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A NOTE ON FORMULATION OF PROPORTIONALITY RESTRAINTS IN LINEAR PROGRAMMING BLENDING PROBLEMS†

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A class of restraints on proportions is observed to be relevant in many applications of linear programming, but problem formulation for these restraints has received little attention in the literature. An approach reported by Taylor [5] is summarized, and an alternative approach is outlined. The alternative is equivalent to the first approach but provides simplicity in problem specification and economies in data preparation.

1 INTRODUCTION

Linear programming procedures receive wide application in the solution of blending problems, which are concerned with the selection of a least-cost mix of components which will satisfy some set of specifications. A common agricultural example of this type of problem is the selection of components for a mixed feed for livestock, as illustrated in discussions by Heady and Candler [3], Taylor [5] and Dent and Casey [2].

The specifications associated with a blending problem typically take the form of minimum, maximum or equality restraints for various properties (such as content of protein, energy, minerals and fibre in a feed). In some cases there may be further specifications that the ratios amongst certain properties of the blended product should be within specified limits, should these properties not correspond to the levels specified as minimum (or maximum) content. For example, specifications for livestock feeds commonly refer to the ratios between calcium and phosphorus. Morrison [4] refers to such a requirement for several classes of livestock, as well as to further proportionality problems such as balances between organic and inorganic forms of phosphorus. Dent and Casey [2] provide as a further example, a problem of balance between protein and riboflavin. Standard textbooks on linear programming such as Heady and Candler [3], Dent and Casey [2] and Beneke and Winterboer [1] discuss many aspects of the formulation of problems in a linear programming framework, but do not discuss procedures for representation of this type

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of restraint on proportions. This note reviews a procedure which has been reported by Taylor [5], and outlines an alternative procedure which simplifies problem formulation and reduces the number of matrix entries which need to be specified.

2 TAYLOR'S PROCEDURE

Let (n) be the number of candidates for use as components of a blended product; (X_j) be the quantity of the j-th candidate used in the blend; and (a_{ij}) be the contribution of the j-th candidate to the i-th restriction. For simplicity we assume two restrictions, expressions (1) and (2), below which specify minimum requirements for two properties of the blended product, for example minimum levels of calcium and of phosphorus in a livestock feed.

$$\sum_{j=1}^{n} a_{1j}X_j \geqslant b_1 \qquad (1)$$

$$\sum_{j=1} a_{2j}X_j \geqslant b_2 \qquad (2)$$

Assume further requirements that the content of the two properties lie within the ratios (b_3) to (b_4). Since the left-hand-sides of expressions (1) and (2) measure the two content levels, the conditions for proportionality correspond to expression (3).

$$b_3 \leqslant \left\{ \sum_{j=}^{n} a_{1j}X_j \middle/ \sum_{j=1}^{n} a_{2j}X_j \right\} \leqslant b_4 \qquad (3)$$

Expression (3) may be considered as dual restrictions which, after cross-multiplying, may be rewritten as expressions (4) and (5).

$$\sum_{j=1}^{n} (b_3a_{2j} - a_{1j}) \, . \, X_j \leqslant 0 \qquad (4)$$

$$\sum_{j=1}^{n} (a_{1j} - b_4a_{2j}) \, . \, X_j \geqslant 0 \qquad (5)$$

The combination of minimum restraints and upper and lower bounds on the ratio between the two properties of the product may be expressed in linear programming terms by imposing (1), (2), (4) and (5) as restraints on the solution.

Taylor [5] has reported a similar derivation of this procedure, and has demonstrated its application in a livestock feed-mix problem. The following section of this note outlines an alternative approach which effectively imposes the same restrictions but reduces the volume of data preparation required for implementation of the model.

3 ALTERNATIVE APPROACHES

Application of the method reported by Taylor [5] and discussed above requires computation of the coefficients $(b_3a_{2j} - a_{1j})$ and $(a_{1j} - b_4a_{2j})$ for each column of the linear programming matrix. Should it be considered desirable to consider variations of the problem with different limits on the proportions, these entries would need to be recomputed and new data cards would need to be prepared. Such variations would involve much less problem revision if the expressions (4) and (5) were rewritten as expressions (6) and (7).

$$\sum_{j=1}^{n} (a_{1j}/a_{2j}) \,.\, X_j \geqslant b_3 \tag{6}$$

$$\sum_{j=1}^{n} (a_{1j}/a_{2j}) \,.\, X_j \leqslant b_4 \tag{7}$$

This variation halves the number of coefficients to be computed, and reduces the number of entries to be revised from $(2n)$ to two if the maximum and minimum ratio restraints are to be modified.

A further substantial reduction in number of computations for data preparation, and in the number of non-zero matrix entries, is available if the approach discussed below is adopted.

A linear programming restraint of the form (8) is equivalent to one of form (9) if the activity (D_i) is a cost-free, non-negative measure of the amount by which a minimum requirement (b_i) is over-satisfied: that is, if activity (D_i) corresponds to a disposal or slack activity (with a coefficient of -1.0) for the minimum restraint.

$$\sum_{j=1}^{n} a_{ij}X_j \geqslant b_i \tag{8}$$

$$\sum_{j=1}^{n} a_{ij}X_j - D_i = b_i \tag{9}$$

Assuming that a feasible solution for the linear programming problem is found, the level of the property indicated by the restraint (8) in the product may be defined either as the left-hand-side of expression (8), or as the term $(b_i + D_i)$

Rather than enter the minimum requirements for two properties of a blend in the forms of equations (1) and (2), we may provide an explicit vector for the disposal (slack) columns and enter equality restraints (10) and (11).

$$\sum_{j=1}^{n} a_{1j}X_j - D_1 = b_1 \tag{10}$$

$$\sum_{j=1}^{n} a_{2j}X_j - D_2 = b_2 \tag{11}$$

The proportionality restraints previously stated in expression (3) may be expressed in the alternative form (12).

$$b_3 \leqslant [(b_1 + D_1) / (b_2 + D_2)] \leqslant b_4 \tag{12}$$

The conditions implied by expression (12) may be expressed separately, after simplifying, as (13) and (14).

$$- D_1 + b_3D_2 \leqslant (b_1 - b_2\, b_3) \tag{13}$$

$$D_1 - b_4D_2 \leqslant (b_2b_4 - b_1) \tag{14}$$

The set of expressions (10), (11), (13) and (14) may therefore be used to represent the same set of restraints as is stated by the set (1), (2), (4) and (5), or the set (1), (2), (6) and (7). However, the approach reported in this section may provide a substantial reduction in data preparation and editing work. The extent of this reduction is indicated in Table 1.

TABLE 1: *Comparison of Data Preparation Work Associated With Alternative Procedures**

	Set of Restraints Adopted		
	1,2,4,5	1,2,6,7	10,11,13,14
Number of restraints	4	4	4
Number of columns	n	n	$n + 2$
Number of non-zero coefficients ..	$4n$	$4n$	$2n + 6$
Number of coefficients to be changed if (b_3, b_4) are changed	$2n$	2	4

* Numbers of entries associated with restraints expressing minimum requirements for two properties of a blend, and expressing minimum and maximum limits for the ratios of these properties, for a problem with (n) activities or candidates for inclusion in the blend.

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