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Intertemporal Risk Aversion
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Wouldn't it be Nice to Tell Whether Robinson Crusoe is Risk Averse?*

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Abstract: The paper introduces a new notion of risk aversion that is independent of the good under observation and its measure scale. The representational framework builds on a time consistent combination of additive separability on certain consumption paths and the von Neumann & Morgenstern (1944) assumptions. In the one-commodity special case, the new notion of risk aversion closely relates to a disentanglement of standard risk aversion and intertemporal substitutability.

Keywords: uncertainty, expected utility, recursive utility, risk aversion, intertemporal substitutability, certainty additivity, temporal lotteries, gauge-freedom, intertemporal risk aversion

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1 Introduction

The paper introduces a new concept of risk aversion that is independent of the good under observation and its measure scale. To these ends, I introduce a new representation theorem for preferences that coincide with the intertemporally additive standard model when restricted to certain consumption paths, and that respect the von Neumann & Morgenstern (1944) axiom for uncertain choices in every period. The representation has a strong symmetry in time and risk and non-linearities can be shifted from one dimension to the other.

1.1 Robinson Crusoe's Motivation

More likely than not, Robinson Crusoe was among the first economic agents you encountered when introduced to economics. Have you ever wondered whether Robinson was risk averse? Have you asked yourself how to measure risk aversion when stranded on a lonely island where money is of no value? Robinson's risk aversion, for example, could be measured with respect to lotteries over coconut consumption. The standard theory works great for coconuts. They are, at least with some effort, an arbitrarily divisible good that comes with a natural unit. And, if we want to measure the coefficient of relative risk aversion, also 'the zero coconut level' is well defined. However, what happens if Robinson also finds litchis on the island? They are similarly well suited for measurement. However, with the standard concept of risk aversion, Robinson might well be risk averse with respect to coconut consumption and turn out risk loving with respect to litchis. If we think of risk aversion as an attitude toward risk, rather than toward coconuts or litchis, this classification of Robinson's risk aversion might be unsatisfactory. It gets worse when Robinson finds that not all coconuts taste the same. Assume he elaborates a chart pinning down a quality distribution of different trees to decide which of them is most worthy climbing up. For coconut quality there is no natural unit, nor is there a naturally given 'zero quality level'. In fact, whether Robinson is judged (Arrow Pratt) risk averse or risk loving with respect to decisions involving coconut quality is completely up to the measure scale he employs in the quality chart.

The quest for such an 'isolated' concept of risk aversion is about much more than an islander's decision making. The developed concept of *intertemporal risk aversion* isolates the measure of an individual's risk attitude from his valuation for money, which is the outcome of a market aggregation depending on other agent's preferences as well as the supply side of the economy.

1.2 Contribution and Relation to the Literature

The paper delivers two equivalent axiomatic characterizations of intertemporal risk aversion. In contrast to Arrow Pratt risk aversion, intertemporal risk aversion is independent of the good under observation or its measure scale. To analyze good and measure scale dependence I analyze changes of the representations under appropriate coordinate transformations on the consumption manifold. I also show that the time and state additive standard model contains the implicit assumption of intertemporal risk neutrality.

The representational framework relates to the seminal work of Kreps & Porteus (1978), who extend the atemporal von Neumann & Morgenstern (1944) setting to a temporal lottery framework. Their representation can be interpreted as an extension of Koopmans' (1960) recursive utility model under certainty to a recursive model for risky settings. The present paper shows that, even when starting from a time-additive model for certain outcomes, the general time consistent model for the evaluation of risky outcomes exhibits recursivity. While Kreps & Porteus' (1978) representation is more general, the present representation has the following attractive features. First, the recursive intertemporal aggregation rules can be characterized by a family of one dimensional functions. Second, it shows a trade-off between linearizing uncertainty aggregation and intertemporal aggregation. Kreps & Porteus' (1978) representation uses expected value to evaluate uncertainty and relies on a nonlinear intertemporal aggregation.[2] The present representation makes it possible to transform nonlinear intertemporal aggregation into nonlinear uncertainty integration. The resulting certainty additive welfare function is helpful for economic intuition and for relating the analysis to e.g. models of ambiguity (Traeger 2010). Third, the representation theorem admits freedom in picking the evaluation function on the certain one period outcomes. I employ this freedom to examine measure scale and good dependence of risk measures.

Epstein & Zin (1989) and Weil (1990) analyze Kreps & Porteus' (1978) representation in a one commodity setting in order to disentangle information about the attitude with respect to risk and with respect to intertemporal substitutability.[3] Taking the model

[2] A nonlinear intertemporal aggregation implies that a welfare gain of one unit today and a welfare gain of another unit in the next period is not the same as welfare gain of two units in a third period. Note that the nonlinearity is different in nature than discounting in a stationary setting, where the discount factor just is part of the welfare evaluation.

[3] Such a distinction between risk aversion and intertemporal substitutability is not possible within a

back to the multi-commodity setting, I study measure scale and good dependence of the Arrow Pratt measure and show the invariance of the *intertemporal risk aversion* measure.

In three accompanying papers I relate the risk aversion concept developed here to different aspects of intertemporal choice, deriving the following insights. First, Kreps & Porteus' (1978) preference for the timing of uncertainty resolution can be explained by a change of absolute intertemporal risk aversion over time (Traeger 2007*b*). Second, the implicit assumption of intertemporal risk neutrality in the standard model removes the bite of risk stationarity. Only for this reason does the standard model contain the free parameter associated with pure time preference (Traeger 2007*a*).[4] Third, ambiguity aversion in the models of smooth ambiguity by Klibanoff, Marinacci & Mukerji (2005, 2009) can be understood as the difference between intertemporal risk aversion with respect to subjective and with respect to objective lotteries. This insight permits an immediate extension of the present model to incorporate ambiguity and extend the smooth ambiguity concept to a more general concept of aversion to the subjectivity of belief (Traeger 2010).

The paper is structured as follows. Section 2 develops the representation. Section 3 discusses measure scale and good dependence when extending Epstein & Zin's (1989) disentanglement of Arrow Pratt risk aversion and intertemporal substitutablity to the multi-commodity setting. Section 4 introduces the concept of intertemporal risk aversion. Section 5 concludes. All proofs are found in the appendix.

2 The Representation

The representation builds on the framework of temporal lotteries introduced by Kreps & Porteus (1978). It is a natural extension of the classical von Neumann & Morgenstern (1944) setting to an intertemporal framework. The employed recursive description of uncertainty is richer than the more widespread framework of atemporal lotteries, where probability measures are defined directly over consumption paths.[5]

standard intertemporally additive expected utility model. There, the Arrow-Pratt measure of relative risk aversion is confined to the inverse of the elasticity of intertemporal substitution (Weil 1990).

[4]This reasoning assumes indifference to the timing of uncertainty resolution in the sense of Kreps & Porteus (1978), which is another implicit assumption of the standard model that is not met in general recursive models.

[5]In Traeger (2007*b*) I discuss the economic differences between these two different settings in detail.

2.1 Setup and Notation

Let Y denote a generic connected compact metric space. Its elements are referred to as outcomes. The set of Borel probability measures on Y is denoted $P = \Delta(Y)$ and equipped with the Prohorov metric (giving rise to the topology of weak convergence). The paper takes uncertainty in form of unique probability measures as given.[6] Extension to a joint axiomatic framework in the sense of Savage (1954) or Anscombe & Aumann (1963) is straightforward, but only obstructs the essential contribution of the paper. A lottery yielding outcome y with probability $p(y) = \lambda$ and outcome y' with probability $p(y') = 1 - \lambda$ is written $\lambda y + (1 - \lambda)y' \in P$. Note that a 'plus' sign between outcomes always characterizes a lottery.[7] The set of degenerate lotteries in P is identified with the set Y of outcomes in the usual way. Preferences defined on P are denoted by $\succeq$ $(\subset P \times P)$.[8] The space of all real valued, continuous functions on Y is denoted by $\mathcal{C}^0(Y)$. For an element $v \in \mathcal{C}^0(Y)$ the notation $\text{range}(v) = [\underline{V}, \overline{V}] = V$ and $\Delta V = \overline{V} - \underline{V}$ is applied.[9]

Introducing time structure, the paper makes use of various compact metric spaces. For all of them the above definitions apply. The primitive connected compact metric space in this paper is denoted X and characterizes welfare determining factors within a period. Its outcomes, i.e. elements $x \in X$, are also referred to as points in consumption space. They can characterize consumption levels or more abstract descriptions of e.g. consumption quality, a state of mood or the state of an ecosystem. Time is discrete with planning horizon $T \in \mathbb{N}$. Individual periods are labeled by time indices $t, \tau \in \{1, ..., T\}$. The space $\mathsf{X}^t = X^{T-t+1}$ denotes the $(T-t+1)-$ fold Cartesian product equipped with the product metric. It characterizes the set of all certain consumption paths from period t to period T.[10] A consumption paths $\mathsf{x} \in \mathsf{X}^t$ is written $\mathsf{x} = (\mathsf{x}_t, \mathsf{x}_{t+1}, ..., \mathsf{x}_T) =$

[6]Well suited for the context of this paper and its applications is the epistemological foundation of probability in the line of Koopman (1940), Cox (1946,1961) and Jaynes (2003), who construct a probabilistic logic.

[7]As Y is only assumed to be a compact metric space there is no immediate addition defined for its elements. In case it is additionally equipped with a vector space or field structure, the vector composition will not coincide with the "+" used here. The "+" sign used here alludes to the additivity of probabilities.

[8]The relations $\succeq$ are required to be reflexive. The asymmetric part is denoted by $\succ$ and interpreted as strict preference. The symmetric part of the relation $\succeq$ is denoted by $\sim$ and interpreted as indifference. Nonindifference is denoted by $\not\sim$ and defined as $\not\sim \equiv P \times P \setminus \sim$.

[9]Compactness of Y and continuity of v assure that the minimum and the maximum are attained.

[10]I do not distinguish different sets of outcomes for different periods. X stands for the union of all

$(x_t, x_{t-1}, ..., x_T)$. Given $\mathsf{x} \in \mathsf{X}^t$, I define $(\mathsf{x}_{-i}, x) = (\mathsf{x}_t, ..., \mathsf{x}_{i-1}, x, \mathsf{x}_{i+1}, ..., \mathsf{x}_T) \in \mathsf{X}^t$ as the consumption path that coincides with x in all but the i^{th} period, in which it renders outcome x. The uncertain choice objects of temporal lotteries are obtained by defining $\tilde{X}_T = X$ and recursively $\tilde{X}_{t-1} = X \times \Delta(\tilde{X}_t)$ for all $t \in \{2, ..., T\}$. Each $\tilde{X}_t$ is equipped with the product metric. I denote $P_t = \Delta(\tilde{X}_t)$ and refer to the elements $p_t \in P_t$ as (period t) lotteries. Observe that in every period the decision maker has a probability distribution over the outcome in the respective period and the probability distribution over the future faced in the next period. Preferences in period t are defined on the set P_t and denoted by $\succeq_t$.

The group of non-degenerate affine transformations is denoted $\mathsf{A} = \{\mathfrak{a} \in \mathcal{C}^0(\mathbb{R}) : \mathfrak{a}(z) = a\,z + b\,,\ a, b \in \mathbb{R}, a \neq 0\}$ with elements $\mathfrak{a} \in \mathsf{A}$. The group of strictly positive affine transformations is denoted $\mathsf{A}^+ = \{\mathfrak{a}^+ \in \mathcal{C}^0(\mathbb{R}) : \mathfrak{a}^+(z) = a\,z + b\,,\ a, b \in \mathbb{R}, a > 0\}$. Furthermore, for a given $a \in \mathbb{R}_{++}$ define $\mathsf{A}^a = \{\mathfrak{a}^a \in \mathcal{C}^0(\mathbb{R}) : \mathfrak{a}^a(z) = a\,z + b\,,\ b \in \mathbb{R}\}$.[11] For compositions of two functions I write $f(g(\cdot)) = f \circ g(\cdot) = fg(\cdot)$.[12]

2.2 Employed Concepts

The first concept employed in the representations is that of a *Bernoulli utility* function. Given a preference relation $\succeq_t$ on P_t for some $t \in \{1, ..., T\}$, I introduce a binary relation $\succeq_t^*$ on X by defining for all $x, x' \in X$

$$x \succeq_t^* x' \quad \Leftrightarrow \quad (\mathsf{x}_{-t}, x) \succeq_t (\mathsf{x}_{-t}, x') \qquad \forall\, \mathsf{x} \in X^t\,.$$[13]

Define the set of Bernoulli utility functions corresponding to the preference relation $\succeq_t$ by $B_{\succeq_t} = \{u_t \in \mathcal{C}^0(X) : x \succeq_t^* x' \Leftrightarrow u_t(x) \geq u_t(x')\, \forall\, x, x' \in X\}$. Bernoulli utility functions represent preferences over certain outcomes within a period.

The second concept relates to the aggregation of utility over uncertainty. Given a strictly monotonic function $f \in \mathcal{C}^0(\mathbb{R})$ I define an *uncertainty aggregation rule* as the

possible outcomes perceivable in any period.

[11] $\mathbb{R}_+ = \{z \in \mathbb{R} : z \geq 0\}$ and $\mathbb{R}_{++} = \{z \in \mathbb{R} : z > 0\}$ denote the sets of all positive, respectively strictly positive, real numbers.

[12] The omission of the composition sign in lengthy expressions shall not create confusion as regular multiplication of functions only appears between fractions.

[13] An assumption of additive separability will turn $\succeq_t^*$ into a complete order on X. For a space $P = \Delta(Y)$ without time structure the definition implies $\succeq^* = \succeq|_Y$.

functional $\mathcal{M}^f : \Delta(Y) \times \mathcal{C}^0(Y) \to \mathrm{I\!R}$ with[14]

$$\mathcal{M}^f(p, v) = f^{-1} \int_Y dp\, f \circ v \ .$$

The uncertainty aggregation rule takes as input the decision maker's perception of uncertainty, expressed by a probability measure p on Y, and a valuation of certain outcomes expressed by a real valued function v on Y. The uncertainty aggregation rule weighs utility values by some function f, aggregates them, and applies the inverse of f to normalize the resulting expression. For certain outcomes an uncertainty aggregation rule returns the value of v itself, i.e. $\mathcal{M}^f(y, v) = v(y)$. The only difference between an uncertainty aggregation rule and a generalized mean is that the former takes the valuation function v as an explicit argument.[15]

The simplest example of an uncertainty aggregation rule corresponds to the expected value operator which is induced by the arithmetic mean ($f = \mathrm{id}$). A widespread non-trivial example is obtained by choosing $f(z) = z^\alpha$ (power mean). Then, for $V \subseteq \mathrm{I\!R}_+$ the following uncertainty aggregation rule obtains:

$$\mathcal{M}^\alpha(p, v) \equiv \mathcal{M}^{\mathrm{id}^\alpha}(p, v) = \left[\int_Y dp\ v^\alpha \right]^{\frac{1}{\alpha}} \ .$$

It is defined for $\alpha \in \mathrm{I\!R}$ with $\mathcal{M}^0(p, v) \equiv \lim_{\alpha \to 0} \mathcal{M}^\alpha(p, v) = \exp\left[\int_Y dp \ln v\right]$.[16] Note that $\mathcal{M}^0(p, v)$ corresponds to a continuous form of the geometric mean, which takes the standard form $\mathcal{M}^0(p, v) = \prod_y v(y)^{p(y)}$ for simple probability measures. In the limit of α going to plus or minus infinity, the uncertainty aggregation rule $\mathcal{M}^\alpha$ only considers the extreme outcomes (abandoning continuity in the probabilities): $\mathcal{M}^\infty(p, v) \equiv \lim_{\alpha \to \infty} \mathcal{M}^\alpha(p, v) = \max_y v(y)$ and $\mathcal{M}^{-\infty}(p, v) \equiv \lim_{\alpha \to -\infty} \mathcal{M}^\alpha(p, v) = \min_y v(y)$. In general it can be shown that the smaller is α, the lower is the certainty equivalent utility that the respective uncertainty aggregation rule brings about (e.g. Hardy, Littlewood & Polya 1964, 26).

The third concept employed in the representation is that of an *intertemporal aggre-*

[14]By continuity of $f \circ v$ and compactness of Y, Lesbeque's dominated convergence theorem ensures integrability (Billingsley 1995, 209).

[15]This correspondence is made precise as follows. Let $p^v \in \Delta(V)$ denote the probability measure induced by p defined on Y through the function $v \in \mathcal{C}^0(Y)$ on its (compact) range V. Then an uncertainty aggregation rule $\mathcal{M}$ is said to be induced by a mean $\overline{\mathcal{M}} : \Delta(V) \to \mathrm{I\!R}$, whenever $\mathcal{M}(p, v) = \overline{\mathcal{M}}(p^v)\ \forall p \in P$. Mean inducedness implies that only the probability of y is used to weigh $v(y)$.

[16]The easiest way to recognize the limit for $\alpha \to 0$ is to note that for any $\alpha > 0$ the function $f_\alpha(z) = \frac{z^\alpha - 1}{\alpha}$ is an affine transformation of $f(z) = z^\alpha$. Affine transformations leave the uncertainty aggregation rule unchanged. Therefore, the fact that $\lim_{\alpha \to 0} \frac{z^\alpha - 1}{\alpha} = \ln(z)$ gives the result.

gation rule. The assumption of additive separability on certain consumption paths will allow me to bring intertemporal aggregation to a similar mean-like form. However, two differences with respect to uncertainty aggregation apply. First, when evaluating lotteries, aggregation over time will generally turn out to be recursive. Second, time aggregation is generally period specific.[17] Using a sequence of time dependent weight functions $\mathbf{g} = (g_t)_{t\in\{1,...,T\}}$ for utility levels in $U_t \subset \mathrm{I\!R}$ with $g_t \in \mathcal{C}^0(U_t)\,\forall\, t \in \{1, ..., T\}$ the aggregation of utility over time can be characterized in the form '$g_t^{-1}\left[g_t(\cdot) + g_{t+1}(\cdot)\right]$'. Given some utility level from consumption in period t and an overall utility level in period $t+1$, both are aggregated with period specific weight functions and, by taking the inverse, normalized back into the period t utility scale. However, the expression as is would be ill defined because values in the ranges of g_t and g_{t+1} generally do not add up to values that lie in the domain of g_t^{-1}. Introducing the necessary normalization yields the *intertemporal aggregation rule* for period t

$$\mathcal{N}_t^{\mathrm{g}} \quad : \; U_t \times U_{t+1} \to \mathrm{I\!R} \tag{1}$$

$$\mathcal{N}_t^{\mathrm{g}}(\cdot,\cdot) = g_t^{-1}\left[\theta_t\, g_t(\cdot) + \theta_t\theta_{t+1}^{-1}\, g_{t+1}(\cdot) + \theta_t\theta_{t+1}^{-1}\,\vartheta_t\right] \;, \tag{2}$$

where $U_t, U_{t+1} \subset \mathrm{I\!R}$ and the normalization constants are defined as

$$\theta_t = \frac{\Delta G_t}{\sum_{\tau=t}^{T}\Delta G_\tau} \quad \text{and} \quad \vartheta_t = \frac{\overline{G}_{t+1}\underline{G}_t - \underline{G}_{t+1}\overline{G}_t}{\Delta G_t}\,. \tag{3}$$

Generally, the representations will allow for a choice where $\underline{G_t} = 0\,\forall\, t$. Then, the intertemporal aggregation rule simplifies to

$$\mathcal{N}_t^{\mathrm{g}}(\cdot,\cdot) = g_t^{-1}\left[\theta_t\, g_t(\cdot) + \theta_t\theta_{t+1}^{-1}\, g_{t+1}(\cdot)\right] \qquad \text{with} \qquad \theta_t = \frac{\overline{G}_t}{\sum_{\tau=t}^{T}\overline{G}_\tau}\,.$$

The constants θ_t characterize the weight of an individual period with respect to the future.[18] In a stationary model they give rise to normalized discount rates, e.g. for $T = 2$, $g_1 = g$ and $g_2 = \beta g$ it is

$$\mathcal{N}_1^{\mathrm{g}}(\cdot,\cdot) = g^{-1}\left[\tfrac{1}{1+\beta}\, g(\cdot) + \tfrac{\beta}{1+\beta}\, g(\cdot)\right]\,.$$

[17] While the independence axiom implies state independence for uncertainty aggregation, additive separability over time does not imply time independence of intertemporal aggregation.

[18] In a representation where the $\underline{G_t}$ are not normalized and positive, the constants ϑ_t characterize an overproportional upwardspread of future weight. Precisely, define constants $\overline{\gamma}$ and $\underline{\gamma}$ by the relations $\overline{G}_{t+1} = \overline{\gamma}\,\overline{G_t}$ and $\underline{G}_{t+1} = \underline{\gamma}\,\underline{G}_t$. Then $\vartheta_t > 0$ is equivalent to $\overline{\gamma} > \underline{\gamma}$.

