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ABSTRACT

The specification and estimation of models of consumer and producer demand with kink points are considered. The presence of kink points divides the demand or production schedule into different regimes. Our approach utilizes the concept of virtual prices. The virtual prices transform binding quantities into nonbinding ones and provide a rigorous justification for structural change in the observed demand functions across regimes. The comparison of virtual prices with market prices determines regime occurences. An application to energy demand in Indonesian manufacturing firms based on the translog cost function is provided.

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1. Introduction.

A number of recent studies (Pitt, 1983a; Deaton and Irish (1982); Strauss (1983)) have used household level data to estimate demand relationships and firm-level data to estimate the derived demand for inputs (Pitt, 1984). These micro data sets are often from developing countries where poor infrastructure and market separation provide price variability in cross-section. Micro data offer a number of important advantages over aggregated data. For example, household age and sex composition are considered major determinants of expenditure patterns but their effects are not easily measured with aggregated data. Unfortunately, many studies using micro data suffer from the lack of an unrestrictive and theoretically consistent approach to dealing with a common attribute of these data, the non- (or otherwise bounded) consumption or production of goods by households or firms. With the increased availability of micro data it is important that this econometric problem be resolved so that the interpretation of results is unclouded by econometric inconsistency.

Zero corner solutions are the special case of a kink point which occurs on the boundary of the choice set of a consumer or producer. Kink points may occur for other reasons, such as block pricing or rationing. Kink points are usually atoms in probability space and hence, for econometric analysis, imply limited dependent variables. However, the estimation of theoretically consistent demand or production structures differs from the well-known limited dependent variable models of Tobin (1958) and Amemiya (1974) in that these structures involve complex structural interactions and cross-equation restrictions.

Recently, Wales and Woodland (1983) have considered the problem of estimating consumer demand systems with binding non-negativity constraints. Their econometric model was derived directly from a random utility function maximized subject to a budget constraint. As is well-known, demand equations can also be derived from an indirect utility function or cost function by application of Roy's identity. Furthermore, any demand system which adds up, is homogeneous of degree zero and has symmetric, negative definite compensated price response is integrable into a theoretically consistent preference ordering (Hurwicz and Uzawa, 1971). This dual approach has proved advantageous in practice. It is easier to specify demand, cost or indirect utility functions than direct utility functions. Systems of demand equations are easily derived from popular flexible functional forms, such as the translog. The dual approach has a particular advantage in the specification and estimation of multiple input-multiple output production structures (McKay, Lawrence and Vlastrin, 1983; Weaver, 1983).

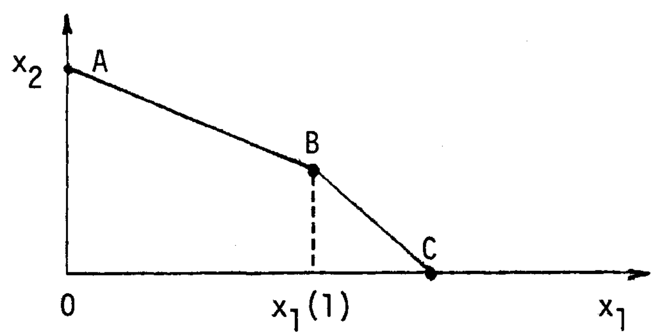
In this paper, we propose a unified approach to the estimation of demand systems with limited dependent variables. Our approach can estimate demand system derived directly from a utility function or indirectly through duality. Contradicting the claim of Wales and Woodland (p. 273) that the indirect utility approach is inappropriate for dealing with non-negativity constraints, we show that such an approach is not only possible but also useful. Our approach is in the tradition of the theory of consumer demand under rationing set forth in Houthakker (1950-51), Pollak (1969, 1970), Howard (1977), Neary and Roberts (1980), and Deaton (1981), and utilizes the concept of virtual prices originated by Rothbarth (1941).

2. The Consumer's Problem With a Convex Budget Set.

Convex budget sets result naturally from binding non-negativity constraints but also from quantity rationing and increasing block pricing. All of these sources of convexity can be analyzed within a common framework. Consider the case of two goods with non-negativity constraints and increasing block pricing for the first good (Figure 1). The marginal unit price for quantities of x_1 less than or equal to $x_1(1)$ is p_{11}, and p_{12}, $(p_{12} > p_{11})$, for quantities greater than $x_1(1)$. With income M, the extended budget line AB is $p_{11}x_1 + p_2x_2 = M$ and the budget line BC is $p_{12}x_1 + p_2x_2 = M + (p_{12} - p_{11})x_1(1)$. The point B is a kink point as are the non-negative boundary points A and C. Quantity rationing with ration $x_1(1)$ can be regarded as the special case of $p_{12} = \infty$.

FIGURE 1.

Two-goods case with increasing block price on x_1



In the general multicommodity case, every commodity may be subject to increasing block pricing. For commodity j, assume there are $I_j (I_j \geq 1)$ different block prices $p_{j1} < p_{j2} < \ldots < p_{jI_j}$ corresponding to knots $x_j(1), \ldots, x_j(I_j - 1)$, $x_1(i) < x_1(i+1)$ for $i = 1, \ldots, I_j - 2$. The case $I_j = 1$ is the standard single price situation. We adopt the convenient conventions $x_j(0) = 0$ and $x_j(I_j) = \infty$ for notational simplicity. The budget segments are $p_{1i_1}x_1 + p_{2i_2}x_2 + \ldots + p_{mi_m}x_m = M_{i_1i_2\ldots i_m}$ if $x_1(i_1 - 1) < x_1 \leq x_1(i_1), \ldots, x_m(i_m - 1) < x_m \leq x_m(i_m)$, where

$$M_{i_1 i_2 \ldots i_m} = M + \Sigma_{j=1}^{m} \Sigma_{\ell=1}^{i_j - 1} (p_{j\ell+1} - p_{j\ell}) x_j(\ell).$$

Let $U(x_1,\ldots,x_m)$ be a utility function which is continuously differentiable, increasing and strictly quasi-concave. The utility maximization problem is

$$\max_{x_1,\ldots,x_m} U(x_1,\ldots,x_m)$$

subject to

$$\text{(2.1)} \qquad \begin{array}{ll} \Sigma_{j=1}^{m} p_{ji_j} x_j \leq M_{i_1 i_2 \ldots i_m}, & i_j = 1,\ldots,I_j; \\ x_j \geq 0, & j = 1,\ldots,m \end{array}$$

where

$$\text{(2.2)} \qquad M_{i_1 i_2 \ldots i_m} = M + \Sigma_{j=1}^{m} \Sigma_{\ell=1}^{i_j - 1} (p_{j\ell+1} - p_{j\ell}) x_j(\ell).$$

This utility maximization problem provides the general framework for the demand analysis of this paper. It includes as special cases quantity rationing and non-negativity constraints. The consumer problem with binding non-negativity constraints and a single price for each commodity is simply

$$\text{(2.3)} \qquad \begin{array}{ll} \text{Max} & U(x_1,\ldots,x_m) \\ \text{subject to} & \Sigma_{j=1}^{m} p_j x_j = M \\ & x_j \geq 0, \;\; j = 1,\ldots,m. \end{array}$$

3. Regime Criteria and Virtual Prices.

For econometric analysis it is necessary to determine the probability that an optimal solution will occur at any kink point (demand regime), given the values of the explanatory variables. For two goods, these conditions are readily obtained diagrammatically. Burtless and Hausman (1978) and Hausman (1979) have considered the optimal solution to the consumers' problem with two goods and a convex budget set based on the location of indifference curves. For the general case of m goods, we derive below regime switching criteria using both Kuhn-Tucker conditions and virtual prices.

