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FITTING PRODUCTION FUNCTIONS TO EXPERIMENTAL DATA

C. D. THROSBY*

Economics Research Officer

1. INTRODUCTION

This article presents in simplified form the mechanical aspects of fitting production functions to experimental data using least squares regression and of deriving production surfaces and economic optima with the aid of the fitted function. The techniques are illustrated step by step with the aid of a worked example which uses a set of actual experimental results. The fitting and extrapolation procedures are shown in detail for a quadratic polynomial function, and brief sections are included showing how these processes are adaptable for use with a Cobb-Douglas, a transcendental and a Spillman function. The article is intended only as an elementary reference on production function techniques, although it covers sufficient ground to enable an agricultural research worker or student to use it as a reasonably comprehensive "recipe-book" in the analysis of suitable data. It is designed to be intelligible to readers with only a basic mathematical or statistical background, and as such may be useful as a supplement to textbooks which presume a higher degree of mathematical sophistication.

Although originally regarded with some suspicion, the production function approach is now receiving increasing attention from agricultural scientists overseas. In this country, agricultural economists have used production functions in cross-sectional farm studies to compute resource productivities,¹ but as yet their application to Australian experimental data has been little explored. To a large extent this has been due to a continuation of the suspicion noted above and as a result many Australian agricultural experiments have been inadequately analysed or have lacked economic significance.² It is hoped that an article such as the present one may, by commencing with first principles and clarifying the mechanics of fitting and

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¹ For example:

J. L. Dillon, "Marginal Productivities of Resources in two Farming areas of N.S.W." *Economic Monograph* No. 188 (Sydney: The Economic Society of Australia and New Zealand), May, 1956.

H. P. Schapper and R. G. Mauldon, "A production function from farms in the wholemilk region of Western Australia", *Economic Record*, Vol. XXXIII, No. 64 (April, 1957), pp. 52-59.

F. G. Jarrett, "Estimation of Resource Productivities as illustrated by a survey of the Lower Murray Valley dairying area", *Australian Journal of Statistics*, Vol. 1, No. 1 (April, 1959), pp. 3-11.

J. H. Dulo, "The Allocation of Resources in the Wool Industry", Paper presented to Third Conference of the Australian Agricultural Economics Society, Canberra, February, 1961.

² For a more detailed account of economic aspects of agricultural experiments, including a treatment of the place of production functions, see Alan G. Lloyd, "Agricultural Experiments and their Economic Significance", this *Review*, Vol. 26, No. 3 (September, 1958), pp. 185-209.

extrapolating from functions, stimulate some agricultural research workers towards adopting a functional approach in the analysis of their experimental data.

2. WHY THE PRODUCTION FUNCTION APPROACH?

It has been traditional in analysing the results of agricultural experiments such as fertilizer or feeding trials to submit the figures to some statistical test of significance such as the analysis of variance. However, once a significant difference between treatments has been demonstrated, the statistical investigation often stops. In pure research this is not necessarily a shortcoming, since the experiment may be orientated towards verifying hypotheses, such as the deficiency of a particular element in a soil, or towards observation of new phenomena, such as the effect of an untried feed component in livestock rations. In addition the qualitative outcome of such research is often of equal or greater importance than the quantitative results. On the other hand, in applied agricultural research, where derivation of farmer recommendations involving a level or levels of a variable factor or factors such as fertilizer, is one of the main aims, significance tests such as the analysis of variance cannot alone extract from the data sufficient information to enable sound recommendations to be made. To illustrate, superphosphate recommendations for some areas of this State have had to be made on the intuition of an extension officer; yet research undertaken to determine the true optimum level for these areas has frequently been of little more economic value than the original intuitive judgment due to shortcomings in the analysis of results. Recommendations made from levels defined as "optimum" in some loose technical sense, or from "optimum" levels calculated only from the raw data at the rates of the independent variables used in the experiment can, if applied unquestioningly, lead to substantial misallocation of farm resources.

Not only agricultural economists but also statisticians have been critical of the inadequacy of significance tests in the analysis of experimental results. Yates has said:

"The emphasis on tests of significance, and the consideration of the results of each experiment in isolation have had the unfortunate consequence that scientific workers have often regarded the execution of a test of significance on an experiment as the ultimate objective. Results are significant or not significant and that is the end of it."³

The functional approach helps to surmount some of these shortcomings. A production function is a mathematical expression defining the way in which a dependent variable changes as a result of changes in one or more independent variables. In regional studies in agriculture aggregate production functions have been used to determine the influence on farm income of the resources used in producing it,⁴ whilst the functional approach

³ F. Yates, "The Influence of *Statistical Methods for Research Workers* on the Development of the Science of Statistics," *Journal of the American Statistical Association*, Vol. 46, No. 253, (March, 1951), p. 33. See also for example in E. J. Williams, *Regression Analysis*, (N.Y.: John Wiley), 1959, Chap. 1, and G. E. P. Box, "The Exploration and Exploitation of Response Surfaces: Some general considerations and examples", *Biometrics*, Vol. 10, Part 1 (1954), pp. 16-60.

⁴ See a summary in E. O. Heady and J. L. Dillon, *Agricultural Production Functions* (Ames: Iowa State University Press, 1961); Chapter 17, "Comparison of Production Function Estimates from Farm Samples over the World", pp. 585-643.

in the experimental field involves relating the inputs varied during an experiment (e.g., fertilizer, feed) to output of product (e.g., pasture, grain, meat).

Given a set of experimental results amenable to a production function analysis, a research worker is able to compute the function which best fits the data. If goodness of fit tests are satisfactory, this function may then be taken as an adequate approximation to the real physical relationship which it represents and extrapolation may be made in a statistically significant manner. This fact is the essence of the argument in favour of the functional approach; i.e., this extrapolation, on the basis of which optimum recommendations, etc., can be calculated, can be made from a mathematically defined function, whereas it cannot be made from an ill-behaved set of raw data, nor from mean or point estimates of the variables.

3. FORMS OF PRODUCTION FUNCTION

It was noted above that, applied to agricultural experimental data, a production function represents a mathematical approximation to the real physical relationship which exists between the dependent variable and the one or more independent variables studied in the experiment. In the real world, the actual production function defining, for example, pasture output may be:

$$(1) \quad Y = f(X_1, X_2, X_3, \dots, X_n)$$

where Y = production of pasture,
 X_1 = seeding rate,
 X_2 = superphosphate rate,
 X_3 = rainfall,

and so on, through all the other variables which have even the slightest affect on pasture growth. In an agricultural experiment designed to determine this relationship, it is obviously not possible to control all the independent variables in equation (1). The research worker therefore varies only one or two and holds the others, or assumes them to be held, constant. In other words he estimates, say,

$$(2) \quad Y = f(X_1, X_2), (X_3, X_4, \dots, X_n \text{ constant}),$$

by growing pasture at a variety of seed and superphosphate rates under otherwise controlled conditions.

The results of this experiment would comprise a set or sets of values for Y corresponding to the levels of X_1 and X_2 used by the experimenter. By a process of trial and error he must now do away with the generality of equation (2) by determining which algebraic form of function out of the many available best characterises his data. On the other hand, he may have intuitive grounds for preferring a particular functional type in which case he may proceed directly with determining whether it does in fact describe his data adequately.

The simplest type of specific functional relationship which could conceivably be applied to agricultural data is a linear function of the form

$$(3) \quad Y = a + bX$$

or, in our pasture example, where there are two independent variables:

$$(4) \quad Y = a + b_1 X_1 + b_2 X_2$$

Equation (3) can be represented graphically in two dimensions as a straight line whose slope⁵ is measured by b and which cuts the vertical axis at a point a units above the origin (i.e. when $X = 0, Y = a$.) Equation (4) is represented in three dimensions by a flat plane.

Functions in more than three variables cannot be represented diagrammatically in full. However two dimensional diagrams of the response curve for one input or three dimensional representations of the production surface for two inputs can be made by holding all other independent variables in such a function constant at some predetermined level.

The functions in equations (3) and (4) are linear and as such are rarely satisfactory for describing agricultural processes. If equation (4), for example, were assumed to be the production function for pasture, then it would follow that infinite production could be obtained from a given area of soil merely by applying infinitely large quantities of seed and fertilizer. Thus, what we require is a functional form which allows for non-linear response, i.e. which permits, amongst other things, the common phenomenon of diminishing returns to be accounted for.

Introduction of transformations of the original variables, such as power or exponential terms, enables the description of non-linear responses. Of the many such functional forms available, the present article is restricted to four of the most commonly used variants, the polynomial, the Cobb-Douglas, the transcendental, and the Spillman-Mitscherlich. All the illustrations are made in terms of the quadratic polynomial form, the relevance of the techniques used to the other three forms being presented largely without illustration, due to space limitations. The characteristics of the various forms of production function, and the criteria to be used in choosing between them have often been discussed in the literature, and the reader is referred elsewhere for details.⁶

A brief outline of the algebraic form of each function is given at this stage.

(i) Quadratic polynomial

Quadratic expansions corresponding to equations (3) and (4) are:

$$(5) \quad Y = a + b_1 X + b_2 X^2$$

and,

$$(6) \quad Y = a + b_1 X_1 + b_2 X_2 + b_3 X_1^2 + b_4 X_2^2 + b_5 X_1 X_2$$

Equation (5) is the second degree polynomial involving one factor X . It contains the linear term in X as well as the term in X raised to the power two. Equation (6) is the second degree polynomial involving two factors X_1 and X_2 . It contains the linear terms in X_1 and X_2 as well as the terms in

⁵ The slope of the line is given as:

$$b = \frac{\delta Y}{\delta X}$$

where δY signifies the change in Y caused by δX , a given change in X .

⁶ For example, Heady and Dillon, *op. cit.*, Ch. 3; (this text also contains a 270-item bibliography of production function literature on pages 645-63); E. O. Heady, "Use and Estimation of Input-Output Relationships or Productivity Coefficients", *Journal of Farm Economics*, Vol. XXXIV, No. 5, (December, 1952), pp. 775-86; R. M. Parish and J. L. Dillon, "Recent Applications of the Production Function in Farm Management Research", this *Review*, Vol. 23, No. 4, (December, 1955), pp. 215-36; etc. Further references for specific functions are given below.

X_1 and X_2 of exponent two, viz. X_1^2 , X_2^2 , and X_1X_2 . Similarly if we had a second degree polynomial involving X_1 , X_2 and X_3 it would involve linear terms in X_1 , X_2 and X_3 as well as six terms with exponent two, viz. X_1^2 , X_2^2 , X_3^2 , X_1X_2 , X_1X_3 and X_2X_3 . The second degree polynomial involving more than three factors may be generalized from this.

Another polynomial of use in describing agricultural phenomena is the square-root form:

$$(7) \quad Y = a + b_1 X + b_2 \sqrt{X}$$

and, for two independent variables,:

$$(8) \quad Y = a + b_1 X_1 + b_2 X_2 + b_3 \sqrt{X_1} + b_4 \sqrt{X_2} + \sqrt{X_1 X_2}$$

Only the quadratic form (equations (5) and (6)) is used here, and is referred to hereafter simply as the "polynomial".

