THEORETICAL ASPECTS OF A DYNAMIC PROGRAMMING MODEL FOR STUDYING THE ALLOCATION OF LAND TO PASTURE IMPROVEMENT

C. D. THROSBY*

1. INTRODUCTION

The study on which this article is based is hoped to be the first of a series investigating the use of dynamic programming in agricultural allocation decisions. The model presented in this article was constructed in simple form to study once-over sequences of allocation of land to some activity of long-run benefit. The activity chosen was pasture improvement, but as will be seen later the section of the model which describes pasture improvement could be altered to make the model specific to some other relevant activity if desired.

The presentation of this article is as follows: First the nature of dynamic programming as an analytical tool is explained. Then the dynamic programming relationships in the model are constructed and a simple numerical example is given to illustrate the computational procedure. The second part of the model, which derives the pasture improvement return functions, is then shown. Some general computational problems are discussed, following which the properties of both parts of the model are investigated in some detail.

---

* Economics Research Officer, N.S.W. Department of Agriculture.

This article is derived from a section of the author's Master's thesis, *Dynamic Programming and the Economics of Pasture Improvement*, (unpublished M. Sc. Agr. thesis : University of Sydney, July, 1962). Grateful acknowledgment is due to Professor K. O. Campbell for assistance at all stages of the work. Views expressed are the responsibility of the author, who is currently studying at the London School of Economics as Thomas Lawrance Pawlett Travelling Scholar of the University of Sydney.

2. OUTLINE OF DYNAMIC PROGRAMMING PRINCIPLES

Conceptually, allocation decisions can generally be reduced to problems of optimization subject to constraints[1]. Under " rigorous " conditions of optimization, agricultural allocation problems have recently been most often attacked using linear programming. There are some allocation decisions, however, for which linear programming is inappropriate, due to the presence of elements in the decision process such as non-linearities, discontinuities, risk, uncertainties, etc. These factors may make it difficult to rationalize the conventional framework of assumptions within which linear programming operates, and the analyst may thus be obliged to seek more suitable algorithms, such as the extensions of linear programming or other branches of mathematical programming. One possibility is dynamic programming which is of particular interest if non-linear and discontinuous relationships are important and/or if sequential decisions are involved.

The peculiar class of decision situation with which dynamic programming is most concerned is that involving multi-stage decision processes. For definitional purposes, consider a physical system whose state at time t is specified by a function $F(x_1, x_2, \ldots, x_m)$ of the set of m parameters x_i. These parameters are called the *state variables.* During the course of time the system is subject to change which is manifested by a sequence of *transformations* of the state variables. If we have power of choice over the transformations which may be applied to the system at any time, the process becomes a *decision process* and the effect of a single *decision* is to cause a single transformation of the parameters. If a sequence of decisions is involved, then the set of transformations describes a *multi-stage decision process.* The choice of transformations is governed by the desire to optimize some function of the final state variables, the *objective function.* Let us define also a *policy* as a feasible sequence of decisions and an *optimal policy* as one which optimizes the objective function. The multi-stage problems with which we are concerned are those in which the outcome of preceding transformations may be used in determining the course of future transformations.

It should be realized at once that dynamic programming does not offer a standard algorithmic method for solving sequential decision problems. Rather it provides a conceptual framework within which a problem can be formulated[2]; the dynamic programming formulation often allows a solution to be reached where in many cases this would not be possible if the problem were being analysed by more " conventional " tools such as calculus, linear programming, routine search, etc. These methods are often useless in multi-stage situations because they are subject to severe limitations on the size of the problem which they can handle as the number of stages is increased, even given unlimited assistance from the fastest electronic computers. The basic reason for dynamic programming's superiority here is that instead of evaluating one specific problem, it imbeds the process under consideration in a family of similar problems. For instance in an N-stage maximization

[1]Indeed, Arrow has remarked that the whole area of decision theory " is mathematically, to a great extent, a branch of the theory of constrained maxima "; see K. J. Arrow, " Decision Theory and Operations Research ", *Operations Research,* Vol. 5, No. 6 (December, 1957), p. 765.

[2]As Bellman says: " Dynamic programming is not so much any fixed set of analytical techniques as a state of mind "; see R. Bellman, *Adaptive Control Processes: A Guided Tour* (Princeton: Princeton University Press, 1961), p. 59.

problem involving a decision p_k over a one-dimensional region at each stage (a series of decisions $p_1, p_2, \ldots, p_N$), " traditional methods " view the process as a single N-dimensional maximization problem; dynamic programming treats this situation as a series of N single-dimensional maximizations. In such a process the effect of decision p_1 is to cause a transformation of the state variables and reduce the N-stage process to an $(N - 1)$ stage process; p_2 causes a further transformation and reduces the number of stages to $(N - 2)$, and so on, to p_N, which is a decision corresponding to a one-stage process. Now, maximization over only one dimension is usually a fairly simple computing matter. Thus, by considering p_N *first* we can derive for the last stage of our N-stage process an optimal decision set corresponding to all likely conditions of the state variables at the end of the second-last stage. Then, moving backwards through time, we can compute at each stage the (single-dimensional) maximization which optimizes the combined effects of the current decision p_k and the already-known results of following an optimal sequence of decisions for the remainder of the process. Mathematically the principle used at each stage is expressed as a *recurrence relation*, which connects decision p_k to the optimized decision sequence for however many stages remain[3].

This concept may be stated formally in terms of Bellman's " Principle of Optimality ": *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*[4] This principle is constantly used in deriving basic dynamic programming equations.

It should be noted that the explanation given above is orientated towards the specific type of allocation problem under consideration here. The technique is not limited to the one computational scheme suggested above, nor must a problem be necessarily " dynamic ", in the temporal sense, for it to be amenable to dynamic programming analysis. For example, a series of *activities* in a single period could be regarded as a series of *stages* for the purposes of a dynamic programming formulation.

[3]Richard Bellman of the RAND Corporation, who coined the term " dynamic programming " and was responsible for formalizing most of the theory behind it, has amassed a large volume of publications on the subject, many of which contain explanations of the basic principles. Of these the most readily understandable are probably " Some Problems in the Theory of Dynamic Programming ", *Econometrica*, Vol. 22, No. 1 (January, 1954), pp. 37-48 and *Dynamic Programming*, RAND, P-787, (January, 1956). Those who found Bellman's Book *Dynamic Programming* (Princeton: Princeton University Press, 1957), heavy going will doubtless welcome the greater lucidity of the more recent book, R. E. Bellman and S. E. Dreyfus, *Applied Dynamic Programming*, (Princeton: Princeton University Press, 1962), which is to be recommended as an authoritative and comprehensive introduction. With the recent increase in interest in dynamic programming, explanatory works have begun to appear, for example, as chapters in operations research textbooks; (see, for instance, S. E. Dreyfus, in R. L. Ackoff, ed., *Progress in Operations Research, Vol. 1*, (New York: Wiley, 1961), pp. 211-42; and M. G. Simpson *et al.*, in B. T. Houlden, *ed.*, *Some Techniques of Operational Research*, (London: English Universities Press, 1962), pp. 56-74). For more detailed lists of dynamic programming publications, reference may be made to a number of bibliographies, such as RAND, *Index of Publications*, (Santa Monica, 1959) pp. 36-41, and *Supplement*, 1961; V. Riley and S. I. Gass, *Linear Programming and Associated Techniques*, (Baltimore: Hopkins, 1959); J. H. Batchelor, *Operations Research: an Annoted Bibliography*, (St. Louis: St. Louis University Press, 1959); and Case Institute of Technology Operations Research Group, *A Comprehensive Bibliography on Operations Research* (New York : Wiley, 1958).

[4]Bellman and Dreyfus, *op. cit.*, p. 15.

Already dynamic programming has been applied to a number of problems in the physical and social sciences[5]. It has been found capable of handling a number of special-case situations such as non-linear[6] and integer[7] problems. More specifically, in economics[8] it has been applied to problems of allocation,[9] inventory optimization[10] (such as warehousing,[11] stock-control[12] etc.) smoothing,[13] bottleneck,[14] and routing,[15] and to problems involving sampling,[16]

---

[5]The fundamental way in which dynamic programming views sequential processes is intuitively obvious, and it seems that several writers have recognized this without the aid of Bellman's formalism. See for example W. C. White " The Determination of an Optimal Replacement Policy for a Continually Operating Egg Production Enterprise ", *Journal of Farm Economics*, Vol. XLI, No. 5, (December, 1959), pp. 1535-45 who, whilst citing Bellman's work, does not adopt his terminology, and R. Schlaifer, *Probability and Statistics for Business Decisions*, (New York: McGraw-Hill, 1959), p. 596 ff who calls the process " backward induction ".

[6]E. L. Peterson, *Statistical Analysis and Optimization of Systems*, (New York: Wiley, 1961) p. 152 ff.

[7]G. B. Dantzig, " Discrete-Variable Extremum Problems ", *Operations Research*, Vol. 5, No. 2, (April, 1957), pp. 266-277 and R. Bellman, " Comment on Dantzig's Paper on Discrete Variable Extremum Problems ", *Operations Research*, Vol. 5, No. 5 (October, 1957) pp. 723-4.

[8]H. M. Wagner and T. M. Whitin, " Dynamic Problems in the Theory of the Firm ", *Naval Research Logistics Quarterly*, Vol. 5, No. 1 (March, 1958) pp. 53-74 and A. B. Bishop and T. H. Rockwell " A Dynamic Programming Computational Procedure for Optimal Manpower Loading in a Large Aircraft Company ", *Operations Research*, Vo. 6, No. 6, (November-December, 1958), pp. 835-48.

[9]Many of Bellman's descriptive articles use an allocation problem as an explanatory example, for example R. Bellman, *Dynamic Programming*, (Princeton: Princeton Univ. Press, 1957), Chap. 1, and R. Bellman, " Some Applications of the Theory of Dynamic Programming—A Review ", *Journal of the Operations Research Society of America*, Vol. 2, No. 3, (August, 1954), pp. 275-88; see also S. Vajda, *Mathematical Programming*, (Reading, Mass.: Addison-Wesley, 1961), Chap. 13, pp. 245-51 and D. W. Miller and M. K. Starr, *Executive Decisions and Operations Research* (Englewood Cliffs: Prentice Hall, 1960), pp. 328-34 for numerical examples.

[10]M. Beckmann, " Economic Applications of Dynamic Programming ", Paper presented to Econometric Society Meeting, December, 1959, abstract in *Econometrica*, Vol. 28, No. 3 (July, 1960), p. 693; R. Bellman and S. E. Dreyfus, *On the Formulation of Dynamic Programming Problems*: *I*, RAND, RM-1888 (April, 1957) Chap. 3, pp. 7-11; M. Sasieni, " Dynamic Programming and Inventory Problems ", *Operational Research Quarterly*, Vol. 11, No. 1-2, (March-June, 1960), pp. 41-9; and A. S. Manne, *Economic Analysis for Business Decisions*, (New York: McGraw-Hill, 1961), Chap. 9, pp. 136-54.

[11]M. Sasieni, A. Yaspan and L. Friedman, *Operations Research—Methods and Problems*, (New York: Wiley, 1959) pp. 274-9.

[12]E. Ventura, " Application of Dynamic Programming to the Control of Stock and to the Calculation of a Maximum Stock Capacity ", *Operational Research Quarterly*, Vol. 12, No. 1, (May, 1961), pp. 66-78.

[13]R. Bellman, *Dynamic Programming and its Application to Variational Problems in Mathematical Economics*, RAND, P-796, (April, 1956) and R. Bellman and S. E. Dreyfus, *On the Computational Solution of Dynamic Programming Processes*, RAND, RM-1749.

[14]R. Bellman, " Bottleneck Problems, Functional Equations and Dynamic Programming ", *Econometrica*, Vol. 23, No. 1 (January, 1955), pp. 73-87.

[15]R. Bellman, *On a Routing Problem*, RAND, P-1000, (December, 1956).