First, consider the consumers' problem with only binding non-negativity constraints (2.3). Assume that the first ℓ goods are not consumed, i.e., $x_i^* = 0$, $i = 1,\ldots,\ell$ and $x_i^* > 0$, $i = \ell+1,\ldots,m$ where $x^{*\prime} = (x_1^*, x_2^*,\ldots,x_\ell^*,x_{\ell+1}^*,\ldots,x_m^*)$ denotes the demanded quantity vector. The Lagrangean function for this problem is

$$(3.1) \qquad L = U(x_1,\ldots,x_m) + \lambda(M - \Sigma_{j=1}^m p_j x_j) + \Sigma_{j=1}^m \psi_j x_j$$

where λ and ψ's are Lagrangean multipliers. The Kuhn-Tucker conditions that characterize this solution x^* are

$$(3.2) \qquad \frac{\partial U(x^*)}{\partial x_i} - \lambda p_i + \psi_i = 0, \quad \psi_i \geq 0, \qquad i = 1,\ldots,\ell$$

$$(3.3) \qquad \frac{\partial U(x^*)}{\partial x_j} - \lambda p_j = 0, \qquad j = \ell+1,\ldots,m$$

$$(3.4) \qquad \Sigma_{j=\ell+1}^m p_j x_j^* = M, \quad \lambda > 0.$$

Wales and Woodland use the inequalities,

$$(3.2)' \qquad \frac{\partial U(x^*)}{\partial x_i} - \lambda p_i \leq 0, \qquad i = 1,\ldots,\ell$$

and equations (3.3) and (3.4) to determine the choice of regime. Under some stochastic utility function specifications they derive the likelihood function for their model. The inequalities (3.2)′ do not have much intuitive content. Below, we demonstrate that the use of virtual prices provides a simple, intuitive interpretation of regime criteria and a deeper insight into the problem.

Virtual prices are simply those prices which support a vector of demands. Neary and Roberts have shown that if the preference function is strictly quasi-concave, continuous and strictly monotonic, any allocation can be supported with virtual prices. Strict monotonicity also implies that support prices will be strictly positive. The virtual price for good i, ξ_i, at x_i^* can be defined as

$$(3.5)\qquad \begin{aligned} \xi_i &= \frac{\partial U(x^*)}{\partial x_i} / \lambda \\ &= p_m \frac{\partial U(x^*)}{\partial x_i} / \frac{\partial U(x^*)}{\partial x_m}, \qquad i = 1,\ldots,\ell. \end{aligned}$$

For the remaining goods, the virtual price ξ_j at x^* is the observed price p_j for $j = \ell+1,\ldots,m$. Hence, the Kuhn-Tucker conditions (3.2) are equivalent to the inequalities

$$(3.2)'' \qquad \xi_i \leq p_i \qquad i = 1,\ldots,\ell$$

which compares the virtual prices of the nonconsumed goods with their corresponding market prices. The virtual price vector ξ, $\xi' = (\xi_1,\ldots,\xi_m)$ with income M, supports the quantity vector x^*, which is the solution to the unconstrained problem $\max_x \{U(x) | \xi' x = M\}$ without non-negativity constraints. Thus, virtual prices are shadow prices. The goods i, $i = 1,\ldots,\ell$, are not consumed because their market prices exceed their corresponding shadow prices.

Before analyzing the problem (2.1), it is instructive to consider the two-goods case with a single kink point and without any binding non-negativity constraints. Consider the regime for which the optimum occurs at the kink point $\bar{x}$, $\bar{x} = (x_1(1), \bar{x}_2)$ where $\bar{x}_2 = (M - p_{11}x_1(1))/p_2$ (see Figure 1). Consider the utility maximization problem,

$$\max_{x_1,\, x_2} \; U(x_1, x_2)$$

subject to

$$(3.6)\qquad \begin{aligned} p_{11}x_1 + p_2x_2 &\le M; \\ p_{12}x_1 + p_2x_2 &\le M_1 \end{aligned}$$

where $M_1 = M + (p_{12} - p_{11})x_1(1)$. The Lagrangean function is

$$L = U(x_1, x_2) + \lambda_1(M - p_{11}x_1 - p_2x_2) + \lambda_2(M_1 - p_{12}x_1 - p_2x_2).$$

The Kuhn-Tucker conditions that characterize the optimum regime at the kink point $\bar{x}$ are

$$(3.7)\qquad \frac{\partial U(\bar{x})}{\partial x_1} - \lambda_1 p_{11} - \lambda_2 p_{12} = 0,$$

$$(3.8)\qquad \frac{\partial U(\bar{x})}{\partial x_2} - (\lambda_1 + \lambda_2)p_2 = 0,$$

$$(3.9)\qquad p_{11}\bar{x}_1 + p_2\bar{x}_2 = M, \quad \lambda_1 \ge 0,$$

$$(3.10)\qquad p_{12}\bar{x}_1 + p_2\bar{x}_2 = M_1, \quad \lambda_2 \ge 0.$$

These conditions are not of direct use since these criteria are expressed as equalities. Define a variable ξ_1 as

$$(3.11)\qquad \xi_1 = \frac{1}{\lambda_1 + \lambda_2}\frac{\partial U(\bar{x})}{\partial x_1}$$

where λ_1 and λ_2 are solutions from (3.7)-(3.10). Equivalently,

$$(3.11)'\qquad \xi_1 = p_2\frac{\partial U(\bar{x})}{\partial x_1} \Big/ \frac{\partial U(\bar{x})}{\partial x_2}.$$

It follows that equation (3.7) can be rewritten as

$$(3.7)' \qquad \xi_1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} p_{11} - \frac{\lambda_2}{\lambda_1 + \lambda_2} p_{12} = 0.$$

Equation (3.7)′ implies that

$$(3.12) \qquad \xi_1 - p_{11} = \frac{\lambda_2}{\lambda_1 + \lambda_2} (p_{12} - p_{11})$$

and

$$(3.13) \qquad \xi_1 - p_{12} = \frac{\lambda_1}{\lambda_1 + \lambda_2} (p_{11} - p_{12}).$$