(ii) Cobb-Douglas Function

The power or Cobb-Douglas function, so named after its originators⁷, takes the general form:

$$(9) \quad Y = a X_1^{b_1} X_2^{b_2} \dots X_n^{b_n}$$

(iii) Transcendental Function

■ This hybrid form⁸, which is more versatile than the preceding ones, but which has associated drawbacks, is represented for n resources by:

$$(10) \quad Y = a X_1^{\frac{b_1}{e}} X_2^{\frac{b_2}{e}} \dots X_n^{\frac{b_n}{e}}$$

where e = base of natural logarithms.

(iv) Spillman Function

Two exponential functions, similar in form, were evolved independently by Spillman and Mitscherlich early in this century. That most often used is the Spillman function⁹, which, for one input, is of the form:

$$(11) \quad Y = M - AR^x$$

where M = maximum total production obtainable by the use of resource X ,

A = increase in output due to X (i.e. $\max. Y - \min. Y$),

R = ratio of successive increments in output to total output. In cases where a zero level of X produces no Y , $M = A$ and the Spillman function devolves to:

$$(12) \quad Y = A(1 - R^x)$$

Generalizing equation (12) to n inputs, we have:

$$(13) \quad Y = A(1 - R_1^{x_1})(1 - R_2^{x_2}) \dots (1 - R_n^{x_n})$$

⁷ C. W. Cobb and P. H. Douglas, "A Theory of Production", *American Economic Review*, Vol. XVIII, (Supplement) (March, 1928), pp. 139-165.

⁸ See A. N. Halter, H. O. Carter and J. G. Hocking, "A note on the Transcendental Production Function", *Journal of Farm Economics*, Vol. XXXIX, No. 4, November, 1957, pp. 966-74. To call this the transcendental function is perhaps misleading since it is only a transcendental function of the form shown in equation (10). Literally "transcendental" means "a priori", or "based on intuition".

⁹ See, for example, W. J. Spillman, *Use of the Exponential Yield Curve in Fertilizer Experiments*, U.S.D.A., Technical Bulletin No. 348, 1933.

4. THE DATA

The source of the data used for illustration in this article is a seed-superphosphate trial which was conducted by Research Agronomist J. Strang at Berry on the South Coast of New South Wales during 1959-60. The subsection of the experiment from which these figures were taken involved four superphosphate levels, three rates of seeding of Clare subterranean clover and two replications. There are thus twenty-four observations of Y , which is measured in cwt. of total dry matter per acre. *Phalaris tuberosa* was sown on all plots at a constant seeding rate as a common grass. The data are shown in Table 1.

TABLE 1

*Pasture Production at Berry as Influenced by
Seed and Superphosphate Rates: cwt. Dry Matter per acre*

X_2 Super- phosphate Rate (cwt. per acre)	X_1 Seeding Rate (lb. per acre)					
	2		8		24	
	Replication		Replication		Replication	
	I	II	I	II	I	II
0	11.7	13.3	17.2	30.7	17.9	24.0
2	31.5	30.9	41.7	33.1	58.0	46.4
4	37.2	23.3	45.6	55.5	51.2	50.4
8	46.5	44.9	55.4	56.2	53.5	55.2

5. FITTING THE POLYNOMIAL FUNCTION

Fitting polynomial, Cobb-Douglas and transcendental functions involves estimating a and the required number of b 's (b_i) in equations (6) and (9), and a , b_i , and c_i in equation (10). The Spillman, which presents some special problems, involves the estimation of M , A and R .

The technique of regression is used to estimate these unknowns or *parameters*. The simple elements of the theory behind regression can best be understood by considering a two-dimensional scatter diagram of observations of Y at various levels of X . If a marked trend is noticeable, a free-hand line could be drawn on the graph. Regression fits a *mathematically defined* line through these points such that the dispersion of the points about the line is as small as possible; i.e. it places the line of "best fit" in a position such that the sum of squares of deviations of points from the line is minimized. Hence the term "least squares regression".¹⁰

¹⁰ It might be argued that "regression function" might be a more apt name for the production function. However, not all production functions need be derived by regression. Thus, not all production functions are regression functions, and, by the same token, not all regression functions are production functions.

For two independent variables, the process fits a three dimensional regression plane such that the dispersion of points around the plane is minimized. This process is generalized for n independent variables by fitting the "plane" of best fit in n -dimensional space.

The equation we aim to derive for illustrative purposes is a polynomial of the form shown in equation (6). For convenience of notation we shall put X_3 instead of X_1^2 , X_4 instead of X_2^2 , and X_5 for $X_1 X_2$. We should also write in an error term, ε , to account for the effect on Y of random variations in inputs not considered. Thus the equation for which we want to find the parameters a, b_i ($i = 1, \dots, 5$) is:¹¹

$$(14) \quad Y = a + b_1 X_1 + b_2 X_2 + b_3 X_3 + b_4 X_4 + b_5 X_5 + \varepsilon$$

where the values of X_i and corresponding observations of Y are as shown in Table 2.

TABLE 2
Raw Data Ready for Fitting Polynomial Function

Y	X_1	X_2	X_3	X_4	X_5
11.7	2	0	4	0	0
13.3	2	0	4	0	0
31.5	2	2	4	4	4
30.9	2	2	4	4	4
37.2	2	4	4	16	8
23.3	2	4	4	16	8
46.5	2	8	4	64	16
44.9	2	8	4	64	16
17.2	8	0	64	0	0
30.7	8	0	64	0	0
41.7	8	2	64	4	16
33.1	8	2	64	4	16
45.6	8	4	64	16	32
55.5	8	4	64	16	32
55.4	8	8	64	64	64
56.2	8	8	64	64	64
17.9	24	0	576	0	0
24.0	24	0	576	0	0
58.0	24	2	576	4	48
46.4	24	2	576	4	48
51.2	24	4	576	16	96
50.4	24	4	576	16	96
53.5	24	8	576	64	192
55.2	24	8	576	64	192

The first task is to calculate the sum, sum of squares and mean for each column of Table 2. These are shown in Table 3, together with the standard deviations. The latter need not be computed at this stage, as they are obtainable more easily later in the calculations.

¹¹ The regression coefficients b_i which are to be derived are in fact only estimates of the true regression coefficients β_i ; it will be shown later that confidence limits can be calculated within which the β_i will be known with given probability to fall.

TABLE 3

Sums, Sums of Squares, Means and Standard Deviations

Quantity	Y	X ₁	X ₂	X ₃	X ₄	X ₅
Sum $\sum_j X$..	931.3	272	84	5,152	504	952
Sum of squares $\sum_j (X)^2$..	41,310	5,152	504	2,687,104	26,208	108,192
Mean $\frac{\sum_j X}{n}$..	38.80	11.33	3.50	214.67	21.00	39.67
Standard deviation (see text)	14.68	9.29	2.96	256.67	25.51	54.17

Note: In calculations throughout this article, more decimal places have been carried than are generally shown, in order to minimize rounding errors. On this point see, for example, G. W. Snedecor, *Statistical Methods*, (5th Edition, Ames: Iowa State College Press 1956), p. 440.

We may now derive the variances for each variable and covariances for each pair of variables. The variances are obtained from the relationship:¹²

$$(15) \quad \sigma_i^2 = \frac{\sum_{j=1}^n (X_j - \bar{X})^2}{n}$$

where σ_i^2 = variance of X_i ,
 X_j = the j^{th} observation of X_i ,
 \bar{X} = mean of X_i ,
 n = number of observations,

or, if n is small a truer estimate of the variance is obtained from:

$$(16) \quad \sigma_i^2 = \frac{\sum_j (X_j - \bar{X})^2}{n - 1}$$

¹² For the sake of simplicity, suffixes in mathematical expressions have been omitted as far as possible. In reference to Table 2, the suffix i denotes columns of the table, j denotes rows, and this notation is maintained in similar fashion throughout following sections.

The co-variance between any pair of variables, for example between Y and X_1 , is given by:

$$(17) \quad \sigma_{yx_1} = \frac{\sum_j (Y_j - \bar{Y})(X_{1j} - \bar{X}_1)}{n}$$

The variances and co-variances can be arranged to form a symmetrical matrix, as is shown below. Direct substitution into equations (15) and (17) is an arduous means of obtaining this matrix; hence an alternative method is used.

The top lines of the right-hand sides of equations (15) and (17) represent the sums of squares (equation (15)) and cross-products (equation (17)) of deviations from the respective means. Call the top line of the right-hand side of equation (15) $\sum x^2$ and that of equation (17) $\sum yx_1$; these quantities which are known as the *product moments* are calculated using equations (18) and (19):

$$(18) \quad \sum x_i^2 = \sum_j (X_j^2) - \frac{(\sum X)^2}{n}$$

$$(19) \quad \sum yx_1 = \sum_j (Y X_1) - \frac{\sum Y \sum X_1}{n}$$

After computing the square term for each variable and the cross-product term for each pair of variables, they can be arranged to form the symmetrical product moment matrix¹³, as shown in Table 4.

TABLE 4

Product Moment Matrix

$\sum y^2$					
$\sum y x_1$	$\sum x_1^2$				
$\sum y x_2$	$\sum x_1 x_2$	$\sum x_2^2$			
.	.	.			
.	.	.			
.	.	.			
$\sum y x_n$	$\sum x_1 x_n$	$\sum x_2 x_n$	$\sum x_n^2$	

¹³ The matrix is symmetrical because the covariance of A with B is identical with the covariance of B with A. i.e. the above-diagonal terms of Tables 4 to 9 represent a "mirror image" of the below-diagonal terms.

To illustrate, we calculate the numerical values for Table 4 which correspond to our set of data. These are shown in Table 5. The examples given in equations (20) and (21) should be sufficient to explain the use of equations (18) and (19) in deriving the values in Table 5.

$$(20) \quad \Sigma y^2 = 41,310.11 - \frac{(931.3)^2}{24} = 5,171.81$$

$$(21) \quad \Sigma y x_1 = 11,720.2 - \frac{931.3 \times 272}{24} = 1,165.47$$

TABLE 5
Product Moment Matrix for Data in Table 2.

5,171.81					
1,165.47	2,069.33				
770.05	0	210.00			
27,905.33	56,362.67	0	1,581,141.33		
5,569.10	0	1,740.00	0	15,624.00	
12,464.03	7,242.67	2,380.00	197,269.33	19,720.00	70,429.33

The variance-covariance matrix, which is shown in Table 6, is obtained by dividing each element of the matrix in Table 5 by n (i.e. by 24).

TABLE 6
Variance-Covariance Matrix

215.49					
48.56	86.22				
32.09	0	8.75			
1,162.72	2,348.44	0	65,880.88		
232.05	0	72.50	0	651.00	
519.33	301.78	99.17	8,219.55	821.67	2,934.55

The square root of each element on the diagonal of the variance-covariance matrix provides the standard deviation of each variable, which may now be entered into Table 3.