2.3 Atemporal Uncertainty and Gauge Freedom

This subsection revisits the atemporal von Neumann-Morgenstern setting. A useful perspective on the study is as follows. Choice in a certain and atemporal setting determines the utility function on the certain outcomes only up to strictly increasing transformations. Introducing uncertainty, von Neumann & Morgenstern (1944) single out a particular cardinal utility function evaluating the certain outcomes, by requiring that expected value maximization should represent choice over lotteries. However, intertemporal considerations can cardinalize utility already in a certain setting. Given a cardinal evaluation of certain outcomes, additive uncertainty aggregation rules no longer suffice to represent all decision rules conforming with the von Neumann-Morgenstern axioms.

For a slightly different perspective, I introduce a notion borrowed from physics. A degree of freedom that has no observable effect within a theory is called gauge freedom. It is a freedom to normalize. Analyzing this freedom, instead of choosing a normalization right away, can deliver deeper insights into the model. I will make use of this idea when analyzing good dependence and invariance of risk measures. In the meanwhile, carrying along the gauge freedom of Bernoulli utility allows to derive different representational forms that will prove useful for different inquiries.

The following theorem is a variation of von Neumann & Morgenstern's (1944) famous representation theorem, here on the general connected compact metric space Y. For the simplest interpretation think of Y as the consumption space X. The proof of the intertemporal representation will employ the theorem recursively with Y standing for the spaces $\tilde{X}_t$. In the present, atemporal version, the set of Bernoulli utility functions is simply $B_\succeq = \{v \in \mathcal{C}^0(Y) : y \succeq y' \Leftrightarrow v(y) \geq v(y') \, \forall \, y, y' \in Y\}$.

Theorem 1 (Variation of von Neumann-Morgenstern): Given a binary relation $\succeq$ on P and a Bernoulli utility function $v \in B_\succeq$ with range V, the relation $\succeq$ satisfies the axioms

A1 (weak order) $\succeq$ is transitive and complete, i.e.:

- transitive: $\forall \, p, p', p'' \in P : \; p \succeq p'$ and $p' \succeq p'' \Rightarrow p \succeq p''$
- complete: $\forall \, p, p' \in P : p \succeq p'$ or $p' \succeq p$

A2 (independence) $\forall \, p, p', p'' \in P :$

$$p \sim p' \quad \Rightarrow \quad \lambda \, p + (1-\lambda) \, p'' \sim \lambda \, p' + (1-\lambda) \, p'' \quad \forall \, \lambda \in [0,1]$$

A3 (continuity) $\forall p \in P : \{p' \in P : p' \succeq p\}$ and $\{p' \in P : p \succeq p'\}$ are closed in P

if and only if, there exists a strictly monotonic and continuous function $f : V \to \mathbb{R}$ such that for all $p, p' \in P$

$$p \succeq p' \Leftrightarrow \mathcal{M}^f(p, v) \geq \mathcal{M}^f(p', v). \tag{4}$$

Moreover, f and f' both represent $\succeq$ in the above sense, if and only if, there exists $\mathfrak{a} \in \mathsf{A}$ such that $f' = \mathfrak{a}f$.

Axioms A1-A3 are standard, for a discussion I refer to Kreps (1988). The indeterminacy of f up to affine transformations does not translate into an indeterminacy of the functional $\mathcal{M}^f$. A non-degenerate affine transformation $f' = \mathfrak{a}f$ yields the same uncertainty aggregation rule as the one implied by f itself, i.e. $\mathcal{M}^f(\cdot, \cdot) = \mathcal{M}^{f'}(\cdot, \cdot)$.[19] In consequence, the theorem can be stated as well using only increasing versions of f. The proof of theorem 1 also shows that for preference relations $\succeq$ satisfying axioms A1-A3 the set of Bernoulli utility functions for $\succeq$ is non-empty.

In the atemporal setting, choice under certainty only renders ordinal information on the Bernoulli utility function v. Theorem 1 states that this gauge freedom for Bernoulli utility v translates into the representing uncertainty aggregation rule through the form of the parameterizing function f. Taking this correspondence the other way round one obtains

Corollary 1: For any strictly monotonic, continuous function $f : \mathbb{R} \to \mathbb{R}$ the following assertion holds:
A binary relation $\succeq$ on P satisfies axioms *A1-A3*, if and only if, there exists a continuous function $v : Y \to \mathbb{R}$ such that

$$\forall\, p, p' \in P : \qquad p \succeq p' \Leftrightarrow \mathcal{M}^f(p, v) \geq \mathcal{M}^f(p', v). \tag{5}$$

Moreover, v and v' both represent $\succeq$ in this sense above, if and only if there exists $\mathfrak{a}^+ \in \mathsf{A}^+$ such that $u = f^{-1}\mathfrak{a}^+ f\, u'$.

For $f = \text{id}$, where $\mathcal{M}^f(p, v) = \mathrm{E}_p\, v$, the corollary states the classical von Neumann & Morgenstern (1944) theorem. Then, v is unique up to positive affine transformations. However, the corollary delivers a similar representation theorem for all uncertainty aggregation rules $\mathcal{M}^f$. For example, setting $f = \ln$, the uncertainty aggregation rule in

[19]This relation holds, because the linearity of the integral implies that the inverse f'^{-1} cancels out the affine displacement caused by f'.

the representation corresponds to the geometric mean. Here, the remaining freedom of v is expressed by the group of transformations $u \to c\,u^d\,,\; c,d \in \mathbb{R}_{++}$.[20] Recall that $f^{-1}\mathfrak{a}^+ f\, u'$ describes the composed function $f^{-1} \circ \mathfrak{a}^+ \circ f \circ u'$ and not a multiplication of values. Note that equation (5) uses f only on the restricted domain U. I could also define $f : U \to \mathbb{R}$ on a nondegenerate interval U and require $u : X \to U$ to be surjective. Then, the representing u in equation (5) is unique (see analysis in section 4).

2.4 Intertemporal Representation

This subsection extends theorem 1 to the intertemporal setting. The two additional assumptions imposed for time structure are additive separability on certain consumption paths and time consistency. For convenience of presentation, I also assume that every period involves essential choice alternatives:

A0 (non-degeneracy) For all $t \in \{1, ..., T\}$ there exist $\mathrm{x} \in \mathrm{X}^1$ and $x \in X$ such that

$$(\mathrm{x}_{-t}, x) \not\sim_1 \mathrm{x}\,.$$

In order to match the predominant time-additive framework for certain intertemporal choice I assume additive separability on certain consumption paths. I employ the axiomatization of Wakker (1988).[21]

A4 (certainty separability)

i) For all $\mathrm{x}, \mathrm{x}' \in \mathrm{X}^1$, $x, x' \in X$ and $t \in \{1, ..., T\}$:

$$(\mathrm{x}_{-t}, x) \succeq_1 (\mathrm{x}'_{-t}, x) \;\Leftrightarrow\; (\mathrm{x}_{-t}, x') \succeq_1 (\mathrm{x}'_{-t}, x')$$

ii) If $T = 2$ additionaly: For all $x_t, x'_t, x''_t \in X$, $t \in \{1, 2\}$

$$(x_1, x_2) \sim_1 (x'_1, x''_2) \;\wedge\; (x'_1, x'_2) \sim_1 (x''_1, x_2) \;\Rightarrow\; (x_1, x'_2) \sim_1 (x''_1, x''_2)$$

Wakker (1988) calls part *i*) of the axiom coordinate independence. It requires that the choice between two consumption paths does not depend on period t consumption, whenever the latter coincides for both paths. Part *ii*) is known as the Thomsen condition.

[20] Setting $f = \ln$ implies the remaining freedom $u = f^{-1}\mathfrak{a}^+ f\, u' = e^{a \ln(u') + b} = u'^a e^b$ with $a > 0$.

[21] Other axiomatizations of additive separability include Koopmans (1960), Krantz, Luce, Suppes & Tversky (1971), Jaffray (1974*a*), Jaffray (1974*b*), Radner (1982), and Fishburn (1992).

It is required only if the model is limited to $T = 2$ periods.[22] Axiom 4 is the main ingredient to allow for a certainty additive representation of the form $\sum_{t=1}^{T} u_t^{ca}(\mathrm{x}_t)$.

Preferences in different periods are related by the following consistency assumption adapted from Kreps & Porteus (1978).

A5 (time consistency) For all $t \in \{1, ..., T-1\}$:

$$(x_t, p_{t+1}) \succeq_t (x_t, p'_{t+1}) \quad \Leftrightarrow \quad p_{t+1} \succeq_{t+1} p'_{t+1} \quad \forall\, x_t \in X,\; p_{t+1}, p'_{t+1} \in P_{t+1}\,.$$

It is a requirement for choosing between two consumption plans in period t, which yield a degenerate lottery with a coinciding entry in the respective period. For these choice situations, axiom A5 demands that in period t, the decision maker prefers the plan that gives rise to the lottery that is preferred in period $t+1$.

The recursive application of theorem 1 under assumptions A4 and A5 renders the intertemporal representation. In every step, the uncertainty aggregation rule is applied to the space $P_t = \Delta(\tilde{X}_t)$ employing a recursively constructed aggregate utility function $\tilde{u}_t \in \mathcal{C}^0(\tilde{X}_t)$ to evaluate the degenerate outcomes $\tilde{x}_t \in P_t$.

Theorem 2: Let a sequence of preference relations $\succeq \equiv (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfy axiom A0. Let a sequence of functions $\boldsymbol{u} \equiv (u_t)_{t\in\{1,...,T\}}$ satisfy $u_t \in B_{\succeq_t}$. Then, the sequence of preference relations $(\succeq_t)_{t\in\{1,...,T\}}$ satisfies

i) A1-A3 for all $\succeq_t, t \in \{1, ..., T\}$ (vNM setting)

ii) A4 for $\succeq_1$ (certainty additivity)

iii) A5 (time consistency)

if and only if, for all $t \in \{1, ..., T\}$ there exist strictly increasing[23] and continuous functions $f_t : U_t \to \mathbb{R}$ and $g_t : U_t \to \mathbb{R}$ such that with defining the functions $\tilde{u}_t : \tilde{X}_t \to \mathbb{R}$ recursively by $\tilde{u}_T(x_T) = u_T(x_T)$ and

$$\tilde{u}_t(x_t, p_{t+1}) = \mathcal{N}_t^{\mathrm{g}}\left(\, u_t(x_t)\,,\; \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1})\,\right) \tag{6}$$

it holds for all $t \in \{1, ..., T\}$ that

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \geq \mathcal{M}^{f_t}(p'_t, \tilde{u}_t) \quad \forall\, p_t, p'_t \in P_t\,. \tag{7}$$

---

[22]In the case of two periods parts *i)* and *ii)* can also be replaced by the single requirement of triple cancellation (see Wakker 1988, 427).

[23]Alternatively the theorem can be stated replacing increasing by monotonic for $(f_t)_{t\in\{1,...,T\}}$ and demanding that either all $(g_t)_{t\in\{1,...,T\}}$ are strictly increasing or that all are strictly decreasing.

Moreover, $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ and $(u_t, f'_t, g'_t)_{t\in\{1,...,T\}}$ both represent $\succeq$ in the above sense, if and only if, for some $a \in \mathbb{R}_{++}$ there exist $\mathfrak{a}^a_t \in \mathsf{A}^a$ and $\mathfrak{a}^+_t \in \mathsf{A}^+$ for all $t \in \{1, ..., T\}$ such that $(f'_t, g'_t) = (\mathfrak{a}^+_t f_t, \mathfrak{a}^a_t g_t)$.

I call a sequence of triples $(\boldsymbol{u}, \boldsymbol{f}, \boldsymbol{g}) \equiv (u_t, f_t, g_t)_{t\in\{1,...,T\}}$ as above a representation in the sense of theorem 2 of the set of preference relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$. The representation theorem recursively constructs an aggregate utility $\tilde{u}_t$ that depends on the utility gained from the outcome in the respective period $u_t(x_t)$ and the aggregate utility derived from a particular lottery p_{t+1} over the future. While the lottery over the future is evaluated by means of the uncertainty aggregation rule, aggregation over time employs the intertemporal aggregation rule $\mathcal{N}^{\mathrm{g}}_t$.

2.5 Gauging

Like in section 2.3, the freedom to choose the Bernoulli utility function renders some gauge freedom to the representation in theorem 2. The following lemma holds.

Lemma 1: Let $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ represent $(\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 2. For all $t \in \{1, ..., T\}$ let $s_t : U_t \to \mathbb{R}$ be a strictly increasing and continuous transformations. Then, also the sequence of triples $(u'_t, f'_t, g'_t) = (s_t \circ u_t, f_t \circ s_t^{-1}, g_t \circ s_t^{-1})_{t\in\{1,...,T\}}$ represents $(\succeq_t)_{t\in\{1,...,T\}}$.

Similar to corollary 1 in section 2.3, I can employ the current lemma to gauge the uncertainty aggregation rules in theorem 2 to any desired form that is parameterized by a sequence of strictly monotonic and continuous functions.

Corollary 2 (f-gauge) :

For any sequence of strictly increasing and continuous functions $\boldsymbol{f} = (f_t)_{t\in\{1,...,T\}}$ with $f_t : \mathbb{R} \to \mathbb{R}$ the following equivalence holds:

A sequence of preference relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfying axiom A0, satisfies axioms A1-A5, if and only if, for all $t \in \{1, ..., T\}$ there exist continuous functions $u_t : X \to \mathbb{R}$ as well as strictly increasing and continuous functions $g_t : U_t \to \mathbb{R}$ such that with defining equation (6) the representation (7) of theorem 2 holds.

Moreover, $(u_t, g_t)_{t\in\{1,...,T\}}$ and $(u'_t, g'_t)_{t\in\{1,...,T\}}$ both represent $\succeq$ in the above sense, if and only if, for some $a \in \mathbb{R}_{++}$ there exist affine transformations $\mathfrak{a}^+_t \in \mathsf{A}^+$ and $\mathfrak{a}^a_t \in \mathsf{A}^a$ for all $t \in \{1, ..., T\}$ such that $(u'_t, g'_t) = (f_t^{-1} \mathfrak{a}^+_t f_t \, u_t \, , \, \mathfrak{a}^a_t \, g_t \, f_t^{-1} {\mathfrak{a}^+_t}^{-1} f_t)$.

Choosing all functions f_t as the identity, corollary 2 yields the normalization, i.e. gauge, implicitly used by Kreps & Porteus (1978). Setting $f_t = \text{id}$ implies that the uncertainty aggregation rule becomes additive, i.e. expected utility, and the characterizing equations (6) and (7) of the representation write as

Kreps Porteus gauge *($f = \text{id}-$gauge)* :

$$\tilde{u}_t(x_t, p_{t+1}) \;=\; \mathcal{N}_t^{\text{g}}\left(\; u_t(x_t)\;,\; \text{E}_{p_{t+1}}\;\tilde{u}_{t+1}\;\right)$$

$$p_t \;\succeq_t\; p_t \quad \Leftrightarrow \text{E}_{p_t}\,\tilde{u}_t \;\geq\; \text{E}_{p'_t}\,\tilde{u}_t\,.$$

Note that Kreps & Porteus (1978) do not demand additive separability on certain consumption paths in the sense of axiom A4. Therefore, they obtain a slightly more general intertemporal aggregation rule. In the notion of Johnsen & Donaldson (1985), Kreps & Porteus (1978) axiomatization implies conditional strong independence, while the axioms of this paper imply unconditional strong independence. The latter step allows me to characterize intertemporal aggregation by a sequence of one dimensional functions $\boldsymbol{g}$. While uncertainty aggregation is linear in the Kreps Porteus gauge, utility between different periods generally has to be aggregated nonlinearly.

Alternatively, I can choose Bernoulli utility in a way to make time aggregation linear. Stepping stone is the following

Corollary 3 (g-gauge) :

For any sequence of strictly increasing and continuous functions $\boldsymbol{g} = (g_t)_{t\in\{1,...,T\}}$ with $g_t : \mathbb{R} \to \mathbb{R}$ the following equivalence holds:

A sequence of preference relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ on $(P_t)_{t\in\{1,...,T\}}$ satisfying axiom A0, satisfies axioms A1-A5, if and only if, for all $t \in \{1, ..., T\}$ there exist continuous functions $u_t : X \to \mathbb{R}$ as well as strictly increasing and continuous functions $f_t : U_t \to \mathbb{R}$ such that with defining equation (6) the representation (7) of theorem 2 holds.

Moreover, $(u_t, f_t)_{t\in\{1,...,T\}}$ and $(u'_t, f'_t)_{t\in\{1,...,T\}}$ both represent $\succeq$ in the above sense, if and only if, for some $a \in \mathbb{R}_{++}$ there exist affine transformations $\mathfrak{a}_t^+ \in \mathsf{A}^+$ and $\mathfrak{a}_t^a \in \mathsf{A}^a$ for all $t \in \{1, ..., T\}$ such that $(u'_t, f'_t) = (g_t^{-1}\mathfrak{a}_t^a g_t\, u_t\,,\, \mathfrak{a}_t^+ f_t\, g_t^{-1}\mathfrak{a}_t^{a-1}\, g_t)$.

Choosing the functions g_t as the identity for all $t \in \{1, ..., T\}$ yields the *certainty additive gauge*. This representation can be simplified by recognizing that the remaining freedom in choosing $\mathfrak{a}_t^a$ can be used to normalize $u_t = [0, \overline{U}_t]$ and the freedom in choosing $\mathfrak{a}_t^+$

can be used to absorb the normalization constants θ_t into the functions f_t.[24] Then, the characterizing equations (6) and (7) of the representation write as

Certainty additive gauge *($g = \text{id}-$gauge)* :

$$\tilde{u}_t(x_t, p_{t+1}) \;=\; u_t(x_t) + \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1})$$

$$p_t \;\succeq_t\; p'_t \quad \Leftrightarrow \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \geq \mathcal{M}^{f_t}(p'_t, \tilde{u}_t)\,.$$

In this gauge, uncertainty aggregation will generally be nonlinear and, thus, differ from taking the expected value.

Another gauge is possible whenever the outcome space is a one-dimensional subset of the reals, i.e. $X \subset \mathbb{R}$, and Bernoulli utility is strictly increasing in the consumption level $x \in X$. Then, the representing Bernoulli utility functions u_t in theorem 2 can be chosen as the identity. The representation corresponding to equations (6) and (7) is characterized by

Epstein Zin gauge *($u = \text{id}-$gauge, one commodity)* :

$$\tilde{u}_t(x_t, p_{t+1}) = \mathcal{N}_t^{\mathrm{g}}\left(\; x_t \;,\; \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) \;\right) \tag{8}$$

$$p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \geq \mathcal{M}^{f_t}(p'_t, \tilde{u}_t)$$

In this representation, Bernoulli utility is not explicit anymore. Such a gauge is used by Epstein & Zin (1989) to distinguish between risk aversion and intertemporal substitutability as explained in the next section.

3 Epstein Zin in General Consumption Space

The section analyzes Epstein & Zin's (1989) disentanglement of (standard) risk aversion and intertemporal substitutability. I discuss the coordinate and good dependence of the risk measure and relate both to the gauge freedom in the representations of section 2.

3.1 Atemporal Risk Aversion & Intertemporal Substitutability

Risk aversion and intertemporal substitutability cannot be distinguished in the standard framework of intertemporally additive expected utility (Weil 1990). In the latter

[24] As $g_t = \text{id}$, the normalization constants are $\theta_t = \frac{\Delta U_t}{\sum_{\tau=t}^T \Delta U_\tau} = \frac{\overline{U}_t}{\sum_{\tau=t}^T \overline{U}_\tau}$ and $\vartheta_t = \frac{\overline{U}_{t+1} \cdot 0 - 0 \cdot \overline{U}_t}{\overline{U}_t} = 0$. Recall that these constants were introduced to make the intertemporal aggregation rule well defined. As the intertemporal aggregation rule is eliminated in the certainty additive gauge, it is no surprise that the constants can be eliminated as well.

approach, the Arrow-Pratt measure of relative risk aversion is confined to the inverse of the intertemporal elasticity of substitution. In their seminal work Epstein & Zin (1989) show that these two characteristics of preference can be disentangled in the more general setting of temporal lotteries.[25] The authors use a one commodity setting and the Epstein Zin gauge derived in section 2.5, where aggregate utility is constructed by the recursion

$$\tilde{u}_t(x_t, p_{t+1}) = \mathcal{N}_t^{\mathrm{g}} \left(\; x_t \; , \; \mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) \; \right) \; . \tag{9}$$

Precisely, the representation assumed by Epstein & Zin (1989) slightly differs from the one supported by my axioms. On the one hand, with respect to the intertemporal aggregation rule, the authors assume the special case where $g(z) = z^\rho$, which renders an intertemporal aggregation with a constant elasticity of intertemporal substitution. On the other hand, they employ a more general uncertainty aggregation rule, which cannot be characterized by a simple function f and, in general, does not comply with von Neumann & Morgenstern's (1944) independence axiom.