[16]R. Kalaba, " Optimum Preventative Sampling via Dynamic Programming ", *Operations Research*, Vol. 6, No. 3, (May-June, 1958) pp. 439-40.

reliability,[17] and replacement.[18] Judging by the success of these exploratory applications, the potential of dynamic programming appears to be very extensive. Vajda has called it a new discipline offering " truly astonishing scope "[19]. In an agricultural context Candler and Musgrave have said " it is anticipated that just over the horizon there are a host of new applications of dynamic programming "[20] some of which we have discussed elsewhere[21]. One such agricultural application has appeared recently[22].

Let us examine briefly some further features of dynamic programming which make it attractive in studying economic optimization problems.

(*a*) The dynamic programming approach is readily adaptable to the incorporation of *stochastic variables*, into the basic model. It is a well-known characteristic of many agricultural processes (including pasture improvement) that the future behaviour of some important physical and economic variables (e.g. rainfall, prices) is not known with certainty, but might be specifiable as a probability distribution. When the functions in a dynamic programming model include stochastic quantities, the effect of a decision may be to determine not a definite outcome, but a probability distribution of outcomes. In such cases the *expected* value of the objective function is optimized.[23]

(*b*) The dynamic programming method can also make some allowance for uncertainty—processes in which the quantitative specifications of the system are not completely known to the decision-maker at the outset. When such situations involve the decision-maker in learning more about the system under his control, they are called *adaptive processes*[24].

[17] R. Bellman and S. E. Dreyfus, " Dynamic Programming and the Reliability of Multi-Component Devices ", *Operations Research*, Vol. 6, No. 2, (March-April, 1958), pp. 200-6.

[18] S. E. Dreyfus, " A Note on an Industrial Replacement Process ", *Operational Research Quarterly*, Vol. 8, No. 4, (December, 1957) pp. 190-3; R. Bellman, " Equipment Replacement Policy ", *Journal of the Society for Industrial and Applied Mathematics*, Vol. 3, No. 3, (September, 1955), pp. 133-6; and, in an agricultural context, W. C. White, *op. cit.*, (footnote 5).

[19] S. Vajda, Review of Bellman, *Dynamic Programming*, in *Econometrica*, Vol. 27, No. 3, (July, 1959) p. 538.

[20] W. Candler and W. F. Musgrave, " A Practical Approach to the Profit Maximization Problems in Farm Management ", *Journal of Agricultural Economics*, Vol. XIV, No. 2, (December, 1960) p.215.

[21] C. D. Throsby, " Some Applications of Dynamic Programming in Agricultural Economics ", (Sydney: Statistical Society of N.S.W.: mimeo), May, 1962; and " Some Dynamic Programming Models for Farm Management Research ", *Journal of Agricultural Economics*, Vol. XVI, No. 1, (June, 1964), pp. 98-110.

[22] O. R. Burt and J. R. Allison, " Farm Management Decisions with Dynamic Programming ", *Journal of Farm Economics*, Vol. 45, No. 1, (February, 1963), pp. 121-36.

[23] See, for example, H. A. Simon, " Dynamic Programming under Uncertainty with a Quadratic Criterion Function ", *Econometrica*, Vol. 24, No. 1, (January, 1956), pp. 74-81. This field has been explored extensively with general programming models (e.g., R. J. Freund, " The Introduction of Risk into a Programming Model ", *Econometrica*, Vol. 24, No. 3 (July, 1956) pp. 253-63), and Simon has called the contribution of dynamic programming to this area of study " far-reaching "; see H. A. Simon, " Theory of Decision-Making in Economics and Behavioural Science ", *American Economic Review*, Vol. XLIX, No. 3 (June, 1959), p. 259.

[24] See, for example, R. Bellman, *Adaptive Control Processes: A Guided Tour*, (Princeton: Princeton Univ. Press, 1961) Chap. XV; S. E. Dreyfus, *op. cit.*, (footnote 3), p. 222 ff.; cf. also S. Reiter, " Surrogates for Uncertain Decision Problems: Minimal Information for Decision Making ", *Econometrica*, Vol. 25, No. 2, (April, 1957) pp. 339-45, and K. J. Arrow, " Utilities, Attitudes, Choices: A Review Note ", *Econometrica*, Vol. 26, No. 1, (January, 1958), p. 10 and pp. 13-14.

(*c*) In agricultural situations the discrete nature of the input-output pattern existing for a class of farm resources (such as buildings and machinery) leads to " lumpiness " in production functions and in cost and return functions. An attractive feature of dynamic programming is that it places no restrictions on the nature of the functions used to specify the structure of a system. Whereas linear programming, for example, requires constraints placed on variables to be linear functions, dynamic programming can handle extremes of non-linearity and discontinuity.

(*d*) The repetitive nature of the calculations involved in dynamic programming makes it amenable to computer solution. Since the mechanics of the calculations are identical at each stage, programming a specific problem is simplified.

(*e*) Dynamic programming analyses and solutions are adaptable to *sensitivity analysis* or the study of variation in the structure of the system under observation. This is partly because, as noted above, the dynamic programming formulation obtains the solution to a specific problem via the solutions to a series of related problems. This is advantageous because of the fact that the *structure* of a problem is what interests us most in many analyses: hence we can examine a series of solutions and determine the stability of the general results.[25]

After the encomiastic remarks of previous paragraphs it is as well to mention some limitations of dynamic programming. Its chief drawback is, paradoxically, one of computation; it is limited by the dimensionality of the optimization required *at each stage*. Increasing the dimensionality in this direction causes exponential increases in the computing burden.[26] (In the present work this difficulty was not encountered, since the model involved maximization over only one dimension at each period.) Thus, although dynamic programming can solve some special linear programming problems of the multi-stage variety,[27] it does not threaten to oust linear programming in the solution of problems to which the latter is peculiarly adapted.[28]

A further caution is of course that, like other forms of programming, the economic content of dynamic programming is nil. It is merely a mathematical method, and " can only help us to find the implications of the economic information which we already have or are willing to assume ".[29]

---

[25]cf. R. Bellman, *Dynamic Programming*, (Princeton: Princeton Univ. Press, 1957), p.6.

[26]See further in R. Bellman, *Multidimensional Maximization and Dynamic Programming*, RAND, P-1086, (May, 1957).

[27]cf. G. B. Dantzig, *On the Status of Multi-stage Linear Programming Problems*, RAND: P-1028, (February, 1957); G. Morton, " An Application of Dynamic Programming ", *Conference on Linear Programming, May* 1954, (Ferranti Ltd., *mimeo*, 1954), pp. 32-40; and C. D. Throsby, " Some Notes on ' Dynamic ' Linear Programming ", this *Review*, Vol. 30, No. 2, (June, 1962), p. 125.

[28]See particularly in S. E. Dreyfus, *A Comparison of Linear Programming and Dynamic Programming*, RAND, P-885, (June, 1956) and R. Bellman, " Functional Equations and Successive Approximations in Linear and Non-Linear Programming ", *Naval Research Logistics Quarterly*, Vol. 7, No. 1, (March, 1960), pp. 63-83.

[29]W. J. Baumol, " Activity Analysis in One Lesson ", *American Economic Review*, Vol. XLVIII, No. 5, (December, 1958), p. 837.

3. THE MODEL

Part I: Derivation of Functional Equations

(i) CONSTRUCTION OF DYNAMIC PROGRAMMING EQUATIONS

We can now proceed with the description of the dynamic programming model representing the pasture improvement decision process. The formulation below bears some resemblance to Bellman's basic allocation model already cited.[30]

In this study pasture improvement is construed as an allocation problem in which the farmer is considered as being faced with a multi-stage decision process, at each stage of which he must decide on the amount of land to be allocated to improved pasture. It is hypothesized that he must take into account at each stage the outcomes of decisions at previous stages in the process and the expected results at future stages, and that his feasible choices are subject to restrictions placed on the supplies of other resources such as capital and labour which he has at his disposal.

Let x denote the number of units of land available for pasture improvement at any stage, and y denote the amount of it actually allocated to pasture improvement at that stage, where $0 \leqslant y \leqslant x$. The amount of land available for alternate uses at that stage is thus $(x - y)$, which is also equivalent to the amount of land available for allocation in the next period, since the y units devoted to pasture improvement are not considered available for reallocation during the life of the pasture; i.e.

$$(x_k - y_k) = x_{k+1}$$

where k is used to refer to any stage in a multi-stage process. Now let—

$g(y)$ = net return resulting from an allocation of y units of land to pasture improvement, and

$h(x - y)$ = net return resulting from the remainder, $(x - y)$ units, being devoted to non-pasture.

Let the initial quantity of x available $= x_0$.

Our problem then is simply to maximize

$$(1) \qquad F(x_0, x_1, x_2, \ldots, x_N) = \sum_{k=1}^{N} [\, g(y_k) + h(x_k - y_k) \,]$$

over the region y_k defined by

$$0 \leqslant y_1 + y_2 + \ldots + y_N \leqslant x_0$$

where

$$0 \leqslant y_k \leqslant x_k$$

and

$$(x_k - y_k) = x_{k+1}$$

Rephrasing the objective function of equation (1) according to dynamic programming precepts will yield two *functional equations*[31] (the first for a one-stage process, the second for an N-stage process) which will form the basic

[30]See also R. Bellman, I. Glicksberg and O. A. Gross, *Some Aspects of the Mathematical Theory of Control Processes*, RAND, R-313, (January, 1958), for an exhaustive treatment of this allocation model.

[31]As R. Bellman puts it: " The basic idea underlying our (dynamic programming) analysis is that of replacing the decision problem by a functional equation ". See " Some Problems in the Theory of Dynamic Programming ", *Econometrica*, Vol. 22, No. 1, (January, 1954), p. 37.

objective function system for our model. Thus, by letting $f_N(x)$ represent the *total return from an N-stage process when an optimal policy is used*, and with $g(y)$ and $h(x - y)$ as defined above, we can derive:

$$f_1(x) = \max_{0 \leqslant y \leqslant x} [g(y) + h(x - y)] \tag{2}$$

and the recurrence relation

$$f_N(x) = \max_{0 \leqslant y \leqslant x} [g(y) + h(x - y) + f_{N-1}(x - y)] \tag{3}$$

These two equations express in general form the criterion of our model. Equation (2) is used to derive the one-stage allocation, then repeated application of the recurrence relation yields for any stage k those values of y which maximize the sum of (*a*) the net return from an allocation of y units to improved pasture in stage k; (*b*) the net return from an allocation of $(x - y)$ units to non-pasture in stage k; and (*c*) the total return obtainable from using an optimal policy with respect to the $(x - y)$ units available for reallocation in stage $k + 1$. Overall, we aim to determine the sequence $\{y_k\}$ for $k = 1, 2, \ldots, N^*$, which maximizes $f_{N^*}(x)$, where N^* = total number of stages we want to consider in the whole process.

(ii) A Numerical Example

In order to clarify the nature of the dynamic programming model and to put a recognizable numerical form to the functions involved, a simple worked example is now presented.

(*a*) *The problem:* We are given x_0 acres of land, and a series of $\frac{x_0}{\beta}$ observations (where β represents the " grid size " of the functions $g(y)$ and $h(x - y)$). Determine the allocation sequence $\{y_k\}$ (i.e., the set of functions $y(k)$) which maximizes over an N-year period the sum of the present values of the income streams resulting from the alternative investment strategies.

(*b*) *The data:* Given x_0 = 500 acres

β = 100 acres

N = 3 years

and given the following $\frac{500}{100} = 5$ observations (plus the zero level) of the present values of the return functions g and h:

(Imagine all measurements to be in £'00).

For y =	$g(y)$ =	and for $(x - y)$ =	$h(x - y)$ =
0	0	0	0
100	14	100	1
200	27	200	2
300	38	300	3
400	47	400	4
500	54	500	5

We want to find a sequence (y_1, y_2, y_3) which optimizes the three-stage objective function:

$$f(x) = \sum_{k=1}^{3} [\, g(y_k) + h(x_k - y_k) \,] \tag{4}$$

(*c*) *The solution*: First we determine the function $y(x)$ which maximizes the one-stage functional equation (2). This is achieved by calculating, for each possible value of x, the function $f(y)$, and selecting its maximum. The collation of the values of y which maximize the set of functions $f(y)$ gives us $y_1(x)$ and the collation of the maxima gives us $f_1(x)$. The function $y_1(x)$ represents the optimal allocations for the *last* of our three stages.