The sample correlation coefficients (r_{ij}) between any pair of variables can be calculated from quantities contained in either of the matrices in Tables 4 or 6. Using variances (or standard deviations) and covariances the sample correlation coefficient between, say, X_1 and X_2 can be computed from:

$$(22) \quad r_{x_1 x_2} = \frac{\sigma_{x_1 x_2}}{\sqrt{\sigma_{x_1}^2 \sigma_{x_2}^2}} = \frac{\sigma_{x_1 x_2}}{\sigma_{x_1} \sigma_{x_2}}$$

where $\sigma_{x_1 x_2}$ = covariance between X_1 and X_2
 $\sigma_{x_1}^2$ = variance of X_1 (similarly for X_2)
 σ_{x_1} = standard deviation of X_1 (similarly for X_2)

Alternatively the same equation can be written as in equation (23), for use with the product moments shown in Tables 4 and 5.

$$(23) \quad r_{x_1 x_2} = \frac{\Sigma x_1 x_2}{\sqrt{\Sigma x_1^2 \Sigma x_2^2}}$$

For example, using equation (22):

$$(24) \quad r_{y x_1} = \frac{48.56}{\sqrt{14.68 \times 9.29}} = 0.356$$

By this means we construct the correlation matrix which, like the variance—covariance matrix, is symmetrical about the diagonal. Each variable is of course perfectly correlated with itself, i.e. $r = 1$ on the diagonal. The correlation matrix in lower triangular form is shown in Table 7.

TABLE 7
Correlation Matrix

1.000					
0.356	1.000				
0.739	0	1.000			
0.309	0.985	0	1.000		
0.620	0	0.961	0	1.000	
0.653	0.600	0.619	0.591	0.595	1.000

The sample correlation coefficients merely measure the degree of association between two variables without holding the others constant. Thus no causal implications should be inferred.

Upon reaching the stage in the sequence of computations of having to hand either the product moment matrix as in Tables 4 and 5, and/or the variance—covariance matrix as in Table 6, there are a number of ways to proceed, each leading ultimately to the regression coefficients. For purposes of illustration three methods are treated here. Firstly, we solve a series of simultaneous equations which yields directly the b_i . The second method involves inverting the variance—co-variance matrix. The third computes the regression coefficients *via* the “Gaussian multipliers”. The first two of these are included here for the sake of completeness; however it will usually be found that use of the Gaussian multipliers is the most satisfactory technique.

(a) The normal equations

The normal equations in their general form are written as in (25), with coefficients taken directly from Table 4.

$$(25) \quad \begin{aligned} b_1 \Sigma x_1^2 + b_2 \Sigma x_2 x_1 + \dots + b_n \Sigma x_n x_1 &= \Sigma x_1 y \\ b_1 \Sigma x_1 x_2 + b_2 \Sigma x_2^2 + \dots + b_n \Sigma x_n x_2 &= \Sigma x_2 y \\ &\vdots \\ b_1 \Sigma x_1 x_n + b_2 \Sigma x_2 x_n + \dots + b_n \Sigma x_n^2 &= \Sigma x_n y \end{aligned}$$

Substituting our own data (from Table 5), we derive the following set of simultaneous equations:

$$\begin{array}{rclclclclclcl}
 (26) & 2,069.33b_1 & + & 0 & + & 56,362.67b_3 & + & 0 & + & 7,242.67b_5 & = & 1,165.47 \\
 & 0 & + & 210.00b_2 & + & 0 & + & 1,740.00b_4 & + & 2,380.00b_5 & = & 770.05 \\
 & 56,362.67b_1 & + & 0 & + & 1,581,141.33b_3 & + & 0 & + & 197,269.33b_5 & = & 27,905.33 \\
 & 0 & + & 1,740.00b_2 & + & 0 & + & 15,624.00b_4 & + & 19,720.00b_5 & = & 5,569.10 \\
 & 7,242.67b_1 & + & 2,380.00b_2 & + & 197,269.33b_3 & + & 19,720.00b_4 & + & 70,429.33b_5 & = & 12,464.03
 \end{array}$$

There are a number of means available for solving such a set of simultaneous equations. One is by successively eliminating terms (e.g. in (26) if the first row is multiplied by $\frac{2,380.00}{7,242.67}$ and subtracted from the second, the b_5 term disappears) until finally left with an expression in only one unknown, then back-substituting sequentially until all the unknowns have been determined. Another is the "Doolittle method", the recipe for which can be found in most statistics textbooks.¹⁴

The equations in (26) were solved by the former method, yielding the following values for the regression coefficients:

$$\begin{array}{rcl}
 b_1 & = & 2.9040 \\
 b_2 & = & 9.4566 \\
 b_3 & = & -0.0835 \\
 b_4 & = & -0.6722 \\
 b_5 & = & -0.0194
 \end{array}$$

¹⁴ See, for example, M. Ezekiel and K. A. Fox, *Methods of Correlation and Regression Analysis*, Third edition (N.Y.: John Wiley 1959), Appendix 2. pp. 489-507.

(b) Inversion of the variance—co-variance matrix

If A is a square matrix and B is another square matrix of the same order (i.e. the same size) having the property that $BA = I$ where I is the *identity matrix* (a square matrix in which all elements are zeros except for one set of diagonal elements, which are ones) then B is defined as the *inverse* of A . For example, a 4×4 matrix when multiplied by its inverse gives the matrix:

$$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}$$

There are several methods for finding the inverse of a matrix. One involves the solution of n sets of simultaneous equations:

$$\begin{array}{l} c_1 a_{11} + c_2 a_{12} + \dots + c_n a_{1n} = 1, 0, \dots, 0 \\ c_1 a_{21} + c_2 a_{22} + \dots + c_n a_{2n} = 0, 1, \dots, 0 \\ \vdots \\ c_1 a_{n1} + c_2 a_{n2} + \dots + c_n a_{nn} = 0, 0, \dots, 1 \end{array} \quad (27)$$

For our problem a_{ij} are elements of the variance—covariance matrix. If the solutions for c are called d_{ij} , then:

$$\begin{array}{l} c_1 = d_{11}, d_{12}, \dots, d_{1n} \\ c_2 = d_{21}, d_{22}, \dots, d_{2n} \\ \vdots \\ c_n = d_{n1}, d_{n2}, \dots, d_{nn} \end{array} \quad (28)$$

where d_{ij} are the elements of the inverse of the variance—covariance matrix.

The regression coefficients are obtained by dividing each element of the first column of the inverse below the diagonal by the first term of the column, and reversing the sign. It can be seen that if only the regression coefficients are required, and not the complete inverse of the variance—covariance matrix, only one of the n sets of equations in (27) need be solved, viz.:

$$\begin{array}{l} c_1 a_{11} + c_2 a_{12} + \dots + c_n a_{1n} = 1 \\ c_1 a_{21} + c_2 a_{22} + \dots + c_n a_{2n} = 0 \\ \vdots \\ c_1 a_{n1} + c_2 a_{n2} + \dots + c_n a_{nn} = 0 \end{array} \quad (29)$$

since this yields $d_{11}, d_{21}, \dots, d_{n1}$ which is the required first column of the inverse.

Using equations (27) and (28) is a laborious way of inverting a *symmetrical* matrix, since a much quicker method is available (analogous with the Doolittle method) which is applicable specifically to symmetrical matrices.¹⁵

¹⁵ See J. Friedman and R. J. Foote, *Computational Methods for Handling Systems of Simultaneous Equations*, U.S.D.A., Agricultural Handbook No. 94, November, 1955, p. 9.

The symmetrical inverse of the variance—co-variance matrix from Table 6 is shown in Table 8.

TABLE 8
Inverse of Variance—Co-variance Matrix

0.02929					
— 0.08503	0.66186				
— 0.27683	0.85621	4.26616			
0.00244	— 0.02131	— 0.02311	0.00073		
0.01969	0.05716	— 0.35087	0.00164	0.03312	
0.00055	— 0.00625	— 0.02026	0.00005	0.00037	0.00134

The regression coefficients are derived in the manner explained above:

$$\begin{aligned} b_1 &= 2.9030 \\ b_2 &= 9.4513 \\ b_3 &= -0.0835 \\ b_4 &= -0.6723 \\ b_5 &= -0.0189 \end{aligned}$$

(c) The Gaussian multipliers

The Gaussian multipliers are derived by removing the first row and column of the matrix in Table 4 and inverting it. To put it more formally, the solution of equation system (30):

$$\begin{aligned} (30) \quad & c_1 \Sigma x_1^2 + c_2 \Sigma x_1 x_2 + \dots + c_n \Sigma x_1 x_n = 1, 0, \dots, 0 \\ & c_1 \Sigma x_2 x_1 + c_2 \Sigma x_2^2 + \dots + c_n \Sigma x_2 x_n = 0, 1, \dots, 0 \\ & \vdots \\ & c_1 \Sigma x_n x_1 + c_2 \Sigma x_n x_2 + \dots + c_n \Sigma x_n^2 = 0, 0, \dots, 1 \end{aligned}$$

yields a series of solutions for c called:

$$\begin{aligned} (31) \quad & c_1 = g_{11}, g_{12}, \dots, g_{1n} \\ & c_2 = g_{21}, g_{22}, \dots, g_{2n} \\ & \vdots \\ & c_n = g_{n1}, g_{n2}, \dots, g_{nn} \end{aligned}$$

where g_{ij} represent the elements of the symmetrical matrix of Gaussian multipliers.

In our example the matrix to be inverted, using one of the methods noted above, is found by removing the first row and column of Table 5. This matrix and its calculated inverse are shown in Table 9.

TABLE 9
Inversion of Matrix to Obtain Gaussian Multipliers

Matrix for Inversion (from Table 5)

2,069.33				
0	210.00			
56,362.67	0	1,581,141.33		
0	1,740.00	0	15,624.00	
7,242.67	2,380.00	197,269.33	19,720.00	70,429.33

Inverse

0.01729				
0.00219	0.06874			
— 0.00059	0	0.00002		
0	— 0.00687	0	0.00083	
— 0.00019	— 0.00063	0	0	0.00006

The regression coefficients are found by substituting in the equations:

$$\begin{aligned}
 (32) \quad & b_1 = g_{11} \Sigma x_1 y + g_{12} \Sigma x_2 y + \dots + g_{1n} \Sigma x_n y \\
 & b_2 = g_{21} \Sigma x_1 y + g_{22} \Sigma x_2 y + \dots + g_{2n} \Sigma x_n y \\
 & \vdots \\
 & b_n = g_{n1} \Sigma x_1 y + g_{n2} \Sigma x_2 y + \dots + g_{nn} \Sigma x_n y
 \end{aligned}$$

For example,

$$\begin{aligned}
 (33) \quad & b_1 = 0.01729 \times 1,165.47 + 0.00219 \times 770.05 + \dots \text{etc.} \\
 & = 2.9036
 \end{aligned}$$

By this means the regression coefficients are calculated as follows:

$$\begin{aligned}
 b_1 &= 2.9036 \\
 b_2 &= 9.4523 \\
 b_3 &= -0.0848 \\
 b_4 &= -0.6725 \\
 b_5 &= -0.0193
 \end{aligned}$$

Whilst any of the above methods might be used to derive the b_i , the last one presents several attractive features which may swing the balance in its favour. It will be noted that the matrices in Table 9 are independent of Y . This means that once the Gaussian multipliers have been found, they are applicable to any number of different sets of values of Y which are dependent on the same set of independent variables. For instance, in the seed-superphosphate trial from which our data are taken, grass and clover fractions and phosphorus and nitrogen contents of the tops were determined for each of the 24 observations of total dry matter yield. Using the Gaussian multipliers, the regression coefficients for functions relating each of these individual quantities to seed and superphosphate are obtainable simply by computing the values of $\Sigma x_i y$ in each case and substituting into equation (32). In addition the Gaussian multipliers can be used directly to determine

the precision of the regression coefficients, as is seen below. On the other hand, method (a) may be preferred in some cases, e.g., when it can be seen by inspection that unknowns can be eliminated quickly from the normal equations, leading to a speedy solution.

