THE ASSIGNMENT TECHNIQUE: SOME AGRICULTURAL APPLICATIONS OF A SIMPLE OPTIMIZING PROCEDURE

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1. INTRODUCTION

Practising farm economists and farm management advisers are frequently confronted with problems relating to the optimal allocation of scarce means between competing uses. The importance of such problems in agriculture accounts for the considerable emphasis now placed on programming methods as a means of identifying optimal resource use patterns. This paper is concerned with the special class of allocation problems where the objective is to find the optimal assignment of a number of facilities to an equal number of alternative uses. Each facility may be allocated to only one use, and the performance of each facility in each use must be known. For example, it might be necessary to assign a number of workers to an equal number of jobs with the aim of obtaining the maximum output from the team as a whole.

Assignment problems of this kind can be solved using the transportation technique.[1] Each facility can be regarded as a source with one unit of goods to be despatched, and each use can be regarded as a destination with unit demand. The least cost or maximum profit shipment pattern can then be determined, corresponding to the optimal assignment of facilities to uses. However, an alternative, simpler method of solution is provided by the so-called "Hungarian method" for the assignment problem, as developed by Kuhn.[2]

The assignment technique involves a series of simple arithmetic operations on the $n \times n$ cost matrix. The matrix is modified in an iterative manner until at least one zero occurs in each row and in each column, such that n of these zeros can be selected to give an optimal assignment. The method can best be described with the aid of simple example.

2. AN ILLUSTRATIVE EXAMPLE

A farmer intends to plant four different crops in each of four equal sized paddocks. The paddocks vary in soil fertility and the crops vary in their nutrient requirements, so that the costs of the fertilizers which must be applied depend on which crop is grown in which paddock. The associated cost matrix showing fertilizer costs for each crop in each paddock is given in Table 1.

Since the paddocks are of equal size it may be assumed that the total revenue will be the same whatever assignment is adopted. The farmer's objective in this decision problem is, therefore, to assign paddocks to crops in a manner which minimizes the total fertilizer cost. Mixed cropping of paddocks is not permitted, nor may the same crop be grown in two paddocks.

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[1] See: G. B. Dantzig, "Application of the Simplex Method to a Transportation Problem", Chapter XXIII in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, Cowles Commission Monograph No. 13, (New York: John Wiley, 1951), pp. 359–73.

[2] H. W. Kuhn, "The Hungarian Method for the Assignment Problem", *Naval Research Logistics Quarterly*, Volume 2, No. 1, (March 1955), pp. 83–97.

TABLE 1

Initial Table of Fertilizer Costs ($)

	Paddocks A	B	C	D
Crops I ..	200	300	100	100
II ..	500	800	300	200
III ..	400	900	600	100
IV ..	800	700	800	400

The steps of the solution method are outlined below. The process is illustrated in Table 2.

TABLE 2

Solution Procedure for the First Example

MATRIX 1:

Initial Matrix of Fertilizer Costs ($100's)

				Row Minima
2	3	1	1	1
5	8	3	2	2
4	9	6	1	1
8	7	8	4	4

$n = 4$

First Step. The smallest element in each row of the initial matrix is determined and is then subtracted from every element in its row. This results in a modified matrix (Matrix 2, Table 2) with at least one zero in each row. A number of horizontal and/or vertical lines are now drawn on the modified matrix, passing through all the zeros. The position of the lines is determined by inspection such that the number required t_2 is as small as possible. (It may be noted that there will be a number of alternative ways of drawing the lines which satisfy this condition.) Let the minimum number of lines for the k-th matrix be t_k. The optimal solution is reached when $t_k = n$. Otherwise the next step in the procedure is followed. At the end of the first step in this example $t_2 = 2$. Since this is less than n (i.e., 4), we proceed to Step 2.

TABLE 2 (*continued*)

MATRIX 2:

1	2	0	0
3	6	1	0
3	8	5	0
4	3	4	0

$t_2 = 2 \ (<n)$

Column Minima	1	2	0	0

Second Step. The smallest element in each column of Matrix 2 is subtracted from every element in its column, and the smallest number of lines t_3 that include all the zeros in the new matrix (Matrix 3, Table 2) is again determined. Since t_3 is still less than n, the third step is followed.