As $p_{12} > p_{11}$, we have $\xi_1 \geq p_{11}$ and $p_{12} \geq \xi_1$. Thus, if the optimum occurs at the kink point, it is necessary that $p_{12} \geq \xi_1 \geq p_{11}$. That these inequalities are sufficient conditions can be shown as follows. Define ω as

$$(3.14) \qquad \omega = \frac{\partial U(\bar{x})}{\partial x_2} / p_2.$$

Since $p_{12} \geq \xi_1 \geq p_{11}$, where $\xi_1 = p_2 \frac{\partial U(\bar{x})}{\partial x_1} / \frac{\partial U(\bar{x})}{\partial x_2} = \frac{\partial U(\bar{x})}{\partial x_1} / \omega$, there exists a $\mu \in [0, 1]$ such that $\xi_1 = \mu p_{12} + (1 - \mu)p_{11}$. Define ω_1 and ω_2 as

$$(3.15) \qquad \omega_1 = \mu\omega$$

and

$$(3.16) \qquad \omega_2 = (1 - \mu)\omega.$$

Obviously, $\omega_1 \geq 0$, $\omega_2 \geq 0$ and $\omega = \omega_1 + \omega_2$. Thus we have

$$\frac{\partial U(\bar{x})}{\partial x_1} - \omega_1 p_{12} - \omega_2 p_{11} = 0$$

$$\frac{\partial U(\bar{x})}{\partial x_2} - (\omega_1 + \omega_2) p_2 = 0$$

$$p_{11} x_1(1) + p_2 \bar{x}_2 = M, \quad \omega_1 \geq 0$$

$$p_{12}x_1(1) + p_2\bar{x}_2 = M_1, \quad \omega_2 \geq 0$$

which are the Kuhn-Tucker conditions which characterize $\bar{x}$ as the solution. Hence, it can be concluded that the regime criteria which determine this regime are the inequalities

(3.17) $$p_{11} \leq \xi_1 \leq p_{12}.$$

From the constructions in (3.14) and (3.11), we have essentially that

(3.18) $$\frac{\partial U(\bar{x})}{\partial x_2} - \omega p_2 = 0$$

(3.19) $$\frac{\partial U(\bar{x})}{\partial x_1} - \omega \xi_1 = 0.$$

Define an "income" C as

(3.20) $$C = M + (\xi_1 - p_{11})x_1(1).$$

It follows that

(3.21) $$\xi_1 x_1(1) + p_2\bar{x}_2 = C.$$

Thus the plane $\{(x_1, x_2)|\xi_1 x_1 + p_2 x_2 = C\}$ is tangent to the indifferent curve at the kink point $\bar{x}$, which is point B in Figure 1, and supports this kink point as the solution given the price vector (ξ_1, p_2) and income C. The price ξ_1 is the virtual price for good 1 at the quantity $x_1(1)$, and C is the corresponding virtual income. The kink point is optimal because its virtual (shadow) price is greater than the first block price but less than the second.

The comparison of virtual prices with market prices can select among regimes in the general problem of the consumer with a convex budget set (2.1, 2.2). We state the following results with detailed proof omitted:

Theorem 1. Let $x^* = (x_1^*,\ldots,x_m^*)$ be the demanded quantity vector. Consider the general regime with the form:

$$x_1^* = x_2^* = \ldots = x_{\ell_1-1}^* = 0;$$

$$(3.22)\qquad x_{\ell_1}^* = x_{\ell_1}(i_{\ell_1}^\circ),\ x_{\ell_1+1}^* = x_{\ell_1+1}(i_{\ell_1+1}^\circ),\ldots,x_{\ell_2-1}^* = x_{\ell_2-1}(i_{\ell_2-1}^\circ);$$

$$x_{\ell_2}(i_{\ell_2}^\circ-1) < x_{\ell_2}^* < x_{\ell_2}(i_{\ell_2}^\circ),\ldots,x_m(i_m^\circ - 1) < x_m^* < x_m(i_m^\circ)$$

where $0 \le \ell_1 \le \ell_2 \le m$ and for some $i_{\ell_1}^\circ,\ i_{\ell_1+1}^\circ,\ldots,i_m^\circ$. The necessary and sufficient conditions for this regime's occurrence are:

$$p_{j1} \ge \xi_j(x^*),\qquad j = 1,\ 2,\ldots,\ell_1-1;$$

$$(3.23)\qquad p_{ji_j^\circ} \le \xi_j(x^*) \le p_{j(i_j^\circ+1)},\qquad j = \ell_1,\ \ell_1+1,\ldots,\ell_2-1;$$

$$x_j(i_j^\circ-1) < x_j^* < x_j(i_j^\circ),\qquad j = \ell_2,\ \ell_2+1,\ldots,m,$$

where $\xi_j(x^*)$ is the virtual price of good j at the optimum point x^*.

In principle, virtual prices can be constructed from either the direct or indirect utility functions. Consider the case where ℓ goods are rationed at the quantities x_i°, $i = 1,\ldots,\ell$. With a utility function $U(x_1,\ldots,x_m)$, a price vector p and income M, the constrained utility maximization problem is

$$(3.24)\qquad \max_x U(x_1,\ldots,x_\ell,\ x_{\ell+1},\ldots,x_m)$$

subject to $\quad x_i = x_i^\circ,\qquad i = 1,\ldots,\ell$

and $\quad p'x = M.$

Implicitly, it is assumed that $M > \Sigma_{i=1}^{\ell}\, p_i x_i^\circ$ so that the problem is well posed. The Lagrangean function is

$$L = U(x) + \lambda(M - p'x) + \Sigma_{i=1}^{\ell} \eta_i(x_i^\circ - x_i)$$

where λ and the η's are Lagrangean multipliers. The solution x^* of (3.24) satisfies the first-order conditions:

$$\text{(3.25)} \quad \frac{\partial U(x^*)}{\partial x_i} - \lambda p_i - \eta_i = 0, \quad x_i^* = x_i^\circ, \qquad i = 1,\ldots,\ell$$

$$\text{(3.26)} \quad \frac{\partial U(x^*)}{\partial x_j} - \lambda p_j = 0, \qquad j = \ell+1,\ldots,m$$

$$\text{(3.27)} \quad p'x^* = M.$$

The demanded quantities x_j^*, $j = \ell+1,\ldots,m$, of the unrationed goods can be solved from (3.26) and (3.27) conditional on the rationed goods at x_i°, $i = 1,\ldots,\ell$;

$$\text{(3.28)} \quad x_j^* = D_j(p_{\ell+1},\ldots,p_m, M - \Sigma_{i=1}^{\ell} p_i x_i^\circ | x_1^\circ,\ldots,x_\ell^\circ) \qquad j = \ell+1,\ldots,m$$

The equations (3.28) are the <u>conditional demand equations</u> for the unrationed goods. The virtual prices ξ_j, $j = 1,\ldots,\ell$ of the rationed goods at $x_1^\circ,\ldots,x_\ell^\circ$ are

$$\text{(3.29)} \quad \begin{aligned} \xi_j(x^*) &= \frac{1}{\lambda}\frac{\partial U(x^*)}{\partial x_j} \\ &= p_m \frac{\partial U(x_1^\circ,\ldots,x_\ell^\circ, x_{\ell+1}^*,\ldots,x_m^*)}{\partial x_j} \Big/ \frac{\partial U(x_1^\circ,\ldots,x_\ell^\circ, x_{\ell+1}^*,\ldots,x_m^*)}{\partial x_m}. \end{aligned}$$

for $j = \ell+1,\ldots,m$. Substituting the conditional demand equations (3.28) into (3.29), the virtual prices can be written as functions of the observed prices for the unrationed goods, income and the rationed quantities:

$$\text{(3.30)} \quad \xi_j(x^*) = \xi_j(p_{\ell+1},\ldots,p_m, M_R; x_1^\circ,\ldots,x_\ell^\circ) \qquad j = 1,\ldots,\ell$$

where M_R, $M_R = M - \Sigma_{i=1}^{\ell} p_i x_i^\circ$, is the income remaining for expenditure on unrationed goods. The virtual prices and the conditional demand equations as functions of p, M and the kink points provide conditions for regime occurrence as in the conditions (3.23).