Having ascertained the b_i , it only remains to calculate a , the constant term, to complete the regression equation. This is achieved using equation (34):

$$(34) \quad a = \bar{Y} - \sum_i b_i \bar{X}_i$$

where \bar{Y} = mean of dependent variable,

\bar{X}_i = means of independent variables (from Table 3).

Hence, using the regression coefficients calculated by method (b),¹⁶ we derive, for our example:

$$(35) \quad \begin{aligned} a &= 38.8042 - (2.9030 \times 11.3333 + 9.4513 \times 3.5000 - \dots \\ &\quad - 0.0189 \times 39.6667) \\ &= 5.6168 \end{aligned}$$

The completed regression equation is shown using our original notation, in (36).

$$(36) \quad Y = 5.6168 + 2.9030 X_1 + 9.4513 X_2 - 0.0835 X_1^2 - 0.6723 X_2^2 - 0.0189 X_1 X_2$$

Equation (36) represents the quadratic polynomial production function which describes the data in Table 1, and as such can be used for prediction purposes, for examining the pattern of substitution between the two inputs studied, for calculating economic optima, etc. Before being justified in proceeding, however, it is necessary to apply several tests of significance to the function to determine (a) how well it describes the original data, (b) whether the individual regression coefficients are significantly different from zero and (c) whether in fact there was a significant increase in Y due to the treatments in the experiment. (a) is achieved by computing the multiple correlation coefficient R , (b) is determined by applying the t -test and (c) *via* the analysis of variance.

Using equation (36), the predicted value of Y , denoted \hat{Y} , can be calculated for any feasible level of X_1 and X_2 . If \hat{Y} is computed for the levels of X_1 and X_2 used in the original experiment, the quantity $\sum_j (\hat{Y}_j - Y_j)^2$, which is termed the residual sum of squares, may be worked out. This measures the deviations of the actual values of Y from the fitted regression line or plane. If the residual sum of squares is expressed as a fraction of the total sum of squares (the sum of deviations of the actual values of Y from the *mean*), then we have a measure of the tendency of the actual values of Y to lie around the fitted line or plane rather than to be scattered randomly around

¹⁶ The differences between the same regression coefficients calculated by the different methods are attributable to rounding errors. The reason for using those from method (b) in the subsequent illustrations is that they were derived entirely by a computer using a programme with built-in routines for minimizing rounding error. The regression coefficients found *via* method (a) were derived entirely by hand, and those from (c) were calculated partly by hand and partly on the computer.

their mean value. In practice this fraction is subtracted from 1 to give R^2 , the square of the multiple correlation coefficient. The closer the Y lie to \hat{Y} rather than to \bar{Y} , the smaller the ratio of residual to total sums of squares, and the closer R^2 and R to unity.

If the inverse of the variance—co-variance matrix has been computed (Table 8) then the residual sum of squares (RSS) is available as n times the reciprocal of the diagonal element corresponding to the dependent variable. i.e.

$$(37) \quad RSS = 24 \times \frac{1}{0.02929} = 819.40$$

whence R^2 is found as:

$$(38) \quad R^2 = 1 - \frac{819.40}{5,171.81} \\ = 0.8416$$

If the residual sum of squares is not available directly, use may be made of equation (39) to obtain R^2 .

$$(39) \quad R^2 = \frac{b_1 \Sigma x_1 y + b_2 \Sigma x_2 y + \dots + b_n \Sigma x_n y}{\Sigma y^2}$$

For example using data from Table 5:

$$(40) \quad R^2 = \frac{2.9030 \times 1,165.47 + 9.4513 \times 770.05 + \dots + 0.0189 \times 12,464.03}{5,171.81} \\ = 0.8416$$

and hence $R = 0.9174$. The significance of this number can be ascertained by referring to a table of values of R for different levels of significance. In this way it is found that the R value for our regression relation is significant at the 1 per cent level.

In order to determine whether the regression coefficients b_i are significantly different from some hypothesized value β_i (usually zero), a t -test is performed; i.e. "Student's" t is determined for each b_i and its level of significance found from a table of t -values. First, however, an estimate must be made of the variance of the error term in the regression relation (i.e. the variance of ϵ in equation (14)). This is called s^2 (being only an *estimate* of σ^2), and is given by:

$$(41) \quad s^2 = \frac{RSS}{(n - k - 1)}$$

where RSS = residual sum of squares as defined above,

n = number of observations,

k = number of independent variables.

For our example:

$$(42) \quad s^2 = \frac{819.40}{18} = 45.522$$

whence $s = 6.747$

A number of means are available for deriving t . If the Gaussian multipliers have been determined (equation (31), Table 9,) the value of t for each of the b_i is obtainable from:

$$(43) \quad t = \frac{b_i - \beta_i}{s\sqrt{g_{ii}}}$$

where g_{ii} is the diagonal element in the matrix of Gaussian multipliers which corresponds to independent variable X_i . In the case of X_1 , for example, taking β_1 as zero:

$$(44) \quad t_1 = \frac{2.9030}{6.747 \sqrt{0.01729}} = 3.273$$

If the inverse of the variance—co-variance matrix is available (equation (28), Table 8), then the relationship equivalent to equation (43) is:

$$(45) \quad t = \frac{b_i - \beta_i}{\frac{s\sqrt{D_{x_i x_i}}}{n}}$$

or, alternatively:

$$(46) \quad t^2 = \frac{(b_i - \beta_i)^2 n}{s^2 D_{x_i x_i}}$$

where

$$(47) \quad D_{x_i x_i} = d_{x_i x_i} - \frac{d^2 x_i y}{dy y}$$

where d are the elements of the inverse of the variance—co-variance matrix corresponding to the row and columns shown.¹⁷

Using X_1 again as the example:

$$(48) \quad D_{x_1 x_1} = 0.66186 - \frac{(-0.08503)^2}{0.02929} = 0.41502$$

and hence, for $\beta_1 = 0$:

$$(49) \quad t_1^2 = \frac{(2.9030)^2 \times 24}{45.522 \times 0.41502} = 10.7057$$

and $t_1 = 3.273$

In a similar fashion the t -value for each regression coefficient may be determined and the probability level with $(n - k - 1)$ degrees of freedom ascertained from the "Student's" distribution table. It will be seen from equation (43) that the t -values represent the ratio between the regression coefficients (assuming $\beta_i = 0$) and their standard errors. It is of interest to have the standard errors of the estimated coefficients β_i given by:

$$s\sqrt{\frac{D_{ii}}{n}} \text{ or } s\sqrt{g_{ii}}$$

since they may be used to calculate the "fiducial" or "confidence" limits

¹⁷ The values $D_{x_i x_i}$ are actually diagonal elements of the inverse of the variance—co-variance matrix of *independent* variables. Since this inverse has not been determined directly, the relevant elements must be obtained from the inverse of the variance—covariance matrix of *all* variables, using equation (47).

of the actual regression coefficients β_i . The fiducial limits for each b_i are given by:

$$(50) \quad b_i \pm t \times (\text{standard error of } b_i)$$

where t is read from the "Student's" distribution table for the desired confidence level and $(n - k - 1)$ degrees of freedom.

For example, the 95 per cent. fiducial limits of b_1 are found as:

$$(51) \quad 2.9030 \pm 2.101 \times 0.887 = 4.767 \text{ and } 1.039$$

Table 10 shows the t -values, approximate probability levels, standard errors and 95 per cent fiducial limits for the regression coefficients of our polynomial function in equation (36).

TABLE 10
*t-values, Approximate Probabilities, Standard Errors,
and 95 per cent Fiducial Limits for Regression Coefficients*

Coefficient	t	Approximate Probability	Standard Error	95 % Fiducial Limits	
				Upper	Lower
b_1	3.273	0.01	0.887	4.767	1.039
b_2	5.343	0.001	1.769	13.168	5.735
b_3	2.768	0.02	0.030	— 0.021	— 0.147
b_4	3.462	0.01	0.194	— 0.265	— 1.080
b_5	0.377	0.80	0.050	0.086	— 0.124

The probability column of Table 10 shows the chance of the difference between b_i and β_i being due to random variation. It is seen that b_1 , b_2 , b_3 and b_4 are all significantly different from zero at least at the 2 per cent level, but that in the case of b_5 the "null hypothesis" (i.e. the hypothesis that $\beta_i - b_i = 0$) cannot be rejected.

However, although a t -test may indicate that a particular regression coefficient is not significantly different from zero, this is not necessarily a sufficient reason for dropping it and its associated term from the regression equation. In the first place, if the production function form under consideration has been chosen on *a priori* grounds as that which might best characterize the production relationship concerned, deletion of terms will cause alteration to the projected response pattern. Secondly:

"The nature of a test of significance is that it evaluates the strength of evidence against a null hypothesis. It is extremely unlikely *a priori* that the true coefficient in question is exactly zero, and even if the evidence against it being zero is meager, the *best estimate* is still that available from the data. If a variable is to be dropped out of the system, a more logical basis for doing so lies in the extent to which the yield estimates and their standard errors are actually changed by its omission. In making this decision, all terms, or at least all terms of a given order, involving this variable should be considered as a group".¹⁸

For this latter reason the b_5 term in our fitted regression equation is retained in the subsequent analysis.

As noted in connection with equations (3) and (4) the regression coefficients measure the influence of each variable on Y . Since the independent variables are usually measured in different units, their regression coefficients

¹⁸ R. J. Hader *et al*, "An Investigation of Some of the Relationships between Copper, Iron, and Molybdenum in the Growth and Nutrition of Lettuce: I. Experimental Design and Statistical Methods for Characterizing the Response Surface," *Proceedings of the Soil Science Society of America*, Vol. 21, (1957), p. 63.

are not *directly* comparable. However, standardizing the regression coefficients enables a direct comparison. The simplest way to standardize them is *via* equation (52):

$$(52) \quad b'_i = b_i \frac{\sigma_{x_i}}{\sigma_y}$$

where b'_i = standardized partial regression coefficient;
 σ_{x_i} = standard deviation of X_i ,
 σ_y = standard deviation of Y .

Equivalently, this equation may be written:

$$(53) \quad b'_i = b_i \sqrt{\frac{\Sigma x_i^2}{\Sigma y^2}}$$

In this way the standardized partial regression coefficients for the b_i in our example are calculated as:

$$\begin{aligned} b'_1 &= 1.8363 \\ b'_2 &= 1.9045 \\ b'_3 &= -1.4601 \\ b'_4 &= -1.1683 \\ b'_5 &= -0.0697 \end{aligned}$$

There are several ways of conducting an analysis of variance on the original data in such a way that the overall significance of the fitted regression is also tested. The method used here serves as a sufficient illustration. The total sum of squares with $n - 1$ degrees of freedom has been calculated already (Table 5). The replicates and treatments sums of squares are derived in the usual manner, as would be done if a simple analysis of variance were being carried out on the raw data resulting from the experiment.¹⁹ As a subsection of the treatments entry the sum of squares due to regression is calculated. This is given as the difference between the total and the residual sums of squares, the former available from Table 5, the latter from equation (37). Equivalently, the sum of squares due to regression is given by the top line of the right-hand side of equation (39). The difference between the treatment and regression sums of squares is a measure of the lack of fit of the regression line or plane. As a measure of the effect of the two inputs and to ascertain the extent of any interaction between them, the quantities which would normally be calculated in an analysis of variance may be computed and inserted at this point if so desired. The last entry in the sums of squares column is the error term, which is found as usual by difference. The sums of squares are divided by the degrees of freedom to give the mean squares, and F values where relevant are computed as ratios between individual mean squares and the error mean square. The values of all these quantities corresponding to our example are shown in Table 11.