Enumerating for $N = 1$, for two of the six possible values of x: (maxima are starred).

For $x = 500$, $y =$	500, and $x - y =$	0, the return $f(y)$ is	$54 + 0 = 54$*
	400	100	$47 + 1 = 48$
	300	200	$38 + 2 = 40$
	200	300	$27 + 3 = 30$
	100	400	$14 + 4 = 18$
	0	500	$0 + 5 = 5$
For $x = 400$, $y =$	400, and $x - y =$	0, the return $f(y)$ is	$47 + 0 = 47$*
	300	100	$38 + 1 = 39$
	200	200	$27 + 2 = 29$
	100	300	$14 + 3 = 17$
	0	400	$0 + 4 = 4$

and so on, for $x = 300, 200, 100, 0$.

When fully calculated the series of observations of $y_1(x)$ and $f_1(x)$ is summarized as:

x	$y_1(x)$	$f_1(x)$
500	500	54
400	400	47
300	300	38
200	200	27
100	100	14
0	0	0

We are now ready to compute the second iteration which uses, as will any subsequent iterations, the recurrence relation from equation (3). The last term of this equation is the function $f(x)$ calculated at the previous iteration. Enumerating, again for only two of the six values of x:

For $x = 500$, $y =$	500 and $x - y =$	0 the return $f(y)$ is	$54 + 0 + 0 = 54$
	400	100	$47 + 1 + 14 = 62$
	300	200	$38 + 2 + 27 = 67$
	200	300	$27 + 3 + 38 = 68$*
	100	400	$14 + 4 + 47 = 65$
	0	500	$0 + 5 + 54 = 59$

For $x = 400$, $y = 400$ and $x - y =$	0 the return $f(y)$ is	$47 + 0 + 0 = 47$
300	100	$38 + 1 + 14 = 53$
200	200	$27 + 2 + 27 = 56$*
100	300	$14 + 3 + 38 = 55$
0	400	$0 + 4 + 47 = 51$

and so on, for $x = 300, 200, 100, 0$. Summarizing for $N = 2$, we have:

x	$y_2(x)$	$f_2(x)$
500	200	68
400	200	56
300	100	43
200	0, 100	29
100	0	15
0	0	0

Another application of the recurrence relation yields the best initial allocations for a three-stage process, and since we are only considering a three period situation, this is the final iteration. The solution for $N = 3$ is:

x	$y_3(x)$	$f_3(x)$
500	100	74
400	0, 100	60
300	0	46
200	0	31
100	0	16
0	0	0

The overall results are best summarized as in Table 1. This table is read as follows: Starting with $x = 500$, the best allocation is $y = 100$ in the first period (read from line 1), leaving 400 for reallocation in period 2. This is best deployed by allocating 200 acres (read from line 2), leaving 200 acres, all of which should be allocated in period 3 (read from line 4). The return achieved by this course of action is 74 units read from the list of $f_3(x)$ above. If there were only 400 acres to begin with, Table 1 indicates two optimal policies, viz. 0, 200, 200 or 100, 100, 200, each leading to a return of 60.

TABLE 1

Solution to Allocation Example

x at Start of Period (Acres)	Optimal Allocation of y: acres		
	Period 1	Period 2	Period 3
500	100	200	500
400	0, 100	200	400
300	0	100	300
200	0	0, 100	200
100	0	0	100
0	0	0	0

(iii) Dynamic Aspects of the Model

The already dynamic nature of this model is complicated by the fact that the returns from an allocation of y in any period are generated not only in that period, but also in as many succeeding periods as constitute the pasture's productive life. Let $t = 1, 2, \ldots, T$ denote the stages in the life of a pasture, measured in some convenient units, such as years; t is a *relative* measure of time, i.e., related to the date of pasture establishment. Suppose also that we fix *absolute* time by defining the stage in an overall N-stage process by the subscript k. A farmer following a feasible policy might allocate y_1, y_2, y_3, ... acres to pasture in periods $k = 1, 2, 3, \ldots$. Thus at any stage k (where $k > 1$), there may exist on his farm pastures at various stages (t) of development. For instance, in year $k = 3$ there might be pastures on the farm for which $t = 3$ (sown in $k = 1$), some for which $t = 2$ (sown in $k = 2$), and so on.

Since we are using present value as the measure of the profitability of investment in pasture improvement, we must discount future money flows at an appropriate discount rate. This is achieved in our generalized model by discounting *relative to t*, i.e. using the assumption that the farmer engaged in a sequential decision process will view the future relative to the time each decision has to be made, and not relative to the very beginning of the overall process.[32] The usual procedure is employed for calculating the present value of the investment, [33] although we make use of the term *discount factor*, represented by d_t; any monetary value in period t is reduced to its present value in $t = 1$ by multiplying it by d_t[34]; i.e. if $d^* =$ discount rate,

$$d_t = \frac{1}{(1 + d^*)^{t-1}} \tag{5}$$

It is assumed that the discount rate is constant, although in some cases there may be special grounds for considering discount rates which fluctuate continuously[35] of discontinuously[36] with time.

[32]There were two main reasons for adopting this form of discounting: (*a*) It fits a conceptual framework in which the farmer is assumed to make decisions at a series of discrete stages, rather than by one overall decision: (*b*) It permits greater generality in interpreting empirical results, since a particular programming run does not have to be committed beforehand to set a number of stages. In any case, empirical testing of both methods showed little difference in patterns of solution between the two methods, even for relatively high discount rates.

[33]viz.

$$PV = NR_1 + \frac{NR_2}{(1 + d^*)^1} + \frac{NR_3}{(1 + d^*)^2} + \ldots + \frac{NR_T}{(1 + d^*)^{T-1}}$$

Where $PV =$ Present Value.

$NR_t =$ Net Return in period t ($t = 1, 2, \ldots, T$).

$d^* =$ discount rate.

[34]c.f. Hicks use of the term *discount ratio*, given as

$$\frac{1}{(1 + d^*)}$$

see J. R. Hicks, *Value and Capital*, (Oxford: The Clarendon Press, 1957), p. 185.

[35]See, for example, R. H. Strotz, " Myopia and Inconsistency in Dynamic Utility Maximization ", *Review of Economic Studies*, Vol. XXIII, (1955-6), p. 175.

[36]c.f. M. W. Hoag, " The Relevance of Costs in Operations Research ", *Operations Research*, Vol. 4, No. 4, (August, 1956) pp. 457-8; for some empirical data on farmers' views of the future, see J. A. Boan, " A Study of Farmers' Reactions to Uncertain Price Expectations ", *Journal of Farm Economics*, Vol. XXXVII, No. 1, (February, 1955), pp. 90-5.

A word might now be said about additivity of $g(y)$, (which is derived over T periods), and $h(x - y)$, (which is derived over only one period). It is a common and justifiable practice in general programming problems to include in the objective functions terms derived over different time periods provided they are expressed in summable units: witness, for example, the additivity of " net cash returns " in the objective function of a dynamic linear programme,[37] where different activities cover different numbers of stages. In our model the resource, land, once allocated, becomes " committed " for T periods under improved pasture; the amount not allocated earns a return expressed by the function $h(x - y)$ and is then available in the next period for allocation to pasture or for the earning of another period's non-pasture return. Thus, in arriving at the maximum at each stage, the model compares (*a*) the returns from T years of improved pasture, with (*b*) the returns from one year of unimproved pasture *plus* the return from following an optimal policy over the remaining stages. This should be apparent from the basic recurrence relation which contains on its right-hand side one term in x and two terms in $(x - y)$.

Part II: Pasture Improvement Section

We may now turn to that part of the model which generates the dynamic programming return functions. This is no more than a simple characterization of the economic processes involved in pasture improvement.

First it must be noted that the pasture improvement model includes only those quantities which are assumed to change over the time period considered, as a result of the introduction of a pasture improvement programme. Thus fixed costs such as rates, rent, etc. and returns and costs associated with enterprises not connected with pasture improvement are omitted, as are such quantities as depreciation on machinery, repairs to plant and equipment, etc. which may, under suitable assumptions, be regarded as unaffected by pasture improvement. Thus the model is not a complete representation of a whole-farm situation; rather it considers only that segment of farm organization with which pasture improvement is directly concerned.

Further, it should be noted that the model does not spread the cost of investment in fixed equipment such as fencing, but considers it as a once-over cash outflow which is a discontinuous function of y. There is thus no conceptual distinction in the model between investment in fixed capital and in, say, an extra unit of labour, although their functional forms differ. These assumptions are by no means necessary—any investment pattern could be assumed.

The receipts and expenses involved over time as a result of an allocation of y acres to pasture improvement are considered to be non-linear functions of y. Call aggregate returns and costs R and Q respectively; then we can write:

$$g(y) = R(y) - Q(y) \tag{6}$$

[37]See, for example L. D. Loftsgard and E. O. Heady, " Applications of Dynamic Programming Models for Optimum Farm and Home Plans ", *Journal of Farm Economics*, Vol. XLI, No. 1, (February, 1959), pp. 51-62; W. Candler, " Reflections on 'Dynamic Programming Models' ", *Journal of Farm Economics*, Vol. XLII, No. 4, (November, 1960), pp. 920-6; C. D. Throsby *op cit.*, (footnote 27); c.f. also E. O. Heady and W. Candler, *Linear Programming Methods*, (Ames: Iowa State College Press, 1958) p. 17.

Now let $r_t(y)$ and $q_t(y)$ denote respectively the individual return and cost functions for period t. Remembering that d_t refers to the discount factor in period t, we can define:

(7) $$R(y) = \sum_{t=1}^{T} d_t \, . \, r_t(y) \text{ and } Q(y) = \sum_{t=1}^{T} d_t \, . \, q_t(y)$$

Before considering $r_t(y)$ and $q_t(y)$ in greater detail we list below the notation used in subsequent equations; with the exception of r, q and i, lower case letters in the following list refer to variables measured on a per acre basis, upper case letters refer to aggregate variables (generally functions of y). All monetary variables should be interpreted as being measured in equivalent units.

Let:—

$S_t(y)$ = return function for sheep in period t;
s_t = net returns per acre of y from sheep in period t;
$W(y)$ = return function for wheat (only relevant in $t = 1$);
y_w = upper bound on the acreage of wheat which may be established in one period;
w = net returns per acre from wheat sown;
z = gross returns per acre from wheat, after deducting marketing and freight charges ($Z(y)$ is used subsequently to indicate aggregate wheat returns);
l_t = labour cost per acre of y, involved in establishment of pasture and/or wheat in period $t = 1$, or in maintenance of improved pasture in period $t > 1$;
m_t = fuel cost per acre of y involved in establishment of pasture and/or wheat in period $t = 1$, or in maintenance of pasture in period $t > 1$;
u = seed cost involved in establishment of pasture and/or wheat;
v_t = fertilizer cost per acre of y involved in establishment of pasture and/or wheat in period $t = 1$, or in maintenance of pasture in period $t > 1$;
h = harvesting cost per acre of wheat ($H(y)$ is used subsequently to indicate aggregate harvest costs).;
$E(y)$ = establishment cost function (only relevant in $t = 1$);
e_f = establishment cost per acre of y when the farmer undertakes establishment himself;
e_c = establishment cost per acre of y when the farmer calls in a contractor to undertake the establishment;
y_e = upper bound on the acreage which the farmer can, with available plant, labour, etc., establish himself in one period;
$A(y)$ = agistment cost function (only relevant in $t = 1$);
$L_t(y)$ = investment function for labour, (or other ancillary investment) in period t;
$M_t(y)$ = investment function for fencing in period t;
$I_t(y)$ = function expressing interest on establishment capital chargeable in period t;
i = rate of interest charged on investment capital;
j_t = variable costs per acre of y incurred in maintaining pasture in period t;
a_t = returns per acre of y from sheep sold in period t;
b_t = cost per acre of y of sheep purchased in period t;
c_t = return per acre of y from wool sold in period t;
n_t = running costs per acre of y of sheep carried in period t;

A prime (′) on the variables s, a, c, and n indicates that they relate to unimproved pasture.