In equation (9), the functions f_t (respectively operators $\mathcal{M}^{f_t}$) are interpreted to characterize risk aversion. The functions g_t are interpreted to characterize intertemporal substitutability. The easiest way to derive these interpretations makes use of gauge lemma 1. In the sense of theorem 2, the above representation corresponds to the sequence of triples $(\mathrm{id}, f_t, g_t)_{t\in\{1,\ldots,T\}}$. First, restricting attention to intertemporal choice under certainty, I replace the second entry by a '$\cdot$' to emphasize its insignificance. By virtue of lemma 1, the representation can be transformed to the form $(g_t, \cdot, \mathrm{id})_{t\in\{1,\ldots,T\}}$, where g_t takes the position of Bernoulli utility and intertemporal aggregation is linear. On certain consumption paths, this latter representation is equivalent to the standard intertemporally additive model where the functional form of utility in period t is g_t. Therefore, g_t is a measure of intertemporal substitutability. In a stationary setting with a discount factor β and a constant elasticity of intertemporal substitution as assumed in Epstein & Zin (1989) it is $g_t(x_t) = \beta^t x_t^\rho$ and the intertemporal elasticity of substitution is $\sigma = \left(-\frac{\ddot{g}(x_t)}{\dot{g}(x_t)} x_t\right)^{-1} = \frac{1}{1-\rho}$. I use the Newtonian dot-notation for the derivative to avoid confusion with the prime that labels changed coordinates or representations. Second, restricting attention to atemporal choice under certainty, I replace the third entry by a '$\cdot$' to emphasize its insignificance. Here, I can rewrite the representing triples as $(f_t, \mathrm{id}, \cdot)$. In the atemporal case the representation is equivalent to the standard expected utility

[25] As I show in Traeger (2007*b*), such a disentanglement can also be achieved in an atemporal lottery setting.

model, where f_t characterizes the utility function and, thus, uncertainty aversion.[26] For a twice differentiable function f_t, the Arrow-Pratt measure of relative risk aversion is defined as $\text{RRA}(x_t) = -\frac{\ddot{f}_t(x_t)}{\dot{f}_t(x_t)} x_t$. The advantage of the Arrow-Pratt-measure as opposed to f_t itself is that it eliminates the affine indeterminacy. For the case of constant relative risk aversion, where $f_t(x_t) = x_t^\alpha$, the Arrow-Pratt coefficient becomes $\text{RRA} = 1 - \alpha$. As pointed out by Normandin & St-Amour (1998, 268) the measures f_t and α characterize "a-temporal" risk attitude, as opposed to the "inter-temporal" information contained in the parametrization of intertemporal substitutability.

In the special case where equation (8) exhibits constant elasticity of substitution and constant relative risk aversion, the framework is also known as the generalized isoelastic model. It has been developed independently as well by Weil (1990). Currently, the latter model represents the predominantly employed framework for disentangling risk aversion from intertemporal substitutability. Its applications range from asset pricing (Attanasio & Weber 1989, Svensson 1989, Epstein & Zin 1991, Normandin & St-Amour 1998, Epaulard & Pommeret 2001) over measuring the welfare cost of volatility (Obstfeld 1994, Epaulard & Pommeret 2003*b*) to resource management[27] (Knapp & Olson 1996, Epaulard & Pommeret 2003*a*, Howitt et al. 2005) and evaluation of global warming scenarios (Ha-Duong & Treich 2004). An overview over the empirical findings for the parameters α and ρ can be found in Giuliano & Turnovsky (2003).

3.2 Measure Scale Dependence of the Risk Measure

The analysis in section 2.5 points out that the Epstein Zin gauge is a particular representation for a setting where a one dimensional scale allows to measure everything relevant to preferences and welfare evaluation. By choosing Bernoulli utility as the identity, this exogenously given scale is used to measure risk aversion and intertemporal substitutability. In the following, I integrate the Epstein Zin model into the general setting with a multidimensional consumption space. To these ends, I assume that X is locally homeomporphic to the n–dimensional Eucledian space, making X an n–manifold.[28] Then,

[26]While theorem 2 is constructed for $T \geq 2$ periods, the atemporal treatment in section 2.3 with the corresponding gauge lemma covers the case.

[27]While Knapp & Olson (1996) and Epaulard & Pommeret (2003*a*) solve theoretical models in order to obtain optimal rules for resource use, Howitt, Msangi, Reynaud & Knapp (2005) try to rationalize observed reservoir management in California, which cannot be explained by means of intertemporally additive expected utility.

[28]X is complete metric and, therefore, a second countable Hausdorff space.

for every ${}^{\circ}x$ in the interior of X, there exists an open neighborhood $N({}^{\circ}x) \subset X$ with a coordinate chart $\Phi : N({}^{\circ}x) \to \mathbb{R}^n$. To simplify the presentation, I discuss consumption changes in the neighborhood of a consumption point ${}^{\circ}\bar{x} \in X$ for which there is a single chart Φ covering a compact subset ${}^{\circ}X \subset X$ with some open neighborhood $N({}^{\circ}\bar{x}) \subset {}^{\circ}X$. It will be sufficient to analyze preferences and representations on the set ${}^{\circ}X$. I denote the codomain of ${}^{\circ}X$ under Φ by ${}^{*}X \subset \mathbb{R}^n$. Then, making use of the coordinate system, the mapping

$$ {}^{\circ}X \xrightarrow{u_t} U_t \, , $$

can be broken up into the steps

$$ {}^{\circ}X \xrightarrow{\Phi} {}^{*}X \xrightarrow{u_t^*} U_t \tag{10} $$

where ${}^{\circ}X \subset X$ describes goods and welfare determining states of the world, ${}^{*}X \subset \mathbb{R}^n$ depicts their coordinate characterization in terms of $n-$tuples of real numbers, and $U_t \subset \mathbb{R}$ is the one dimensional codomain of the Bernoulli utility function. The function $u_t^* = u_t \circ \Phi^{-1}$ denotes a Bernoulli utility function defined on the coordinate space. Making use of the coordinate system, a representation of preferences $\succeq$ on ${}^{\circ}X$ in the sense of theorem 2 can be written as the triples

$$ (u_t, f_t, g_t)_{t\in\{1,\dots,T\}} = (u_t^* \circ \Phi, f_t, g_t)_{t\in\{1,\dots,T\}} \; . $$

The point of departure of most economic models is not the space of consumption goods itself, but the coordinate space ${}^{*}X$. Taking as given some exogenous coordinate system Φ, the models represent the implied preferences that are defined directly on the coordinate values, i.e. on the space ${}^{*}X$. This approach leads to a *reduced representation* by the triples

$$ (u_t^*, f_t, g_t)^{\Phi}_{t\in\{1,\dots,T\}} \; , $$

which I label by the exogenous coordinate system Φ.[29]

I now restrict attention to a one dimensional variation in consumption space along a one dimensional submanifold ${}^{\circ}X^1$. I assume that the coordinates are picked such that the first component of the coordinate chart Φ coincides with the consumption variation, i.e. $\Phi_i({}^{\circ}x) = {}^{*}\bar{x}^i \equiv \Phi_i({}^{\circ}\bar{x})$ for $i > 1$ and ${}^{\circ}x \in {}^{\circ}X^1$ while the first coordinate varies in

[29]For a given coordinate system Φ, the functions f_t and g_t are the same in the 'complete' and in the 'reduced' representation (up to their affine indeterminacy). The reduced representation is a representation in the sense of theorem 2 for the 'implied preferences' $\succeq|_{{}^{*}X}$ defined on ${}^{*}X$. These 'implied preferences' can formally be defined as the binary relation $\succeq|_{{}^{*}X} \equiv \{({}^{*}x^a, {}^{*}x^b) \in {}^{*}X \times {}^{*}X : \exists (x^a, x^b) \in X \times X$ with $(x^a, x^b) \in \succeq|_{{}^{\circ}X}$ and ${}^{*}x^a = \Phi(x^a), {}^{*}x^b = \Phi(x^b)\}$. As Φ is a coordinate system of ${}^{\circ}X$, all necessary conditions for the representation theorem carry over.

${}^*X^1 = \Phi_1({}^\circ X^1)$. It is the space ${}^*X^1$ that is taken as point of departure in the Epstein Zin representation. Restricting the map (10) to the one dimensional consumption variation and choosing the restricted one dimensional Bernoulli utility functions $u_t^*|_{{}^*X^1}$ as the identity yields

$${}^\circ X^1 \xrightarrow{\Phi_1} {}^*X^1 \xrightarrow{\text{id}} U_t = {}^*X^1 .$$

The representation in the sense of theorem 2 that uses the above map to represent preferences over the restricted part of the commodity space ${}^\circ X^1$ is $(\Phi_1, f_t, g_t)_{t\in\{1,...,T\}}$. In its reduced form on ${}^*X^1$ it becomes

$$(\text{id}_{{}^*X^1}, f_t, g_t)^{\Phi_1}_{t\in\{1,...,T\}} . \tag{11}$$

Equation (9) corresponds to such a representation.

A general *change of the measure scale* for the one dimensional good or consumption variation depicted by the model corresponds to a change of the first component of the coordinate chart by some strictly increasing continuous transformation $s : \mathbb{R} \to \mathbb{R}$ yielding the new coordinates $\Phi'_1 = s \circ \Phi_1$. By gauge lemma 1, the coordinate transformation implies a representation change to the triples $(\Phi'_1, f_t \circ s^{-1}, g_t \circ s^{-1})_{t\in\{1,...,T\}}$. Defining ${}^*X^{1\prime} = s({}^*X^1)$, $f'_t = f_t \circ s^{-1}$ and $g'_t = g_t \circ s^{-1}$ the new reduced form representation becomes

$$(\text{id}_{{}^*X^{1\prime}}, f'_t, g'_t)^{\Phi'}_{t\in\{1,...,T\}} .$$

Assuming twice differentiability of f_t and f'_t, I compare the Arrow-Pratt measure in the old $\left(\text{RRA}_t({}^*x) = -\frac{\ddot{f}_t({}^*x)}{\dot{f}'_t({}^*x)}\,{}^*x\right)$ and in the new $\left(\text{RRA}'_t({}^*x') = -\frac{\ddot{f}'_t({}^*x')}{\dot{f}'_t({}^*x')}\,{}^*x'\right)$ coordinates. To evaluate both at the same point ${}^\circ x$ in consumption space, the new measure has to be evaluated at ${}^*x' = s({}^*x)$ yielding[30]

$$\text{RRA}'_t({}^*x')|_{{}^*x'=s({}^*x)} = -\frac{s({}^*x)}{\dot{s}({}^*x)}\left[\frac{\ddot{f}_t\,({}^*x)}{\dot{f}_t\,({}^*x)} - \frac{\ddot{s}\,({}^*x)}{\dot{s}\,({}^*x)}\right] = \frac{s({}^*x)}{\dot{s}({}^*x)}\left[\frac{\text{RRA}_t({}^*x)}{{}^*x} + \frac{\ddot{s}\,({}^*x)}{\dot{s}\,({}^*x)}\right] . \tag{12}$$

Equation (12) states that the Arrow-Pratt measure of relative risk aversion generally depends on the measure scale.

Proposition 1: Whether an agent is Arrow Pratt risk averse or risk loving in the Epstein Zin model depends on the measure scale of the good (coordinate system). For a given preference relation and a given one dimensional variation in consumption space, the Arrow Pratt measure of relative risk aversion RRA in the Epstein

[30] See proof of proposition 1. The relation between ${}^*x'$ and ${}^*x$ follows from ${}^*x' = \Phi'_1({}^\circ x) = s \circ \Phi_1({}^\circ x) = s({}^*x)$.

Zin setting can be set to any desired real value by an appropriate choice of the coordinate system.

Some goods come with a natural concept of doubling and a natural meaning of a ‘zero level’. For these goods, a natural coordinate system can be singled out by requiring that it preserves scalar multiplication and maps the ‘zero level’ into $0 \in \mathbb{R}$. Then, the Arrow Pratt measure for a good with respect to its natural coordinate system is determined uniquely.[31] However, many if not most of our welfare influencing factors are not equipped with such a natural vector space structure. A ubiquitous example is the quality of goods including taste, appearance, and e.g. environmental quality.[32]

3.3 Commodity Dependence of the Risk Measure

Next I consider a second variation in consumption space along the one dimensional submanifold $^\circ\!X^2$ and assume that this second variation is captured by the second coordinate Φ_2, i.e. $^*\!x^2$ varies in $^*\!X^2 = \Phi_2(^\circ\!X^2)$ while $\Phi_i(^\circ\!x) = {}^*\!\bar{x}^i$ for $i \neq 2$ and $^\circ\!x \in {}^\circ\!X^2$. Moreover, let the variations described by Φ_1 and Φ_2 characterize a quantitative change of two consumption goods with a naturally given vector space structure, i.e. there exists a natural zero level, a natural concept of doubling, and a natural unit (e.g. coconuts and litchi quantity). Let Φ be a natural coordinate system preserving the natural vector (sub)space structure for the goods. In the following, I analyze Bernoulli utility on these given coordinates (suppressing the fixed coordinates $\Phi_i(x) = {}^*\!\bar{x}^i, i > 2$). Let $u^*_{t\,\mathrm{id}^1}$ be a Bernoulli utility function on $^*\!X$ satisfying $u^*_{t\,\mathrm{id}^1}|_{^*X^1} = \mathrm{id}_{^*X^1}$. It gives once more rise to the representation $(\mathrm{id}_{^*X^1}, f_t, g_t)^\Phi_{t\in\{1,\ldots,T\}}$ corresponding to equation (11).

Whenever the two consumption goods are not perfect substitutes, this Bernoulli utility will not coincide to the identity for variations along $^*\!X^2$ (keeping $\Phi_1(^\circ\!x) = {}^*\!\bar{x}^1$), i.e. $u^*_{t\,\mathrm{id}^1}|_{^*X^2} \neq \mathrm{id}_{^*X^2}$. The following mapping describes a representation using $u^*_{t\,\mathrm{id}^1}$ to describe variations along $^*\!X^2$

$$^\circ\!X^2 \xrightarrow{\Phi_2} {}^*\!X^2 \xrightarrow{u_{\mathrm{id}^1}|_{^*X^2}} U_t \,.$$

In order to obtain an Epstein Zin representation for variations of the second good I have

[31]The well known independence of the measure of *relative* risk aversion from the unit of measurement is observed from equation (12) by setting $s = ax$ with $a > 0$.

[32]Note that even in a setting where preferences would be defined on wealth, the choice of the ‘zero level’ is somewhat arbitrary. Should it include the estimated value of an individiuals material goods, his human capital, the value of his health state or his access to public goods? All would change the agents Arrow Pratt measure of relative risk aversion.

to employ gauge lemma 1, resulting in the representation

$$(\mathrm{id}_{^*X^2}, f_t \circ s^{-1}, g_t \circ s^{-1})^{\Phi_2}_{t\in\{1,\ldots,T\}} \quad \text{with} \quad s = u_{\mathrm{id}^1}|_{^*X^2} \ .$$

Then, the implied Arrow-Pratt risk measure at $^\circ\bar{x}$ for this one dimensional change along the second coordinate is related to the Arrow-Pratt measure for a change along the first coordinate the same way as are RRA′ and RRA in equation (12). However, in section 3.2 s was determined by a change in the coordinate system. In the current section I have assume a given natural coordinate system and s is now determined by the preference relation. Within the set of preferences described by the axioms of this paper, all functions s can hold. As a particular consequence the following proposition holds.

Proposition 2: For a given coordinate system, sign and magnitude of the Arrow Pratt measure of relative risk aversion in the Epstein Zin model generally depend on the good or consumption variation under observation.

For example, take a decision maker who exhibits isoelastic preferences with $f(z) = z^2$, $g(z) = z^\rho$ and Bernoulli utility described by $u^*(x_1, x_2) = x_1^{1/4} x_2^{3/4}$ with respect to the natural coordinates. With respect to variations of the first good, the decision maker is risk averse with an Arrow Pratt measure of relative risk aversion of RRA $= \frac{1}{2}$. With respect to variations of the second good, the decision maker is risk loving with an Arrow Pratt measure of relative risk aversion of RRA $= -\frac{1}{2}$.[33]

3.4 Coordinate and Good Independence of $f_t \circ g_t^{-1}$

This subsection identifies a candidate for a risk measure that is not coupled to a particular consumption good or its measure scale, but rather to preference directly. In section 3.2, a change in measure scale corresponds to a change of Φ that translates into a change of u_t^*. Regauging u_t^* to the identity, in order to conserve the Epstein Zin interpretation, changes the representing funtions f_t and g_t characterizing Arrow Pratt risk attitude and intertemporal substitutability. In section 3.3, changing the good under observation gave rise to a similar change in u_t^*. Again, by regauging u_t^*, this change was carried over into the functions characterizing Arrow Pratt risk attitude and intertemporal substitutability. Both changes, in the measure scale and in the good under observation,

[33] In the extension of atemporal risk aversion to multiple commodities, as developed by Kihlstrom & Mirman (1974), this finding corresponds to a decision maker, who pays a positive risk premium for lotteries of one good, but a negative risk premium for lotteries over another good.

affect the Arrow Pratt characterization of risk (by means of f_t) as a consequence of the necessity to regauge Bernoulli utility u_t^*. Conversely, if a function in the representation of a preference relation $\succeq$ is not affected by changes in the representing Bernoulli utility function it will not affected by changes in measure scale or the good under observation either. The following proposition identifies such a gauge-invariant function.

Proposition 3: Let the preference relation $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ satisfy the axioms of theorem 2 and let $\mathsf{u} = (u_t)_{t\in\{1,...,T\}}$ and $\mathsf{u}' = (u'_t)_{t\in\{1,...,T\}}$ satisfy $u_t, u'_t \in B_{\succeq_t}$. Then, there exist representations $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ and $(u'_t, f'_t, g'_t)_{t\in\{1,...,T\}}$ in the sense of theorem 2 such that $f_t \circ g_t^{-1} = f'_t \circ {g'}_t^{-1}$ for all $t \in \{1, ..., T\}$.

The proposition states that the functions $(f_t \circ g_t^{-1})_{t\in\{1,...,T\}}$ are independent of the coordinate system and the good under observation. It can be restated as the fact that the sequence of functions $(f_t \circ g_t^{-1})_{t\in\{1,...,T\}}$ is gauge invariant. Because of the affine freedom of $f_t \circ g_t^{-1}$ in the representation only the class $H_{\succeq} = \{h_t \in \mathcal{C}^0(\mathrm{I\!R}) : \exists \mathfrak{a}, \mathfrak{a}' \in \mathsf{A}^+ \text{ s.th. } h_t = \mathfrak{a} f_t \circ g_t^{-1} \mathfrak{a}'\}$ and not the function $f_t \circ g_t^{-1}$ itself is uniquely determined by $\succeq$. The next section shows that the functions $(f_t \circ g_t^{-1})_{t\in\{1,...,T\}}$ in fact are measures of risk aversion.

4 Intertemporal Risk Aversion

The section introduces the concept of intertemporal risk aversion (IRA), relates it to the invariant found at the end of the preceding section, and gives conditions for the uniqueness of the measures of absolute and relative intertemporal risk aversion.

4.1 IRA – Axiomatic characterization

This subsection introduces the axiom of intertemporal risk aversion. I give two alternative formulations that turn out equivalent in the present preference framework. The first formulation employs the lottery $\sum_{i=t}^{T} \frac{1}{T-t+1} (\mathsf{x}_{-i}, \mathsf{x}'_i)$. It yields with equal probability the consumption paths $(\mathsf{x}_{-i}, \mathsf{x}'_i), i \in \{t, ..., T\}$. The lottery can also be described as follows. Starting from a consumption path x I switch one of its entries x_i by the entry x'_i of a consumptin path x'. The lottery draws with equal probability the period i in which consumption is changed from x_i to x'_i.