¹⁹ For example, the treatment sum of squares is given by:

$$\left[\frac{(11.7 + 13.3)^2}{2} + \frac{(31.5 + 30.9)^2}{2} + \dots + \frac{(53.5 + 55.2)^2}{2} \right] - \frac{931.3^2}{24}$$

and the replicate sum of squares by:

$$\left[\frac{(11.7 + 31.5 + \dots + 53.5)^2}{12} + \frac{(13.3 + 30.9 + \dots + 55.2)^2}{12} \right] - \frac{931.3^2}{24}$$

TABLE 11
Analysis of Variance of Replicates, Treatments, Regression and Raw Data

Source of Variation	d.f.	Sums of Squares	Mean Square	F
Total	23	5171.81	224.86	..
Replicates	1	0.51	0.51	..
Treatments	11	4807.38	437.03	13.21*
Regression	5	4351.56	870.31	26.31*
Lack of Fit	6	455.82	75.97	..
Superphosphate	3	3525.13	1175.04	35.52*
Seed	2	976.83	488.42	14.76*
Super x Seed interaction	6	305.42	50.90	1.54
Error	11	363.92	33.08	..

* Significant at the 0.1 per cent level.

6. FITTING THE COBB-DOUGLAS, TRANSCENDENTAL, AND SPILLMAN FUNCTIONS

(i) Cobb-Douglas Function

The generalized Cobb-Douglas function was given in equation (9). It will be seen that this equation is linear in logarithms and hence must be converted to log form before using regression to estimate the parameters. For instance, to fit a Cobb-Douglas function to the data in Table 1 would require estimation of a , b_1 and b_2 in the equation:

$$(54) \quad Y = a X_1^{b_1} X_2^{b_2}$$

Representing this in either natural or common logarithms gives equation (55):

$$(55) \quad \log Y = \log a + b_1 \log X_1 + b_2 \log X_2$$

and the problem would then involve simply using the above regression techniques to calculate b_1 , b_2 and the constant term in a system containing two independent variables, $\log X_1$, and $\log X_2$, with dependent variable $\log Y$.

It should be noted that in equation (54) if any X is zero, then Y is also zero. This may not always be realistic. For instance, one of the treatments in our example involves a zero level of superphosphate yet this does not lead to zero production of pasture. This difficulty can be overcome easily by adding a constant to all values of the independent variable in question. Thus, in drawing up the values of the variables for this problem as was done in Table 2 for fitting the polynomial, the arbitrarily selected constant 1 is added to all X_2 making them 1, 3, 5 and 9, instead of 0, 2, 4 and 8. Care should be taken to remember this when using the function for prediction purposes.

Thus, the equation we would derive by regression²⁰ is:

$$(56) \quad Y' = a' + b_1 X'_1 + b_2 X'_2$$

where $Y' = \log Y$
 $a' = \log a$
 $X'_1 = \log X_1$
 $X'_2 = \log (X_2 + 1)$

²⁰ The Cobb-Douglas function derived for the data of Table 1 was:
 $Y = 3.107 X_1^{0.9301} (X_2 + 1)^{0.4771}$ with $R = 0.9036$.

(ii) Transcendental Function

Although this function has not often been used to describe agricultural experimental data, it is interesting to note that when applied to the data of Table 1 in our present example, it gave an R -value which was more significant than that derived for the polynomial or the Cobb-Douglas function. Its chief disadvantage is not so much in the fitting of the function as in its subsequent use in prediction. Equation (10) gave the transcendental function for n resources. For a given problem of, say, two inputs, we would require to estimate the parameters of equation (57).

$$(57) \quad Y = a X_1^{b_1} e^{c_1 X_1} X_2^{b_2} e^{c_2 X_2}$$

As with the Cobb-Douglas function this may be achieved by first converting it to linear form. Taking logarithms, we derive:

$$(58) \quad \log Y = \log a + b_1 \log X_1 + c_1 X_1 \log e + b_2 \log X_2 + c_2 X_2 \log e$$

which is a linear function in the transformations of the variables. Thus, our task resolves itself into estimating a' and b'_i in:

$$(59) \quad Y' = a' + b'_1 X'_1 + b'_2 X'_2 + b'_3 X'_3 + b'_4 X'_4$$

where $Y' = \log Y$,

$$a' = \log a,$$

$$b'_1 = b_1,$$

$$X'_1 = \log X_1,$$

$$b'_2 = c_1 \log e,$$

$$X'_2 = X_1, \text{ etc.,}$$

which is a simple problem in least squares regression. The table corresponding to Table 2 would have five columns, $\log Y$, $\log X_1$, X_1 , $\log X_2$, X_2 and the regression procedures thenceforward are exactly the same as described for the polynomial.²¹

(iii) Spillman Function

The techniques of least squares regression cannot be used to estimate functions which are non-linear in the parameters or which cannot be converted to a linear function *via* some simple transformation of the variables. Exponential functions such as the Spillman-Mitscherlich function are of this nature, and to estimate them, tedious iterative procedures are required, which will not be discussed here. In fact, Heady and Dillon go as far as to say:

"Since any production surface can be fitted reasonably well by an easily calculated polynomial type function, there seems little justification for persevering with functions requiring complex iterative procedures."²²

²¹ The transcendental function derived for the data of Table 1 was:

$$Y = 12.410 X_1^{0.370} e^{-0.022 X_1} (X_2 + 1)^{0.785} e^{-0.089(X_2 + 1)}$$

with $R = 0.9310$.

²² Heady and Dillon, *op cit.*, p. 126.

There is, however, one possible method for estimating a Spillman function *via* least squares regression, which may prove of limited value.²³ By re-arranging and taking logarithms of the simple Spillman function of equation (11), we may derive:

$$(60) \quad \log (M - Y) = \log A + X \log R$$

If an estimate is made of M , the maximum Y obtainable from any level of X , then a simple regression equation:

$$(61) \quad Y' = a + b X$$

may be fitted, in which

$$Y' = \log (M - Y)$$

$$a = \log A$$

$$b = \log R$$

For example, the figures in Table 12 were taken from a section of the same trial as were those in Table 1; Y again measures pasture output in cwt. per acre, and X is superphosphate rate in cwt. per acre.

TABLE 12
Pasture Production Data for Fitting Spillman Function:
cwt. Dry Matter per acre

X Super- phosphate Rate (cwt. per acre)	Replication			
	I	II	III	IV
0	8.3	13.7	4.6	4.9
2	18.6	13.5	12.6	6.5
4	25.2	21.1	18.8	18.2
8	28.9	26.0	26.4	25.5

An estimate of $M = 29.00$ was made and the following function fitted:

$$(62) \quad Y = 29.00 - 16.68 (0.7958^X)$$

with a correlation coefficient for the simple regression of 0.975 (significant at 0.1 per cent. level). Predictions made from equation (62) are shown in Table 13.

TABLE 13
Pasture Production Predicted from Spillman Function

Superphosphate Level, cwt. per acre	0	1	2	3	4	5	6	7	8
Pasture output, cwt. dry matter per acre	12.32	15.73	18.44	20.59	22.31	23.68	24.76	25.63	26.32

This procedure might be invalidated if a sufficiently large error were made in estimating M .

²³ After Snedecor, *op. cit.*, pp. 451-2.

7. EXTRAPOLATION FROM THE FITTED FUNCTION

The value of the production function approach in enabling maximum use to be made of a given set of experimental data has been noted already. We shall now examine some of the uses to which a fitted production function can be put.

The level of output from any combination of inputs can be predicted by substituting for all X_i into a given production function. Although *any* feasible value of the resource levels can be used for prediction, one can only have faith in predictions made within the range of levels used in the original experiment. This will become more obvious later.

Predictions of pasture output made from the polynomial production function in equation (36) are shown in Table 14. The columns of Table 14 represent the response of pasture to superphosphate at various seeding rates; similarly the rows denote the changes in output caused by increasing seeding rates at constant levels of superphosphate. Either the columns and/or the rows of such a table of predictions may be depicted graphically if so desired. For example, the columns of Table 14 are shown in Figure 1. The effects on yield of two factors together can only be shown diagrammatically in three dimensions, as explained in connection with equation (4). A perspective drawing of the production surface (also called "response surface", "prediction surface") constructed from the figures in Table 14 is shown in Figure 2. A diagram such as this is of illustrative value only; for analytical purposes we must use more precise representation.

TABLE 14

Predicted Dry Matter Yield of Pasture: cwt. per acre

Superphosphate Rate (cwt. per acre)	Seeding Rate (lb. per acre)				
	2	6	10	14	18
0	11.1	20.0	26.3	29.8	30.7
2	27.2	36.0	42.1	45.5	46.2
4	38.0	46.6	52.5	55.8	56.3
6	43.4	51.9	57.6	60.7	61.1
8	43.4	51.7	57.3	60.3	60.5

This is achieved by deriving a "contour" diagram of the production surface. Suppose contour lines are drawn on a model of a production surface; since each line is the same height from the base throughout its length, and since the vertical axis measures output, each contour line indicates one level of production. Now, a viewer who looks down from directly above the production surface will see two input axes, and a series of contour lines each denoting a certain level of output. Since this view only takes in two dimensions, it may be graphed accurately in two dimensions.