Three sets of equations express the relationships between these variables in the formation of $r_t(y)$ and $q_t(y)$ and one set shows the derivation of $h(x - y)$.

(i) RETURNS TO IMPROVE PASTURE IN PERIOD $t = 1$

The first set of equations shows gross returns in the first period as the sum of the aggregate returns to sheep and wheat[38], the latter being subject to a restriction imposed on the acreage which may be sown to wheat in any one period.

Thus, we have:

(8) $$r_1(y) = S_1(y) + W(y)$$

where

(9) $$W(y) = \begin{cases} wy & \text{for } 0 \leqslant y \leqslant y_w \\ wy_w & \text{for } y_w < y \leqslant x_0 \end{cases}$$

with

(10) $$w = z - (l + m + u + v + h)$$

(ii) COST OF IMPROVED PASTURE IN $t = 1$

The second set of equations simply expresses costs incurred in period $t = 1$ as the sum of expenditures on establishment of pasture, on additional fencing and labour or other ancillary investment necessary in the first period and on agistment of sheep.

We have, then:

(11) $$q_1(y) = E(y) + A(y) + L_1(y) + M_1(y)$$

where

(12) $$E(y) = \begin{cases} e_f y & \text{for } 0 \leqslant y \leqslant y_e \\ e_f y_e + e_c(y - y_e) & \text{for } y_e < y \leqslant x_0 \end{cases}$$

with

(13) $$e_f = l_1 + m_1 + u + v_1$$

(iii) RETURNS AND COSTS ASSOCIATED WITH PASTURE IN PERIODS $t > 1$

The following equations show the net money returns in period t as equal to the net returns from the sheep (i.e. gross returns from the sale of wool and sheep, less sheep purchase and sheep running costs) *minus* the costs of maintaining the pasture improvement investment (i.e. variable maintenance costs incurred by topdressing, expenditure on ancillary investments, and interest on establishment capital).

We have, then:

(14) $$r_t(y) - q_t(y) = s_t y - j_t y - L_t(y) - M_t(y) - I(y)$$

where

(15) $$s_t = a_t - b_t + c_t - n_t$$

and

(16) $$j_t = l_t + m_t + v_t$$

and

(17) $$I(y) = \frac{i}{100} \cdot E(y)$$

[38]For wholly grazing properties, or other farms where wheat is not relevant, the wheat costs and returns in subsequent expressions should be regarded as taking zero values.

(iv) RETURN FUNCTION FOR NON-IMPROVED PASTURE

$h(x - y)$ is a simple linear function defined as:

(18) $$h(x - y) = s' \cdot (x - y)$$

where

(19) $$s' = a' + c' - n'$$

In the generalized model above, $g(y)$ is assumed repetitive, i.e., having the same value at all stages in a multi-stage process. When $g(y)$ appears in this form, an assumption is necessary that physical input-output relationships involved, as well as the economic climate, are the same at all stages. If it is desired to relax this assumption and consider variations in $g(y)$ (and $h(x - y)$) between periods, the subscript k must be written in to all the above equations. For example, equation (6) becomes

(20) $$g_k(y) = R_k(y) - Q_k(y)$$

where

(21) $$R_k(y) = \sum_{t=1}^{T} r_{t,\,k}(y)$$

etc.

4. COMPUTATIONAL ASPECTS OF THE MODEL

Let us now look further at some computational aspects of the model.

It should be apparent from the empirical example given earlier that each step of the calculations involves the location of the maximum of $f(y)$ for given x.

(*a*) If $g(y)$ and $h(x - y)$ are continuous functions and it eventuates that $f(y)$ is always differentiable, calculus can be used to find an exact maximum, although boundary points will still have to be examined separately. In more complex cases, (for example when another dimension is introduced), Lagrange multipliers can be employed.[39]

(*b*) If $g(y)$ and $h(x - y)$ are not well-behaved but $f(y)$ can be *approximated* by a continuous function, we can proceed as in (*a*).

(*c*) Whatever the nature of $g(y)$ and $h(x - y)$, they can be approximated by a series of discrete observations, and $f(y)$ will then also be similarly described. Locating the maximum of $f(y)$ will simply involve a search through the discrete observations of $f(y)$. This is precisely the technique used on the above example and, indeed, throughout the present study. Besides computational simplicity, its chief advantage is that it permits handling of functions which are otherwise quite beyond mathematical definition. This would appear to be a fairly common way in which dynamic programming problems of low dimension are attacked,[40] (i.e. problems in which maximization is over only one or two dimensions at each stage).

In our example we had, at most, six observations of $f(y)$ to examine at a time. If this number were six hundred or six thousand it would take some time to locate the maximum. By starting with a " coarse " grid, however, one can frequently define the *region* of the maximum and then search

[39]c.f. R. E. Bellman and S. E. Dreyfus, *op. cit.*, (footnote 3), p. 47 ff.

[40]S. E. Dreyfus, " Computational Aspects of Dynamic Programming ", *Operations Research* Vol. 5 No. 3 (June, 1957), p. 411.

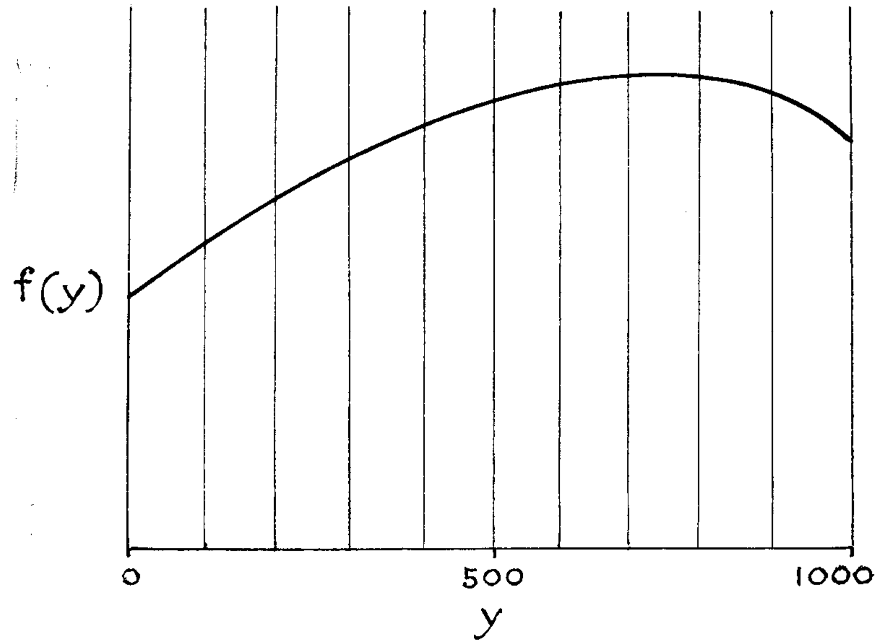

Fig. 1.

thoroughly in this area for its exact value. For example, if one had one-thousand observations of the function $f(y)$ shown in Figure 1, one could determine, by looking at $f(y)$ only at the points $y = 0, 100, 200, \ldots, 1{,}000$, that the maximum lies within the range $700 < y < 800$. One could then search through the points $y = 710, 720, 730, \ldots$, and establish a smaller area in which the maximum exists, and so on. This is known as *approximation in function space*,[41] and greatly reduces computing times for large problems.[42] It should be remembered, however, that if $f(y)$ possesses a series of local maxima, approximation in function space is liable to error.

An analagous approximation procedure can be applied to the functions $y(x)$ which define optimal policies at each stage of the process. Where $y(x)$ is a single-valued function (which was not always found to be the case in our present study) an initial guess at its value can be successively refined in a similar manner to the above, until an optimal solution is reached. This has been termed *approximation in policy space*.[43]

Although the technique of defining a function by a table of discrete observations is extremely useful, it is not without its drawbacks. The chief one is the lack of a clear definition of an extremum in some cases; a computer might, for example, output two adjacent grid points as the values of y

[41]Approximating an exact solution and refining it by successive steps is a common procedure; see, for example, H. S. Houthakker, " On the Numerical Solution of the Transporation Problem ", *Journal of the Operations Research Society of America*, Vol. 3, No. 2, (May, 1955), pp. 210-14; see also R. Bellman, *op cit.*, (footnote 2), Chap. V *et seq.*

[42]In the example given, a search through 10^3 alternatives was reduced to one through 3 x 10 alternatives.

[43]An excellent numerical example of this procedure is given in G. E. Kimball and R. A. Howard, " Sequential Decision Processes ", Massachussetts Institute of Technology (O. R. Centre), *Notes on Operations Research*, 1959. (Massachussetts Technology Press, 1959), pp. 153-78.

maximizing a given $f(y)$, where in fact the true maximum might lie uniquely at some point between them. The easiest solution to this problem is to set the grid size small enough to make it irrelevant which one is selected as the maximum. In empirical practice a compromise must be struck between (*a*) smallness of grid size; (*b*) availability of data to specify functions accurately over small ranges; and (*c*) expected computational times.

Progress has been made in developing dynamic programming algorithms. For instance, Bellman and Dreyfus have published a series of papers under the general title " On the Computational Solution of Dynamic Programming Processes "[44] which outline routines adaptable for specific dynamic programming formulations. Howard[45] developed a " value-iteration method " (for processes of short duration) and a " policy-iteration method " (for lengthier problems); both are computational techniques similar in some respects to the approximation methods discussed above, which may be capable of wider generalisation. In our present study it was found that specifying the functions g and h as a set of discrete approximations, determining $y(x)$ by computing a series of $f(y)$ and searching for their maxima, and tabulating at each iteration $y(x)$ and $f(x)$, proved the most satisfactory method.

Finally in this section on computational aspects we note the existence of a certain " duality " in the formulation of our model. In computing optimal allocation sequences throughout the present study, we always compute the last stage first, working backwards through the system until optimal policies from every point within the system to the end of the N-stage process have been determined. But an equivalent solution can be obtained by working forwards iteratively, determining at each stage the best path *to* (instead of *from*) a given point. At each stage we would obtain a table showing the optimum y to be allocated in this period in order to have given amounts of $(x - y)$ available for the next period. To read off an optimal policy at the end of the calculations, however, we would have to work backwards through our table of results (c.f. working forwards through Table 1). This is somewhat tedious for the allocation process, although for a number of situations (for example, network problems) it would be equally easy to work in either direction. In choosing between the alternative methods for the present study, several factors had to be borne in mind. Conceptually it appeared less easy to handle the forward allocation method. It seemed more desirable to have as the results of a computation the optimal allocation sequences for a whole range of initial resource availabilities at the start of the process, rather than for a range of " left-overs " at the end. It was found, too, that solutions forward through time were subject to exactly the same computational difficulties as was backward allocation. Thus, the method as described above was chosen.

5. SOME ASSUMPTIONS

Before examining the properties of this model, it will be useful to summarize the assumptions upon which it is based. It should be apparent from the formulation of the model in previous pages that we are assuming the farmer to be making decisions under conditions of certainty, and that he is rational,

[44]R. Bellman and S. E. Dreyfus, *On the Computational Solution of Dynamic Programming Processes*, RAND, RM-1745 to 1752, (1956 to 1958); see also S. E. Dreyfus, *On the Computational Solution of Dynamic Programming Processes, XV*: RAND, RM-2134, (March, 1958) and *Dynamic Programming Algorithms and Formulations*, RAND, P-1527, (October, 1958).

[45]R. A. Howard, *Dynamic Programming and Markov Processes*, (Massachussetts: Tech. Press of M.I.T., 1960), Chapters 3-9, especially Chapters 3 and 4.

i.e. is attempting to maximize his utility, the latter being directly measurable as net financial gain. The system is assumed " closed " and the only variable under the farmer's direct control is the amount of land allocated to pasture improvement each year. A particularly important group of assumptions contains those dealing with the decision maker's outlook on the future. The use of present values enables the returns from each pasture improvement allocation to be expressed in terms of its utility to the farmer in the year in which it is undertaken. Thus, despite the fact that pastures laid down in the later periods will be continuing beyond the N-th stage, their utility to the decision maker will have been reduced to terms of their present value in their establishment year which will be at or before the N-th stage. Anticipating a little, it was deduced by empirical analysis that this assumption could be defended where the series of allocations extended over a long period and the overall best policy was being sought; however, if the time planning horizon of the farmer was considered to be short, an alternative would be preferable, e.g. *strictly* limiting the system to a given number of stages.[46] This question is pursued further later.