A decision maker is said to exhibit *weak intertemporal risk aversion* in period $t < T$, if and only if the following axiom is satisfied:

A6$^{\text{w}}$ (weak intertemporal risk aversion) For all $x, x' \in X^t$ holds

$$x \sim_t x' \quad \Rightarrow \quad x \;\succeq_t\; \sum_{i=t}^{T} \frac{1}{T-t+1} \,(x_{-i}, x'_i).$$

A decision maker is said to exhibit *strict intertemporal risk aversion* in period $t < T$, if and only if the following axiom is satisfied:

A6$^{\text{s}}$ (strict intertemporal risk aversion) For all $x, x' \in X^t$ holds

$$x \sim_t x' \quad \wedge \quad \exists\, \tau \in \{t, ..., T\} \text{ s.th. } x_\tau \;\not\sim^*_\tau\; x'_\tau$$

$$\Rightarrow \quad x \;\succ_t\; \sum_{i=t}^{T} \frac{1}{T-t+1} \,(x_{-i}, x'_i).$$

I start with the interpretation of the strict axiom.[34] The first part of the premise states that a decision maker is indifferent between the certain consumption paths x and x'. The second part of the premise requires that there exists a period τ, in which the decision maker is not indifferent between the outcome delivered by consumption path x and the one delivered by consumption path x'. Without loss of generality assume that outcome x_τ is strictly preferred to outcome x'_τ, i.e. $x_\tau \succ^*_\tau x'_\tau$. Then, by the first part of the premise, there also exists a period τ' in which $x'_{\tau'} \succ^*_\tau x_{\tau'}$. Thus, the premise implies that there exists a consumption path $(x_{-\tau'}, x'_{\tau'})$ that is judged superior as well as a consumption path $(x_{-\tau}, x'_\tau)$ that is judged inferior with respect to the consumption path x. Overall, the paths (x_{-i}, x'_i) with $i \in \{t, ..., T\}$ that are judged superior and those that are judged inferior with respect to x balance each other in the sense of the intertemporal trade-off given in the first part of the premise. The second line of axiom A6$^{\text{s}}$ states that for consumption satisfying the above conditions, an intertemporal risk averse decision maker prefers the consumption path x with certainty over the lottery that yields with equal probability any of the consumption paths (x_{-i}, x'_i), some of which make him better off and some of which make him worse off.

The interpretation for the *weak* axiom A6$^{\text{w}}$ is analogous, only that the consumption path x is allowed to coincide with x', and the implication only requires that the lottery is not strictly preferred over the certain consumption path. If axiom A6$^{\text{s}}$ [A6$^{\text{w}}$] is satisfied with $\succ_t$ [$\succeq_t$] replaced by $\prec_t$ [$\preceq_t$], the decision maker is called a strong [weak] *intertemporal risk seeker*. If a decision maker's preferences satisfy weak intertemporal

[34]In the interpretation I assume that preferences satisfy the axioms introduced for the representation in theorem 2.

risk aversion as well as weak intertemporal risk seeking, the decision maker is called *intertemporal risk neutral.*

Before stating the theorem that characterizes intertemporal risk aversion in terms of the representation of theorem 2, I offer an alternative axiomatic characterization of intertemporal risk aversion, which only involves a lottery over two consumption paths. To these ends, define for $x, x' \in X^t$ the consumption paths $x^{\text{high}}(x, x'), x^{\text{low}}(x, x') \in X^t$ by

$$(x^{\text{high}}(x, x'))_\tau = \begin{cases} x'_\tau \text{ if } x'_\tau \succ^*_\tau x_\tau \\ x_\tau \text{ if } x_\tau \succeq^*_\tau x'_\tau \end{cases}$$

and

$$(x^{\text{low}}(x, x'))_\tau = \begin{cases} x'_\tau \text{ if } x_\tau \succeq^*_\tau x'_\tau \\ x_\tau \text{ if } x'_\tau \succ^*_\tau x_\tau \end{cases}$$

for $\tau \in \{t, ..., T\}$. The consumption path $x^{\text{high}}(x, x')$ collects the better outcomes of every period of the two consumption paths x and x', while $x^{\text{low}}(x, x')$ collects the inferior outcomes of every period. The definition of *weak intertemporal risk aversion* in period $t < T$ can also be stated as follows:

A6$^{\text{w}}_*$ (weak intertemporal risk aversion) For all $x, x' \in X^t$ holds

$$x \sim_t x' \quad \Rightarrow \quad x \ \succeq_t \ \tfrac{1}{2}\, x^{\text{high}}(x, x') + \tfrac{1}{2}\, x^{\text{low}}(x, x').$$

And *strict intertemporal risk aversion* in period $t < T$ can be written as:

A6$^{\text{s}}_*$ (strict intertemporal risk aversion) For all $x, x' \in X^t$ holds

$$x \sim_t x' \quad \wedge \quad \exists\, \tau \in \{t, ..., T\} \text{ s.th. } x_\tau \ \not\sim^*_\tau \ x'_\tau$$

$$\Rightarrow \quad x \ \succ_t \ \tfrac{1}{2}\, x^{\text{high}}(x, x') + \tfrac{1}{2}\, x^{\text{low}}(x, x').$$

The interpretations are analogous to those of axioms A6$^{\text{w}}$ and A6$^{\text{s}}$. However, the 'worse off' versus 'better off' trade-off in the lottery can be observed more directly. For long time horizons, the formulation in axioms A6$^{\text{w}}_*$ and A6$^{\text{s}}_*$ reduces the consumption paths offered by the lottery significantly. In the case of two periods, both axioms A6$^{\text{w}}$ and A6$^{\text{w}}_*$ respectively A6$^{\text{s}}$ and A6$^{\text{s}}_*$ coincide. Theorem 3 in the next subsection proves the general equivalence of the two formulations within the preference setup of representation theorem 2.

4.2 IRA – Functional Characterization

The following theorem relates the concept of intertemporal risk aversion to the invariant found in proposition 3 of the preceding section. The set Γ_t is defined as $\Gamma_t = \big(\underline{G}_t, \overline{G}_t\big)$.

Theorem 3: Let the sequence of triples $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ represent the preferences $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ in the sense of theorem 2. For $t \in \{1, ..., T-1\}$ the following assertions hold:

a) A decision maker is strictly intertemporal risk averse [seeking] in period t in the sense of axiom A6$^{\text{s}}$, if and only if, $f_t \circ g_t^{-1}(z)$ is strictly concave [convex] in $z \in \Gamma_t$.

b) A decision maker is weakly intertemporal risk averse [seeking] in period t in the sense of axiom A6$^{\text{w}}$, if and only if, $f_t \circ g_t^{-1}(z)$ is concave [convex] in $z \in \Gamma_t$.

c) A decision maker is intertemporal risk neutral in period t, if and only if, $f_t \circ g_t^{-1}(z)$ is linear in $z \in \Gamma_t$.

d) Assertions *a*-*c*) hold when replacing axiom A6$^{\text{s}}$ by A6$^{\text{s}}_{*}$ and axiom A6$^{\text{w}}$ by A6$^{\text{w}}_{*}$.

Theorem 3 characterizes intertemporal risk attitude in period t by the curvature of the functions $f_t \circ g_t^{-1}$. Concavity of the composition $h \equiv f_t \circ g_t^{-1}$ can be paraphrased as f_t being concave with respect to g_t (Hardy et al. 1964). This interpretation stands out more clearly when rewriting the relation as $f_t = h \circ g_t$. Then f_t is seen to be a concave transformation of g_t. In the one dimensional Epstein Zin analysis with f_t and g_t being twice differentiable, $f_t \circ g_t^{-1}$ concave is equivalent to the Arrow Pratt measure of relative risk aversion dominating the aversion to intertemporal substitution, i.e. equivalent to $-\frac{\ddot{f}_t(x)}{\dot{f}_t(x)}\Big|_{g_t^{-1}(x)} > -\frac{\ddot{g}_t(x)}{\dot{g}_t(x)}\Big|_{g_t^{-1}(x)}$.[35] However, the function $f_t \circ g_t^{-1}$ is well defined also in the multi-commodity setting. Moreover, as seen in proposition 3, in difference to f_t and g_t taken individually, the composition is uniquely determined by the preference relation (up to affine transformations). The composition expresses that an intertemporal risk averse decision maker is more averse to substitute consumption into a risky state than

[35] Here aversion to intertemporal substitutability is measured as the inverse of the intertemporal elasticity of substitution. The relation derives as follows

$$\frac{d^2}{dx^2} f_t \circ g_t^{-1}(x) < 0 \Leftrightarrow \frac{\frac{d^2}{dx^2} f_t(x)}{\frac{d}{dx} f_t(x)} < -\frac{\frac{d^2}{dx^2} g_t^{-1}(x)}{\left(\frac{d}{dx} g_t^{-1}(x)\right)^2} \Leftrightarrow -\frac{\frac{d^2}{dx^2} f_t(x)}{\frac{d}{dx} f_t(x)}\Bigg|_{g_t^{-1}(x)} > -\frac{\frac{d^2}{dx^2} g_t(x)}{\frac{d}{dx} g_t(x)}\Bigg|_{g_t^{-1}(x)} .$$

to substitute it into a certain future.[36] For period T, the function $f_T \circ g_T^{-1}$ is determined by the underlying preferences $\succeq = (\succeq_t)_{t\in\{1,\ldots,T\}}$ to the same degree as the compositions $f_t \circ g_t^{-1}$ for any other period. Therefore, theorem 3 can be used to extend the definition of intertemporal risk aversion to the last period of the planning horizon. A decision maker, and only a decision maker who is intertemporal risk neutral in all periods can be described by the intertermporally additive expected utility standard model.[37] Then f_t equals g_t up to affine transformations and, in the Epstein Zin setting, his Arrow Pratt risk aversion is determined completely by choices under certainty.

An interesting interpretation of theorem 3 and the axioms of intertemporal risk aversion arises in the *certainty additive gauge.* This representational form is particularly well suited to give Bernoulli utility the interpretation of welfare, in the sense that a unit of welfare more in one period and a unit of welfare less in another period bring about the same aggregate welfare. For example axiom A6$^{\text{w}}_{*}$ gains the following interpretation. The premise requires that for two consumption paths, x and x', the per period welfare adds up to the same overall welfare. The consumption path $\mathrm{x}^{\text{high}}(\mathrm{x},\mathrm{x}')$ collects for every period the outcome x_t or x'_t that renders the comparatively higher welfare, while the consumption path $\mathrm{x}^{\text{low}}(\mathrm{x},\mathrm{x}')$ collects the outcome x_t or x'_t that yields the comparatively lower welfare. By construction, the lottery in axiom A6$^{\text{s}}_{*}$ between these 'high welfare' and 'low welfare' consumption paths renders in expectation the same welfare as the certain consumption path x. A decision maker who is weakly intertemporal risk averse is defined by preferring the certain consumption path x over the welfare lottery that leaves him with equal probability either worse or better off, and yields the same welfare as the certain consumption path in expectation. With such an interpretation of certainty additive Bernoulli utility as welfare, *intertemporal risk aversion can be understood as* risk aversion with respect to welfare gains and losses or just as *risk aversion on welfare.* That interpretation is immediate as well from theorem 3. For the certainty additive gauge, the latter states that intertemporal risk aversion is characterized by the concavity of f_t alone. The difference between f_t being a measure of intertemporal risk

[36] This intuition is formulated slightly more precise in a situation where a decision maker has the possibility to either smooth consumption over time or over risk. Whenever the intertemporal risk neutral decision maker is indifferent between the two options, the intertemporal risk averse decision maker prefers to smooth consumption over the risky states, while the intertemporal risk seeking decision maker prefers to keep the risk but smooth consumption over time.

[37] Take the certainty additive gauge ($g_t = \text{id}$) and note that by part c) of theorem 3 f_t becomes linear. Thus, intertemporal aggregation is additive and the uncertainty aggregation rule coincides with the expected value operator.

aversion instead of a standard risk aversion in the Arrow Pratt (or Epstein Zin) sense is that welfare replaces a one dimensional consumption argument.

4.3 Measures of IRA

This section establishes quantitative measures of intertemporal risk aversion. The natural candidate is the construction of an analogue to the coefficient of relative risk aversion in the atemporal setting. For a twice differentiable function $f_t \circ g_t^{-1} : \Gamma_t \to \mathbb{R}$ define a measure of *relative intertemporal risk aversion in period* t as the function

$$\text{RIRA}_t : \Gamma_t \to \mathbb{R}$$

$$\text{RIRA}_t(z) = -\frac{\frac{d^2}{dz^2} f_t \circ g_t^{-1}(z)}{\frac{d}{dz} f_t \circ g_t^{-1}(z)}\, z\,.$$

To evaluate the measure of relative intertemporal risk aversion for a particular point in consumption space define

$$\text{RIRA}_t[\tilde{x}_t] = \text{RIRA}_t(z)|_{z=g_t \circ \tilde{u}_t(\tilde{x}_t)}\ .$$

The so defined numerical measure is not uniquely determined by preferences. In difference to the Arrow Pratt measure of atemporal risk aversion, however, the indeterminacy of RIRA_t is not caused by a dependence on the Bernoulli utility function employed in the representation. The indeterminacy is caused by the affine freedom prevailing in the representations. Define $\mathfrak{a}(z) = az + b$ and $\tilde{\mathfrak{a}}(z) = \tilde{a}z + \tilde{b}$ with $a, \tilde{a} > 0$ and let $f'_t = \mathfrak{a} f_t$ and $g'_t = \tilde{\mathfrak{a}} g_t$. The transformation corresponds to the freedom of f_t and g_t in theorem 2. For the new choice f'_t and g'_t in the representation, the coefficient[38] of relative intertemporal risk aversion, evaluated for the same outcome $\tilde{x}_t$ as RIRA_t, calculates to

$$\text{RIRA}'_t(z')|_{z'=\tilde{a}z+\tilde{b}} = -\frac{\frac{d^2}{dz'^2} f'_t \circ g'^{-1}_t(z')}{\frac{d}{dz'} f'_t \circ g'^{-1}_t(z')}\, z' \Bigg|_{z'=\tilde{a}z+\tilde{b}} = -\frac{\frac{d^2}{dz^2} f_t \circ g_t^{-1}(z)}{\frac{d^2}{dz^2} f_t \circ g_t^{-1}(z)}\, \frac{\tilde{a}\, z + \tilde{b}}{\tilde{a}}\,. \quad (13)$$

While the affine indeterminacy corresponding to the transformation $f_t \circ g_t^{-1} \to \mathfrak{a} f_t \circ g_t^{-1}$ leaves the coefficient of relative intertemporal risk aversion unchanged, an affine change corresponding to a non-zero $\tilde{b}$ in $f_t \circ g_t^{-1} \to f_t \circ g_t^{-1} \tilde{\mathfrak{a}}^{-1}$ changes the coefficient.

The economic interpretation of this indeterminacy is best understood in the certainty additive gauge, where I interpreted certainty additive Bernoulli utility as welfare. Here, intertemporal risk aversion turns into risk aversion with respect to welfare gains and

[38] I adopt the word coefficient also for the case where the function is non-constant and, thus, 'the' coefficient is a function of z.

losses. However, the intertemporal trade-off determines the respective welfare function only up to affine transformations. In order to obtain a measure of *relative* risk aversion, the zero level has to be defined. This reasoning is analogous to that on measure scale dependence in section 3. However, intertemporal risk aversion is measured with respect to the abstract concept of welfare, whose measure scale is determined up to affine transformations. As soon as a zero welfare level is fixed, the coefficients of relative intertemporal risk aversion are uniquely defined. Formally, in equation (13) the choice of a zero welfare level eliminates $\tilde{b}$.

A similar reasoning applies for the definition of a measure of *absolute intertemporal risk aversion in period* t as the function

$$\text{AIRA}_t : \Gamma_t \to \mathbb{R}$$
$$\text{AIRA}_t(z) = -\frac{\frac{d^2}{dz^2} f_t \circ g_t^{-1}(z)}{\frac{d}{dz} f_t \circ g_t^{-1}(z)} \;,$$

with absolute intertemporal risk aversion at point $\tilde{x}_t$ in consumption space defined by $\text{AIRA}_t[\tilde{x}_t] = \text{AIRA}_t(z)|_{z=g_t \circ \tilde{u}_t(\tilde{x}_t)}$. Thinking of intertemporal risk aversion as risk aversion on (certainty additive) welfare gives rise to the insight that the intertemporal trade-off leaves the unit of welfare measurement undetermined. While the indeterminateness of the unit is irrelevant for relative measures of risk aversion, it is required for absolute measures. Formally, under the same transformation of f_t and g_t as above, the new coefficient of absolute intertemporal risk aversion, evaluated for the same outcome, calculates to

$$\text{AIRA}'_t(z')|_{z'=\tilde{a}z+\tilde{b}} = -\left.\frac{\frac{d^2}{dz'^2} f'_t \circ g'^{-1}_t(z')}{\frac{d}{dz'} f'_t \circ g'^{-1}_t(z')}\right|_{z'=\tilde{a}z+\tilde{b}} = -\frac{\frac{d^2}{dz^2} f_t \circ g_t^{-1}(z)}{\frac{d}{dz} f_t \circ g_t^{-1}(z)} \, \frac{1}{\tilde{a}} \,. \tag{14}$$

Again, the affine indeterminacy corresponding to the transformation $f_t \circ g_t^{-1} \to \mathfrak{a} f_t \circ g_t^{-1}$ leaves the coefficient of absolute intertemporal risk aversion unchanged. However, a linear change corresponding to $\tilde{a}$ in $f_t \circ g_t^{-1} \to f_t \circ g_t^{-1} \tilde{\mathfrak{a}}^{-1}$ changes the coefficient. Fixing the unit of welfare measurement eliminates the constant $\tilde{a}$.

Note that, no matter what gauge, it is always the information obtained from the evaluation of intertemporal trade-offs that has to be enriched in order to render either of the measures well defined. In the certainty additive gauge, this information is characterized by the certainty additive Bernoulli utility (which I identified with welfare). In general gauges it is characterized by the composition $g_t \circ u_t$. The following proposition states the general premises to make the numerical measures RIRA_t and AIRA_t well defined.

Proposition 4: Let a sequence of preference relations $\succeq = (\succeq_t)_{t\in\{1,...,T\}}$ be represented in the sense of theorem 2 with twice differentiable functions $(f_t \circ g_t^{-1})_{t\in\{1,...,T\}}$.

a) Choose $\bar{x}_t \in X$ and fix $g_t \circ u_t(\bar{x}_t) = 0$ for all $t \in \{1, ..., T\}$. Then, the risk measures $\text{RIRA}_t[\tilde{x}_t]$ are uniquely determined and coincide for all representations of $\succeq$ in the sense of theorem 2 .

b) Choose two outcomes $\acute{x}_{t^*}, \grave{x}_{t^*} \in X$ with $\acute{x}_{t^*} \succ^*_{t^*} \grave{x}_{t^*}$, a strictly positive number $\bar{w}$ and fix $g_{t^*} \circ u_{t^*}(\acute{x}_{t^*}) - g_{t^*} \circ u_{t^*}(\grave{x}_{t^*}) = \bar{w}_{t^*}$ for some arbitrary period $t^* \in \{1, ..., T\}$. Then, the risk measures $\text{AIRA}_t[\tilde{x}_t]$ are uniquely determined and coincide for all representations of $\succeq$ in the sense of theorem 2.

A particularly convenient form of fixing the risk measures uniquely is choosing the certainty additive gauge and $\underline{U}_t = 0$ for all $t \in \{1, ..., T\}$ and $\overline{U}_1 = 1$. Then, the zero welfare level is fixed to be the worst outcome in every period and the unit of welfare corresponds to the welfare difference between the best and the worst outcome in the first period.

4.4 Revisiting the Castaway

What have we learned about characterizing Robinson's risk attitude? Taking into account his intertemporal decisions, the measure of intertemporal risk aversion allows us to uniquely tell whether Robinson likes risk or dislikes risk. Identifying some point in consumption space with a zero welfare level,[39] Robinson's numerical measure of relative intertemporal risk aversion is uniquely determined for every point in (physical[40]) consumption space. The measure can be derived from decisions over coconuts as well as over litchies or from his decisions involving the coconut quality chart. The numerical measure of absolute intertemporal risk aversion is uniquely determined for every point in consumption space, whenever two (non-indifferent) points in consumption space are identified with a unit difference in welfare.

[39]Welfare is understood as Robinson's certainty additive Bernoulli utility function.

[40]That is, for every physical state of the world and Robinson's 'real world consumption' independent of the coordinate system applied to describe the consumption or the state of the world.

5 Conclusions

I derived the general time consistent model that yields an additive evaluation of certain consumption paths, and respects the von Neumann Morgenstern axioms in every period. The representation admits the freedom to pick strictly increasing transformations of Bernoulli utility, the evaluation function for certain outcomes within a period. I related good and measure scale dependence of risk measures to the behavior of the representation under transformations of Bernoulli utility. This relation served to identify good and measure scale (in)dependence of risk measures. Moreover, the freedom in picking Bernoulli utility allowed for a simultaneous derivation of different representational forms, including a new certainty additive representation and a multicommodity extension of the Epstein Zin representation.