TABLE 2 (*continued*)

MATRIX 3:

0	0	0	0
2	4	(1)	0
2	6	5	0
3	1	4	0

$t_3 = 2 \ (<n), \quad c' = 1$

Third Step. The smallest element c' not included in the lines t_3 is subtracted from every element in the matrix (c_{ij}) but is then added back to those elements which lie along one of the lines. Note that for an element c_{ij} lying along one such line,

$$c_{ij} - c' + c' = c_{ij},$$

and for an element lying at the intersection of two lines

$$c_{ij} - c' + c' + c' = c_{ij} + c'.$$

In this case $c' = 1$, so that 1 is subtracted from the nine elements not included in the lines t_3 and is added to the element at the intersection of the two lines in square ID. This step gives rise to the modified matrix numbered Matrix 4 in Table 2. The number of lines required to include all the zeros is now 4, so that $t_4 = 4 = n$ and the optimal solution has been reached. The optimal assignment is obtained by selecting $n = 4$ of the zeros such that there is only one in each row and column, as indicated in Matrix 4. The corresponding assignment of paddocks to crops is shown in Table 3. Had t_4 not been equal to n, this last step would have been repeated until this condition was found to be satisfied.

TABLE 2 (*continued*)

MATRIX 4:

0*	0	0	1
1	3	0*	0
1	5	4	0*
2	0*	3	0

$t_4 = 4 = n$

Optimal solution reached, as indicated by asterisks

TABLE 3

Optimal Solution for the First Example

	Fertilizer Cost $
Assign paddock A to crop I	200
Assign paddock B to crop IV ..	700
Assign paddock C to crop II ..	300
Assign paddock D to crop III ..	100
Total	1,300

3. DISCUSSION OF THE METHOD

The assignment problem can be stated formally as follows. Given an $n \times n$ cost matrix $[c_{ij}]$, where c_{ij} is the individual cost of allocating the j-th facility to the i-th use, find, among all $n!$ permutations of the set of integers $(1, 2, \ldots, n)$, that permutation $(i_1, i_2, \ldots, i_n)$, for which

$$c_{i_1}1 + c_{i_2}2 + \ldots + c_{i_n}n$$

is a minimum.

The algorithm illustrated above depends upon the fact that the subtraction (or addition) of a constant from any row or any column of an assignment cost matrix does not affect the choice of an optimal assignment. This is because the total cost of every feasible assignment must contain one and only one element from every row and every column. Thus, if a constant c' is subtracted from one row or column, the total cost of every assignment of facilities to uses is reduced by c', and the difference in total cost between one assignment on another is unchanged. Therefore an assignment that minimizes the total cost for the modified matrix will also minimize the total cost for the original matrix.

The effect of the arithmetic of the method is to reduce the elements in the cost matrix to their lowest non-negative values. Steps 1 and 2 of the solution procedure together result in a modified cost matrix with at least one zero element in every row and in every column. If we can find an assignment of all facilities to uses among these zero cost elements, we have identified an optimal solution. Such an assignment would have zero total modified cost, and since all the elements of the modified cost matrix are equal to or greater than zero, no other assignment could have a lower cost.

It can prove difficult to determine whether or not there exists such an assignment of all faculties to uses among the zero elements, especially with a large cost matrix. Although fully systematic algorithms to resolve this difficulty are available, they are too complex for inclusion here.[4] Trial and error is usually entirely adequate for the relatively small matrices commonly encountered in practice. The problem is somewhat simplified by drawing lines through the zero elements, as described. It can be shown that if t_k lines can be chosen that include all the zeros, then the largest number of facilities that can be allocated to uses at zero cost is t_k. Thus, so long as $t_k < n$, an optimal assignment of all facilities to uses has not been identified.[5] In such a case, further modification of the cost matrix is necessary. Since there must be some assignment policy which is optimal, there must exist at least one modified cost matrix for which this optimal assignment has zero total modified cost. The effect of step three of the method is to reduce the cost matrix further, creating at least one new zero element.