It might be noted in passing that since such importance may hinge on the present value assumptions of this model, a *variable* discount rate (mentioned already on page 159) might well be appropriate here, allowing the " future-future " (present values at later stages) to be discounted much more heavily than the " present-future " (present values at earlier stages). This raises the interesting question of how these discount rates might be estimated in practice, but we shall conveniently sidestep this issue at this point.

In the following sections a number of theoretical properties of both parts of the model are derived. The knowledge about its workings gained thus is useful in two ways (*a*) it indicates the nature of actual empirical solutions which might be expected if real data were applied to the model; and, more importantly, (*b*) it gives an insight into the *structure* of the decision process under study and of the optimal policies generated by the model. As mentioned earlier, in many economic analyses quantitative information about the structure of a system is often of greater value than specific numerical results[47]. The methodology used here is to draw up simple hypotheses about the nature of each component of the pasture improvement section of the model; in this way it is possible to specify fairly closely the expected nature of the return functions $g(y)$ and $h(x - y)$. The expected forms of these functions are then " plugged in " to the dynamic programming system and general properties of the solutions established. When this analysis was being carried out, computer time was a strictly limited quantity and we were confined to constructing only *one* set of assumptions about these components of $g(y)$ and $h(x - y)$; the same restriction limited the range of properties of the overall model which could be examined.

6. PROPERTIES OF THE MODEL

Part I: Structure of the Pasture Improvement Section

Remember that $g(y)$ covers the whole range $0 \leqslant y \leqslant x_0$ where x_0 is the total initial amount of land available. In other words for any farm it expresses the profitability of committing to pasture in a given year, areas of land

[46]Under this system $g(y)$ for period N would cover only $t = 1$, for period $N-1$ it would cover only $t = 1$ and 2, and so on.

[47]c.f. further K. J. Cohen and R. M. Cyert, " Computer Models in Dynamic Economics ", *Quarterly Journal of Economics*, Vol. LXXV, No. 1, (February, 1961) p. 115.

ranging from a few acres to the whole farm. Land not committed this year is available for re-allocation next year. Let us examine six components of $g(y)$ which were introduced in Section 3 and consider their likely influence of $g(y)$. The components to be considered are: (i) costs of establishing improved pasture in $t = 1$; (ii) costs and returns associated with sheep in $t = 1$; (iii) costs and returns associated with improved pasture and sheep in $t = 2$ to T; (iv) lumpy components: investments in fencing and labour; (v) interest on capital; and (vi) the procedure of discounting.

(i) Costs of Establishment of Pasture

As noted in Section 3 there are assumed to be two major components of the function $E(y)$, which expresses the costs of establishment of pasture as a function of the number of acres sown. These are the variable costs of the farmer's own establishment activities (incurred up to a limit of y_e acres), and, beyond this amount, the price which he must pay out to a contractor. It is postulated here that (*a*) both of these quantities are approximately linear functions of the number of acres established, and (*b*) that the cost of the contractor's services are greater per unit than those incurred by the farmer doing the same job himself, since the contractor includes in his charge per acre allowances for a profit margin, depreciation and maintenance of his plant, transport to the farm, etc. This leads to an expected shape for $E(y)$ as shown in Figure 2.

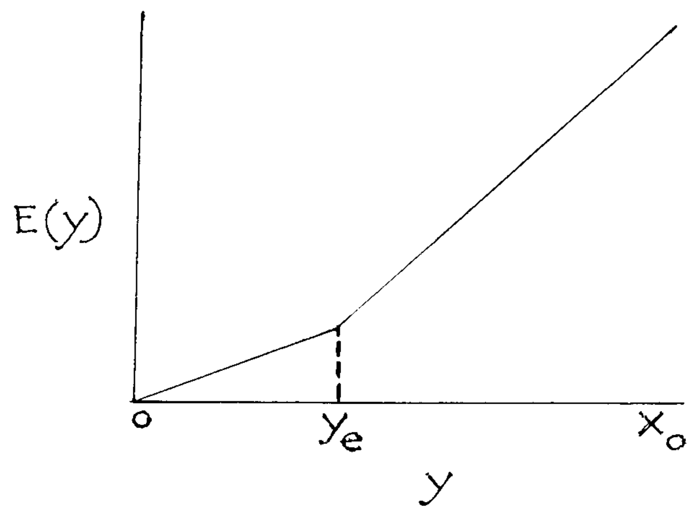

Fig. 2.

(ii) Costs and Returns Associated with Sheep in $t = 1$

Sheep contribute two components to $g(y)$ in period $t = 1$, viz. $S_1(y)$, the returns from sheep displaced from land put under establishment, and $A(y)$ the cost of any agistment which has to be sought. It is reasonable to suppose that if only a small area of improved pasture is sown in any one year the small number of sheep displaced by this area being put out of production could be carried on remaining land on the farm without ill effects either to themselves or to the stock already carried there. In fact, since it appears that pastures, both natural and improved, are frequently understocked,[48] the number of extra sheep which this remaining land could carry might be quite considerable

[48]This statement is made on the basis of discussions with Departmental agronomists and fieldworkers.

especially in a good season. It is likely, however, that as the stocking rate on the remainder of the farm is increased, the returns per sheep displaced will begin to decline. It is hypothesized, for present purposes, that the stocking rate on remaining pastures could be doubled, but that by this stage returns per sheep displaced will have fallen to zero. In other words, for the first acre of y returns per sheep displaced will be the same as they were earning on natural pasture, and for the last acre of y before $0{\cdot}5\ x_0$ returns per sheep displaced will be zero. If, in the absence of any other information, we assume a linear trend between these two reference points, it will be seen that total returns take the form shown in Figure 3.

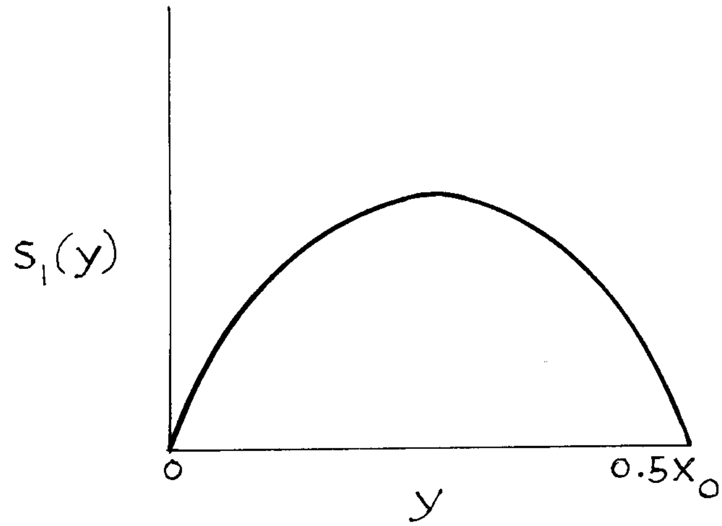

Fig. 3.

For values of y between $0{\cdot}5x_0$ and x_0 (i.e. when more than half of the farm is under establishment in one period), sheep will be displaced with nowhere to go. For simplicity it is assumed the farmer's only possibility is to buy agistment. (An alternative would be to sell the displaced sheep and buy them back when pastures have become established.) Under these circumstances we would be confronted with a similar analytical situation to that described above for the returns from sheep. As y increases, the first few sheep put on agistment might not have to remain there for long: seasonal trends in pasture production on the area of the farm not under establishment would perhaps permit the agisted sheep to be brought back after a few months. But as y increases further this possibility would become less and less available, until the point where $y = x_0$ where *all* the sheep on the farm must be moved off to make way for pasture establishment; they can only be brought back when the newly established pastures are capable of carrying them. It seems reasonable to assume that this will not be until $t = 2$. Between the referenced points $y = 0{\cdot}5x_0$ where agistment costs are zero, and $y = x_0$ where agistment costs per head are at a maximum we assume a linear relationship between sheep displaced and costs of agistment per sheep, in a similar fashion to that used with sheep returns when $y < 0{\cdot}5x$. It is found under these assumptions that as y increases, $A(y)$, the agistment cost function, exhibits the shape shown in Figure 4. Combining these two functions (sheep returns and agistment costs) gives a contribution by sheep to the $t = 1$

component of $g(y)$ shaped approximately as in Figure 5. It is obvious that the above scheme is quite idealised, and that a variety of circumstances might in fact prevail in the real world. However, the functional shape shown in

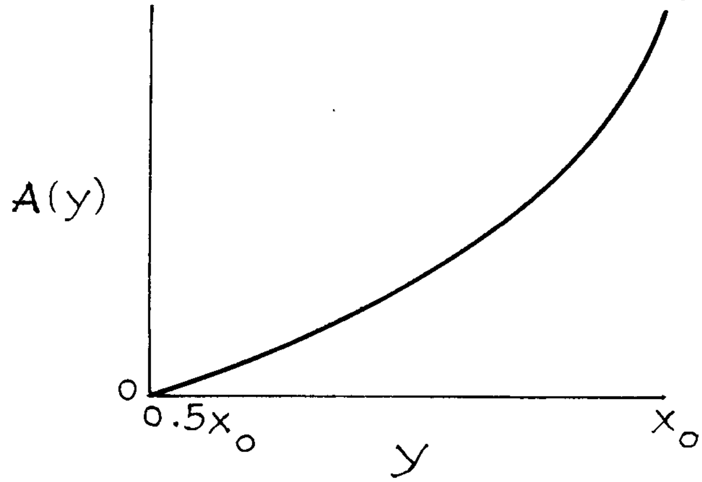

Fig. 4.

Figure 5 does seem intuitively reasonable for the purposes of " plugging in " to the model as formulated here. It should be borne in mind, of course, that even if the general shape of Figure 5 is granted, the exact point at which the curve would cut the y axis might still vary quite widely.

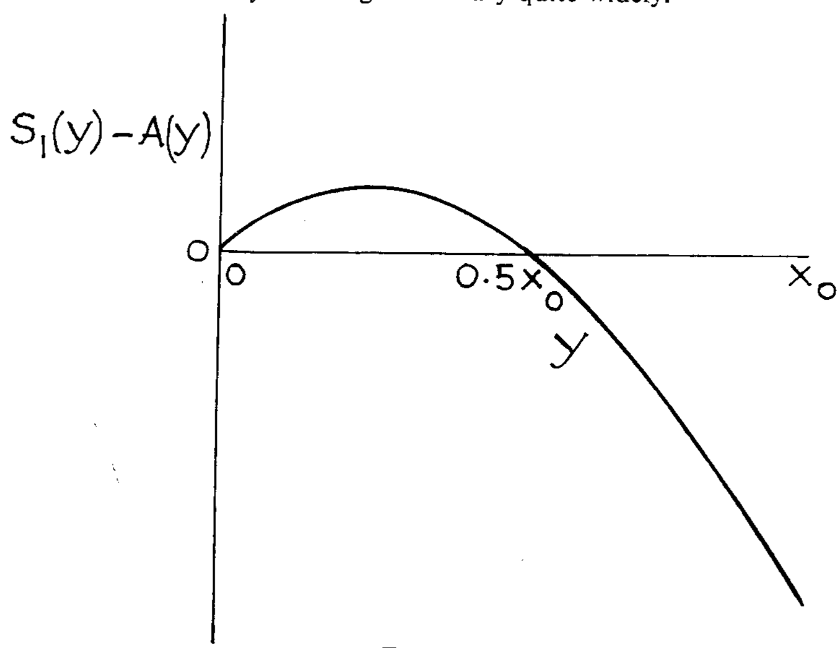

Fig. 5

(iii) COSTS AND RETURNS ASSOCIATED WITH SHEEP IN $t = 2$ TO T

Once pastures have been established and become productive it is hypothesized that provided they are uniform in quality over the whole farm, there would be no financial " destabilizer " in $r_t(y) - q_t(y)$ other than " lumpy " expenditures considered in the next section. In other words if the above uniformity assumption holds, carrying capacities can be assumed the same over all areas of improved pasture at the same stage of development. and hence the costs of any sheep bought to reach the required stocking rates, the gross returns from wool and meat sold and the gross costs of maintaining sheep and pasture, are likely to be approximately directly proportional to y, i.e. the main components of $r_t(y) - q_t(y)$ for $t > 1$ are likely to be close to linear. Thus we adopt the hypothesis that returns from the sale of sheep and wool derived from improved pasture and the costs associated with pasture maintenance (fertilizer, labour, fuel) and sheep (purchases to make up numbers and running expenses) are all linear components of $r_t(y) - q_t(y)$ for $t > 1$.