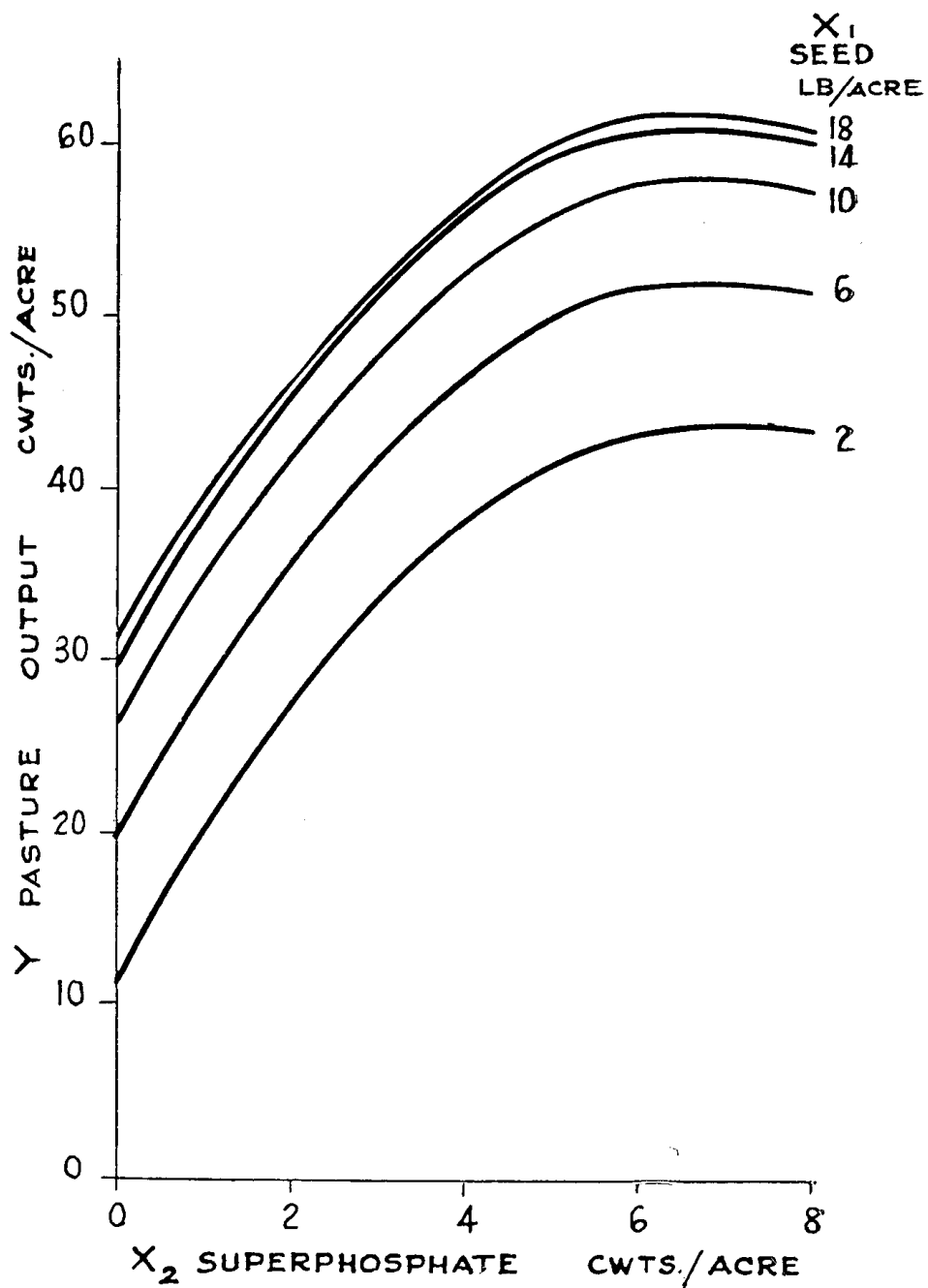


Fig. 1. Superphosphate Response Curves for Various Seeding Levels.

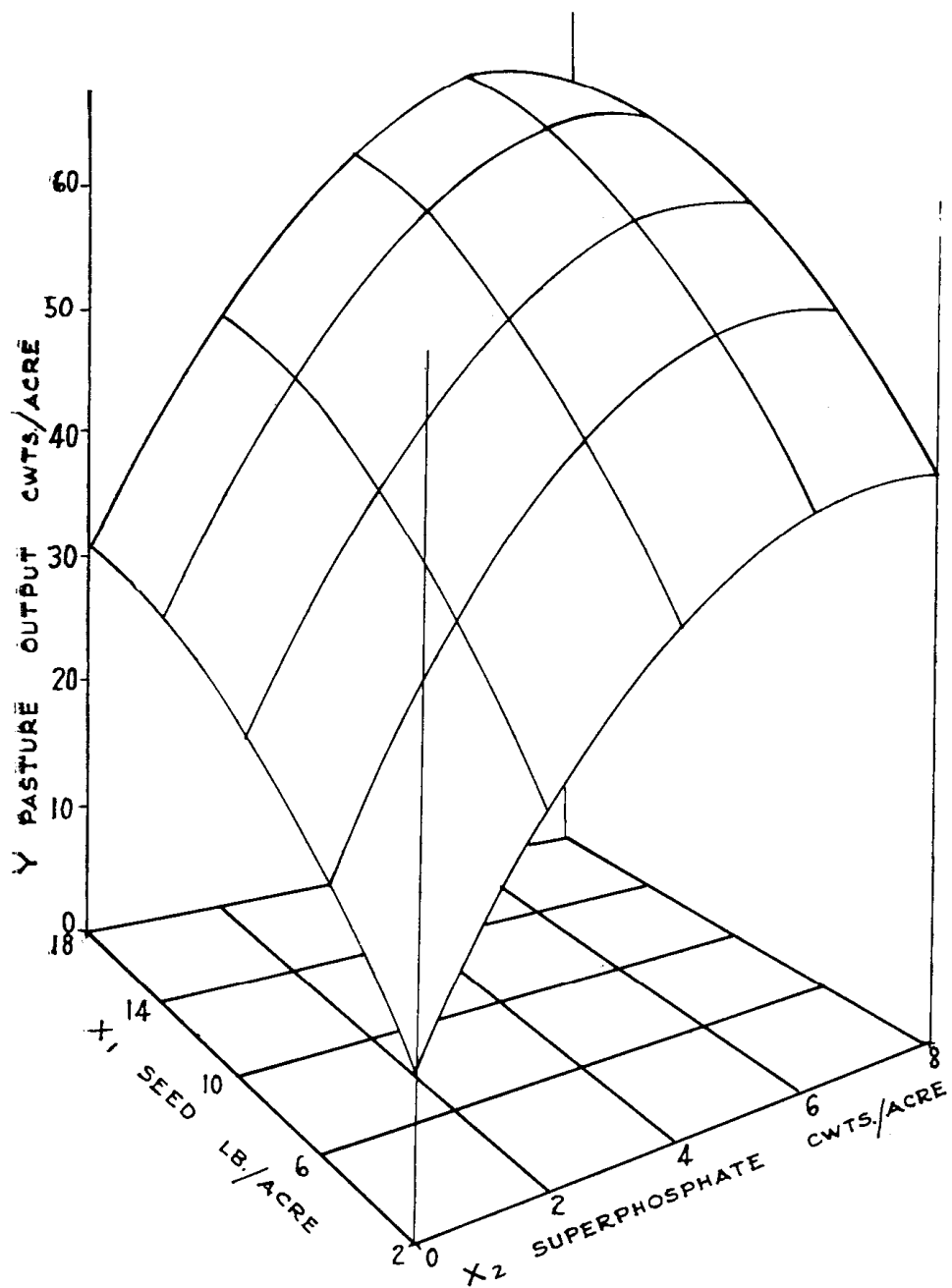


Fig. 2. Production Surface Depicting a Three-dimensional View of Pasture Yield Response to Seed and Superphosphate Application

These lines indicating levels of production are known as *isoquants* and may be derived graphically or mathematically. If a horizontal line is drawn on Figure 1 at, say, 40 cwt. dry matter production, it intersects the 18, 14, 10, 6, and 2 lb. seeding rate lines at superphosphate levels of 1.0, 1.2, 1.6, 2.6, and 4.5 cwt. respectively; i.e. combinations of seed (lb.) and superphosphate (cwt.) of 18 and 1.0, 14 and 1.2, 10 and 1.6, etc. all produce a yield of 40 cwt. A series of such horizontal lines yields a series of isoquants, which may be drawn on to another graph.

Mathematical derivation of the isoquants is much more satisfactory. Consider the polynomial function shown in general form in equation (6). Rearranging the terms, we have:

$$(63) \quad b_3 X_1^2 + (b_1 + b_5 X_2) X_1 + (a + b_2 X_2 + b_4 X_2^2 - \hat{Y}) = 0$$

Since this is a quadratic equation in X_1 , we may use the general formula:

$$(64) \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to solve it, where a, b, c are the coefficients of the general quadratic equation $ax^2 + bx + c = 0$. Applying equation (64) to (63) we derive:

$$(65) \quad X_1 = \frac{-(b_1 + b_5 X_2) \pm \sqrt{(b_1 + b_5 X_2)^2 - 4b_3(a + b_2 X_2 + b_4 X_2^2 - \hat{Y})}}{2b_3}$$

This is the isoquant equation and may be solved for X_1 using various levels of \hat{Y} and X_2 . It should be remembered that a production surface, isoquants, etc., can be derived for any pair of independent variables in a production function with *more* than two inputs, by holding the other inputs constant at some predetermined level and predicting \hat{Y} for a range of the two variables under consideration.

Substituting into equation (65) the values of a and b_i shown in equation (36) leads to:

$$(66) \quad X_1 = \frac{-2.9030 + 0.0189 X_2 \pm \sqrt{10.3034 + 3.0470 X_2 - 0.2241 X_2^2 - 0.3340 \hat{Y} - 0.1670}}{-0.1670}$$

which could be solved using any desired levels of X_2 and \hat{Y} . To illustrate, equation (66) was solved for X_1 with $Y = 30, 40, 50, 60$ cwt. and X_2 ranging from 0 to 10 cwt. The isoquants thus derived are shown in Figure 3.

The limits of experimental observation (2 — 24 lb. seed, 0 — 8 cwt. superphosphate) are drawn onto Figure 3. The positions of the isoquants outside the rectangle formed by the experimental limits cannot be trusted, since any trends in this region were not measured by the experiment.

It will be seen from Figure 3 that a yield of, for example, 50 cwt. can be obtained by combining 23 lb. of seed with 3 cwt. of superphosphate. Obviously it would be irrational to operate at this point since the same production can be obtained from the same superphosphate rate by using only $12\frac{1}{2}$ lb. of seed. Similarly, 7 lb. of seed and 8 cwt. of superphosphate is an irrational combination, because the same results can be achieved using 7 lb. and 5 cwt. The two points where a given isoquant is closest to each

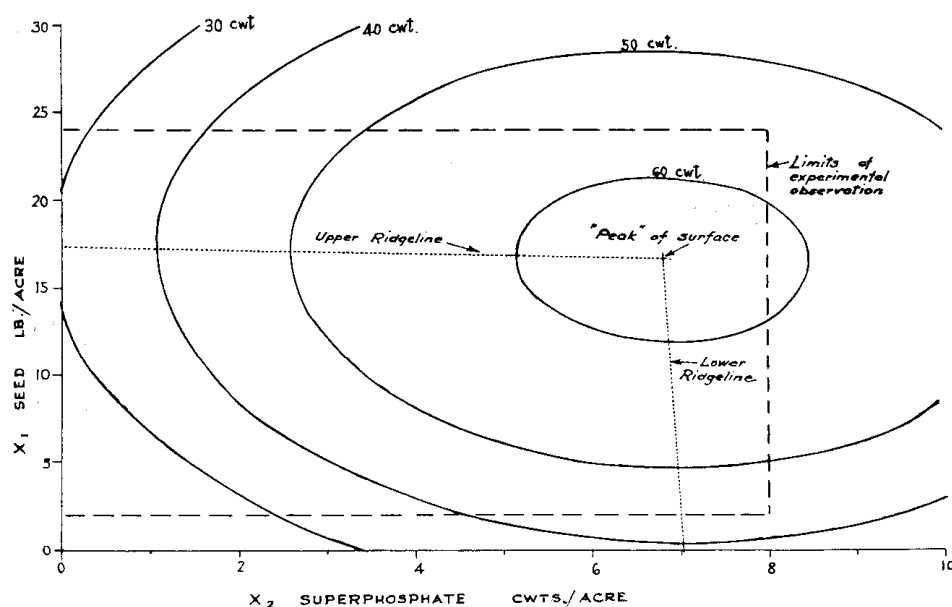


Fig. 3. Yield Isoquants for Pasture Outputs of 30, 40, 50 and 60 cwt. per acre in Relation to Seed and Superphosphate Rates.

of the two axes of Figure 3 represent the edges of the rational phase of production for that isoquant, and the two lines joining these points on different isoquants are called the *ridgelines*. The ridgelines enclose the rational area of production, and may be fitted by eye, or derived mathematically, as explained at a later stage.