It should be noted that with $t_k < n$, the minimum element c' is subtracted from a greater number of elements than it is added to. Therefore, so long as $t_k < n$, this third step in the process results in a net reduction to the sum of all the elements. However, when $t_k = n$,

[4] See H. W. Kuhn, *op. cit.* Alternatively, as already noted, large problems may be solved by the transportation technique.

[5] This was demonstrated as long ago as 1916. See D. König "Uber Graphen und ihre Anwendung auf Determinatentheorie und Mengenlehre", *Mathematische Annalen*, Volume 77, (1916), pp. 453–65.

no further reduction in the sum of the elements can be made, since n rows or columns of n elements each must be increased by c' before the same number of $n \times n$ elements can be reduced by c'. Conversely, therefore, when no further reduction can be made to the sum of the elements in a modified cost matrix, there must be at least one feasible assignment of all facilities to uses giving zero modified total cost, and the optimal solution has been reached. Furthermore, since the cost matrix can be reduced in this way at most a finite number of times, an optimum must be reached in a finite number of steps.

4. A MAXIMIZATION PROBLEM

If an assignment problem is one of maximization rather than minimization it can be converted to a minimization problem, amendable to solution in the manner outlined above, by subtracting each element in the initial profit matrix from the largest element. This transformation can be combined with the first step of the optimizing procedure by subtracting each element from the largest element in its row, as illustrated in the next example.

A regional director of an agricultural advisory service is concerned to find the optimal assignment of his staff of five advisers to the five districts in his region. Each district has its special agricultural problems and each adviser has particular strengths and weaknesses in his technical knowledge. The director is able to ascribe a mark out of a possible total of 20 to the degree of success he would expect each adviser to achieve if he were to be posted to each district. The marks awarded are shown in Table 4, Matrix 1. The objective is to assign advisers to districts to obtain the largest possible total mark.

The steps in the solution procedure are shown in Matrices 1 to 5 in Table 4 and the optimal assignment derived is shown in Table 5.

TABLE 4

The Assignment of Advisers to Districts

MATRIX 1:

	Advisers					
Districts	A	B	C	D	E	Row Maxima
I	13	18	14	5	12	18
II	12	6	19	13	11	19
III	8	9	7	7	6	9
IV	2	11	8	3	5	11
V	10	15	7	2	11	15

$n = 5$

TABLE 4—*continued*

Step 1: *Row elements subtracted from row maxima*

MATRIX 2:

5	0	4	13	6
7	13	0	6	8
1	0	2	2	3
9	0	3	8	6
5	0	8	13	4

$t_2 = 2$ ($<n$)

Column minima	1	0	0	2	3

Step 2: *Column minima subtracted from column elements*

MATRIX 3:

4	0	4	11	3
6	13	0	4	5
0	0	2	0	0
8	0	3	6	3
4	0	8	11	(1)

$t_3 = 3$ ($<n$), $c' = 1$

TABLE 4—*continued*

Step 3: *Minimum element subtracted from all elements and added back to rows and columns covered by* t_3

MATRIX 4:

(3)	0	3	10	2
6	14	0	4	5
0	1	2	0	0
7	0	2	5	2
3	0	7	10	0

$t_4 = 4$ (<n), $c' = 3$

Step 4: *Minimum element subtracted from all elements and added back to rows and columns covered by* t_4

MATRIX 5:

0*	0	3	7	2
3	14	0*	1	5
0	4	5	0*	3
4	0*	2	2	2
0	0	7	7	0*

$t_5 = 5 = n$

Optimal solution reached, as indicated by asterisks

TABLE 5

Optimal Assignment of Advisers to Districts

	Marks Awarded
Post adviser *A* to district I	13
Post adviser *B* to district IV	11
Post adviser *C* to district II	19
Post adviser *D* to district III	7
Post adviser *E* to district V	11
Total	61%

5. EXTENDING THE APPLICABILITY OF THE TECHNIQUE

It will be noted that the form of the problems illustrated so far has been somewhat limited, in that the number of facilities has been exactly equal to the number of uses, and it has been assumed that each facility could be allocated to every use. These restrictions can be relaxed by the use of certain computational devices.