(iv) LUMPY COMPONENTS: INVESTMENT IN FENCING AND LABOUR IN $t = 1$ TO T.

It is postulated that as y increases there will be an increasing need for investment in permanent improvements such as fencing, water points, buildings, machinery, and in other resources such as labour, at some time during the pasture improvement programme. Not only the quantity but also the timing of this investment will be dependant on the size of y. If large areas of pasture are sown down in one year, it is feasible that investment in, say, fencing, to enable efficient management of the pasture, will be required sooner than if only a small area were established. It is also hypothesized that this investment is not likely to be a continuous function of y, nor even readily approximable by a continuous function. Investment items such as buildings and tractors are " lumpy " inputs, and are largely indivisible. For example, it may be that if the area sown to pasture in one year exceeds a certain level, say y_a, an extra permanent hand will become necessary, requiring a large single expenditure in that year for levels of y beyond y_a.

We have included in the model two components to allow for these ancillary investments, viz. $L_t(y)$ called the " labour investment " and $M_t(y)$ called the " fencing investment ". It is stressed that it is not intended that these quantities be thought of as referring *solely* to expenditure on those two items; rather it is intended that they be construed as covering any similar item of investment necessitated by the adoption of a pasture improvement programme in any application of the model to empirical situations. (They have been called labour and fencing in our formulation because these were the only two investment possibilities which we treated in subsequent empirical investigations).

These investment functions might have a variety of forms. Two typical shapes for these investment functions (which were examined in the empirical section of this project) are shown in Figure 6.

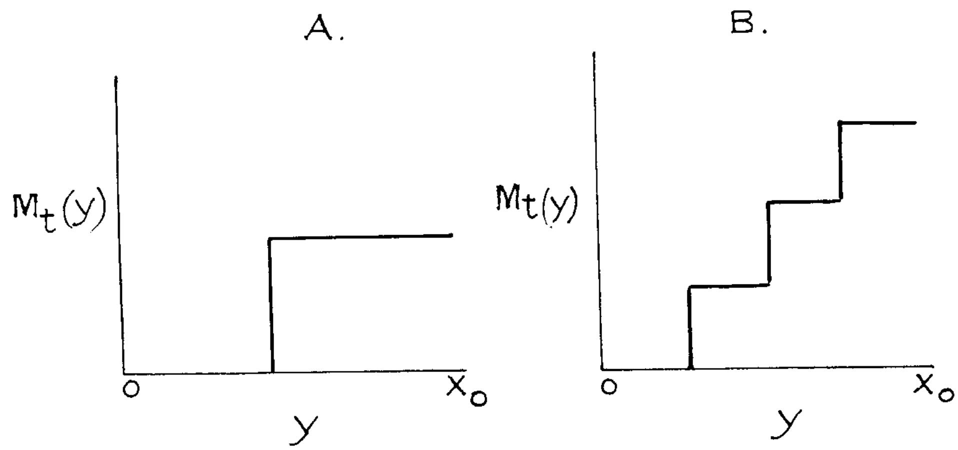

Fig. 6.

(v) Interest on Establishment Capital

The effect of the function $I_t(y)$ on the shape of $r_t(y) - q_t(y)$ is like'y to be small, because of its relatively small numerical size. Being related to $E(y)$ (see Figure 2) its only contribution to $g(y)$ will be reinforcing the effects of $E(y)$.

(vi) Effects of Discounting

We have seen already that discounting of future money flows is assumed in the model to be effected relative to t. The implications of this for the shape of the functions discussed above is, of course, that it reduces the influence of return functions on $g(y)$ as t increases. (Conversely the importance of the establishment year and of the early years of the pasture's life is increased). For example if the " steps " in Figure 6B are generated in sequential time periods, approaching the present as y increases, then discounting tends to give the function a rising trend as shown in Figure 7.

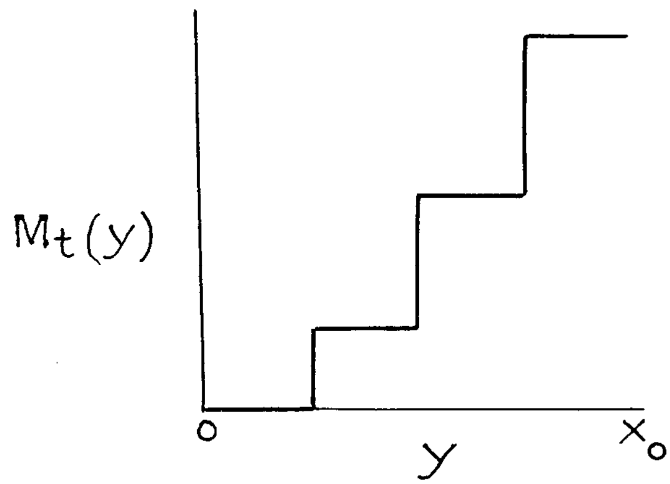

Fig. 7.

We may now postulate the expected shape of $g(y)$. Given the formulation of our model and the above hypotheses about the nature of the various functions of which $g(y)$ is composed, it is found that:

(*a*) $g(y)$ is *concave to the origin* (i.e., shows diminishing returns to scale).

(*b*) for low values of T, $g(y)$ possesses an interior maximum; and

(*c*) as T increases, $g(y)$ takes higher values and tends towards linearity.

Figure 8 shows these conclusions graphically, with discontinuities smoothed out. Even though Figure 8 gives ground for some interesting speculation about the structure of the decision process under study, a true picture cannot be obtained until the functions derived above are incorporated into the rest of the model.

Let us turn now to the theoretical shape of $h(x - y)$. In the formulation contained in Section 3 $h(x - y)$ was constructed as a linear function. This simplifies the analytical and empirical analysis although the nature of the model by no means necessitates such an assumption: $h(x - y)$, like $g(y)$, could assume any desired form. Nevertheless the linearity assumption may not be inconsistent with the nature of the real-world relationship which it represents. This return to $(x - y)$ acres of unimproved land involves receipts and costs associated with sheep grazing natural pasture. Fixed costs *per se* are not included in the model, as noted earlier, but only those quantities which vary with the number of sheep carried (receipts from the sale of wool and meat and sheep running expenses). Since it is reasonable to suppose that these receipts and costs are, in the aggregate, directly proportional to sheep numbers and since stocking rates are assumed uniform over all areas of unimproved land on the farm, it will be realized that $h(x - y)$ might indeed be close to linear.

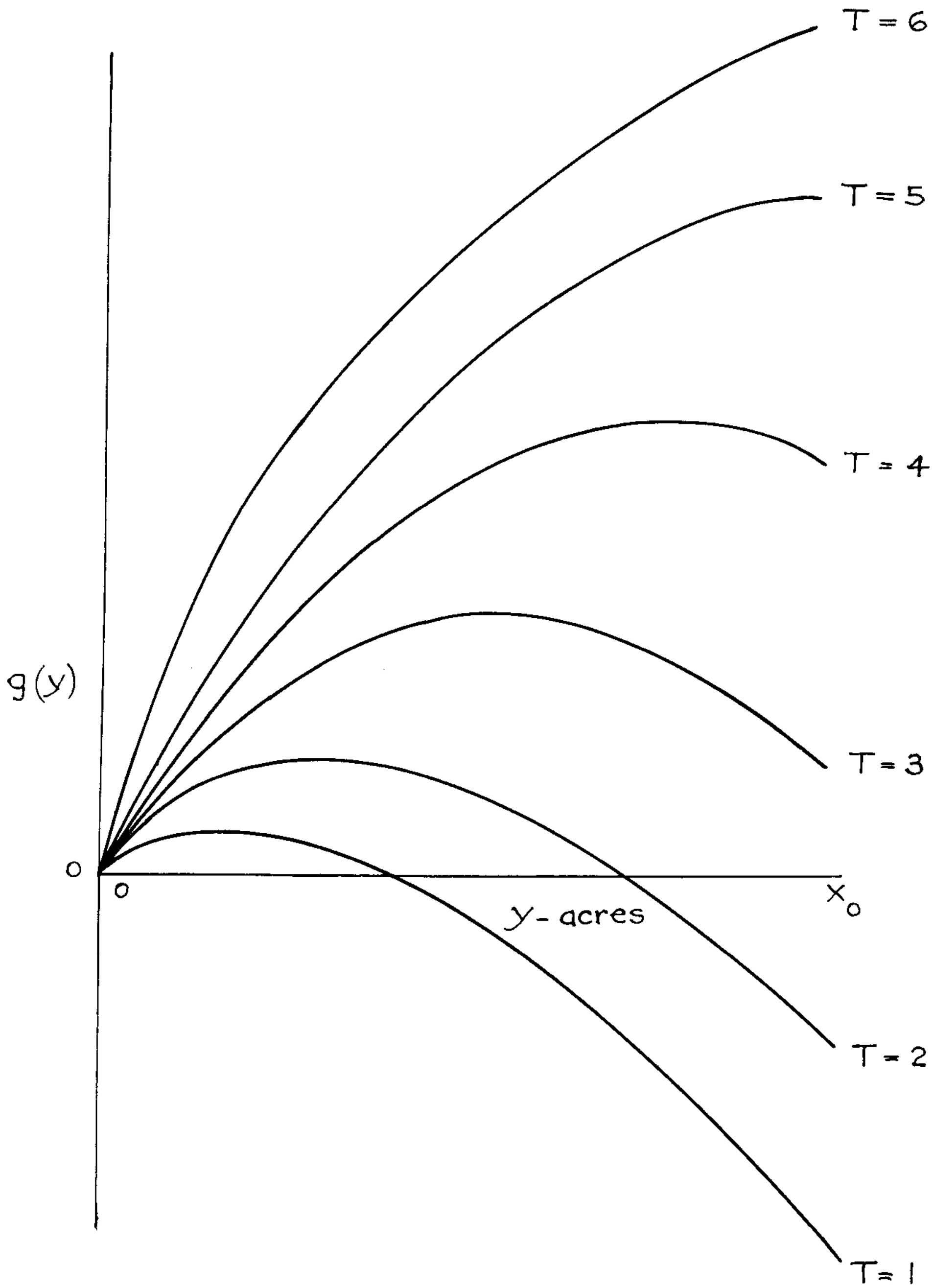

Fig. 8.

Part II: Structure of Optimal Solutions

Given some definite notions about the shape of the main component functions in the pasture improvement section of the model, let us consider the effects of using such functions in the other part of the model. More precisely, the questions now posed are: what is the general nature of solutions generated by the model, and how much are they dependant on the form of the functions g and h? The term " solution " is used to include:

$f(y)$, the returns for the range of y at a given x, from which the maximum is selected;

$f(x)$, the collation of all these maxima to make up the optimal return for the complete range of x available at any stage;

$y(x)$, the optimal allocations of y as a function of the amount of land available at any stage; and

$y(k)$, the overall allocation sequence yielding the optimal return for a given initial amount of x.

In the following discussion we establish six general properties of these functions which give *in toto* the overall resolution of the decision problem under study as generated by our model. These properties concern:

(i) the shape of $f(y)$;
(ii) the behaviour of $f(y)$;
(iii) the shape of $y(x)$;
(iv) the shape and behaviour of $f(x)$;
(v) the behaviour of $y(x)$; and
(vi) the shape and behaviour of $y(k)$.