Virtual prices for the rationed goods can also be derived from unconditional (notional) demand equations. The unconditional (notional) demand functions $D_i(p, M)$, $i = 1,\ldots,m$, are the solutions to the unconstrained utility maximization problem $\max \{U(x)|p'x = M\}$. With the goods i rationed at quantities x_i°, $i = 1, 2,\ldots,\ell$, the virtual prices which support the commodity vector $x^* = (x_1^\circ,\ldots,x_\ell^\circ, x_{\ell+1}^*,\ldots,x_m^*)$ as an unconstrained utility maximum are characterized by the demand relations:

$$(3.31) \qquad x_i^\circ = D_i(\xi_1, \xi_2,\ldots, \xi_\ell, p_{\ell+1},\ldots, p_m, c), \qquad i = 1,\ldots,\ell;$$

$$(3.32) \qquad x_j^* = D_j(\xi_1, \xi_2,\ldots, \xi_\ell, p_{\ell+1},\ldots, p_m, c), \qquad j = \ell+1,\ldots,m$$

where

$$(3.33) \qquad c = M + \Sigma_{i=1}^{\ell} (\xi_i - p_i)x_i^\circ$$

is the virtual income, and $\Sigma_{i=1}^{\ell} \xi_i x_i^\circ + \Sigma_{j=\ell+1}^{m} p_j x_j^* = c$ is the corresponding budget tangent plane. The virtual prices ξ_i of the rationed goods are solutions to the equations (3.31) to (3.33). The virtual prices for the unrationed goods are the market prices (Neary and Roberts). The virtual price approach allows for a wide choice of functional forms because it does not necessarily require the specification of the direct utility function. As we demonstrate below, the use of an indirect utility function representation of preference is also attractive.

The demanded quantities x_j^*, $j = \ell+1,\ldots,m$ for the unrationed goods satisfy the conditional demand equations (3.28). With the introduction of virtual prices ξ_i, $i = 1,\ldots,\ell$ for the rationed goods, x_j^* satisfies the unconditional demand equations,

$$(3.34) \qquad x_j^* = D_j(\xi_1, \xi_2,\ldots, \xi_\ell, p_{\ell+1},\ldots, p_m, M_R + \Sigma_{i=1}^{\ell} \xi_i x_i^\circ), \qquad j = \ell+1,\ldots,m$$

where $M_R = M - \Sigma_{i=1}^{\ell} p_i x_i^\circ$. By substituting the virtual price equations (3.30) into (3.34), one obtains the conditional demand equations (3.28). Thus the conditional demand equations can be derived from the unconditional demand equations via the virtual prices and vice-versa.[5/] As a function of prices $p_{\ell+1},\ldots,p_m$, remaining income M_R and the rationed quantities x_i°, $i = 1,\ldots,\ell$, the demand equations (3.34) can be interpreted as conditional demand equations conditional on x_i°, $i = 1,\ldots,\ell$.

4. Econometric Model Specification: Binding Non-Negative Constraints Case.

Estimation of the notional demand equations requires the specification of a functional form with a finite number of unknown parameters plus stochastic components. These components reflect random preferences or other unexplained factors. Let θ be the vector of unknown parameters and ε the vector of random components. The stochastic notional demand equations are

$$(4.1) \qquad q_i = D_i(v;\, \theta,\, \varepsilon)$$

where $v = p/M$ is a vector of normalized prices. These demand equations can be derived either by maximizing the direct utility function subject to the budget constraint as in (2.3) or from an indirect utility function through Roy's Identity. Let $H(v;\, \theta,\, \varepsilon)$ be an indirect utility function defined as

$$(4.2) \qquad H(v;\, \theta,\, \varepsilon) = \max_q \{U(q;\, \theta,\, \varepsilon) \,|\, vq = 1\}.$$

Applying Roy's Identity, the notional demand equations are

$$(4.3) \qquad q_i = \frac{\partial H(v;\, \theta,\, \varepsilon)}{\partial v_i} \Big/ \Sigma_{j=1}^{K}\, v_j\, \frac{\partial H(v;\, \theta,\, \varepsilon)}{\partial v_j} \qquad i = 1,\ldots,K.$$

In the analysis of quantity rationing, Deaton (1981) has noted that it may be difficult to analytically derive virtual price functions from most flexible function forms for the indirect utility function. For the case of binding non-negativity constraints, all of the restricted demands are zero rather than positive as in the rationing literature. With zero restricted demands, the derivation of virtual prices is considerably simplified as the denominator in Roy's Identity (4.3) drops out of the virtual price functions. If demands for the first L goods are zero, the virtual prices i are solved from the equations

$$(4.4) \qquad 0 = \partial H(\xi_1,\ldots,\, \xi_L,\, \bar{v};\, \theta,\, \varepsilon)/\partial v_i \qquad i = 1,\ldots,L$$

and the remaining (positive) demands are

$$(4.5) \qquad x_i = \frac{\partial H(\xi_1,\ldots,\xi_L,\bar{v};\theta,\varepsilon)}{\partial v_i} \Big/ \Sigma_{j=1}^K v_j \frac{\partial H(\xi_1,\ldots,\xi_L,\bar{v};\theta,\varepsilon)}{\partial v_j},$$

$$i = L+1,\ldots,K.$$

As an illustration of the notional demand approach, consider the translog indirect utility function of Christensen, Jorgenson and Lau (1975),

$$(4.6) \qquad H(v;\theta,\varepsilon) = \Sigma_{i=1}^K \alpha_i \ln v_i + \frac{1}{2}\Sigma_{i=1}^K \Sigma_{j=1}^K \beta_{ij} \ln v_i \ln v_j + \Sigma_{i=1}^K \varepsilon_i \ln v_i$$

where ε is a K-dimensional vector of normal variables $N(0, \Sigma)$.6/ A convenient normalization is $\Sigma_{i=1}^K \alpha_i = -1$ and $\Sigma_{i=1}^K \varepsilon_i = 0$.7/ The notional share equations derived from Roy's Identity are

$$(4.7) \qquad v_i q_i = \frac{\alpha_i + \Sigma_{j=1}^K \beta_{ij} \ln v_j + \varepsilon_i}{D} \qquad i = 1,\ldots,K$$

where $D = -1 + \Sigma_{i=1}^K \Sigma_{j=1}^K \beta_{ij} \ln v_j$. Consider the regime for which the quantity demanded for one of the goods is zero and positive for all others, i.e., $x_1 = 0$, $x_2 > 0,\ldots, x_K > 0$. The virtual price ξ_1 as a function of $v_2,\ldots,v_K$, is

$$\ln\xi_1 = -(\alpha_1 + \Sigma_{j=2}^K \beta_{1j} \ln v_j + \varepsilon_1)/\beta_{11}.$$

The remaining positive share equations are

$$(4.8) \qquad v_i x_i = \frac{\alpha_i - \alpha_1 \frac{\beta_{i1}}{\beta_{11}} + \Sigma_{j=2}^K (\beta_{ij} - \beta_{1j}\frac{\beta_{i1}}{\beta_{11}})\ln v_j + \varepsilon_i - \frac{\beta_{i1}}{\beta_{11}}\varepsilon_1}{\Sigma_{j=2}^K (\beta_{\cdot j} - \beta_{\cdot 1}\frac{\beta_{1j}}{\beta_{11}})\ln v_j - (1 + \frac{\alpha_1}{\beta_{11}}\beta_{\cdot 1}) - \frac{\beta_{\cdot 1}}{\beta_{11}}\varepsilon_1},$$

$$i = 2,\ldots,K$$

where $\beta_{\cdot j} = \Sigma_{i=1}^K \beta_{ij}$. Note from the above equations that ε_i can be expressed as functions of x_i and ε_1. The switching conditions for this demand regime are

$$\varepsilon_1 \geq -(\alpha_1 + \Sigma_{j=1}^{K} \beta_{1j} \ln v_j)$$

and $x_i > 0$, $i = 2,\ldots,K$.