I employed this framework to introduce a concept of intertemporal risk aversion that is independent of the good under observation or its measure scale. I provided two alternative axiomatic characterizations of intertemporal risk aversion. I derived measures of relative and absolute intertemporal risk aversion and discussed there uniqueness properties. An intertemporal risk averse decision maker has a stronger propensity to smooth consumption over risk than to smooth consumption over time. The widespread modeling framework of intertemporally additive expected utility implicitly assumes intertemporal risk neutrality. Conversely, intertemporal risk neutrality uniquely pins down the standard model within the more general framework. In the certainty additive form of the representation, which permits a time additive conceptualization of welfare, intertemporal risk aversion can be interpreted as risk aversion with respect to welfare gains and losses. Areas of application of the concept of intertemporal risk aversion comprise any field where time and uncertainty play an important role.

Appendix

A Proofs for Section 2

Notational Remark: Some proofs employ the additional notation that $\mathrm{x}^t \in \mathrm{X}^t$ denotes a consumption path from period t to period T. As before, x^t_τ denotes the period τ entry of consumption path x^t.

Proof of theorem 1: Sufficiency: *i)* As X is a compact metric space it is Polish and, thus, separable. Therefore, by theorem 3 of Grandmont (1972) axioms A1-A3 imply the existence of an expected utility representation.
ii) Denote a general representation in the sense of theorem 1, equation (4), by (v, f). The expected utility representation corresponds to the special case (v^0, id), for some $v^0 \in \mathcal{C}^0(X)$. Obviously v^0 is a Bernoulli utility function and it holds $v^0(x^1) \geq v^0(x^2) \Leftrightarrow x^1 \succeq x^2 \Leftrightarrow v(x^1) \geq v(x^2)$ for all $x^1, x^2 \in X$. Therefore, a strictly increasing transformation s relates the function v stated in the theorem to the one in the expected utility form: $v = s \circ v^0$.
iii) To find that continuity of v and v^0 imply continuity of $s : V^0 \to V$, define $V^0 \equiv \mathrm{range}(v^0)$ and $V \equiv \mathrm{range}(v)$. The preimage of any closed subset $A \subset U$ under s is closed:
As v is continuous the preimage of A under v, $B = v^{-1}(A)$, is closed. Moreover, a closed subset of a compact space B is compact and the image of a compact set under the continuous function v^0 is compact (Schofield 2003, 111). In consequence the resulting image $v^0(B)$, which is the sought for preimage of A under s,[41] is closed. Hence, s is continuous.
iiii) If a tuple (v, f) represents $\succeq$ in the sense of theorem 1, then so does the tuple $(s \circ v, f \circ s^{-1})$ for any $s : V \to \mathrm{I\!R}$ strictly increasing and continuous:
The second tuple denotes the representation $sf^{-1}\left[\int_X (fs^{-1})(sv\, dp\right] = sf^{-1}\left[\int_X fv\, dp\right]$. It a strictly increasing transformation of the representation $\mathcal{M}^f(p, v)$ for $\succeq$ and hence a representation for $\succeq$ itself. Moreover $s \circ v$ and $f \circ s^{-1}$ are continuous and the latter is strictly monotonic.[42]
Therefore, the tuple $(s \circ v^0, s^{-1})$ represents $\succeq$. Defining $f \equiv s^{-1}$ yields the desired

[41] To confirm that $v^0(B)$ is the preimage of A under s note that $s \circ v^0(B) = s \circ v^0(v^{-1}(A)) = v(v^{-1}(A)) = A$.

[42] Continuity of s^{-1} follows from the fact that the inverse of a strictly monotonic function on an interval is continuous (Heuser 1988, 231).

representation (v, f) for $\succeq$.
Necessity: *i*) First let f be strictly increasing and (v, f) represent $\succeq$ in the sense of theorem 1. By *iiii*) in the sufficiency part of the proof with $s = f$ strictly increasing and continuous find that $(f \circ v, \text{id})$ represents $\succeq$. But with $v^0 \equiv f \circ v$ the latter is an expected utility representation. Therefore, theorem 3 of Grandmont (1972) verifies that axioms A1-A3 are satisfied.
ii) For f strictly decreasing note that $\mathcal{M}^f = \mathcal{M}^{-f}$ and hence the above reasoning can be applied to the representing tuple $(v, -f)$ with $-f$ strictly increasing.
Uniqueness: As is well known, in the expected utility presentation (v^0, id) the function v^0 is unique up to positive affine transformations. Thus, the uniqueness up to positive affine transformations of f follows from $v^0 = f \circ v$.

□

Proof of corollary 1: Sufficiency: As in the proof of theorem, 1 axioms A1-A3 imply the existence of a representation (u^0, id) for $\succeq$. By part *iii*) in the latter proof, also $(f^{-1}v^0, f)$ represents $\succeq$. Due to continuity of f^{-1} (see footnote 42) and v^0, the function $u \equiv f^{-1}v^0$ is a continuous function for which the representation of corollary 1 holds.
Necessity: As v in equation (4) is a Bernoulli utility function, this part of the proof is implied by necessity in theorem 1.
Uniqueness: Assume v and v' both represent $\succeq$: Equation (5) implies for degenerate lotteries the existence of a strictly increasing function s such that $v' = s \circ v$. As in *iii*) of the proof of theorem 1 it follows that s is continuous. By *iiii*) in the proof of theorem 1 it follows that with $(v', f) = (s \circ v, f)$ also $(s^{-1} \circ s \circ v, f \circ s) = (v, f \circ s)$ is a representation of $\succeq$. Comparing the latter with the representation (v, f) the uniqueness part of theorem 1 implies the existence of $\mathfrak{a} \in \mathsf{A}$ such that $f = \mathfrak{a} f s$. From the fact that s is strictly increasing I can infer that also $\mathfrak{a}$ has to be strictly increasing and therefore $\mathfrak{a} = \mathfrak{a}^+ \in \mathsf{A}^+$. It follows $fv = \mathfrak{a}^+ fsv \Rightarrow fv = \mathfrak{a}^+ fv' \Rightarrow v = f^{-1}\mathfrak{a}^+ fv'$.
Assume it exists $\mathfrak{a}^+ \in \mathsf{A}^+$ such that $v = f^{-1}\mathfrak{a}^+ fv'$: First let f be increasing. If (u, f) is a representation of $\succeq$ then by theorem 1 also $(u, \mathfrak{a}^+ f)$ is a representation. By part *iiii*) of the proof of theorem 1 it follows that also $([\mathfrak{a}^+ f]u, \mathfrak{a}^+ f[\mathfrak{a}^+ f]^{-1})$ is a representation. Using part *iiii*) of the proof of theorem 1 once more yields the result that $(f^{-1}\mathfrak{a}^+ fu, f)$ is a representation of $\succeq$. As in the necessity part of the proof of theorem 1, for f decreasing use the representation $(u, -\mathfrak{a}^+ f)$. By a similar reasoning as above, the tuples $([-\mathfrak{a}^+ f]u, \text{id})$, $(f^{-1}\{-[-\mathfrak{a}^+ f]u\}, -f)$, $(f^{-1}\mathfrak{a}^+ fu, -f)$ and $(f^{-1}\mathfrak{a}^+ fu, f)$ are represenations of $\succeq$. □

Proof of theorem 2: The proof is divided into four parts. The first part gives a representation for certain consumption paths. Part two derives a corresponding recursive formulation, still only for certain consumption paths. Finally, part three elaborates the general representation for temporal lotteries as given in the theorem. Part four verifies that the derived representation implies all axioms.

Sufficiency: **Part I:** First, the axioms imply the existence of an additive representation of $\succeq_1 |_{\mathrm{X}^1}$. Hereto note that, if the sets $\{p'_1 \in P_1 : p'_1 \succeq_1 \mathrm{x}\}$ and $\{p'_1 \in P_1 : \mathrm{x} \succeq_1 p'_1\}$ are closed in P_1 for all $\mathrm{x} \in \mathrm{X}^1 \subset P_1$, then the sets $\{p'_1 \in P_1 : p'_1 \succeq_1 \mathrm{x}\} \cap \mathrm{X}^1 = \{\mathrm{x}' \in \mathrm{X}^1 : \mathrm{x}' \succeq_1 \mathrm{x}\}$ and $\{p'_1 \in P_1 : \mathrm{x} \succeq_1 p'_1\} \cap \mathrm{X}^1 = \{\mathrm{x}' \in \mathrm{X}^1 : \mathrm{x} \succeq_1 \mathrm{x}'\}$ are closed in X^1 endowed with the relative topology for all $\mathrm{x} \in \mathrm{X}^1$. Moreover the relative topology on X^1 is the product topology on X^T. Therefore, by Wakker (1988, theorem III.4.1) axioms A0, A1, A3 and A4' bring about the existence of a sequence $u_t^{ca} \in \mathcal{C}^0(X), t \in \{1, ..., T\}$, such that $\sum_{t=1}^T u_t^{ca}$ represents $\succeq_1 |_{\mathrm{X}^1}$.

Second, note that certainty additivity for $\succeq_1$ carries over to $\succeq_t$ for all t with *coinciding* Bernoulli utility functions $u_\tau^{ca}, _{\tau \geq t}$. The argument works inductively. Given that $\succeq_t |_{\mathrm{X}_1}$ has a certainty additive representation with Bernoulli utility functions $u_\tau^{ca}, _{\tau \geq t}$, it follows from time consistency A5 that for all $\mathrm{x}^{t+1}, \mathrm{x}'^{t+1} \in \mathrm{X}^{t+1}$ and any $x_t \in X$:

$$\begin{array}{rccc} & \mathrm{x}^{t+1} & \succeq_{t+1} & \mathrm{x}'^{t+1} \\ \Leftrightarrow & (x_t, \mathrm{x}^{t+1}) & \succeq_t & (x_t, \mathrm{x}'^{t+1}) \\ \Leftrightarrow & u_t^{ca}(x_t) + \sum_{\tau=t+1}^T u_\tau^{ca}(\mathrm{x}_\tau^{t+1}) & \geq & u_t^{ca}(x_t) + \sum_{\tau=t+1}^T u_\tau^{ca}(\mathrm{x}_\tau'^{t+1}) \\ \Leftrightarrow & \sum_{\tau=t+1}^T u_\tau^{ca}(\mathrm{x}_\tau^{t+1}) & \geq & \sum_{\tau=t+1}^T u_\tau^{ca}(\mathrm{x}_\tau'^{t+1}). \end{array}$$

Therefore $\succeq_{t+1}$ has a certainty additive representation which uses the same Bernoulli utility functions u_τ^{ca} for $\tau \geq t+1$ as does the above representation for $\succeq_t$. In the following u_t^{ca} continues to denote the above utility index derived from certainty additivity, while u_t denotes the period t (Bernoulli-) utility function given in the theorem.

Third, I show that for every pair of utility functions u_t^{ca} and u_t there exists a strictly increasing, continuous transformation g_t such that $u_t = g_t \circ u_t^{ca}$. By $u_\tau \in B_{\succeq_t}$ I have:

$$\begin{array}{rcccl} & u_t(x_t) & \geq & u_t(x'_t) & \\ \Leftrightarrow & x_t & \succeq_t^* & x'_t & \\ \Leftrightarrow & (x_t, x_{t+1}, ..., x_T) & \succeq_t & (x'_t, x_{t+1}, ..., x_T) & \forall\, x_{t+1}, ..., x_T \in X \\ \Leftrightarrow & u_t^{ca}(x_t) + \sum_{\tau=t}^T u_\tau^{ca}(x_\tau) & \geq & u_t^{ca}(x'_t) + \sum_{\tau=t}^T u_\tau^{ca}(x_\tau) & \forall\, x_{t+1}, ..., x_T \in X \\ \Leftrightarrow & u_t^{ca}(x_t) & \geq & u_t^{ca}(x'_t) & \end{array}$$

Hence u_t is a stricly[43] monotonic transformation of u_t^{ca} and it exists a strictly increasing function $g_t : U_t \to \mathrm{I\!R}$ such that $u_t^{ca} = g_t \circ u_t$. For the fact that continuity of u_t^{ca} and u_t imply continuity of g_t consult the proof of theorem 1.

Forth, I give a representation over certain consumption paths in terms of the Bernoulli utility functions $u_t, t \in \{1, ..., T\}$, given in the theorem. This is merely a task of combining the two results derived above which yield for all t and all $\mathrm{x}^t, \mathrm{x}'^t \in \mathrm{X}^t$:

$$\begin{array}{lcc} \mathrm{x}^t & \succeq_t & \mathrm{x}'^t \\ \Leftrightarrow \sum_{\tau=t}^T u_\tau^{ca}(\mathrm{x}_\tau^t) & \geq & \sum_{\tau=t}^T u_\tau^{ca}(\mathrm{x}_\tau'^t) \\ \Leftrightarrow \sum_{\tau=t}^T g_\tau \circ u_\tau(\mathrm{x}_\tau^t) & \geq & \sum_{\tau=t}^T g_\tau \circ u_\tau(\mathrm{x}_\tau'^t). \end{array}$$

Part II: In this part, I construct the recursive analogue to the above representation for certain consumption paths. It employs the intertemporal aggregation rules defined in equations (1) and (2). The first step is to show that the normalization constants defined in equation (3) ensure that the domain of g_t^{-1} in the intertemporal aggregation rule is $[\underline{G}_t, \overline{G}_t]$. To this purpose, note that

$$\overline{G}_{t+1} + \vartheta_t = \frac{\overline{G}_{t+1}(\overline{G}_t - \underline{G}_t) + \overline{G}_{t+1}\underline{G}_t - \underline{G}_{t+1}\overline{G}_t}{\Delta G_t} = \frac{\Delta G_{t+1}}{\Delta G_t}\, \overline{G}_t \quad \text{and}$$
$$\underline{G}_{t+1} + \vartheta_t = \frac{\underline{G}_{t+1}(\overline{G}_t - \underline{G}_t) + \overline{G}_{t+1}\underline{G}_t - \underline{G}_{t+1}\overline{G}_t}{\Delta G_t} = \frac{\Delta G_{t+1}}{\Delta G_t}\, \underline{G}_t\,.$$

The maximal value of the argument of $g_t^{-1}[\,\cdot\,]$ in $\mathcal{N}_t^{\mathrm{g}}$ is taken on for $\overline{G}_t = g_t(\overline{U}_t)$ and $\overline{G}_{t+1} = g_{t+1}(\overline{U}_{t+1})$ which yields

$$\begin{aligned} \theta_t \left[g_t(\cdot) + \theta_{t+1}^{-1}\left\{g_{t+1}(\cdot) + \vartheta_t\right\}\right]^{\max} &= \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau}\left[\overline{G}_t + \frac{\sum_{\tau=t+1}^T \Delta G_\tau}{\Delta G_{t+1}}\left\{\overline{G}_{t+1} + \vartheta_t\right\}\right] \\ &= \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau}\left[\overline{G}_t + \frac{\sum_{\tau=t+1}^T \Delta G_\tau}{\Delta G_{t+1}}\left\{\frac{\Delta G_{t+1}}{\Delta G_t}\,\overline{G}_t\right\}\right] \\ &= \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau}\left[\frac{\overline{G}_t \Delta G_t + \overline{G}_t \sum_{\tau=t+1}^T \Delta G_\tau}{\Delta G_t}\right] = \overline{G}_t. \end{aligned}$$

The minimal value of the argument of $g_t^{-1}[\,\cdot\,]$ in $\mathcal{N}_t^{\mathrm{g}}$ is taken on for $\underline{G}_t = g_t(\underline{U}_t)$ and $\underline{G}_{t+1} = g_{t+1}(\underline{U}_{t+1})$. In this case, the same equation holds true with $\overline{G}_t$ replaced by $\underline{G}_t$. Hence the expression defining the intertemporal aggregation rule $\mathcal{N}_t^{\mathrm{g}}$ is well defined.

For the second step, I introduce the notation ${}^t\mathrm{x}^{t-1}$ to denote the continuation of the consumption path $\mathrm{x}^{t-1} \in \mathrm{X}^{t-1}$ from period t on, i.e. $\mathrm{x}^{t-1} = (\mathrm{x}_{t-1}^{t-1}, {}^t\mathrm{x}^{t-1})$. Then, define the aggregate intertemporal utility functions *for certain consumptions paths* by setting $\tilde{u}_T = u_T$ and for $1 < t \leq T$ recursively:

$$\begin{aligned} \tilde{u}_{t-1}(\mathrm{x}^{t-1}) &\equiv \tilde{u}_{t-1}(\mathrm{x}_{t-1}^{t-1}, {}^t\mathrm{x}^{t-1}) = \mathcal{N}_{t-1}^{\mathrm{g}}\left(u_{t-1}(\mathrm{x}_{t-1}^{t-1}), \tilde{u}_t({}^t\mathrm{x}^{t-1})\right) \\ &= g_{t-1}^{-1}\left[\theta_{t-1}\, g_{t-1} \circ u_{t-1}(\mathrm{x}_{t-1}^{t-1}) + \theta_{t-1}\theta_t^{-1}\, g_t \circ \tilde{u}_t({}^t\mathrm{x}^{t-1}) + \theta_{t-1}\theta_t^{-1}\vartheta_{t-1}\right]. \end{aligned}$$

[43] The strictness follows from the fact that the transformation work in both directions and negation.

From the first step in this part it follows that range$(\tilde u_t) = [\underline{U}_t, \overline{U}_t]$.

Third, I show that there exist constants ξ_t, such that the following equation holds for all $t \in \{1, .., T\}$:

$$\theta_t^{-1}\, g_t \circ \tilde u_t(\mathrm{x}^t) = \textstyle\sum_{\tau=t}^T g_\tau \circ u_\tau(\mathrm{x}_\tau^t) + \xi_t\,. \tag{15}$$

As $\theta_T = 1$ this relation obviously holds for $t = T$ (with $\xi_T = 0$). The following manipulation shows that the equation holds by (backwards) induction for all t:

$$\begin{aligned}
&\theta_{t-1}^{-1}\, g_{t-1} \circ \tilde u_{t-1}(\mathrm{x}^{t-1}) \\
&\quad = \theta_{t-1}^{-1}\, g_{t-1} \circ g_{t-1}^{-1}\left[\theta_{t-1}\, g_{t-1} \circ u_{t-1}(\mathrm{x}_{t-1}^{t-1}) + \theta_{t-1}\theta_t^{-1}\, g_t \circ \tilde u_t({}^t\mathrm{x}^{t-1}) + \theta_{t-1}\theta_t^{-1}\vartheta_{t-1}\right] \\
&\quad = g_{t-1} \circ u_{t-1}(\mathrm{x}_{t-1}^{t-1}) + \underbrace{\theta_t^{-1}\, g_t \circ \tilde u_t({}^t\mathrm{x}^{t-1})} + \theta_t^{-1}\vartheta_{t-1} \\
&\quad = g_{t-1} \circ u_{t-1}(\mathrm{x}_{t-1}^{t-1}) + \textstyle\sum_{\tau=t}^T g_\tau \circ u_\tau(\mathrm{x}_\tau^{t-1}) + \underbrace{\xi_t + \theta_t^{-1}\vartheta_{t-1}} \\
&\quad = \textstyle\sum_{\tau=t-1}^T g_\tau \circ u_\tau(\mathrm{x}_\tau^{t-1}) \qquad + \qquad \xi_{t-1}\quad .
\end{aligned}$$

But (15) states that on certain consumption paths $\tilde u_t$ is a (strictly) increasing transformation of $\sum_{\tau=t}^T g_\tau \circ u_\tau$ and hence *a representation of* $\succeq_t |_{\mathrm{X}^t}$.

Part III: The extension of the representation to uncertainty recursively employs theorem 1 and the following proposition:

> In the setup of theorem 1 there exists a certainty equivalent y^p for all $p \in P$ satisfying $v(y^p) = \mathcal{M}^f(p, v)$.

Proof: Pick an arbitrary $p \in P$. I show that the set of *certainty equivalents* $\{y \in Y : v(y) = \mathcal{M}^f(p, v)\}$ is nonempty. As Y is connected compact and v is continuous, the range is a closed interval $v(X) = [\underline{V}, \overline{V}]$. Moreover $\underline{V} = \min_y v(y) = f^{-1} \int \big(\min_y f \circ v(y)\big)\, dp \le \mathcal{M}^f(p, v) \le f^{-1} \int \big(\max_y f \circ v(y)\big)\, dp = \max_y v(y) = \overline{V}$. Therefore, $v^{-1}\left(\mathcal{M}^f(p, v)\right)$ is nonempty for all $p \in P$ (q.e.d.).

The induction hypothesis in proving theorem 2 is the following: For some $t \in \{1, ..., T\}$ and $\tilde u_t$ defined as in the theorem

H1 $\exists f_t : U_t \to \mathrm{I\!R}$ s.th. $p_t \succeq_t p'_t \;\Leftrightarrow\; \mathcal{M}^{f_t}(p_t, \tilde u_t) \ge \mathcal{M}^{f_t}(p'_t, \tilde u_t) \quad \forall\, p_t, p'_t \in P_t$.