Consider the trivial case of a farmer who is concerned to allocate three land-using activities, I, II, and III between three paddocks *A*, *B*, and *C* in a way which maximizes the total gross margin. Assume that activity III may not be undertaken in paddock *C* for some reason such as unsuitability of soil. The initial profit matrix for this problem may be set up as shown in Matrix 1, Table 6.

The method of solution remains the same as that illustrated in the previous example, but the penalty cost imposed on activity III in paddock *C* ensures that this combination is excluded from the optimal assignment.

TABLE 6

The Assignment of Paddocks to Activities

MATRIX 1:

Paddock

Activity	A	B	C	Row Maxima
I	120	150	100	150
II	200	175	80	200
III	140	165	$-\infty$	165

n = 3

TABLE 6—*continued*

Step 1: *Row elements subtracted from row maxima*

MATRIX 2:

30	0	50
0	25	120
25	0	∞

$t_2 = 2\ (<n)$

Column Minima	0	0	50

Step 2: *Column minima subtracted from column elements*

MATRIX 3:

30	0	0*
0*	25	70
25	0*	∞

$t_3 = 3 = n$

Optimal solution reached, as indicated by asterisks

Further flexibility can be achieved in appropriate cases by the introduction of dummy facilities or dummy uses. For example, a farmer might have a set of several buildings available which could be devoted to a number of different uses. Suppose there are four buildings available, *A*, *B*, *C*, and *D*, and five possible uses as follows:

I. store implements
II. store wheat
III. store fodder
IV. keep cattle
V. keep pigs.

Only four of these five possible uses can be accommodated in the available buildings. One use must therefore be foregone, but the farmer insists that accommodation must be provided for cattle. In addition, not all buildings are suitable for every use. Buildings *B* and *D* have entrances too narrow to enable them to be used for the storage of implements, while *A* and *C* are too remote from a water supply to be used for stock. Finally, building *D* cannot satisfactorily be adapted for pig keeping. All these conditions, together with the estimated financial gains from the remaining feasible allocations of buildings to the various uses, are incorporated in the initial profit matrix shown in Table 7. This matrix includes a dummy facility to allow one use to be foregone, and infeasible allocations of facilities to uses are marked with penalty costs. Again, the same method of solution may be employed, an optimum being reached in four steps. This solution is given in Table 8.

TABLE 7

Initial Profit Matrix for Assigning Buildings to Various Uses

	Building A	B	C	D	Dummy
Use I	50	$-\infty$	30	$-\infty$	0
II	220	200	300	275	0
III	160	150	180	175	0
IV	$-\infty$	450	$-\infty$	300	$-\infty$
V	$-\infty$	390	$-\infty$	$-\infty$	0

TABLE 8

Optimal Assignment of Buildings to Uses

	Profit $
Assign building *A* to use III (fodder storage) ..	160
Assign building *B* to use V (pig keeping)	390
Assign building *C* to use II (wheat storage) ..	300
Assign building *D* to use IV (cattle rearing) ..	300
Total	1,150

6. CONCLUSION

Further refinements can be made to assignment models to accommodate a variety of other planning stiuations. For example, subdivision of facilities or uses will allow multiple allocations to be considered. However, the examples described above should be sufficient to demonstrate the scope of the assignment technique and its applicability to decision-making in agriculture. Clearly, the number of real planning decisions which can be formulated in a way which makes them amendable to solution by this method is rather limited. Also, problems of data estimation could be quite severe in practice. On the other hand, the assignment technique has the important advantage that the computations involved are reasonably straight-forward. This would enable the sensitivity of solutions to changes in the assumptions to be investigated relatively easily. Similarly, the value of the technique is enhanced by the fact that it could be used by advisers in the field, or even by farmers themselves. The relative ease with which the method could be applied to solve suitable practical decision problems contrasts favourably with some of the other optimizing procedures currently being advocated for use in agriculture, for which ready access to a computer is usually essential.