Let $f(\varepsilon_1)$ be the density function of ε_1 and $g(\varepsilon_2,\ldots, \varepsilon_{K-1}|\varepsilon_1)$ be the conditional density function, conditional on ε_1. The Jacobian transformation $J_1(x, \varepsilon_1)$ from $(\varepsilon_2,\ldots, \varepsilon_{K-1})$ to $(x_2,\ldots, x_{K-1})$, which can be derived from (4.8), is a function of x and ε_1. The likelihood function for this demand regime for one observation is

$$\int_{-(\alpha_1 + \Sigma_{j=1}^{K} \beta_{1j} \ln v_j)}^{\infty} J_1(x, \varepsilon_1) g(\varepsilon_2,\ldots, \varepsilon_{K-1}|\varepsilon_1) f(\varepsilon_1) d\varepsilon_1$$

where ε_i, $i = 2,\ldots,K-1$ are functions of x and ε_1 from (4.8). Now consider the demand regime in which the demands for the first two commodities are zero and all remaining demands are positive. The virtual prices ξ_1 and ξ_2 as functions of $v_3,\ldots,v_K$ are

$$\begin{pmatrix} \ln\xi_1 \\ \ln\xi_2 \end{pmatrix} = -B^{-1} \begin{pmatrix} \alpha_1 + \Sigma_{j=3}^{K} \beta_{1j} \ln v_j \\ \alpha_2 + \Sigma_{j=3}^{K} \beta_{2j} \ln v_j \end{pmatrix} - B^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

where $B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$. The remaining positive shares are

$$(4.9) \qquad v_i x_i = \frac{\alpha_i + \beta_{i1} \ln\xi_1 + \beta_{i2} \ln\xi_2 + \Sigma_{j=3}^{K} \beta_{ij} \ln v_j + \varepsilon_i}{-1 + \beta_{\cdot 1} \ln\xi_1 + \beta_{\cdot 2} \ln\xi_2 + \Sigma_{j=3}^{K} \beta_{\cdot j} \ln v_j}, \qquad i = 3,\ldots,K.$$

The ε_i, $i = 3,\ldots,K$, can be expressed (from (4.9)), as functions of x, ε, and ε_2. The regime switching conditions are

$$B^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \geq -\begin{pmatrix} \ln v_1 \\ \ln v_2 \end{pmatrix} - B^{-1} \begin{pmatrix} \alpha_1 + \Sigma_{j=3}^{K} \beta_{1j} \ln v_j \\ \alpha_2 + \Sigma_{j=3}^{K} \beta_{2j} \ln v_j \end{pmatrix}.$$

Let $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = B^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$. Furthermore, let $g(\varepsilon_3,\ldots, \varepsilon_{K-1}|\eta_1, \eta_2)$ be the conditional density function of $(\varepsilon_3, \ldots, \varepsilon_{K-1})$, conditional on η_1 and η_2, and $f(\eta_1, \eta_2)$ be the marginal density of η_1 and η_2. The Jacobian transformation $J_2(x, \eta_1, \eta_2)$ from $(\varepsilon_3, \ldots, \varepsilon_{K-1})$ to $(x_3, \ldots, x_{K-1})$ can be derived from (4.9) and is a function of x and η_1, η_2. The likelihood function for this regime for one observation is

$$\int_{s_2}^{\infty} \int_{s_1}^{\infty} J_2(x, \eta_1, \eta_2) g(\varepsilon_3, \ldots, \varepsilon_{K-1} | \eta_1, \eta_2) f(\eta_1, \eta_2) d\eta_1 d\eta_2$$

where $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = - \begin{pmatrix} \ln v_1 \\ \ln v_2 \end{pmatrix} - B^{-1} \begin{pmatrix} \alpha_1 + \Sigma_{j=3}^{K} \beta_{1j} \ln v_j \\ \alpha_2 + \Sigma_{j=3}^{K} \beta_{2j} \ln v_j \end{pmatrix}$ and ε's are functions of x and η_1, η_2 derived from (4.9). The likelihood function for other regimes can similarly be derived.

Let $I_i(c)$ be a dichotomous indicator such that $I_i(c) = 1$ if the observed consumption pattern for individual i is the demand regime c, zero otherwise. Let $\ell_c(x_i; \theta)$ denote the likelihood function for regime c for sample i. The likelihood function for an independent sample with N observations is

$$L = \Pi_{i=1}^{N} \Pi_{c} [\ell_c(x_i; \theta)]^{I_i(c)}$$

5. The Likelihood Function of the General Demand System.

Consider a general (notional) demand system with m goods

$$x_i = D_i(p_1, p_2, \ldots, p_m, M; \theta) + \varepsilon_i, \qquad i = 1, \ldots, m \tag{5.1}$$

where θ is a vector of unknown parameters and ε_i is the disturbance with zero mean. The budget constraint implies that $\Sigma_{i=1}^m p_i\varepsilon_i = 0$. The disturbances ε_i, $i = 1, \ldots, m$ are correlated and are heteroscedastic. Let $x^* = (x_1^*, \ldots, x_m^*)$ be the observed demand quantity vector. Without loss of generality, consider the general regime in Theorem 1. The criteria for the determination of this regime are the conditions (3.23). The virtual prices $\xi_1, \xi_2, \ldots, \xi_{\ell_2-1}$ are determined by the following equations:

$$(5.2)\quad \begin{aligned} 0 &= D_j(\xi_1, \xi_2, \ldots, \xi_{\ell_2-1}, p_{\ell_2}, \ldots, p_m, M_R + \Sigma_{k=\ell_1}^{\ell_2-1} \xi_k x_k(i_k^\circ); \theta) + \varepsilon_j, \\ &\qquad j = 1, \ldots, \ell_1-1; \\ x_j(i_j^\circ) &= D_j(\xi_1, \xi_2, \ldots, \xi_{\ell_2-1}, p_{\ell_2}, \ldots, p_m, M_R + \Sigma_{k=\ell_1}^{\ell_2-1} \xi_k x_k(i_k^\circ); \theta) + \varepsilon_j, \\ &\qquad j = \ell_1, \ldots, \ell_2-1 \end{aligned}$$

where $M_R = M_{1\ldots1 i_{\ell_1}^\circ \ldots i_{\ell_2-1}^\circ i_{\ell_2}^\circ \ldots i_m^\circ} - \Sigma_{k=\ell_1}^{\ell_2-1} p_k x_k(i_k^\circ)$. The remaining demanded quantities are

$$(5.3)\quad \begin{aligned} x_j^* &= D_j(\xi_1, \xi_2, \ldots, \xi_{\ell_2-1}, p_{\ell_2}, \ldots, p_m, M_R + \Sigma_{k=\ell_1}^{\ell_2-1} \xi_k x_k(i_k^\circ); \theta) + \varepsilon_j, \\ &\qquad j = \ell_2, \ell_2+1, \ldots, m. \end{aligned}$$