All these properties have been established by enumeration rather than by formal mathematics. Enumeration is an inductive method which infers general properties of a system from the results of a series of carefully chosen empirical tests. It should be noted that the following properties apply in full only when $g(y)$ is a *continuous* concave function. Discontinuity and deviations from a well-behaved shape produce some anomalies. Occasionally fairly small abnormalities and complexities can be sufficient to lead to results inconsistent in one or another respect with these general properties.[49] (In specific empirical circumstances, of course, one would include such deviations from a well-behaved pattern as were relevant, and would be able to observe departures in solutions from the regular pattern.) It should be remembered also that the results below only apply if $g(y)$ and $h(x - y)$ are the same at each iteration. (i.e. do not vary from stage to stage in the overall process) and are both positive.

(i) SHAPE OF $f(y)$

Given that $g(y)$ is concave and $h(x - y)$ is linear, it is possible to show that that $f(y)$ is also concave, and may possess an interior maximum. i.e. $f(y)$ may assume a shape such as in Figure 9 in which case its maximum does not lie at an end point of the curve. It can be shown that if $g(y)$ is convex (i.e. shows *increasing* returns to scale) and $h(x - y)$ linear, $f(y)$ will be convex also and its maximum will always lie at an end point, i.e. where $y = 0$ or x, as Figure 10 shows. Under such a system an optimal policy always involves allocating " all or nothing " at any given stage. But since we are not concerned with a convex $g(y)$ this point is only of academic interest here.[50]

[49]c.f. R. Bellman, *op. cit.*, (footnote 25), p. 25.

[50]Both of these properties have been established rigorously by Bellman for the model already cited (see particularly, Bellman, *op. cit.*, (footnote 25) pp. 19-25). The present author has confirmed by enumeration that Bellman's conclusion is applicable also to our model.

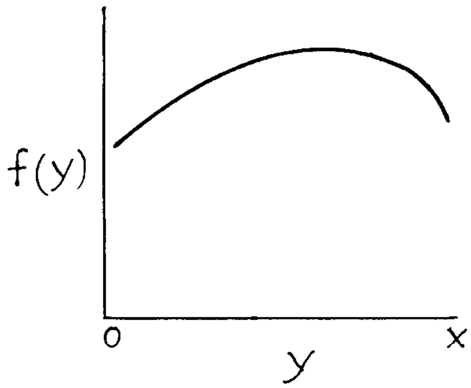

Fig. 9.

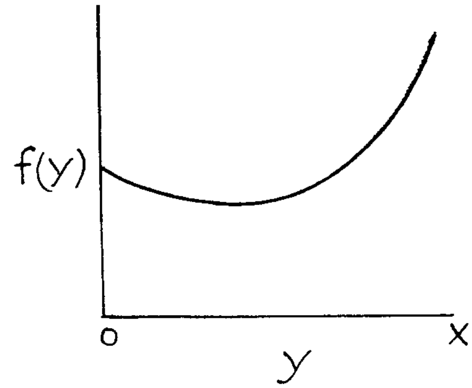

Fig. 10.

(ii) Behaviour of $f(y)$

For a given value of x, $f(y)$ grows larger at each iteration and the values of y at its maximum tend towards zero. Showing graphically, as the

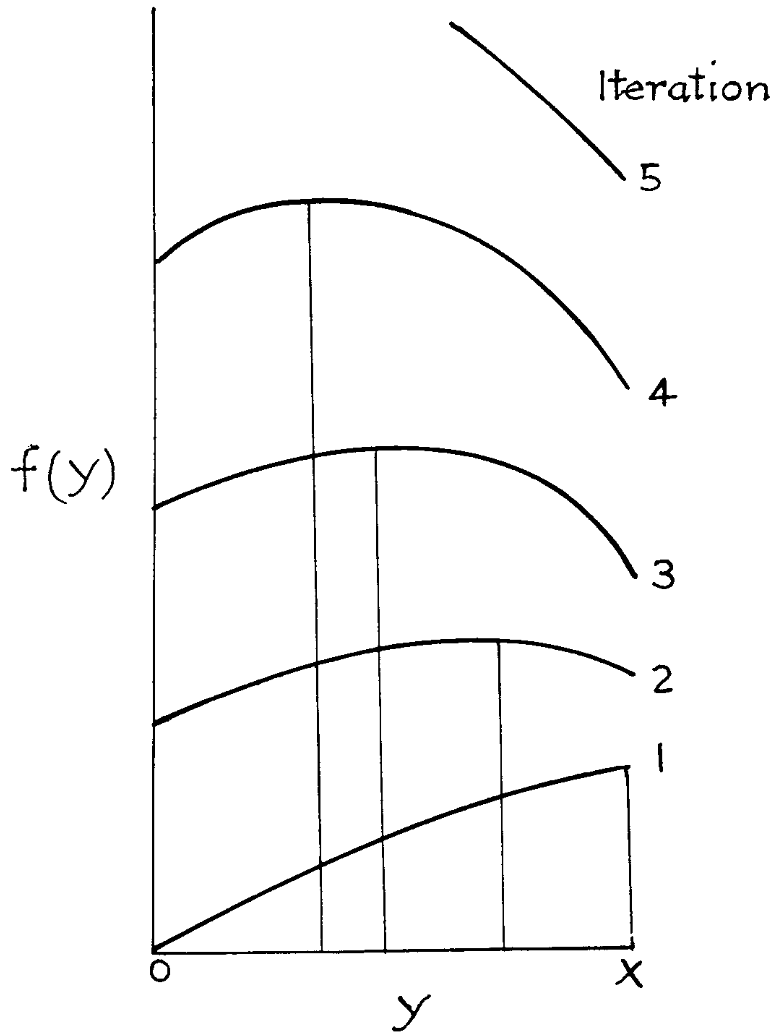

Fig. 11.

number of stages is increased $f(y)$ tends to move as shown in Figure 11. Remember there is a different set of these curves for each value of x.

It might be noted that this property could be used in optimizing the search for the maximum of $f(y)$: the maximum always lies at values of y less than the maximizing value for the previous iteration, (provided the continuity conditions etc. hold).

(iii) Shape of $y(x)$

It can be shown (and it follows in part from (ii)) that $y(x)$ at each iteration is equal to or smaller than its values at the last iteration.

Showing graphically it exhibits the pattern described in Figure 12. We shall call the triangular space enclosed by the first iteration the " decision region ".

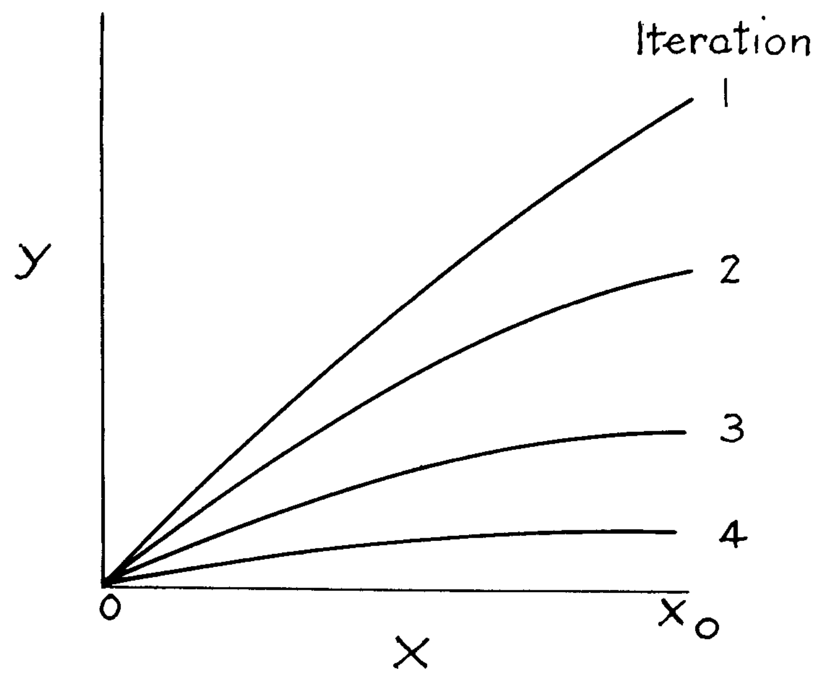

Fig. 12.

(iv) Shape and behaviour of $f(x)$

Firstly, for any stage $f(x)$ never decreases as x increases. This is because, according to our model, it is never unprofitable to own *some* land, and hence the same or greater profit can always be made by having more of the resource.

Secondly, if N (the overall number of stages) is left unspecified, the objective function of our model is unbounded, i.e. can increase indefinitely. The function $y(x)$ reaches a steady state at zero (cf. Figure 12), but $f(x)$ continues to increase.[51] The reason for this is that of the recurrence relation in the model deals with a " once-over " allocation, and does not include a term for " feeding back " land into the system when pastures expire. For

[51]It is interesting to compare this result with that derived for Bellman's model mentioned above (see footnote 49). In Bellman's model $f(x)$ converges.

problems where the number of iterations required for $y(x)$ to reach zero is less than or equal to T, this does not present any problem.[52] In such cases, as mentioned earlier, either the present value assumptions are adequate to take care of the problem, or the overall process can be *strictly* curtailed to N periods. (See footnote [46]). In the general case the objective function can be constrained, by presetting the parameter N. This is by no means an *ad hoc* procedure. There may be reasons for ascribing it a particular value before computing a programme; on the other hand, if it is desired to find the best general policy, it seems most appropriate to let the running of the programme itself compute N as the number of iterations required before $y(x)$ reaches a steady state at zero. If N is greater than this, optimal policies will advocate doing nothing for several periods,[53] if it is less there may be surplus resource at the end of the process. We call this number of iterations N^*. In other words, N^* defines the number of periods required to allocate all the available resource using an overall optimal policy, given the structure of the model.

(v) BEHAVIOUR OF $y(x)$

It can be shown that the behaviour of $y(x)$ within the decision region, and the shape of the decision region itself, are dependent on the nature of $g(y)$ and $h(x - y)$. The following are the five most important properties as established by enumeration:

(*a*) If $g(y)$ is linear and greater than $h(x - y)$, the optimal solution is to allocate all the available land to pasture at one stage. (If $g(y)$ is linear and less than $h(x - y)$ the optimal solution is never to allocate any land to pasture).

(*b*) As diminishing returns become more marked in $g(y)$—i.e. as it becomes more sharply concave—N^* increases (provided the value of $g(y)$ at its maximum remains the same). Under these circumstances the decision region may become smaller.

(*c*) If $g(y)$ has an interior maximum where, say, $y = y_m$ its position in the region $y_m < y \leqslant x_0$ has no effect on optimal solutions.

(*d*) As the values of y at which $g(y)$ reaches a maximum decrease (provided the value of $g(y)$ at the maximum remains the same) the decision region becomes smaller, but N^* increases.

(*e*) For a given $g(y)$, as $h(x - y)$ approaches zero the decision region gets larger (approaches $g(y)$), and N^* increases.

These conclusions are perhaps the most important properties of our model in that they give a direct picture of the effects of the nature of the return functions for improved and non-improved land on optimal pasture improvement policies. They are discussed with a more practical orientation in the following section.

---

[52]In actual empirical cases examined this was almost always found to be the case. There are also other ways in which this problem might be handled, besides incorporating feedback terms into the basic model. For instance, a " rolling-planning " time outlook could be assumed, in which the decision maker's current planning horizon is progressively replaced by a new one as land begins to become re-available as pastures expire.

[53]For example if N were made 100 years, an optimal policy might dictate something such as " Do nothing for 96 years, then allocate all the land to pasture within four years ".

(vi) SHAPE AND BEHAVIOUR OF $y(k)$

The optimal allocations at each stage form a step function when expressed relative to k, the stages in the overall multi-stage process, since k is measured in discrete time periods. It was found in all cases studied that this function, $y(k)$, was an increasing function, i.e. exhibited a shape such as that known in Figure 13. In other words optimal policies generated under the assumptions noted earlier advocate increasing allocations to pasture improvement as time progresses. This was largely confirmed by the later analysis using real-farm data. However, it should be noted that for programmes with N strictly constrained in the manner described earlier (see footnote [46]), $y(k)$ is no longer necessarily an increasing function.