Let us turn now to the derivation of economic optima from a fitted production function, still using the quadratic polynomial as our example. Taking partial derivatives of the function in equation (6) gives:

$$(67) \quad \frac{\partial Y}{\partial X_1} = b_1 + 2b_3 X_1 + b_5 X_2$$

$$\text{and } \frac{\partial Y}{\partial X_2} = b_2 + 2b_4 X_2 + b_5 X_1$$

whence:

$$(68) \quad \frac{\partial X_2}{\partial X_1} = \frac{b_1 + 2b_3 X_1 + b_5 X_2}{b_2 + 2b_4 X_2 + b_5 X_1}$$

Equation (68) represents the exact marginal rate of substitution between X_2 and X_1 ; it should strictly be negative, but since it is negative for all rational areas of production, the minus sign is often omitted.²⁴ Theoretically the optimum combination of two factors X_1 and X_2 is given when the marginal rate of substitution of X_1 for X_2 ($\partial X_2 / \partial X_1$) equals the ratio of prices of X_1 to X_2 ²⁵; i.e., where:

$$(69) \quad \frac{\partial X_2}{\partial X_1} = \frac{p_{X_1}}{p_{X_2}}$$

²⁴ See E. O. Heady: *Economics of Agricultural Production and Resource Use*, (Englewood Cliffs: Prentice Hall, 1957), p. 141.

²⁵ *ibid.* p. 171 ff.

Let $p_{x_1}/p_{x_2} = P$; then the optimum is given where:

$$(70) \quad P = \frac{b_1 + 2 b_3 X_1 + b_5 X_2}{b_2 + 2 b_4 X_2 + b_5 X_1}$$

Solving this for one factor, say X_1 , gives:

$$(71) \quad X_1 = \frac{b_1 - b_2 P + (b_5 - 2 b_4 P) X_2}{b_5 P - 2 b_3}$$

which the reader may verify algebraically if desired. By substituting a given price ratio P into equation (71) the *isocline* for that price ratio can be found. An isocline is a line joining a series of isoquants through points at which the isoquants have the same slope. Thus the isocline for price ratio P joins the points where the isoquants have slope measured by equation (69), and represents the least cost expansion path for that price ratio.

Where the marginal rates of substitution of X_1 for X_2 and X_2 for X_1 are zero, the ridgelines are found; i.e. where:

$$(72) \quad \frac{b_1 + 2 b_3 X_1 + b_5 X_2}{b_2 + 2 b_4 X_2 + b_5 X_1} = 0$$

Hence, the ridgelines are given by the equations:

$$(73) \quad X_1 = \frac{-b_1 - b_5 X_2}{2 b_3} \text{ and } X_1 = \frac{-b_2 - 2 b_4 X_2}{b_5}$$

which are derived from equation (72), and which the reader may check if desired.

Equations (71) and (73) show that the isoclines and ridgelines are linear. On the production surface they converge towards the peak. Thus the point where they all intersect on Figure 3 represents the maximum output possible from any combination of X_1 and X_2 . The levels of X_1 and X_2 required to produce the maximum Y can be read from Figure 3 when the isoclines have been drawn in, or may be calculated using:

$$(74) \quad X_1 = \frac{b_2 b_5 - 2 b_1 b_4}{4 b_3 b_4 - b_5^2} \text{ and } X_2 = \frac{2 b_2 b_3 - b_1 b_5}{b_5^2 - 4 b_3 b_4}$$

of which the expression for X_2 is derived by equating the right-hand sides of the two equations in (73), and that for X_1 from equation (72) in a similar fashion.

Let us now illustrate the points made since equation (67) by referring back to our example. The marginal rate of substitution equation is calculated from (68) as:

$$(75) \quad \frac{\partial X_2}{\partial X_1} = \frac{2.9030 - 0.1670 X_1 - 0.0189 X_2}{9.4513 - 1.3446 X_2 - 0.0189 X_1}$$

Assume that the price of superphosphate is likely to range from 12s. to 16s. per cwt. and the price of seed from 5s. to 15s. per lb. Thus the price ratio, P , might be expected to vary from about 0.25 to 1.25. For five possible price ratios, 0.25, 0.50, 0.75, 1.00 and 1.25, isoclines are derived using equation (71). For example, for $P = 0.50$, the isocline equation is:

$$(76) \quad X_1 = 4.1486 X_2 - 11.5727$$

The ridgelines are found from equation (73), for example, the upper ridge-line is given by:

$$(77) \quad X_1 = \frac{-2.9030 + 0.0189 X_2}{-0.1670}$$

The maximum yield of pasture of 61.85 cwt. is achieved by combining 16.62 lb. of seed and 6.80 cwt. of superphosphate, calculated from equation (74).

The isoclines and ridgelines are shown on a diagram of the relevant area of the production surface, in Figure 4. The derivation of a diagram such as Figure 4 is of significance for extension purposes, not only for individual farmer recommendations, but also as an easily understandable aid when discussing experimental results with farmer groups.

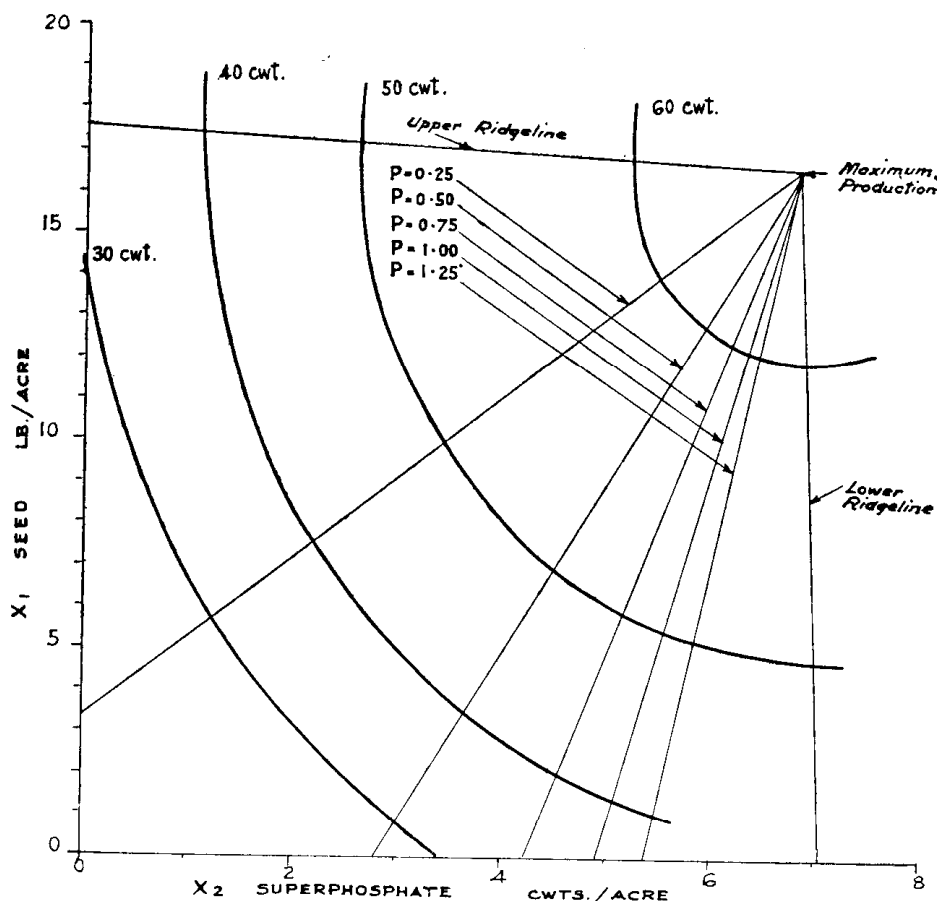


Fig. 4. Yield Isoquants for Pasture Outputs of 30, 40, 50 and 60 cwt. per acre and Isoclines for Various Price Ratios.

The isocline analysis indicates the minimum cost combinations of seed and superphosphate when capital available for their purchase is limited. As increasing amounts of capital become available the use of the two factors should be expanded along the isocline corresponding to the ruling price ratio. If capital is non-limiting, the farmer would be interested in expanding

production to the maximum profit point. This will vary not only with the costs of the two inputs but also with the price of the product. The profit maximization point is found where:

$$(78) \quad \frac{\partial Y}{\partial X_1} = \frac{P_{X_1}}{P_y} \text{ and } \frac{\partial Y}{\partial X_2} = \frac{P_{X_2}}{P_y}$$

Thus, by substituting equation (67) into (78) we obtain:

$$(79) \quad b_1 + 2 b_3 X_1 + b_5 X_2 = \frac{P_{X_1}}{P_y}$$

$$\text{and } b_2 + 2 b_4 X_2 + b_5 X_1 = \frac{P_{X_2}}{P_y}$$

which can be solved simultaneously for X_1 and X_2 for any desired ratio between the prices of the three variables. By substituting the values so obtained for X_1 and X_2 back into the original production function, the output of the dependent variable at the point of maximum profit can be predicted. This process can be repeated for a range of feasible price levels, and the total costs and returns, and net returns can be calculated in each case.

A significant feature of the production function approach in the analysis of agricultural data should now be evident. The series of isoquants derived from the experimental figures is of course independent of any economic factors, being merely a representation of an underlying physical and biological relationship. Hence, economic optima can be derived for *any* price and cost conditions simply by superimposing ruling monetary values on to the physical model. This generality of the production function analysis, enabling its application to any feasible economic regime, gives it a considerable advantage over less versatile analytical methods.

Computing the combinations of seed and superphosphate which maximize profit in our example raises a difficulty which can be of importance in some production function studies: what price or range of prices should be placed on the product in the experiment? For trials studying commodities placed directly on the market, such as animal products, grains etc., the likely range of market prices can easily be ascertained²⁶; however, if the product is not sold but is used on the farm in the production of another commodity, a price must be imputed for it. From an Australian point of view perhaps the most important product in this category is pasture, which is also probably the hardest to value. The simplest approach is to use its market price as hay, but this may overlook important factors influencing its value to an individual farmer if he anticipates using it for grazing. Another method is to apply a series of conversion factors changing the dependent variable from cwt. of pasture to lb. of wool or meat, or gallons of milk; this approach however, is highly susceptible to error *via* unreal assumptions.

We take the simplest course in calculating the profit maximization points for our example, and assume that the pasture is to be sold as hay. Two further complications are met. Firstly the demand for hay can vary markedly with seasonal conditions, etc., leading to instability in its price and making price forecasts difficult. Secondly, the experimental results were expressed in dry matter terms, whereas baled hay may contain up to 15 per cent moisture; thus an allowance should be made for this in estimating its price.

²⁶ For example, see W. O. McCarthy, "Production Function Analysis of a Fertilizer Trial on Barley", *Australian Journal of Agricultural Economics*, Vol. 3 No. 2, (December, 1959,) pp. 1-11.

For the purposes of illustration we assume prices of 5s., 15s. and 25s. per cwt. of pasture dry matter, corresponding roughly to hay prices of £5, £15 and £25 per ton respectively.