These equations provide an implicit function from the disturbance vector $(\varepsilon_1, \ldots, \varepsilon_{m-1})$ to the vector $(\xi_1, \xi_2, \ldots, \xi_{\ell_2-1}, x_{\ell_2}^*, x_{\ell_2+1}^*, \ldots, x_{m-1}^*)$. As $\Sigma_{i=1}^m p_i\varepsilon_i = 0$, the equation x_m^* is functionally dependent on the other equations and is redundant. A specified joint density function for $(\varepsilon_1, \ldots, \varepsilon_{m-1})$ implies a joint density function for $(\xi_1, \ldots, \xi_{\ell_2-1}, x^*_{\ell_2}, x^*_{\ell_2+1}, \ldots, x^*_{m-1})$, which can be derived straightforwardly since the Jacobian matrix is easily derived from (5.2) and (5.3). Let $f(\xi_1, \ldots, \xi_{\ell_2-1}, x^*_{\ell_2}, x^*_{\ell_2+1}, \ldots, x^*_{m-1})$ denote the implied joint density function. It follows that the contribution of this regime to the likelihood function for an observation is

$$\int_{p_{\ell_2-1} i^\circ_{\ell_2-1}}^{p_{\ell_2-1}(i^\circ_{\ell_2-1}+1)} \cdots \int_{p_{\ell_1} i^\circ_{\ell_1}}^{p_{\ell_1}(i^\circ_{\ell_1}+1)} \int_{-\infty}^{p_{\ell_1-1,1}} \cdots \int_{-\infty}^{p_{11}} f(\xi_1, \xi_2, \ldots, \xi_{\ell_2-1}, x^*_{\ell_2}, x^*_{\ell_2+1}, \ldots, x^*_{m-1}) \, d\xi_1 d\xi_2 \ldots d\xi_{\ell_2-1}.$$

The evaluation of the likelihood function may be cumbersome and expensive if there are integrals with dimensions more than two. It is an open question whether computationally simple estimation methods can be derived.

6. The Firm's Problems.

Our analyses, which have focused on consumer demand models, can be extended to the analyses of kink points in production economics. The firm's problem differs from that of the consumer in the constraints of the primal problem and the observability of profit but not utility. Kink points may occur because of binding non-negativity constraints or inputs or outputs in a multiple input or output technology or as a result of rationing or block pricing.

Consider the profit maximization problem subject to quantities constraints:

$$\max_{x,\, q} \; p'q - r'x \tag{6.1}$$

$$\text{subject to} \quad F(q, x) = 0, \quad \bar{q} \geq q \geq 0, \quad \bar{x} \geq x \geq 0$$

where x and q are $k \times 1$ and $M \times 1$ vectors of inputs and outputs respectively, and $\bar{x}$ and $\bar{q}$ are the upper quantity limits. The production function F is an increasing function of q's and a decreasing function of x's. Other standard regularity conditions on F such as differentiability and strict quasi-concavity are assumed. To illustrate the construction of virtual prices from the production technology F, let us consider a rather simple regime with $x^* = (0, x_2^*, \ldots, x_K^*)'$ and $q^* = (\bar{q}_1, q_2^*, \ldots, q_M^*)'$ where the first input is not utilized and the first output is produced at the quota level. The Lagrangean function is

$$L = p'q - r'x + \lambda(0 - F(q, x)) + \phi'q + \psi'x + \delta'(\bar{q} - q) + \omega'(\bar{x} - x)$$

where ϕ, ψ, δ and ω are vectors of Lagrangean multipliers. This regime is characterized by the following Kuhn-Tucker conditions:

$$-r_1 - \lambda\frac{\partial F(q^*, x^*)}{\partial x_1} + \psi_1 = 0, \quad \psi_1 \geq 0;$$

$$-r_i - \lambda\frac{\partial F(q^*, x^*)}{\partial x_i} = 0, \qquad i = 2, \ldots, K;$$

$$p_1 - \lambda\frac{\partial F(q^*, x^*)}{\partial q_1} - \delta_1 = 0, \quad \delta_1 \geq 0;$$

$$p_j - \lambda\frac{\partial F(q^*, x^*)}{\partial q_j} = 0, \qquad j = 2,\ldots,M.$$

$$F(q^*, x^*) = 0, \quad q^* \geq 0, \quad x^* \geq 0.$$

Define the virtual price ξ_{d1} for input 1 and virtual price ξ_{s1} for output 1 at $(x_1 = 0, \bar{q}_1)$ as

$$\xi_{d1} = -\lambda \frac{\partial F(q^*, x^*)}{\partial x_1}$$

and

$$\xi_{s1} = \lambda\frac{\partial F(q^*, x^*)}{\partial q_1}.$$

Since $\frac{\partial F(q^*, x^*)}{\partial x_1} < 0$ and $\frac{\partial F(q^*, x^*)}{\partial q_1} > 0$, ξ_{d1} and ξ_{s1} are strictly positive. It follows that $\psi_1 = r_1 - \xi_{d1}$ and $\delta_1 = p_1 - \xi_{s1}$. Therefore this regime is characterized by

$$r_1 \geq \xi_{d1}, \quad 0 < x_i^* < \bar{x}_i, \qquad i = 2,\ldots,K$$

and $$p_1 \geq \xi_{s1}, \quad 0 < q_j^* < \bar{q}_j, \qquad j = 2,\ldots,M.$$

Input 1 is not used because the market price is too high and output 1 is produced up to the quota limit because the virtual price is not. This technique can be easily generalized to other regimes.

The case of increasing block prices in inputs can be reformulated into the framework (6.1). Consider the simple case of a single output x with production function $q = f(x)$. Assume the price of input x is r_1 if the purchased amount is not greater than $x_1(1)$ but a higher price r_2 for amounts in excess of $x_1(1)$. Hence the cost $c(x)$ is

$$c(x) = r_1 x, \qquad , \text{ if } x \le x_1(1);$$

$$= r_1 x_1(1) + r_2(x - x_1(1)), \text{ if } x > x_1(1).$$

The problem $\max_x \{pq - c(x) | q = f(x),\ x \ge 0\}$ can be rewritten into an identical problem with two perfectly substitutable inputs:

$$\max_{x_1,\, x_2} \ pq - r_1x_1 - r_2x_2$$

$$\text{subject to } q = f(x_1 + x_2),\ 0 \le x_1 \le x_1(1),\ x_2 \ge 0.$$

As the price of x_1 is less than x_2, x_1 will always be purchased first. x_2 will be purchased only if x_1 has been purchased up to its upper limit $x_1(1)$. $x_1(1)$ is a kink point in this model.