The induction step will move backwards from $t = T$ to $t = 1$. The proof uses (and also proves recursively) an additional hypothesis claiming that for every lottery there exists a certainty equivalent that is a certain consumption path:

H2 For all $p_t \in P_t$ there exists $\mathrm{x}^{p_t} \in \mathrm{X}^t$ such that $\mathrm{x}^{p_t} \sim p_t$.

First, I verify that induction hypothesis H1 and H2 are satisfied for $t = T$. Setting $Y = X$, $y = x_T = \mathrm{x}_T$ and $v = \tilde u_T = u_T$, H1 is an immediate consequence of theorem 1

and H2 is an immediate consequence of the above proposition.
Given H1 and H2 for period t, I proceed to show that the induction hypotheses also hold for $t-1$. To this end, note that $\mathcal{M}^{f_t}(p_t, \tilde{u}_t) = \mathcal{M}^{f_t}(\mathrm{x}^{p_t}, \tilde{u}_t) = \tilde{u}_t(\mathrm{x}^{p_t})$ and find that the following equivalence holds:

$$\begin{array}{rccc} & (x_{t-1}, p_t) & \succeq_{t-1} & (x'_{t-1}, p'_t) \\ \Leftrightarrow & (x_{t-1}, \mathrm{x}^{p_t}) & \succeq_{t-1} & (x'_{t-1}, \mathrm{x}^{p'_t}) \\ \Leftrightarrow & \tilde{u}_{t-1}(x_{t-1}, \mathrm{x}^{p_t}) & \geq & \tilde{u}_{t-1}(x'_{t-1}, \mathrm{x}^{p'_t}) \\ \Leftrightarrow & \mathcal{N}^{\mathrm{g}}_{t-1}\left(u_{t-1}(x_{t-1}), \tilde{u}_t(\mathrm{x}^{p_t})\right) & \geq & \mathcal{N}^{\mathrm{g}}_{t-1}\left(u_{t-1}(x'_{t-1}), \tilde{u}_t(\mathrm{x}^{p'_t})\right) \\ \Leftrightarrow & \mathcal{N}^{\mathrm{g}}_{t-1}\left(u_{t-1}(x_{t-1}), \mathcal{M}^{f_t}(p_t, \tilde{u}_t)\right) & \geq & \mathcal{N}^{\mathrm{g}}_{t-1}\left(u_{t-1}(x'_{t-1}), \mathcal{M}^{f_t}(p'_t, \tilde{u}_t)\right) \\ \Leftrightarrow & \tilde{u}_{t-1}(x_{t-1}, p_t) & \geq & \tilde{u}_{t-1}(x'_{t-1}, p'_t)\,, \end{array}$$

where $\tilde{u}_{t-1}$ is the aggregate intertemporal utility function for degenerate period $t-1$ lotteries as given in the theorem. $\tilde{u}_{t-1} \in \mathcal{C}^0(X_{t-1} \times P_t)$ satisfies $(x_{t-1}, p_t) \succeq_{t-1} (x'_{t-1}, p'_t) \Leftrightarrow \tilde{u}_{t-1}(x_{t-1}, p_t) \geq \tilde{u}_{t-1}(x'_{t-1}, p'_t)$ for all $(x_{t-1}, p_t), (x'_{t-1}, p'_t) \in X_{t-1} \times P_t$. Therefore, applying theorem 1 on the compact metric space $Y = X_{t-1} \times P_t$ with the preference relation $\succeq_{t-1}$ and $v = \tilde{u}_{t-1}$ implies the existence of $f_{t-1} : U_{t-1} \to \mathrm{I\!R}$ such that:

$$p_{t-1} \succeq_{t-1} p'_{t-1} \;\Leftrightarrow\; \mathcal{M}^{f_{t-1}}(p_{t-1}, \tilde{u}_{t-1}) \geq \mathcal{M}^{f_{t-1}}(p'_{t-1}, \tilde{u}_{t-1}) \quad \forall\, p_{t-1}, p'_{t-1} \in P_{t-1}.$$

Hence, H1 also holds for $t-1$. Moreover, as shown in the above proposition, for every lottery $p_{t-1} \in P_{t-1}$ there exists a certainty equivalent $\tilde{x}^c = (x^c_{t-1}, p^c_t) \in X_{t-1} \times P_t$ such that $p_{t-1} \sim_{t-1} \tilde{x}^c$. Moreover, given that induction hypothesis H2 holds for t, there exists a certain consumption path $\mathrm{x}^{p^c_t}$ with $\mathrm{x}^{p^c_t} \sim_t p^c_t$. Therefore, by time consistency $\mathrm{x}^{p_{t-1}} \equiv (x^c_{t-1}, \mathrm{x}^{p^c_t})$ is a certain consumption path which satisfies $\mathrm{x}^{p_{t-1}} \sim_{t-1} p_{t-1}$. Hence, the second induction hypothesis H2 is satisfied for $t-1$. Recursion implies that H1 and H2 are satisfied for all $t \in \{1, ..., T\}$, which proofs the sufficiency of the axioms for the representation.

Part IV: Necessity:

A1 (weak order): Transitivity and completeness are trivial.

A2 (independence): Let $p_t \sim_t p'_t$. Then for any $p''_t \in P_t, a \in [0,1]$ it follows:

$$\begin{array}{rccc} & p_t & \sim_t & p'_t \\ \Leftrightarrow & f_t^{-1} \int f_t \tilde{u}_t \, dp_t & = & f_t^{-1} \int f_t \tilde{u}_t \, dp'_t \\ \Leftrightarrow & \int f_t \tilde{u}_t \, dp_t & = & \int f_t \tilde{u}_t \, dp'_t \\ \Leftrightarrow & a \int f_t \tilde{u}_t \, dp_t + (1-a) \int f_t \tilde{u}_t \, dp''_t & = & a \int f_t \tilde{u}_t \, dp'_t + (1-a) \int f_t \tilde{u}_t \, dp''_t \\ \Leftrightarrow & f_t^{-1} \int f_t \tilde{u}_t \, d(a\, p_t + (1-a)\, p''_t) & = & f_t^{-1} \int f_t \tilde{u}_t \, d(a\, p'_t + (1-a)\, p''_t) \\ \Leftrightarrow & a\, p_t + (1-a)\, p''_t & \sim_t & a\, p'_t + (1-a)\, p''_t . \end{array}$$

A3 (continuity): Using the topology of weak convergence on P_t, the functional $\mathcal{M}^{f_t}(\cdot, \tilde{u}_t) : P_t \to \mathrm{I\!R}$ is continuous. For all $p_t \in P_t$ define the numbers $U^{p_t} \in \mathrm{I\!R}$ by $U^{p_t} = \mathcal{M}^{f_t}(p_t, \tilde{u}_t)$. Then, the sets $\{p'_t \in P_t : p'_t \succeq_t p_t\}$ and $\{p'_t \in P_t : p_t \succeq_t p'_t\}$ are the inverse image of the closed intervals $[U^{p_t}, \overline{U}]$ and $[\underline{U}, U^{p_t}]$ under $\mathcal{M}^{f_t}(\cdot, \tilde{u}_t)$ and as such they are closed.

A4 (certainty additivity): Defining $u_\tau^{ca} = g_\tau \circ u_\tau$ for all $\tau \in \{1, ..., T\}$ find that for all $\mathsf{x}, \mathsf{x}' \in \mathsf{X}^T$:

$$\begin{array}{rccc} & \mathsf{x} & \succeq & \mathsf{x}' \\ \Leftrightarrow & \tilde{u}_t(\mathsf{x}) & \geq & \tilde{u}_t(\mathsf{x}') \\ \Leftrightarrow & \sum_{\tau=t}^T g_\tau \circ u_\tau(\mathsf{x}_\tau) & \geq & \sum_{\tau=t}^T g_\tau \circ u_\tau(\mathsf{x}'_\tau) \\ \Leftrightarrow & \sum_{\tau=t}^T u_\tau^{ca}(\mathsf{x}_\tau) & \geq & \sum_{\tau=t}^T u_\tau^{ca}(\mathsf{x}'_\tau). \end{array}$$

A5 (time consistency): For all $t \in \{1, ..., T\}$ find for all $x_t \in X_t$ and $p_{t+1}, p'_{t+1} \in P_{t+1}$:

$$\begin{array}{rl} & (x_t, p_{t+1}) \quad \succeq_t \quad (x_t, p'_{t+1}) \\ \Leftrightarrow & g_t^{-1} \left[\theta_t \, g_t \circ u_t(x_t) + \theta_t \theta_{t+1}^{-1} \, g_{t+1} \circ M^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) + \theta_t \theta_{t+1}^{-1} \vartheta_t \right] \\ & \geq g_t^{-1} \left[\theta_t \, g_t \circ u_t(x_t) + \theta_t \theta_{t+1}^{-1} \, g_{t+1} \circ M^{f_{t+1}}(p'_{t+1}, \tilde{u}_{t+1}) + \theta_t \theta_{t+1}^{-1} \vartheta_t \right] \\ \Leftrightarrow & M^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) \quad \geq \quad M^{f_{t+1}}(p'_{t+1}, \tilde{u}_{t+1}) \\ \Leftrightarrow & p_{t+1} \quad \succeq_{t+1} \quad p'_{t+1}. \end{array}$$

Uniqueness: Let $(u_t, f_t, g_t)_{t \in \{1,...,T\}}$ and $(u_t, f'_t, g'_t)_{t \in \{1,...,T\}}$ both represent $\succeq$: By Wakker (1988, theorem III.4.1) and part one of the proof, it follows that there exists $\mathfrak{a}_t^a \in \mathsf{A}^a$ such that $g'_t = \mathfrak{a}_t^a g_t$. The fact that there exists $\mathfrak{a}_t^+ \in \mathsf{A}^+$ such that $f'_t = \mathfrak{a}_t^+ f_t$, rather than $\mathfrak{a}_t \in \mathsf{A}$ as in theorem 1, follows from the fact that in theorem 2 I restricted the parameterizing functions f_t of the uncertainty aggregation rule to increasing versions.

Let there be given $a \in \mathrm{I\!R}_{++}$ as well as $\mathfrak{a}_t^a \in \mathsf{A}^a$ and $\mathfrak{a}_t^+ \in \mathsf{A}^+$ for all $t \in \{1, ..., T\}$ such that $(f'_t, g'_t) = (\mathfrak{a}^+_t f_t, \mathfrak{a}^a_t g_t)$: If (g_t, f_t) represents $\succeq$, so does $(g_t, \mathfrak{a}^+_t f_t)$ as $\mathcal{M}^{\mathfrak{a}^+_t f_t}(p_t, \tilde{u}_t) = f_t^{-1} \mathfrak{a}_t^{+-1} \int \mathfrak{a}^+_t f_t \tilde{u}_t \, dp_t = \mathcal{M}^{f_t}(p_t, \tilde{u}_t)$. Similarly it holds that $(f_t, \mathfrak{a}^a_t g_t)$ and, thus, (f'_t, g'_t) is a representation of $\succeq$ as $\mathcal{N}^{\mathrm{g}}_t = \mathcal{N}^{\mathrm{g}'}_t$: For the $\boldsymbol{g}'$-scenario the normalization constants change as follows.

$$\begin{aligned}
\theta'_t &= \frac{\Delta G'_t}{\sum_{\tau=t}^T \Delta G'_\tau} = \frac{a\Delta G_t}{\sum_{\tau=t}^T a\Delta G_\tau} = \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau} = \theta_t \quad \text{and} \\
\vartheta'_t &= \frac{\overline{G'}_{t+1}\underline{G'}_t - \underline{G'}_{t+1}\overline{G'}_t}{\Delta G'_t} = \frac{(a\overline{G}_{t+1}+b_{t+1})(a\underline{G}_t+b_t)-(a\underline{G}_{t+1}+b_{t+1})(a\overline{G}_t+b_t)}{a\Delta G_t} \\
&= a\vartheta_t + \frac{b_{t+t}a(\underline{G}_t-\overline{G}_t)+b_t a(\overline{G}_{t+1}-\underline{G}_{t+1})}{a\Delta G_t} + \frac{b_{t+1}b_t - b_{t+1}b_t}{a\Delta G_t} = a\vartheta_t - b_{t+1} + b_t\frac{\Delta G_{t+1}}{\Delta G_t}
\end{aligned}$$

for $t \in \{1, ..., T\}$.[44] Hence, noting that $g_t'^{-1}(\cdot) = g_t^{-1}\left[a_t^{-1}\{(\cdot) - b_t\}\right]$, the intertemporal aggregation rule transforms as

$$\begin{aligned}
\mathcal{N}^{\mathrm{g}'}_t(\cdot,\cdot) &= g_t'^{-1}\left[\theta'_t g'_t(\cdot) + \theta'_t\theta'^{-1}_{t+1} g'_{t+1}(\cdot) + \theta'_t\theta'^{-1}_{t+1}\vartheta'_t\right] \\
&= g_t^{-1}\Big[a^{-1}\Big\{\theta_t(ag_t(\cdot)+b_t) + \theta_t\theta^{-1}_{t+1}(ag_{t+1}(\cdot)+b_{t+1}) \\
&\qquad\qquad +\theta_t\theta^{-1}_{t+1}(a\vartheta_t - b_{t+1} + b_t\tfrac{\Delta G_{t+1}}{\Delta G_t}) - b_t\Big\}\Big] \\
&= g_t^{-1}\Big[\theta_t g_t(\cdot) + \theta_t\theta^{-1}_{t+1}g_{t+1}(\cdot) + \theta_t\theta^{-1}_{t+1}\vartheta_t + a^{-1} \\
&\qquad \Big\{\theta_t b_t + \theta_t\theta^{-1}_{t+1}b_{t+1} + \theta_t\theta^{-1}_{t+1}(-b_{t+1} + b_t\tfrac{\Delta G_{t+1}}{\Delta G_t}) - b_t\Big\}\Big] \\
&= g_t^{-1}\Big[\theta_t g_t(\cdot) + \theta_t\theta^{-1}_{t+1}g_{t+1}(\cdot) + \theta_t\theta^{-1}_{t+1}\vartheta_t\Big] \;=\; \mathcal{N}^{\mathrm{g}}_t(\cdot,\cdot)\,.
\end{aligned}$$

To arrive at the last line I have used the relation

$$\begin{aligned}
\theta_t\theta^{-1}_{t+1} &= \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau}\frac{\sum_{\tau=t+1}^T \Delta G_\tau}{\Delta G_{t+1}} = \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau}\frac{\sum_{\tau=t}^T \Delta G_\tau}{\Delta G_{t+1}} - \frac{\Delta G_t}{\sum_{\tau=t}^T \Delta G_\tau}\frac{\Delta G_t}{\Delta G_{t+1}} \\
&= (1-\theta_t)\frac{\Delta G_t}{\Delta G_{t+1}}.
\end{aligned}$$

Proof of lemma 1: For the last period it holds $\tilde{u}'_T = s_T \circ \tilde{u}_T$ and $\mathcal{M}^{f'_T}(p_T, \tilde{u}'_T) = s_T \circ f_T^{-1}\left[\int f_T \circ s_T^{-1} \circ s_T \circ \tilde{u}_T \, dp_T\right] = s_T \circ \mathcal{M}^{f_T}(p_T, \tilde{u}_T)$. Moreover, recursively for period $t \in \{1, ..., T-1\}$ find that the new aggregate intertemporal utility function becomes[45]

$$\begin{aligned}
\tilde{u}'_\tau(x_t, p_{t+1}) &= s_t g_t^{-1}\left[\theta_t\, g_t s_t^{-1} s_t u_t(x_t) + \theta_t\theta^{-1}_{t+1}\, g_{t+1}\mathcal{M}^{f_{t+1}}(p_{t+1}, \tilde{u}_{t+1}) + \theta_t\theta^{-1}_{t+1}\vartheta_t\right] \\
&= s_t \circ \tilde{u}_t(x_t, p_{t+1})\,.
\end{aligned}$$

and that the uncertainty aggregation rule changes to

$$\mathcal{M}^{f'_t}(p_t, \tilde{u}'_t) = s_t \circ f_t^{-1}\left[\int f_t \circ s_t^{-1} \circ s_t \circ \tilde{u}_t \, dp_t\right] = s_t \circ \mathcal{M}^{f_t}(p_t, \tilde{u}_t).$$

As the latter is a strictly increasing transformation of $\mathcal{M}^{f_t}(p_t, \tilde{u}_t)$, it represents $\succeq_t$. □

[44] Where b_{T+1} and ΔG_{T+1} are treated as zero to render $\vartheta'_T = 0$.

[45] As the range of g_τ and g'_τ are the same, the normalization constants do not change.

Proof of corollary 2: Sufficiency: By Wakker (1988, theorem III.4.1) the axioms imply that the sets of Bernoulli utility functions are nonempty. Therefore, theorem 2 implies the existence of a representation $(u_t^0, f_t^0, g_t^0)_{t\in\{1,...,T\}}$. Define the functions $s_t = f_t^{-1} f_t^0$ which are strictly increasing and continuous (see footnote 42). Lemma 1 implies that $([f_t^{-1} f_t^0] u_t^0, f_t^0 [f_t^{-1} f_t^0]^{-1}, g_t^0 [f_t^{-1} f_t^0]^{-1})_{t\in\{1,...,T\}} = (f_t^{-1} f_t^0 u^0, f_t, g_t^0 f_t^{0^{-1}} f_t)_{t\in\{1,...,T\}}$ is a representation of $\succeq$, with f_t characterizing the uncertainty aggregation rule.
Necessity: Necessity of the axioms to hold is implied by theorem 2.
Uniqueness: Let $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ and $(u_t', f_t, g_t')_{t\in\{1,...,T\}}$ be representations in the sense of the corollary: For every t there exist strictly increasing and continuous functions s_t such that $u_t' = s_t \circ u_t$. Lemma 1 implies that with $(u_t', f_t, g_t')_{t\in\{1,...,T\}} = (s_t\, u_t, f_t, g_t')_{t\in\{1,...,T\}}$ being a representation of $\succeq$, so is the sequence of triples $(u_t, f_t s_t, g_t' s_t)_{t\in\{1,...,T\}}$. Comparing the latter to the representation $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$, the uniqueness part of theorem 2 implies the existence of $a \in \mathrm{I\!R}_{++}$ and the existence of affine transformations $\mathfrak{a}_t^+ \in \mathsf{A}^+$ and $\mathfrak{a}_t^a \in \mathsf{A}^a$ for all $t \in \{1, ..., T\}$, such that

$$f_t = \mathfrak{a}_t^+ f_t\, s_t \quad \Leftrightarrow \quad s_t^{-1} = f_t^{-1} a_t^+ f_t \quad \text{and} \tag{16}$$

$$g_t = \mathfrak{a}_t^a g_t' s_t. \tag{17}$$

Substituting the relation for s_t in equation (16) into the equations for g_t and u_t renders

$$g_t = \mathfrak{a}_t^a g_t' f_t^{-1} a_t^{+^{-1}} f_t \quad \text{and}$$
$$u_t = s_t^{-1} u_t' = f_t^{-1} a_t^+ f_t\, u_t'.$$

Let $(u_t, f_t, g_t)_{t\in\{1,...,T\}}$ be a representation of $\succeq$ and let $a \in \mathrm{I\!R}$, $\mathfrak{a}_t^+ \in \mathsf{A}^+$ and $\mathfrak{a}_t^a \in \mathsf{A}^a$ for all $t \in \{1, ..., T\}$: Then, by theorem 2, the sequence $(u_t, \mathfrak{a}_t^+ f_t, \mathfrak{a}_t^a\, g_t)_{t\in\{1,...,T\}}$ is a representation of $\succeq$. By lemma 1 it follows that also $([\mathfrak{a}_t^+ f] u_t, \mathfrak{a}_t^+ f_t [\mathfrak{a}_t^+ f_t]^{-1}, \mathfrak{a}_t^a g_t [\mathfrak{a}_t^+ f_t]^{-1})_{t\in\{1,...,T\}}$ $= (\mathfrak{a}_t^+ f_t u_t, \mathrm{id}, \mathfrak{a}_t^a g_t f_t^{-1} \mathfrak{a}_t^{+^{-1}})_{t\in\{1,...,T\}}$ is a representation of $\succeq$. Applying lemma 1 once again yields the result that the sequence $(f_t^{-1} \mathfrak{a}_t^+ f_t u_t, f_t, \mathfrak{a}_t^a g_t f_t^{-1} \mathfrak{a}_t^{+^{-1}} f_t)_{t\in\{1,...,T\}}$ is a representation of $\succeq$. □

Proof of corollary 3: Imitates the proof of corollary 2. In the uniqueness part instead of equations (16) and (17) find

$$f_t = \mathfrak{a}_t^+ f_t'\, s_t \quad \text{and}$$
$$g_t = \mathfrak{a}_t^a g_t s_t \quad \Leftrightarrow \quad s_t^{-1} = g_t^{-1} \mathfrak{a}_t^a g_t\,.$$

Substituting the result for the functions s_t into the equations for f_t and u_t renders

$$f_t = \mathfrak{a}_t^+ f_t' \, g_t^{-1} \mathfrak{a}_t^{a-1} g_t \quad \text{and}$$
$$u_t = s_t^{-1} u_t' = g_t^{-1} \mathfrak{a}_t^a \, g_t \, u_t' \, . \qquad \square$$

B Proofs for Section 3

Proof of proposition 1: First, I confirm equation (12). Let $z' = s(z) \Leftrightarrow z = s^{-1}(z')$ and note that $\frac{d}{dz'}s^{-1}(z') = [\frac{d}{dz}s(z)]^{-1}$ and

$$\begin{aligned}
\frac{\frac{d^2}{dz'^2}s^{-1}(z')}{\frac{d^2}{dz'^2}s^{-1}(z')} &= -\frac{\frac{d^2}{dz^2}s(z)}{\left(\frac{d}{dz}s(z)\right)^2}\,, \\
\frac{d}{dz'}f'_t(z') &= \frac{d}{dz'}f_t \circ s^{-1}(z') = \frac{d}{dz}f_t(z) \cdot \frac{d}{dz'}s^{-1}(z')\,, \\
\frac{d^2}{dz'^2}f'_t(z') &= \frac{d^2}{dz^2}f_t(z) \cdot \left[\frac{d}{dz'}s^{-1}(z')\right]^2 + \frac{d}{dz}f_t(z) \cdot \frac{d^2}{dz'^2}s(z')^{-1}\,, \\
\frac{\frac{d^2}{dz'^2}f'_t(z')}{\frac{d}{dz'}f'_t(z')} &= \frac{\frac{d^2}{dz^2}f_t(z)}{\frac{d}{dz}f_t(z)}\,\frac{d}{dz'}s^{-1}(z') + \frac{\frac{d^2}{dz'^2}s(z')^{-1}}{\frac{d}{dz'}s(z')^{-1}} \\
&= \frac{1}{\frac{d}{dz}s(z)}\left[\frac{\frac{d^2}{dz^2}f_t(z)}{\frac{d}{dz}f_t(z)} + \frac{\frac{d^2}{dz^2}s(z)}{\frac{d}{dz}s(z)}\right]
\end{aligned}$$

Thus, for $z = {}^*x$ the risk measure RIRA_t' evaluated at $z' = {}^*x' = \tilde{\Phi}_1({}^\circ x) = s \circ \Phi_1({}^\circ x) = s({}^*x) = s(z)$ is given by equation (12).