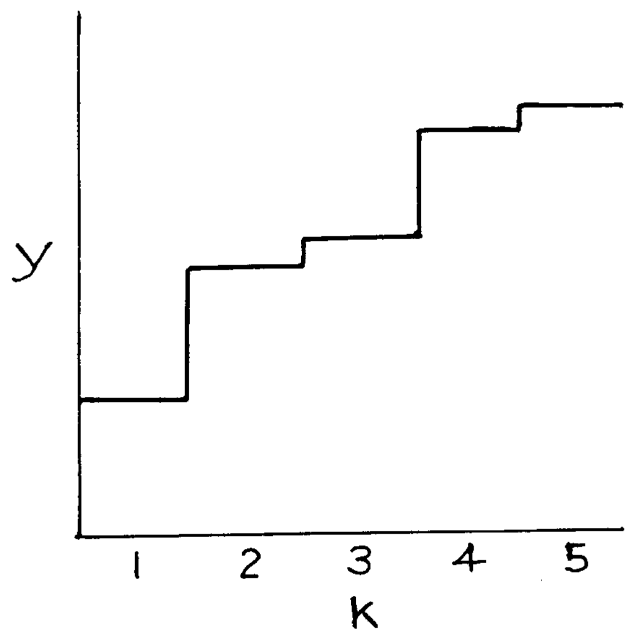

Fig. 13.

7. CONCLUDING REMARKS

(i) IMPLICATIONS OF RESULTS

An empirical study of the use of this model was conducted as a part of this project, based on case studies of five farms in New South Wales. The return functions derived and the nature of the dynamic programming solutions obtained confirmed in general the theoretical findings discussed above. To conclude the present article, we consider briefly some real-world implications of some of the above theoretical results as suggested by the empirical analyses.

(*a*) It was hypothesized above that the pasture improvement benefit function shows diminishing returns to scale due mainly to the effects of non-linear components in the establishment year, and to the necessity for increased ancillary investment during the programme if y is large. It was shown that the degree to which diminishing returns are evident has a marked influence on the optimal policies generated by the model. If $g(y)$ is markedly concave the optimal allocation sequence is spread over a number of time periods, involving relatively small optimal allocations at each stage. This would appear to be characteristic of *smaller* farms because the *non-linear* elements mentioned above seem likely to be relatively larger numerically than for larger farms. As the extent of diminishing returns in the pasture improvement return function becomes less marked so does it become more profitable to allocate greater amounts to pasture improvement at a time. This was found to be more characteristic of larger farms, since here the linear components of $g(y)$ were found to be relatively stronger. If $g(y)$ is linear, the model recommends allocating all land to pasture in one stage. It is suggested that this situation might be approached when aerial sowing of pastures is involved, although no substantial empirical evidence was available to support this hypothesis.[54]

(*b*) It was also observed above that optimal policies generated by the model involve increasing allocations of land to pasture over time. Such policies, if N^* were small, would lead to a sudden severe financial drain several periods after the start of the programme, with the likelihood that capital constraints would prohibit the attainment of an optimum-optimorum. Budgeting methods analysing the adoption of optimal policies on the case-study farms confirmed this, and produced income streams over time resembling those observed by Campbell and Shand.[55] It was found of course that the severity of this problem was reduced if capital were able to be built up in the earlier stages of the programme, for instance via the increased returns and cost-sharing achieved by sowing pastures under a cash crop such as wheat. This was confirmed strikingly by one Central-western farm of less than 1,000 acres, where despite capital and other limitations the farmer was able to establish an average of about 200 acres of pasture annually, sown under wheat or oats.

(*c*) It was found that variations in some parameters causing a whole shift in $g(y)$ are likely not to have such a significant effect on optimal allocation sequences as on expected returns. For instance it was observed that for a given farm situation variations in establishment methods with different productivity patterns had significant effects on returns, but not such a marked effect on optimal policies. Similar remarks apply to the cost components of one particular establishment method. In other words emphasis on individual establishment cost components *per se* as determinants of optimal pasture improvement policies might be misplaced if a long run optimization is required. On the other hand optimal allocation sequences were found to be quite sensitive to parameter variation which affects the shape of the return functions, for example by influencing some non-linear elements more than linear ones, or by affecting only a section of $g(y)$ and $h(x - y)$. In particular, interesting results were obtained by considering variations in the time planning horizon of the farmer, either through the rate at which future income streams are discounted or via the placement of strict limitations on the number of time periods the farmer is assumed to account for in formulating plans; for example, applying discounting relative to the start of the overall process, rather than relative to each stage, causes optimal allocations at the beginning

[54]This situation was approximated on one of the farms studied where large areas of pasture could be established by air at relatively low seeding rates; here the non-linear element in $t = 1$ was not so important, hence g(y) tended towards linearity making it feasible to sow down a major portion of the property at one time. Of course the two factors modifying this picture in reality are capital availability and risk considerations; in the case of the farm in question, however, it was apparent that neither of these was of great importance to the farmer.

[55]K. O. Campbell and R. T. Shand, *An Economic Study of Pasture Improvement on Some Farms in New South Wales*, (Sydney: University of Sydney, Department of Agricultural Economics, Mimeographed Report No. 2), 1958.

of the process to increase, whereas increasing the size of the discount rate leads to increased allocations at the end. The extent of the differences, however, is not great, and choice of discount rates and methods of application can be dictated by individual situations. In general it was found that such restrictions, since they involve the derivation of a sub-optimum rather than an optimum-optimorum, lead to reductions in optimal allocations and expected returns.

(ii) Uses and Limitations of Dynamic Programming

Let us now consider briefly some uses and limitations of dynamic programming as brought out by this study.

Perhaps the greatest advantage of dynamic programming in this context is the way in which it can handle non-linear and discontinuous functions. If the pasture improvement problem considered here were formulated for analysis by linear programming, it would be necessary to assume that the function $g(y)$ was linear subject to linear constraints or at least that it was some well-behaved non-linear form such as quadratic. It has been shown not only that this particular function is likely to be non-linear, but also that its shape has important influences on optimal policies.

The second important feature of dynamic programming is the way in which it treats optimization over time. The problem considered here involved decisions at a series of stages, where the effect of each decision itself influenced a series of stages. This doubly dynamic nature of the decision process was easily reducible to terms which dynamic programming could handle. The emphasis throughout such a dynamic programming analysis as this is on the sequential aspects of the decision process: this appears to be a more versatile representation of time-dependant decision problems than that afforded by the " once-over " approach of, say, a dynamic linear programme.

This leads to the third major attractive characteristic of dynamic programming, the generality of the results which it yields. It is possible to compute in one single dynamic programming run, optimal policies for a complete range of number of stages and resource availabilities. The results thus exhibit a generality of application which linear programming lacks. This feature also enables the experimenter to examine for specific problems a variety of " sub-optima " where it is thought that an overall optimum is unattainable due to risk aversion, capital restrictions, or other factors relevant for a particular situation.

Turning to major limitations of dynamic programming in problems of agricultural allocation, the most important is the severe restriction on the number of alternative allocations which can be considered at each stage of the programme. In our problem, maximization was over the domain of only one variable at each stage, and hence computing difficulties were not struck. However, it would not have been possible to extend this over more than about two or three dimensions if feasible computing times were not to be exceeded. In other words if it were desired to optimize multi-stage allocations of land to pasture improvement by, say, several different establishment methods, this type of dynamic programming model would be inapplicable if more than about three methods were being considered. Hence for multi-activity multi-resource optimizations linear programming and its variants probably remain the most appropriate techniques. Dynamic programming is only feasible if the number of alternatives at each stage can be refined down to a small number without losing too much realism.

Secondly, feedback problems may present difficulties. In single-dimensional deterministic models these obstacles may be sidestepped by assumption or simple manipulation of the functions (as above) or by incorporating a feedback term into recurrence relations (see further below). In stochastic or multi-dimensional models, however, the existence of feedback can complicate model construction and increase computing times, possibly to the extent of making such models computationally infeasible.

(iii) Further Development

In this study we have constructed a simple model and examined in some detail the structure of the decision process it represents. To conclude, let us consider the question: what is the potential for extending this sort of analysis? Essentially there are two possibilities: one is to seek more realistic models, the other is to find better ways of quantifying theoretical models we already have. There seems no clearcut choice. Certainly the high data demands of dynamic programming where non-linear functions and stochastic models are used suggest a *prime facie* case for improved data collection and assembly. This is reinforced by the fact that current tools such as linear programming are proving successful in handling an increasingly wide range of dynamic and other problems. This has been demonstrated in the case of pasture improvement, for instance, by the recent work of Pearse.[56] However, it is not necessarily axiomatic that building more complicated, more realistic models makes data requirements more complicated or more extensive. For example, research currently being undertaken by the present author suggests that some dynamic programming models can be organized to make better use of the existing amount of data needed in some linear programming analyses. In any case there seems considerable scope for the use of dynamic programming models as a means of clarifying the *structure* of decision processes with which farmers are faced. In other words it is argued that building and testing more comprehensive models is likely to increase our conceptual (if not as yet our real-world-empirical) understanding of some aspects of normative decision making at the farm level.

As a suitable coda to this paper we consider finally some ways in which the dynamic programming section of the above simple model might be extended towards a more realistic representation of the pasture improvement process, following the lines suggested in the last paragraph. Some recurrence relations are given below whose empirical workings the reader should be able to deduce, using the principles outlined earlier in this paper.

Equation (22) shows the straightforward incorporation of a feedback term into the original recurrence relation of equation (3). Notation is the same as for (3):

$$(22) \qquad f_N(x) = \max_{y} \left[g(y) + h(x - y) + f_{N-1}(x - y) + f_{N-r}(y) \right]$$

where r = number of periods land is " tied up " under the one pasture sward.

---

[56]R. A. Pearse, " An Example of the Use of Linear Programming to Study Credit Requirements for Pasture Improvement ", (Paper presented to the Annual Conference of the Australian Agricultural Economics Society, Sydney, February, 1963) and " Financial Returns and Capital Requirements for Optimum Pasture Improvement Plans ", this *Review*, Vol. 31, No. 4, (December, 1963). See also the earlier work of F. H. Gruen, " Pasture Improvement—The Farmer's Economic Choice ", *Australian Journal of Agricultural Economics*, Vol. 3, No. 2, (December, 1959), pp. 19-44.

Note that in this and subsequent recurrence relations, " infeasible " terms drop out; for example when $N-r \leqslant 0$, the term $f_{N-r}(y)$ becomes "infeasible" and disappears (c.f. the reduction of equation (3) to equation (2), where $f_{N-1}(x - y)$ becomes infeasible in the last stage of the process).

This formulation could be extended further to encompass two different methods of pasture establishment:

$$(23) \qquad f_N(x) = \max_{y,\, z} \left[g(y) + g'(z) + h(x - y - z) + \dots \right]$$

where $g'(z)$ = return function for an allocation of z units of improved pasture using the second establishment method.

However, a more satisfactory representation can be obtained by recasting the recurrence relation completely so as to account for a *series* of " activities " (establishment methods) in a series of stages. This disposes of the awkward two-dimensionality of equation (23), yet allows any number of establishment methods to be considered. However, some new computational problems are introduced, for instance this model must be specifically constrained in order that certain types of infeasible policies are not generated. The recurrence relation (using a revised notation) is shown in (24):

$$(24) \quad f_{jk}(x_k) = \max_{y} \left[g_{jk}(y) + f_{j+1,k}(x_k - y) + f_{1,k+r_j}(y) \right]$$

where $f_{jk}(x)$ = total return over establishment methods $j, j + 1, \dots, J$ and periods $k, k + 1, \dots, K$ beginning in period k with a quantity of land x_k and using an optimal allocation policy;

$g_{jk}(y)$ = return function for an allocation of y units of land to pasture via establishment method j in period k;

r_j = number of periods for which pasture remains down when established by method j.

The rephrasing involved in this model overcomes the " dimensionality " problem to the extent that computational time is only linearly related to the number of establishment methods and number of periods considered, and these quantities may thus be regarded as virtually unconstrained. One is therefore free to incorporate a further *resource* into the model (leading to a two-dimensional maximization at each iteration). In the case of pasture improvement, this extra resource might be capital, and return functions would then be expressed in matrix form, the general element g_{pq} being the total net return for a given establishment method in a given year for p units of land and q units of capital. The special theoretical and computational problems raised by such models will be treated in a later article.

Sydney: V. C. N. Blight, Government Printer—1964