Similarly, the decreasing block prices in outputs can also be formulated in the framework (6.1). Consider a single output case that the output quantity q can be sold at price p_1 if the quantity is not greater than the quota amount $q(1)$; however, quantities in excess of $q(1)$ can only be sold at a lower price p_2. The revenue function will be

$$R(q) = p_1 q \qquad , \text{ if } q \le q(1);$$

$$= p_1 q(1) + p_2(q - q(1)), \text{ if } q > q(1).$$

The profit maximization problem $\max_{x,\, q} \{R(q) - rx | q = f(x)\}$ can be rewritten indentically as a model with two perfectly substitutable outputs:

$$\max_{q_1,\, q_2,\, x} \ p_1q_1 + p_2q_2 - rx$$

$$\text{subject to } q_1 + q_2 = f(x),\ 0 \le q_1 \le q(1),\ q_2 \ge 0.$$

The quantity $q(1)$ is a kink point in this model.

Instead of the direct approach of profit maximization with a specified production technology, one can also consider dual approaches through the specification of profit or cost functions. Application of Shephard's lemma or the Hotelling-McFadden lemma (see, e.g., McFadden [1978]) provides (notional) input demand and output supply functions. Virtual prices for the kink points can be solved directly from these functions. In the next two sections, the specification and estimation of a translog profit and cost functions is described in order to clearly illustrate this approach.

7. Econometric Models With Translog Profit and Cost Functions: Corner Solution Cases.

The variable translog profit function with fixed factors $z = (z_1,\ldots,z_L)'$ is

$$(7.1)\quad \begin{aligned} \ln\pi &= \alpha_{00} + \Sigma_{i=1}^{K+M} \alpha_{i0} \ln v_i + \frac{1}{2} \Sigma_{i=1}^{K+M} \Sigma_{j=1}^{K+M} \alpha_{ij} \ln v_i \ln v_j + \Sigma_{\ell=1}^{L} \gamma_{\ell 0} \ln z_\ell \\ &+ \frac{1}{2} \Sigma_{\ell=1}^{L} \Sigma_{j=1}^{L} \gamma_{\ell j} \ln z_\ell \ln z_j + \Sigma_{i=1}^{K+M} \Sigma_{\ell=1}^{L} \beta_{i\ell} \ln v_i \ln z_\ell \end{aligned}$$

where $v_i = r_i$, $i = 1,\ldots,K$ and $v_i = p_{i-K}$ for $i = K+1,\ldots, M+K$. For computational tractability, random elements are introduced in the linear terms, as follows

$$(7.2)\quad \begin{aligned} \ln\pi &= \alpha_{00} + \Sigma_{i=1}^{K+M} (\alpha_{i0} + \varepsilon_i)\ln v_i + \frac{1}{2} \Sigma_{i=1}^{K+M} \Sigma_{j=1}^{K+M} \alpha_{ij} \ln v_i \ln v_j \\ &+ \Sigma_{\ell=1}^{L} \gamma_{\ell 0} \ln z_\ell + \frac{1}{2} \Sigma_{\ell=1}^{L} \Sigma_{j=1}^{L} \gamma_{\ell j} \ln z_\ell \ln z_j + \Sigma_{i=1}^{K+M} \Sigma_{\ell=1}^{L} \beta_{i\ell} \ln v_i \ln z_\ell \\ &+ \varepsilon_0 \end{aligned}$$

where $\varepsilon = (\varepsilon_0, \varepsilon_1,\ldots, \varepsilon_{K+M})$ is assumed to be multivariate normal $N(0, \Omega)$. The Hotelling-McFadden lemma implies that the notional cost shares of profit are

$$(7.3)\quad -\frac{r_i x_i}{\pi} = \alpha_{i0} + \Sigma_{j=1}^{K+M} \alpha_{ij} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{i\ell} \ln z_\ell + \varepsilon_i \qquad i = 1,\ldots,K$$

and the notional revenue shares of profit are

$$(7.4)\quad \frac{p_i q_i}{\pi} = \alpha_{i+K0} + \Sigma_{j=1}^{K+M} \alpha_{i+K,j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{i+K,\ell} \ln z_\ell + \varepsilon_{i+K} \qquad i = 1,\ldots,M.$$

Virtual prices corresponding to zero inputs and outputs can be solved from the share equations. Without loss of generality, consider the regime where the first two inputs are zero. The corresponding virtual prices are v_1^* and v_2^*, defined by the following equations:

$$0 = \alpha_{10} + \alpha_{11} \ln v_1^* + \alpha_{12} \ln v_2^* + \Sigma_{j=3}^{K+M} \alpha_{1j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{1\ell} \ln z_\ell + \varepsilon_1$$

$$0 = \alpha_{20} + \alpha_{21} \ln v_1^* + \alpha_{22} \ln v_2^* + \Sigma_{j=3}^{K+M} \alpha_{2j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{2\ell} \ln z_\ell + \varepsilon_2.$$

The solutions are linear in ε_1 and ε_2:

$$\begin{pmatrix} \ln v_1^* \\ \ln v_2^* \end{pmatrix} = -\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{10} + \Sigma_{j=3}^{K+M} \alpha_{1j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{1\ell} \ln z_\ell + \varepsilon_1 \\ \alpha_{20} + \Sigma_{j=3}^{K+M} \alpha_{2j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{2\ell} \ln z_\ell + \varepsilon_2 \end{pmatrix}.$$

The remaining non-zero share equations are:

$$(7.5) \qquad \frac{v_i \bar{s}_i}{\bar{\pi}} = \alpha_{i0} - (\alpha_{i1}\ \alpha_{i2}) \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{10} + \Sigma_{j=3}^{K+M} \alpha_{1j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{1\ell} \ln z_\ell + \varepsilon_1 \\ \alpha_{20} + \Sigma_{j=3}^{K+M} \alpha_{2j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{2\ell} \ln z_\ell + \varepsilon_2 \end{pmatrix}$$
$$+ \Sigma_{j=3}^{K+M} \alpha_{ij} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{i\ell} \ln z_\ell + \varepsilon_i \qquad i = 3,\ldots,K+M$$

where $\bar{s} = (-\bar{x}', \bar{q}')'$ and are linear in ε. The conditional profit function for this regime is

$$\ln\bar{\pi} = \alpha_{00} + (\alpha_{10} + \varepsilon_1)\ln v_1^* + (\alpha_{20} + \varepsilon_2)\ln v_2^* + \Sigma_{i=3}^{K+M} (\alpha_{i0} + \varepsilon_i)\ln v_i$$
$$(7.6) \qquad + \tfrac{1}{2} \Sigma_{i=1}^{2} \Sigma_{j=1}^{2} \alpha_{ij} \ln v_i^* \ln v_j^* + \tfrac{1}{2} \Sigma_{i=1}^{2} \Sigma_{j=3}^{K+M} \alpha_{ij} \ln v_i^* \ln v_j$$
$$+ \tfrac{1}{2} \Sigma_{i=3}^{K+M} \Sigma_{j=1}^{2} \alpha_{ij} \ln v_i \ln v_j^* + \tfrac{1}{2} \Sigma_{i=3}^{K+M} \Sigma_{j=3}^{K+M} \alpha_{ij} \ln v_i \ln v_j$$
$$+ \Sigma_{\ell=1}^{L} \gamma_{\ell 0} \ln z_\ell + \tfrac{1}{2} \Sigma_{\ell=1}^{L} \Sigma_{j=1}^{L} \gamma_{\ell j} \ln z_\ell \ln z_j + \Sigma_{i=1}^{2} \Sigma_{\ell=1}^{L} \beta_{i\ell} \ln v_i^* \ln z_\ell$$
$$+ \Sigma_{i=3}^{K+M} \Sigma_{\ell=1}^{L} \beta_{i\ell} \ln v_i \ln z_\ell + \varepsilon_0$$

which is nonlinear in ε_1 and ε_2 but linear in ε_0, ε_i, $i = 3,\ldots,K+M$. The first two regime switching conditions are

$$-\begin{pmatrix} \ln v_1 \\ \ln v_2 \end{pmatrix} - \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{10} + \Sigma_{j=3}^{K+M} \alpha_{1j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{1\ell} \ln z_\ell \\ \alpha_{20} + \Sigma_{j=3}^{K+M} \alpha_{2j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{2\ell} \ln z_\ell \end{pmatrix} \leq \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