Second, let $\text{RIRA}_t = -\frac{\frac{d^2}{dz^2}f_t(z)}{\frac{d}{dz}f_t(z)}z = c^o$ characterize risk aversion for some coordinate system at point ${}^\circ x$ with $z = \Phi_1({}^\circ x)$. Let $z' = s(z)$ be the first coordinate of ${}^\circ x$ after the change of coordinate system described in the text. Let c^n be the desired value of risk aversion at ${}^\circ x$. Choosing $s = \text{id} - \Phi_1({}^\circ x) + \frac{c^n}{c^o}$ yields

$$\begin{aligned}
\text{RIRA}_t'(z')|_{z'=\Phi_1'({}^\circ x)} &= -\frac{\frac{d^2}{dz'^2}f'_t(z')}{\frac{d}{dz'}f'_t(z')}z'|_{z'=\Phi_1'({}^\circ x)} \\
&= -\frac{s(z)}{\frac{d}{dz}s(z)}\left[\frac{\frac{d^2}{dz^2}f_t(z)}{\frac{d}{dz}f_t(z)} + \frac{\frac{d^2}{dz^2}s(z)}{\frac{d}{dz}s(z)}\right]|_{z=\Phi_1({}^\circ x)} \\
&= -\frac{\text{id} - \Phi_1({}^\circ x) + \frac{c^n}{c^o}}{1}\left[c^o + 0\right]|_{z=\Phi_1({}^\circ x)} = c^n\,. \qquad \square
\end{aligned}$$

Proof of proposition 2: Follows immediately from equation (12) and the fact that preferences cannot, in general, be represented by Bernoulli utility functions that are linear in all arguments at all points.

Proof of proposition 3: Because $u_t, u'_t \in B_{\succeq_t}$, there exist strictly increasing and continuous transformations s_t such that $u'_t = s_t \circ u_t$ for all $t \in \{1, ..., T\}$. By lemma 1 the sequence $(s_t \circ u_t, f_t \circ s_t^{-1}, g_t \circ s_t^{-1})_{t\in\{1,...,T\}} = (u'_t, f_t \circ s_t^{-1}, g_t \circ s_t^{-1})_{t\in\{1,...,T\}} = (u'_t, f'_t, g'_t)_{t\in\{1,...,T\}}$ is a representation of $\succeq$ in the sense of theorem 2 with $f'_t \circ g'^{-1}_t = f_t \circ g_t^{-1}$ for all $t \in \{1, ..., T\}$. □

C Proofs for Section 4

Proof of theorem 3: The proof is divided into five parts. In the first, I translate axiom A6$^{\text{s}}$ into the representation of theorem 2. In the second part, I show that the equation derived in the first part locally implies strict concavity of $f_t \circ g_t^{-1}$. Part three extends this result to strict concavity on the entire set Γ_t. Part four proofs the necessity of axiom A6$^{\text{s}}$ for the strict concavity of $f_t \circ g_t^{-1}$. Together, parts one through four proof assertion a) of the theorem for the case of strict intertemporal risk aversion. For the case of strict intertemporal risk seeking just change the signs in the inequalities and replace concave by convex. Part five lays out how assertions b-d) follow from the proof of assertion a).

Part I ("⇒"): In part one I translate axiom A6$^{\text{s}}$ into the representation of theorem 2. I start with the first line, i.e the premise, and use equation (15) to find:

$$\begin{array}{ccc} \mathrm{x}^t & \sim_t & \mathrm{x}'^t \\ \Rightarrow g_t^{-1}\left[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}^t_\tau) + \xi_t\right] & = & g_t^{-1}\left[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}'^t_\tau) + \xi_t\right]. \end{array} \tag{18}$$

The existence of $\tau \in \{t, ..., T\}$ such that $\mathrm{x}^t_\tau \not\sim^*_\tau \mathrm{x}'^t_\tau$, translates into

$$g_\tau u_\tau(\mathrm{x}^t_\tau) \neq g_\tau u(\mathrm{x}'^t_\tau) \text{ for some } \tau \in \{t, ..., T\}. \tag{19}$$

The second line of axiom A6$^{\text{s}}$ becomes

$$\begin{array}{ccc} \mathrm{x}^t & \succ_T & \sum_{i=t}^T \frac{1}{T-t+1} (\mathrm{x}^t_{-i}\mathrm{x}'^t_i). \\ \Rightarrow g_t^{-1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau(\mathrm{x}^t_\tau) + \xi_t\right] & > & \\ & & f_t^{-1}\left[\sum_{i=t}^T \frac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau\left((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)_\tau\right) + \xi_t\right]\right] \end{array}$$

$$\Rightarrow f_t g_t^{-1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau(\mathrm{x}^t_\tau) + \xi_t\right] > \sum_{i=t}^T \frac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau\left((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)_\tau\right) + \xi_t\right].$$

Using equation (18) the left hand side can be transformed as follows:

$$f_t g_t^{-1}\left[\tfrac{T-t}{T-t+1}\left[\theta_t \textstyle\sum_{\tau=t}^T g_\tau u_\tau(\mathsf{x}_\tau^t)+\xi_t\right]+\tfrac{1}{T-t+1}\left[\theta_t \textstyle\sum_{\tau=t}^T g_\tau u_\tau(\mathsf{x}_\tau'^t)+\xi_t\right]\right] >$$
$$\textstyle\sum_{i=t}^T \tfrac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \textstyle\sum_{\tau=t}^T g_\tau u_\tau\left((\mathsf{x}_{-i}^t \mathsf{x}_i'^t)_\tau\right)+\xi_t\right]$$
$$\Rightarrow f_t g_t^{-1}\left[\tfrac{1}{T-t+1}\left[\theta_t \textstyle\sum_{i=t}^T\sum_{\tau=t}^T g_\tau u_\tau\left((\mathsf{x}_{-i}^t \mathsf{x}_i'^t)_\tau\right)+\xi_t\right]\right] >$$
$$\textstyle\sum_{i=t}^T \tfrac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \textstyle\sum_{\tau=t}^T g_\tau u_\tau\left((\mathsf{x}_{-i}^t \mathsf{x}_i'^t)_\tau\right)+\xi_t\right]$$
$$\Rightarrow f_t g_t^{-1}\left[\textstyle\sum_{i=t}^T \tfrac{1}{T-t+1}\left[\theta_t \textstyle\sum_{\tau=t}^T g_\tau u_\tau\left((\mathsf{x}_{-i}^t \mathsf{x}_i'^t)_\tau\right)+\xi_t\right]\right] > \tag{20}$$
$$\textstyle\sum_{i=t}^T \tfrac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \textstyle\sum_{\tau=t}^T g_\tau u_\tau\left((\mathsf{x}_{-i}^t \mathsf{x}_i'^t)_\tau\right)+\xi_t\right].$$

Let me define the function $\tilde{z} : \mathsf{X}^t \to \Gamma_t$ by $\tilde{z}(\mathsf{x}^t) = \theta_t \sum_{\tau=t}^T g_\tau u_\tau(\mathsf{x}_\tau^t) + \xi_t$. Compare part two of the proof of theorem 2 to see that, when restricting the domain to those consumption paths satisfying equation (19),[46] the function $\tilde{z}$ is onto $\Gamma_t = \left(\underline{G}_\tau\,,\,\overline{G}_\tau\right) = \left(\theta_t \sum_{\tau=t}^T \underline{G}_\tau + \xi_t\,,\,\theta_t \sum_{\tau=t}^T \overline{G}_\tau + \xi_t\right)$. In particular define $z_i = \tilde{z}\big((\mathsf{x}_{-i}^t \mathsf{x}_i'^t)\big)$. In this notation equation (20) becomes

$$f_t g_t^{-1}\left(\textstyle\sum_{i=t}^T \tfrac{1}{T-t+1}\, z_i\right) \quad > \quad \textstyle\sum_{i=t}^T \tfrac{1}{T-t+1} f_t g_t^{-1}(z_i). \tag{21}$$

If equation (21) had to hold for all $z_i \in \Gamma_t$ it would be a straight forward condition for strict concavity of $f_t \circ g_t^{-1}$. However, axiom A6$^{\mathrm{s}}$ does not immediately imply that the equation has to be met for every choice $(z_i)_{i\in\{t,\ldots,T\}}, z_i \in \Gamma_t$. *Equation* (21) has to hold *only for sequences* $(z_i)_{i\in\{t,\ldots,T\}}$ that are stemming from consumption paths $(\mathsf{x}_{-i}^t \mathsf{x}_i'^t)$ for which $\mathsf{x}'^t \in \mathsf{X}^t$ and $\mathsf{x}_\tau^t \in \mathsf{X}^t$ *satisfy the premise of axiom A6*$^{\mathrm{s}}$. *In what follows, I proceed to show that this restricted demand is enough to imply strict concavity of* $f_t \circ g_t^{-1}$ *on* Γ_t.

Part II ("⇒"): Let $z^o \in \Gamma_t$. In this part I show that for every such z^o there exists an open neighborhood $N_{z^o} \subset \Gamma_t$ such that equation (21) implies strict concavity of $f_t \circ g_t^{-1}$ on N_{z^o}.

In the first step I define a certain consumption path $\mathsf{x}^{ot} \in \mathsf{X}^t$ with $\tilde{z}(\mathsf{x}^{ot}) = z^o$. It will satisfy the additional characteristic that none of its outcomes is extremal. Define $(G_\tau^o)_{\tau\in\{t,\ldots,T\}}$ to be a sequence with $\underline{G}_\tau < G_\tau^o < \overline{G}_\tau \forall\tau$ and $\theta_t \sum_{\tau=t}^T G_\tau^o + \xi_t = z^o$. Such a sequence has to exist as $z^o \in \Gamma_t$ implies $\theta_t \sum_{\tau=t}^T \underline{G}_\tau + \xi_t < z^o < \theta_t \sum_{\tau=t}^T \overline{G}_\tau + \xi_t$. Moreover by connectedness of X and continuity of $g_\tau \circ u_\tau$ there exists for every $\tau \in \{t,\ldots,T\}$ an outcome $x_\tau^o \in u_\tau^{-1}\left[g_\tau^{-1}\left(G_\tau^o\right)\right]$ such that $G_\tau^o = u_\tau g_\tau(x_\tau^o)$. Define $\mathsf{x}_\tau^{ot} = (x_t^o, \ldots, x_T^o)$.

In the second step I define deviation paths $\mathsf{x}^{\mu t}$ around x^{ot}. Set $\epsilon_\tau = \min\{G_\tau^o - \underline{G}_\tau, \overline{G}_\tau - G_\tau^o\}$ for $\tau \in \{t,\ldots,T\}$ and let $\epsilon = \min_{\tau\in\{t,\ldots,T\}} \epsilon_\tau$. By construction of x^{ot} it is

[46] It is for the latter restriction that the theorem is considering the *open* set Γ_t.

$\epsilon > 0$. For *any* sequence $\mu = (\mu_\tau)_{\tau\in\{t,...,T\}}$ with $\mu_\tau \in (-\epsilon, \epsilon)$ define $G^\mu_\tau = G^o_\tau + \mu_\tau$ for all $\tau \in \{t, ..., T\}$. Then each G^μ_τ is element of $(G^o_\tau - \epsilon, G^o_\tau + \epsilon) \subset (\underline{G}_\tau, \overline{G}_\tau)$ and hence there exists $x^{\mu t}_\tau \in u_t^{-1}\left[g_t^{-1}(G^\mu_\tau)\right]$. Define $\mathrm{x}^{\mu t} = (x^\mu_t, ..., \mathrm{x}^\mu_T)$.

Third, I calculate the $z^\mu_i \in \Gamma_t$ corresponding to the consumption paths $(\mathrm{x}^{ot}_{-i}\mathrm{x}^{\mu t}_i)$ and restate the condition $x^{ot} \sim_t x^{\mu t}$ in terms of z^o and $(z^\mu_i)_{i\in\{t,...,T\}}$. It is

$$\begin{aligned} z^\mu_i = \tilde z\big((\mathrm{x}^{ot}_{-i}\mathrm{x}^{\mu t}_i)\big) &= \theta_t \textstyle\sum_{\tau=t}^T g_\tau u_\tau\left((\mathrm{x}^{ot}_{-i}\mathrm{x}^{\mu t}_i)_\tau\right) + \xi_t \\ &= \theta_t\left((\textstyle\sum_{\tau=t}^T G^o_\tau) - G^o_i + G^\mu_i\right) + \xi_t \\ &= z^o + \theta_t(G^\mu_i - G^o_i). \end{aligned}$$

Hence $z^\mu_i = \tilde z\big((\mathrm{x}^{ot}_{-i}\mathrm{x}^{\mu t}_i)\big)$ as a fuction of μ_i is *onto* $(G^o_\tau - \theta_t\epsilon\,,\, G^o_\tau + \theta_t\epsilon)$. The equation also implies that the condition $[\mathrm{x}^{ot}_\tau] \not\sim_\tau [\mathrm{x}^{\mu t}_\tau] \Leftrightarrow g_\tau u_\tau(x^\eta_\tau) \neq g_\tau u(x^\mu_\tau) \Leftrightarrow G^o_\tau \neq G^\mu_\tau$ for some $\tau \in \{t, ..., T\}$ is equivalent $z^\mu_i \neq z^o$ for some τ. Using equation (18) I further find that $x^{ot} \sim_t x^{\mu t}$ translates into

$$\begin{aligned} &\theta_t \textstyle\sum_{\tau=t}^T G^o_\tau + \xi_t = \theta_t \sum_{\tau=t}^T G^\mu_\tau + \xi_t \\ \Rightarrow\ &\theta_t \textstyle\sum_{\tau=t}^T G^o_\tau + \xi_t = \frac{T-t}{T-t+1}\left(\theta_t \sum_{\tau=t}^T G^o_\tau + \xi_t\right) + \frac{1}{T-t+1}\left(\theta_t \sum_{\tau=t}^T G^\mu_\tau + \xi_t\right) \\ \Rightarrow\ &\theta_t \textstyle\sum_{\tau=t}^T G^o_\tau + \xi_t = \frac{1}{T-t+1}\sum_{i=t}^T\left(\theta_t\left((\sum_{\tau=t}^T G^o_\tau) - G^o_i + G^\mu_i\right) + \xi_t\right) \\ \Rightarrow\ &\qquad z^o = \tfrac{1}{T-t+1}\textstyle\sum_{i=t}^T z^\mu_i. \end{aligned}$$

Summarizing steps one to three I have shown that equation (21) has to hold for all sequences $(z_i)_{i\in\{t,...,T\}}$ with $z_i \in (z^o - \theta_t\epsilon, z^o + \theta_t\epsilon)$ satisfying $\frac{1}{T-t+1}\sum_{i=t}^T z_i = z^o$ (and not all $z_i = z^o$). However, due to the restriction that the weighted average has to equal z^o this requirement is not enough to guarantee concavity of $f_t g_t^{-1}$ on $z_i \in (z^o - \theta_t\epsilon, z^o + \theta_t\epsilon)$. *Define* $N_{z^o} = (z^o - \frac{\theta_t\epsilon}{2}, z^o + \frac{\theta_t\epsilon}{2})$. In the following I proceed to show that (21) has to hold for *all* non-constant sequences $(z_i)_{i\in\{t,...,T\}}$ with $z_i \in N_{z^o}$. The latter will be sufficient to guarantee strict concavity of $f_t g_t^{-1}$ on the open set N_{z^o}.

In step four, take any $z^* \in N_{z^o}$. I construct a corresponding consumption path x^{*t} with $z^* = \tilde z(\mathrm{x}^{*t})$ as well as a perturbation $\mathrm{x}^{\eta t}$ around it. Define

$$G^*_\tau = G^o_\tau + \tfrac{z^*-z^o}{\theta_t(T-t+1)} \in \left(G^o_\tau - \tfrac{-\theta_t\epsilon}{2\theta_t(T-t+1)}\,, G^o_\tau + \tfrac{-\theta_t\epsilon}{2\theta_t(T-t+1)}\right) \subset (G^o_\tau - \epsilon, G^o_\tau + \epsilon)\,.$$

Then there exists $x^*_\tau \in u_t^{-1}\left[g_t^{-1}(G^*_\tau)\right]$. Define the consumption path $\mathrm{x}^{*t} = (x^*_t, ..., x^*_T)$ and find that indeed

$$\begin{aligned} \tilde z(\mathrm{x}^{*t}) &= \theta_t \sum_{\tau=t}^T G^*_\tau + \xi_t = \theta_t\left(\sum_{\tau=t}^T G^o_\tau + \tfrac{z^*-z^o}{\theta_t(T-t+1)}\right) + \xi_t \\ &= z^o + z^* - z^o\left(\sum_{\tau=t}^T \tfrac{1}{(T-t+1)}\right) = z^* \end{aligned}$$

Aim of the following construction is to make sure that the perturbations $\mathrm{x}^{\eta t}$ around x^{*t} account for all sequences $(z_i)_{i\in\{t,...,T\}}$ with $z_i \in N_{z^o}$ that satisfy $\frac{1}{T-t+1}\sum_{i=t}^T z_i = z^*$. Define $\epsilon^*_- = \epsilon-(G^o_i-G^*_i)$ and $\epsilon^*_+ = \epsilon+(G^o_i-G^*_i)$. For any sequence $\eta = (\eta_\tau)_{\tau\in\{t,...,T\}}$ with $\eta_\tau \in (-\epsilon^*_-, \epsilon^*_+)$ let $G^\eta_\tau = G^o_\tau+\eta_\tau$ for all $\tau \in \{t, ..., T\}$. Then each G^η_τ is in $(G^o_\tau-\epsilon, G^o_\tau+\epsilon) \subset (\underline{G}_\tau, \overline{G}_\tau)$ and hence there exists $x^{\eta t}_\tau \in u_t^{-1}\left[g_t^{-1}(G^\eta_\tau)\right]$. Let $\mathrm{x}^{\eta t} = (x^\eta_t, ..., \mathrm{x}^\eta_T)$.