The two equations corresponding to (79) are shown in (80):

$$(80) \quad \begin{aligned} 2.9030 - 0.1670 X_1 - 0.0189 X_2 &= \frac{p_{x_1}}{p_y} \\ 9.4513 - 1.3446 X_2 - 0.0189 X_1 &= \frac{p_{x_2}}{p_y} \end{aligned}$$

By substituting a range of values for p_{x_1} , p_{x_2} , and p_y into these equations, the profit maximizing rates of seed and superphosphate are found. These are shown, together with the costs, total returns and net returns in Table 15.

The figures in Table 15 show that for a fairly wide range of price and cost conditions profits are maximized by applying between 5 and 6 cwt. of superphosphate and 10 to 15 lb. of seed.

It is not proposed here to discuss the implications of applying experimental results to actual farming practice. General statements such as that in the preceding paragraph would certainly appear justifiable but the limits to the accuracy with which recommendations can be made from experimental results are by no means clear.²⁷

8. EXTRAPOLATION FROM THE COBB-DOUGLAS, TRANS-CENDENTAL AND SPILLMAN FUNCTIONS

Although the principles applied in deriving isoquants, isoclines and economic optima from logarithmic or exponential functions are similar to those used above for the polynomial, the mathematics involved is somewhat more complicated. The reader is referred to the literature for a more detailed account²⁸, and only a brief outline is presented here.

(i) Cobb-Douglas Function

The isoquants are obtained by rearranging equations (54) or (55) such that X_1 can be found by assuming a range of values for X_2 and Y . The marginal rate of substitution between X_1 and X_2 can be obtained again by partial differentiation with respect to each factor followed by division, as was done with equations (67) and (68). It will be found that the marginal rate of substitution derived from the Cobb-Douglas function (equation (81)) is somewhat simpler than that for the polynomial (equation (68)); viz:

$$(81) \quad \frac{\partial X_2}{\partial X_1} = \frac{b_1 X_2}{b_2 X_1}$$

The isoclines, which diverge outwards from the origin, are found by equating (81) to the price ratio p_{x_1}/p_{x_2} . The profit maximizing rates are obtained by solving simultaneously:

$$(82) \quad \begin{aligned} b_1 X_2 &= \frac{p_{x_1}}{p_y} \\ b_2 X_1 &= \frac{p_{x_2}}{p_y} \end{aligned}$$

²⁷ See, for example, E. R. Swanson, "Problems of Applying Experimental Results to Commercial Practice", *Journal of Farm Economics*, Vol. XXXIX, No. 2 (May, 1957), pp. 382-9; also Lloyd op. cit., pp. 201-3.

²⁸ See particularly Heady and Dillon, op. cit., Chapter 3.

TABLE 15
*Profit Maximizing Rates of Seed and Superphosphate in Pasture Production,
 for Various Price and Cost Conditions*

Price of:			Profit Maximizing level of:		Output of Pasture (Y)	Cost of:		TOTAL RETURNS	NET RETURNS
Hay (Py)	Seed (Px ₁)	Super-phosphate (Px ₂)	Seed (X ₁)	Super-phosphate (X ₂)		Seed	Super-phosphate		
Shillings/ cwt.	Shillings/ lb.	Shillings/ cwt.	lb./acre	cwt./acre	cwt./acre	Shillings/ acre	Shillings/ acre	Shillings/ acre	Shillings/ acre
5	5	12	10.8	5.1	56.8	54	61	284	169
5	10	12	4.8	5.2	48.1	48	62	241	131
5	5	14	10.9	4.8	56.1	55	67	281	159
5	10	14	4.9	4.9	47.4	49	69	237	119
15	5	12	14.7	6.2	61.2	74	74	918	770
15	10	12	12.7	6.3	60.2	127	76	903	700
15	15	12	10.7	6.3	58.6	161	76	879	642
15	5	14	14.7	6.1	61.1	74	85	917	758
15	10	14	12.7	6.2	60.2	127	87	903	689
15	15	14	10.7	6.2	58.5	161	87	878	630
25	5	12	15.5	6.5	61.5	78	78	1538	1382
25	15	12	13.1	6.5	60.6	197	78	1515	1240
25	5	14	15.5	6.4	61.5	78	90	1538	1370
25	15	14	13.1	6.4	60.6	197	90	1515	1228

(ii) Transcendental Function

If the transcendental function for two resources is expressed in *natural* logarithmic form as:

$$(83) \quad \log Y = a + b_1 \log X_1 + b_2 X_1 + b_3 \log X_2 + b_4 X_2$$

then the isoquant equation is found by rearranging to give:

$$(84) \quad b_1 \log X_1 + b_2 X_1 = \log Y - a - b_3 \log X_2 - b_4 X_2$$

A range of values is assumed for Y and X_2 and equation (84) is solved for X_1 . This requires iterative procedures to find an exact solution. However an approximate solution is obtainable graphically by plotting the left-hand side of (84) for a range of X_1 and then reading off particular values of the right hand side. The marginal rates of substitution equation, which can be used to derive isoclines, profit maximizing points, etc., is:

$$(85) \quad \frac{\partial X_2}{\partial X_1} = \frac{X_2 (b_1 + b_2 X_1)}{X_1 (b_3 + b_4 X_2)}$$

(iii) Spillman Function

If the Spillman function shown in equation (13) is written for two inputs, the isoquant equation can be derived by taking logarithms and rearranging to give:

$$(86) \quad X_1 = \frac{\log \left[1 - \frac{Y}{A(1 - R_2^{X_2})} \right]}{\log R_1}$$

The marginal rate of substitution equation for this function is given as:

$$(87) \quad \frac{\partial X_2}{\partial X_1} = \frac{(1 - R_2^{X_2}) (R_1^{X_1} \log_e R_1)}{(1 - R_1^{X_1}) (R_2^{X_2} \log_e R_2)}$$

9. SUMMARY OF PROCEDURES

Following is a summary of the *minimum* amount of calculations which would be required to conduct a single production function analysis on a set of experimental data.

(a) The regression relation to be fitted is drawn up in its simplest "linear" form and the observations of the dependent variable and the complete set of independent variables tabulated. (Equation (14) ; Table 2.)

(b) The sums, sums of squares and means are calculated. (Table 3.)

(c) The sums of squares and cross products of deviations from the respective means (product moments) are computed. (Equations (18), (19) ; Table 4.)

(d) Next the matrix of product moments corresponding to the *independent* variables is inverted to find the Gaussian multipliers. (Equations (30), (31) ; Table 9.)

(e) The regression coefficients and the constant term of the regression relation are calculated. (Equations (32), (34).)

(f) Statistical tests indicating the significance of the regression relation are conducted. The multiple correlation coefficient is found; a *t*-test is performed on the regression coefficients; and an analysis of variance is carried out. (Equations (39), (41), (43) ; Table 11.)

(g) Assuming the significance tests to be satisfactory, predictions of the dependent variable are made for any desired levels of the independent variables, by substitution into the fitted production function. From these data the response surface may be drawn. (Table 14; Figure 2.)

(h) A series of isoquants for two inputs is obtained by substituting a range of values for the dependent variable and one of the inputs into the fitted production function, and solving for the other input. Any other resources included in the function must be held constant. (Equation (65); Figures 3, 4.)

(i) The marginal rate of substitution between two inputs is obtained by partial differentiation of the production function with respect to each input followed by division to eliminate the δY term. From this, the isoclines, indicating least cost expansion paths, the ridgelines, indicating the boundaries of the rational area of production, and the profit maximizing combinations of inputs, are computed. (Equations (67) to (73), (78), (79); Figure 4; Table 15.)

10. CONCLUSION

A large number of agricultural experiments are designed specifically to study the response of a product to variations in one or more resources. In the past, analysis of the results of such experiments have often used only the classical analysis of variance model, thereby disregarding the fact that the underlying physical or biological response is continuous rather than discrete.

By adopting a functional approach in the analysis of data from many experiments, both the technical understanding and the reliability and extent of recommendations can be increased. This approach does not presume to supersede the analysis of variance; rather, it supplements it. One of its major advantages is that the regression model allows quantitative description of the response over the whole of the relevant range of the variables; thus derivation of economic optima is more accurate, and the application of a variety of price and cost conditions to the production model is made possible.

APPENDIX

Some Short Cuts in Computations

Coding

If an experiment involves equally spaced levels of one or more independent variables coding may be useful to simplify computation. For example, if an experiment studied fertilizer at rates of 0, 56, 112, 168, 224 lb. per acre the figures actually used in the regression could be made 0, 1, 2, 3, 4; when extrapolating from the fitted function, results will have to be decoded, which requires a small amount of additional labour.

The Use of Computers

With the increasing availability of electronic computers, more and more research workers are taking advantage of their speed and versatility to reduce the labour and widen the range of statistical calculations.

Most computers will have a multiple regression programme in their library which will compute the regression coefficients for a given set of data and will also supply the user with most of the additional information he will require (means, standard deviations, co-variances, correlation coefficients, etc.). Such programmes may sacrifice speed for completeness; i.e., whilst they may provide almost any of the quantities which the user may want, they may also be relatively slower than routines designed to carry out one specific task, such as matrix inversion. Thus, if expenditure on computer time is restricted, a research worker may find it more satisfactory to calculate most of the regression by hand, and only use the computer for the difficult or lengthy sections.

A suggested procedure is to work out the product moments terms (Table 4 and 5) by hand—a simple matter with a desk calculator—then to use a computer to invert the product moment matrix corresponding to the independent variables; the Gaussian multipliers so obtained can be used to find the regression coefficients. Matrix inversion, which is an arduous and time-consuming task when done by hand, can be carried out by a computer in a matter of minutes. For example, the inversion of the matrix in Table 9 took the University of Sydney's SILLIAC one minute, including input of programme and data and output of results.

Since most computers only work with any number n which is within the range $-1 \leq n \leq 1$, data will usually have to be *scaled* beforehand by the user, so that all numbers lie within this range.²⁹

In this context, scaling of a number greater than unity involves successive division by 10 (i.e., left-shifting the decimal point) until it is just less than unity. This has no effect on the digital component of the interim or final results of fitting a regression equation; its only effect is on the position of the decimal point. Hence results as supplied by the computer will have to be *descaled* to bring them back to their original state. The following rules of thumb should assist in descaling data:

(a) n left-shifts of the decimal point in X necessitate n right-shifts of the decimal point in the sum, mean and standard deviation of X ; $2n$ right-shifts in the sum of squares, product moment and variance of X ; and $2n$ left-shifts of the corresponding diagonal element of the inverted variance—covariance matrix and of the Gaussian multipliers matrix.

(b) n left-shifts of the decimal point in X_1 and m in X_2 necessitate $(n + m)$ right-shifts in the product moment and the covariance between X_1 and X_2 and $(n + m)$ left-shifts in the corresponding elements of the inverted variance—covariance matrix and of the Gaussian multipliers matrix.

A computer can be particularly useful in the calculation of isoquant and isocline equations, especially with the more mathematically complex functions. For instance isoquants can be plotted more accurately and for a greater range of Y ; isoclines and profit maximization points can be computed for a greater number of price and cost assumptions.

²⁹Some computer programmes offer the facility of "floating point" operation, which eliminates the necessity of scaling. Such programmes, however, are usually extremely slow.