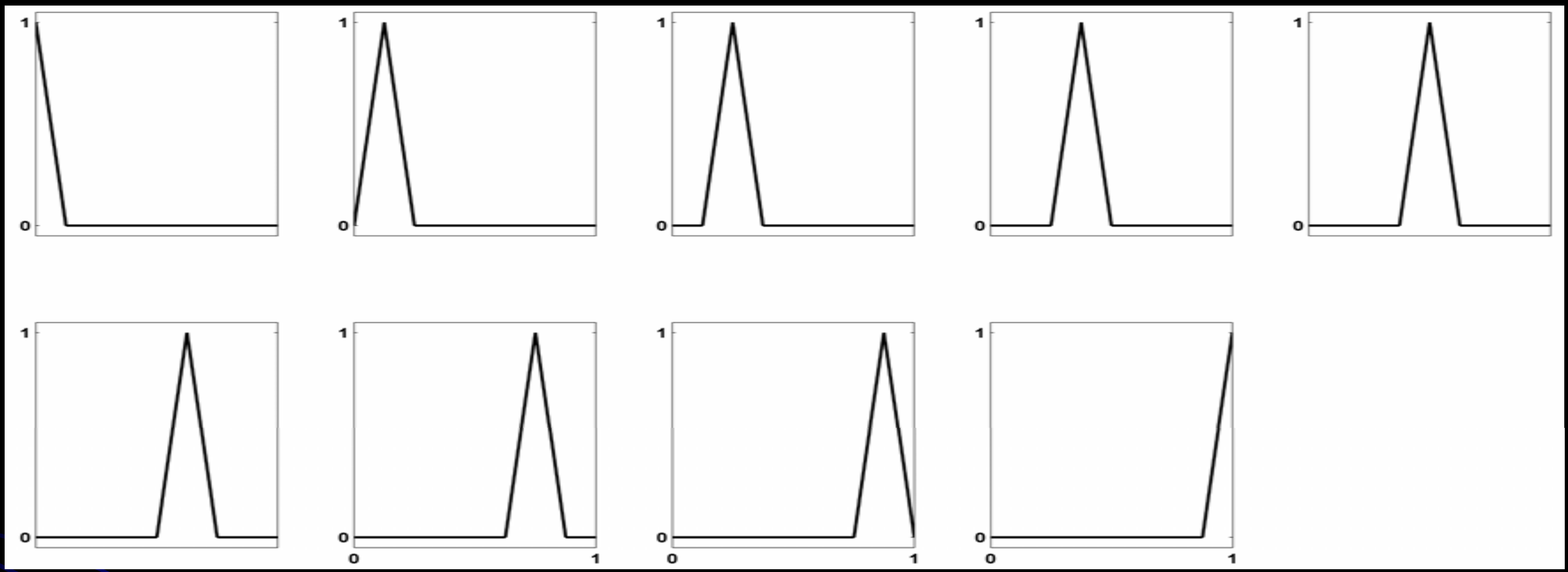
Contract Summary

Contract Type	Expected Utility	Premium
"Standard" on Original Index	–14.7953	50.626
"Standard" on Fitted Index	–14.8080	47.708
Optimal on Original Index	–14.7946	56.855

Practical Implications

- Insurance companies cannot offer contracts with payoffs tailored for each individual risk
- Optimal payoff functions generally cannot be approximated by “standard” contracts
- Possible solution
  - an arbitrary function can be approximated by linear splines

Practical Implications



- Linear splines are essentially "butterfly spreads"
- Insureds can use such "butterfly spread"-like contracts to replicate payoffs of contracts optimal for their specific risks

Conclusions

- An optimal payoff structures can be derived for individual risks, but results are hard to generalize
- A “standard” contract is not the best payoff structure for index contracts
- However, “butterfly spread”-like index contracts can be used as building block to replicate an arbitrary payoff scheme

Conclusions

- Copulas provide a mechanism to estimate joint distribution of risk variable and index using historical data
- Advantages
  - no need to assume linear relationship
  - expected utility framework can be used
- Issues
  - different copulas result in different joint distributions
  - kernel copulas are sensitive to choice of kernels and bandwidth