In step five, I calculate the $z^\eta_i = \tilde{z}\big((\mathrm{x}^{*t}_{-i}\mathrm{x}^{\eta t}_i)\big)$ corresponding to the consumption paths $(\mathrm{x}^{*t}_{-i}\mathrm{x}^{\eta t}_i)$ and restate the condition $x^{*t} \sim_t x^{\eta t}$ in terms of z^* and $(z^\eta_i)_{i\in\{t,...,T\}}$. It is

$$\begin{aligned} z^\eta_i = \tilde{z}\big((\mathrm{x}^{*t}_{-i}\mathrm{x}^{\eta t}_i)\big) &= \theta_t \textstyle\sum_{\tau=t}^T g_\tau u_\tau\left((\mathrm{x}^{*t}_{-i}\mathrm{x}^{\eta t}_i)_\tau\right) + \xi_t \\ &= \theta_t \left((\textstyle\sum_{\tau=t}^T G^*_\tau) - G^*_i + G^\eta_i \right) + \xi_t \\ &= z^* + \theta_t(G^\eta_i - G^*_i). \end{aligned} \tag{22}$$

As before with x^{ot} and $x^{\mu t}$ the condition $[\mathrm{x}^{*t}_\tau] \not\sim_\tau [\mathrm{x}^{\eta t}_\tau]$ for some $\tau \in \{t, ..., T\}$ is equivalent to $z^\mu_i \neq z^o$ for some i and equations (18) and (22) translate $x^{*t} \sim_t x^{\eta t}$ into

$$z^* = \frac{1}{T-t+1}\sum_{i=t}^T z^\eta_i \,.$$

In step six it is shown that the z^η_i calculated in the previous step can generate any sequence $(z_i)_{i\in\{t,...,T\}}$ with elements $z_i \in N_{z^o}$ that satisfies $\frac{1}{T-t+1}\sum_{i=t}^T z^\eta_i = z^*$. To verify this fact find that each $z^\eta_i = z^* + \theta_t(G^\eta_i - G^*_i)$ can take any[47] of the values in

$$\begin{aligned} &\Big(z^* + \theta_t(-\epsilon^*_-)\ ,\ z^* + \theta_t\epsilon^*_-\Big) \\ =\ &\Big(z^o + (z^* - z^o) - \theta_t\big(\epsilon - (G^o_i - G^*_i)\big)\ ,\ z^o + (z^* - z^o) + \theta_t\big(\epsilon + (G^o_i - G^*_i)\big)\Big) \\ =\ &\Big(z^o + (z^* - z^o) - \theta_t\epsilon - \theta_t\tfrac{z^*-z^o}{\theta_t(T-t+1)}\ ,\ z^o + (z^* - z^o) + \theta_t\epsilon - \theta_t\tfrac{z^*-z^o}{\theta_t(T-t+1)}\Big) \\ =\ &\Big(z^o - \theta_t\epsilon + (z^* - z^o)\left(1 - \tfrac{1}{T-t+1}\right)\ ,\ z^o + \theta_t\epsilon + (z^* - z^o)\left(1 - \tfrac{1}{T-t+1}\right)\Big) \end{aligned}$$

which due to $z^* \in N_{z^o} = (z^o - \frac{\theta_t\epsilon}{2}, z^o + \frac{\theta_t\epsilon}{2})$ is a superset of

$$\begin{aligned} \supseteq\ &\Big(z^o - \theta_t\epsilon + \tfrac{\theta_t\epsilon}{2}\left(1 - \tfrac{1}{T-t+1}\right)\ ,\ z^o + \theta_t\epsilon - \tfrac{\theta_t\epsilon}{2}\left(1 - \tfrac{1}{T-t+1}\right)\Big) \\ \supseteq\ &\Big(z^o - \tfrac{\theta_t\epsilon}{2}\ ,\ z^o + \tfrac{\theta_t\epsilon}{2}\Big). \end{aligned}$$

Therefore the z^η_i can take on any value in N_{z^o} as long as the sequence satisfies $z^* = \frac{1}{T-t+1}\sum_{i=t}^T z^\eta_i$. Hence equation (21) also has to hold for all non-constant sequences $(z_i)_{i\in\{t,...,T\}}$ with $z_i \in N_{z^o}$ and $\frac{1}{T-t+1}\sum_{i=t}^T z_i = z^*$.

Finally, I show that $f_t g_t^{-1}$ has to be strictly concave on N_{z^o}. Equation (21) has to hold for all non-constant sequences $(z_i)_{i\in\{t,...,T\}}$ with $z_i \in N_{z^o}$ and $\frac{1}{T-t+1}\sum_{i=t}^T z_i = z^*$. But z^*

[47] Of course all z_i together have to sum up to $(T-t+1)z^*$ and not all z_i can be equal to z^*. These however are the only restrictions.

was an arbitrary element of N_{z^o} and steps four to six hold for any $z^* \in N_{z^o}$. Therefore equation (21) has to hold for all sequences $(z_i)_{i\in\{t,...,T\}}$ with $z_i \in N_{z^o}$ except for the constant sequences with $z_i = z_j \, \forall \, i,j \in \{t,...,T\}$).[48] Now pick any $l \in \{t,...,T-1\}$ and define $\lambda = \frac{l-t+1}{T-t+1} > 0$. Furthermore for any pair $z_a, z_b \in N_{z^o}$ select $z_t = ... = z_l = z_a$ and $z_{l+1} = ... = z_T = z_b$. Then equation (21) becomes

$$f_t g_t^{-1}\big(\lambda z_a + (1-\lambda) z_b\big) \quad > \quad \lambda f_t g_t^{-1}(z_a) + (1-\lambda) f_t g_t^{-1}(z_b)$$

and has to hold for all $z_a, z_b \in N_{z^o}, z_a \neq z_b$. But due to the continuity of $f_t \circ g_t^{-1}$ this implies strict concavity of $f_t \circ g_t^{-1}$ on N_{z^o} (Hardy et al. 1964, 74,75).

Part III ("⇒"): In this part I show that the local strict concavity of $f_t \circ g_t^{-1}$ on N_{z^o} for all $z^o \in N_{z^o}$ as derived in the second part implies strict concavity on Γ_t.[49] I will first demonstrate that weak concavity extends to Γ_t and then that local strict concavity together with global weak concavity imply strict concavity of $f_t \circ g_t^{-1}$ on all of Γ_t.

First, note that a concave function $h_t = f_t \circ g_t^{-1}$ on N_{z^o} has non-increasing right-continuous right-derivatives h'_{t+} as well as non-increasing left-continuous left-derivatives h'_{t-} at every point in N_{z^o} (van Tiel 1984, 4,5). Moreover there are at most countably many points in N_{z^o} where h_t is not differentiable (van Tiel 1984, 5). Take any closed interval $[z^l, z^u] \subset \Gamma_t$. Then already a finite number of open sets N_{z^o} with $z^o \in I \subseteq \Gamma_t$, I finite, cover $[z^l, z^u]$ (Heine-Borel-theorem). Hence there are just countably many points where h_t is not differentiable on $[z^l, z^u]$. Denote the countable set where h_t is not differentiable by A. Then on $[z^l, z^u] \backslash A$ it is $h'_{t-} = h'_{t+}$ and due to the left-continuity of the left-derivative and right-continuity of the right-derivative ${h_t}'$ is continuous on $[z^l, z^u] \backslash A$. Moreover for all points in A the left- and right-derivative exist. But for such an almost everywhere continuously differentiable function the fundamental theorem of calculus applies (Königsberger 1995, 217). Therefore the relation $h_t(z) = h_t(c) + \int_c^z h'_{t+}(z')\, dz$, $c, z \in [z^l, z^u]$ holds. By van Tiel (1984, 9) such an integral representation with a right-continuous non-increasing integrand is a sufficient condition for weak concavity of h_t on $[z^l, z^u]$. Moreover any open set $\Gamma_t \subset \mathbb{R}$ is exhaustible by compact sets, i.e there exists an isotone sequence of closed intervals $[z_n^l, z_n^u]_{n\in\mathbb{N}}$ such that $\Gamma_t = \bigcup_{n\in\mathbb{N}} [z_n^l, z_n^u]$. Hence h_t has to be weakly concave on Γ_t.

Second, I show that local strict concavity together with global weak concavity implies strict concavity on Γ_t. Take any pair of points $z_a, z_b \in \Gamma_t, z_a < z_b$. Let $z_c \in N_{z_b}$ be a

[48] Any such sequence yields a weighted arithmetic mean that lies within N_{z^o}.

[49] I have to show that concavity does not only hold for convex combinations within a particular set N_{z^o} but for all convex combinations within Γ_t.

point satisfying $z_a < z_c < z_b$. Moreover define $\lambda \in (0,1)$ by $z_c = \lambda z_a + (1-\lambda) z_b$ and let $\mu = \frac{1}{2\lambda}$. Then the following inequality holds for any pair $z_a \neq z_b$ in Γ_t (as $z_a < z_b$ is wlog):

$$\begin{aligned}
f_t g_t^{-1}\big(\tfrac{1}{2} z_a + \tfrac{1}{2} z_b\big) &= f_t g_t^{-1}\big(\mu\,\lambda\, z_a + (1-\mu\,\lambda)\, z_b\big) \\
&= f_t g_t^{-1}\big(\mu\,\lambda\, z_a + \big(\mu\,(1-\lambda) + (1-\mu)\big)\, z_b\big) \\
&= f_t g_t^{-1}\big(\mu\,\underbrace{(\lambda\, z_a + (1-\lambda)\, z_b)}_{z_c} + (1-\mu)\, z_b\big) \\
&> \mu\; f_t g_t^{-1}\big(\lambda z_a + (1-\lambda)\, z_b\big) + (1-\mu)\; f_t g_t^{-1}(z_b) \\
&\geq \mu\; \big(\lambda\, f_t g_t^{-1}(z_a) + (1-\lambda)\, f_t g_t^{-1}(z_b)\big) + (1-\mu)\; f_t g_t^{-1}(z_b) \\
&= \mu\,\lambda\, f_t g_t^{-1}(z_a) + \big(\mu\,(1-\lambda) + (1-\mu)\big)\, f_t g_t^{-1}(z_b) \\
&= \mu\,\lambda\, f_t g_t^{-1}(z_a) + (1-\mu\lambda)\, f_t g_t^{-1}(z_b) \\
&= \tfrac{1}{2} f_t g_t^{-1}(z_a) + \tfrac{1}{2}\, f_t g_t^{-1}(z_b)\,.
\end{aligned}$$

Therefore $f_t g_t^{-1}$ is strictly concave on Γ_t (Hardy et al. 1964, 75).

Part IV ("⇐"): It is left to proof that strict concavity on Γ_t implies axiom A6$^{\text{s}}$. As in part one of this proof the prerequisite of A6$^{\text{s}}$ becomes

$$\begin{aligned}
& \qquad\qquad \mathrm{x}^t \qquad\qquad\qquad \sim_t \qquad\qquad\qquad \mathrm{x}'^t \\
\Rightarrow\ & g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}^t_\tau) + \xi_t\Big] \;=\; g_t^{-1}\Big[\theta_t \sum_{\tau=t}^{T} g_\tau u_\tau(\mathrm{x}'^t_\tau) + \xi_t\Big]\,.
\end{aligned} \tag{23}$$

The existence of $i \in \{t, ..., T\}$ such that $\mathrm{x}^t_i \not\sim^*_i \mathrm{x}'^t_i$ translates into

$$\begin{aligned}
& \qquad\qquad g_\tau u_\tau(\mathrm{x}^t_i) \qquad\qquad \neq \qquad\qquad g_\tau u(\mathrm{x}'^t_i) \\
\Leftrightarrow\ & \theta_t \sum_{\substack{\tau=t\\ \tau\neq i}}^{T} g_\tau u(\mathrm{x}^t_\tau) + \theta_t g_i u_\tau(\mathrm{x}^t_i) + \xi_t \neq \sum_{\substack{\tau=t\\ \tau\neq i}}^{T} g_\tau u(\mathrm{x}^t_\tau) + \theta_t g_\tau u(\mathrm{x}'^t_i) + \xi_t \\
\Leftrightarrow\ & \qquad\qquad \tilde{z}(\mathrm{x}^t) \qquad\qquad \neq \qquad\qquad \tilde{z}\big((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)\big)
\end{aligned} \tag{24}$$

for some $i \in \{t, ...T\}$. But then due to strict concavity of $f_t \circ g_t^{-1}$, the fact that $\tilde{z}\big((\mathrm{x}^t_{-i}\mathrm{x}'^t_i)\big)$

cannot be the same for all i,[50] and using equation (23) it has to hold that

$$\begin{aligned}
& f_t g_t^{-1}\left[\sum_{i=t}^T \tfrac{1}{T-t+1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau\left((\mathrm{x}_{-i}^t \mathrm{x}_i'^t)_\tau\right)+\xi_t\right]\right] > \\
& \qquad \sum_{i=t}^T \tfrac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau\left((\mathrm{x}_{-i}^t \mathrm{x}_i'^t)_\tau\right)+\xi_t\right] \\
\Rightarrow\ & f_t g_t^{-1}\left[\tfrac{T-t}{T-t+1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau(\mathrm{x}_\tau^t)+\xi_t\right]+\tfrac{1}{T-t+1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau(\mathrm{x}_\tau'^t)+\xi_t\right]\right] > \\
& \qquad \sum_{i=t}^T \tfrac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau\left((\mathrm{x}_{-i}^t \mathrm{x}_i'^t)_\tau\right)+\xi_t\right] \\
\Rightarrow\ & g_t^{-1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau(\mathrm{x}_\tau^t)+\xi_t\right] > \\
& \qquad f_t^{-1}\left[\sum_{i=t}^T \tfrac{1}{T-t+1} f_t g_t^{-1}\left[\theta_t \sum_{\tau=t}^T g_\tau u_\tau\left((\mathrm{x}_{-i}^t \mathrm{x}_i'^t)_\tau\right)+\xi_t\right]\right] \\
\Rightarrow\ & \mathrm{x}^t \quad \succ_T \quad \sum_{i=t}^T \tfrac{1}{T-t+1}\left(\mathrm{x}_{-i}^t \mathrm{x}_i'^t\right).
\end{aligned}$$

Note that the flow of manipulations is laid out in more detail (going backwards) in part two of the proof.

Part V: Assertion b) is obtained by replacing A6$^{\mathrm{s}}$ by A6$^{\mathrm{w}}$and the strict inequaties by their weak counterparts.[51] A decision maker is intertemporal risk neutral if his preferences satisfy weak risk seeking as well as weak risk aversion. Therefore, assertion b) implies that the function $f_t \circ g_t^{-1}$ has to be concave and convex at the same time and, thus, linear. On the other hand, a representation featuring a linear composition $f_t \circ g_t^{-1}$ yields indifference between the certain consumption path and the lottery and, therefore, satisfies weak risk seeking as well as weak risk aversion (compare part four of the proof). In consequence, assertion c) holds. The proof of assertion d) is completely analogous to that of assertion a). Equation (21) becomes

$$f_t g_t^{-1}\left(\tfrac{1}{2}\, z^{\text{high}}+\tfrac{1}{2}\, z^{\text{low}}\right) \quad > \tfrac{1}{2} f_t g_t^{-1}(z^{\text{high}})+\tfrac{1}{2} f_t g_t^{-1}(z^{\text{low}})\,,$$

implying that the last step ("Finally...") in part three of the proof can be omitted. □

Proof of proposition 4: Let the triples $(u_t, f_t, g_t)_{t\in\{1,\dots,T\}}$ and $(u_t', f_t', g_t')_{t\in\{1,\dots,T\}}$ be arbitrary representations for the set of preference relations $\succeq = (\succeq_t)_{t\in\{1,\dots,T\}}$ in the sense of theorem 2. For all $t \in \{1,\dots,T\}$ there exist strictly increasing continuous functions s_t such that $u_t' = s_t \circ u_t$. By lemma 1 and the uniqueness part of theorem 2, there exist $a \in R_{++}$ and affine transformations $\mathfrak{a}_t^+ \in \mathsf{A}^+$ and $\mathfrak{a}_t^a \in \mathsf{A}^a$ such that $f_t' = \mathfrak{a}_t^+ f_t \circ s_t^{-1}$ and $g_t' = \mathfrak{a}_t^a\, g_t \circ s_t^{-1}$ for all $t \in \{1,\dots,T\}$.

To compare the measures of intertemporal risk aversion at the same point in consump-

[50]This is implied by equation (24) as again $\tilde{z}(\mathrm{x}^t)$ equals the weighted average $\frac{1}{T-t+1}\sum_{i=t}^T \tilde{z}\big((\mathrm{x}_{-i}^t \mathrm{x}_i'^t)\big)$.

[51]In this case the second step in part three becomes redundant.

tion space $\tilde{x}_t$, find that

$$z' = g'_t \circ \tilde{u}'_t(\tilde{x}_t) = \mathfrak{a}^a_t \, g_t \circ s_t^{-1} \circ s_t \circ u_t(\tilde{x}_t) = \mathfrak{a}^a_t \, g_t \circ u_t(\tilde{x}_t) = az + b_t \, .$$

a) The requirement $g_t \circ u_t(\bar{x}_t) = \tilde{g}_t \circ \tilde{u}_t(\bar{x}_t) = 0$ for all $t \in \{1, ..., T\}$ yields

$$\begin{aligned} 0 &= \tilde{g}_t \circ \tilde{u}_t(\bar{x}_t) = a g_t \circ s_t^{-1} s_t u_t(\bar{x}_t) + b_t = a g_t \circ u_t(\bar{x}_t) + b_t \\ &= a \cdot 0 + b_t = b_t \, . \end{aligned}$$

In consequence, for twice differentiable functions $f_t \circ g_t^{-1}$, it follows by equation (13) that

$$\tilde{\text{RIRA}}_t(\tilde{z})\Big|_{\tilde{z}=az} = -\frac{\left(\tilde{f}_t \circ \tilde{g}_t^{-1}\right)''(\tilde{z})}{\left(\tilde{f}_t \circ \tilde{g}_t^{-1}\right)'(\tilde{z})} \, \tilde{z}\Big|_{\tilde{z}=az} = -\frac{\left(f_t \circ g_t^{-1}\right)''(z)}{\left(f_t \circ g_t^{-1}\right)'(z)} \, z = \text{RIRA}_t(z) \, .$$

Thus, the measures of relative intertemporal risk aversion RIRA_t are independent of the particular choice of the triples $(u_t, f_t, g_t)_{t \in \{1,...,T\}}$ representing the underlying preferences $\succeq = (\succeq_t)_{t \in \{1,...,T\}}$.

b) The requirement implies

$$\begin{aligned} \bar{w} &= g'_{t^*} \circ u'_{t^*}(\acute{x}_{t^*}) - g'_{t^*} \circ u'_{t^*}(\grave{x}_{t^*}) \\ &= \mathfrak{a}^a_{t^*} \, g_{t^*} \circ s_{t^*}^{-1} \circ u'_{t^*}(\acute{x}_{t^*}) - \mathfrak{a}^a_{t^*} \, g_{t^*} \circ s_{t^*}^{-1} \circ u'_{t^*}(\grave{x}_{t^*}) \\ &= \mathfrak{a}^a_{t^*} \, g_{t^*} \circ u_{t^*}(\acute{x}_{t^*}) - \mathfrak{a}^a_{t^*} \, g_{t^*} \circ u_{t^*}(\grave{x}_{t^*}) \\ &= a g_{t^*} \circ u_{t^*}(\acute{x}_{t^*}) + b_{t^*} - a g_{t^*} \circ u_{t^*}(\grave{x}_{t^*}) - b_{t^*} = a\bar{w} \, . \end{aligned}$$

Therefore, $a = 1$ and, as the multiplicative constant is the same for all periods, the remaining freedom of the expression $f_t \circ g_t^{-1}$ corresponds to transformations $f_t \circ g_t^{-1} \rightarrow \tilde{f}_t \circ \tilde{g}_t^{-1} = \mathfrak{a}^+_t f_t \circ g_t^{-1} \mathfrak{a}_t^{1^{-1}}$, where $\mathfrak{a}_t^{1^{-1}}$ denotes the inverse of $\mathfrak{a}_t^{a=1}$, i.e. $\mathfrak{a}_t^{1^{-1}}(z) = z - b_t$. In consequence, evaluating the twice differentiable functions $f_t \circ g_t$ and $f'_t \circ g'_t$ at the same point in consumption space yields by equation (14) that

$$\tilde{\text{AIRA}}_t(\tilde{z})\Big|_{\tilde{z}=z+b_t} = -\frac{\left(\tilde{f}_t \circ \tilde{g}_t^{-1}\right)''(\tilde{z})}{\left(\tilde{f}_t \circ \tilde{g}_t^{-1}\right)'(\tilde{z})}\Big|_{\tilde{z}=z+b_t} = -\frac{\left(f_t \circ g_t^{-1}\right)''(z)}{\left(f_t \circ g_t^{-1}\right)'(z)} = \text{AIRA}_t(z) \, .$$

Thus, the measures of absolute intertemporal risk aversion AIRA_t are independent of the particular choice of the triples $(u_t, f_t, g_t)_{t \in \{1,...,T\}}$ representing the underlying preferences $\succeq = (\succeq_t)_{t \in \{1,...,T\}}$. □

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