To guarantee that the estimated variable profit function is linear homogeneous in prices, or equivalently that the estimated share equations are homogeneous of degree zero, the following restrictions on parameters and disturbances must hold:

$$\Sigma_{i=1}^{K+M} \alpha_{i0} = 1, \quad \Sigma_{j=1}^{K+M} \alpha_{ij} = 0, \quad \alpha_{ij} = \alpha_{ji} \quad \text{for all} \quad i, j; \quad \Sigma_{i=1}^{K+M} \beta_{i\ell} = 0,$$

$$\text{for all} \quad \ell; \quad \Sigma_{i=1}^{K+M} \varepsilon_i = 0.$$

The latter equality implies that the stochastic components are statistically linearly dependent. Therefore, one of the non-zero share equations can be deleted in the formulation of the likelihood. Define the random variables ω_1 and ω_2 as

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}.$$

Let $f(\varepsilon_0, \varepsilon_3, \ldots, \varepsilon_{K+M-1} | \omega_1, \omega_2)$ be the conditional density function, conditional on ω_1 and ω_2, and $g(\omega_1, \omega_2)$ be the marginal bivariate density function. As the Jacobian of the transformation $(\varepsilon_0, \varepsilon_3, \ldots, \varepsilon_{K+M-1})$ to $(\ln\bar{\pi}, \frac{v_3\bar{s}_3}{\bar{\pi}}, \ldots, \frac{v_{K+M-1}\bar{s}_{K+M-1}}{\bar{\pi}})$ is unity, the likelihood function for this regime with one observation is

$$(7.7) \qquad \int_{-c_2}^{\infty} \int_{-c_1}^{\infty} f\left(\ln\bar{\pi}, \frac{v_3\bar{s}_3}{\bar{\pi}}, \ldots, \frac{v_{K+M-1}\bar{s}_{K+M-1}}{\bar{\pi}} \Big| \omega_1, \omega_2\right) g(\omega_1, \omega_2)\, d\omega_1 d\omega_2$$

where
$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \ln v_1 \\ \ln v_2 \end{pmatrix} + \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{10} + \Sigma_{j=3}^{K+M} \alpha_{1j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{1\ell} \ln z_\ell \\ \alpha_{20} + \Sigma_{j=3}^{K+M} \alpha_{2j} \ln v_j + \Sigma_{\ell=1}^{L} \beta_{2\ell} \ln z_\ell \end{pmatrix}.$$

Likelihood functions for other regimes can similarly be derived, as well as the likelihoods for cost functions.

The likelihood (7.7) utilizes both the share equations and the profit function to derive the full information estimator. If only the share equations are used, the estimation procedure is less efficient but the corresponding likelihood function is simpler because share equations are linear in the disturbances. Nevertheless, the likelihood function will still involve multivariate normal probabilities.

8. An Application: Estimation of an Energy Cost Function.

In this section, we apply the econometric model set out above to the estimation of a translog energy cost function. The production structure used in deriving energy demand relationships parallels that of Fuss (1977) and Pindyck (1979). First, it is assumed that the production function is weakly separable in energy inputs. Thus, the cost-minimizing mix of energy inputs is independent of the mix of other factors. Second, the energy aggregate is assumed homothetic in its components so that cost minimization becomes a two-stage procedure: optimize the mix of fuels which make up the energy aggregate and then choose the cost minimizing mix of the energy aggregate, capital, labor, materials, and other factors. Here we will only estimate the energy aggregator function from which interfuel substitution elasticities can be derived.

The data used in the estimation came from the raw data tapes of the annual industrial surveys (Survey Perusahaan Industri) of Indonesia. The sample consists of establishments manufacturing fabricated metal products, machinery and equipment (ISIC classification 38). There are 1410 observations. Three fuels are identified: (purchased) electricity, fuel oils and other fuels.[8/]

Local market price data for energy inputs were available for all firms. The substantial spatial variation of prices characteristic of island Indonesia, as well as the large sample size, make it possible to estimate price response from cross-section data with reasonable precision.

All three fuels went unconsumed by a substantial number of firms (Table 1) and many firms consumed only one of the three. Many firms in Indonesia use prime movers or generate all or some of their electricity in-plant, which is why only 60% of firms purchased electricity. We expect that fuel oils and other petroleum fuels, which are used to power electric generators and prime movers, are close substitutes for purchased electricity.

The (unobserved) price index for a unit of energy is the unit translog cost function

$$(8.1) \qquad \ln P_E = \alpha_{00} + \Sigma_i \alpha_{i0} \ln p_i + \frac{1}{2} \Sigma_i \Sigma_j \alpha_{ij} \ln p_i \ln p_j$$

where i, j = 1 (electricity), 2 (fuel oil) and 3 (other petroleum fuels), and p_i is the price of the $i^{\underline{th}}$ fuel. Randomness as well as firm specific characteristics are allowed to influence demands by writing the parameters α_{i0} as

$$(8.2) \qquad \alpha_{i0} = \gamma_{i0} + \Sigma_j \gamma_{ij} C_j + \varepsilon_i \qquad i = 1, 2, 3$$

where C_j is the $j^{\underline{th}}$ firm characteristic and ε is a three-dimensional vector of normal variables $N(0, \Sigma)$. The firm characteristics C include dichotomous variables for Java/outside Java and urban/rural location and the year the establishment began operation. The share equations corresponding to the cost function (8.1, 8.2) are

$$(8.3) \qquad s_i = \gamma_{i0} + \Sigma_j \gamma_{ij} C_j + \Sigma_j \alpha_{ij} \ln p_j + \varepsilon_i \qquad i = 1, 2, 3$$

from which virtual prices for zero demands are readily solved for.

The maximum likelihood estimates of the parameters of the cost function, obtained using the quadratic hill-climbing methods in GQOPT package of Goldfeld and Quandt, are presented in Table 2. The asymptotic t-ratios reported in the table suggest that all three firm characteristics are significant determinants of energy demand. A likelihood ratio that supports this contention ($x^2(6) = 354.13$).

Table 3 provides estimates of interfuel (partial) fuel price elasticities. Own and cross-price elasticities are fairly large in magnitude. Electricity is a substitute with both fuel oil and other fuels. Its large cross-price elasticities suggest how close electricity is for alternative fuels. On the other hand, fuel oil and other fuels are complements.

TABLE 1

Sample Characteristics

Fuels used		Number of firms
Electricity only		97
Fuel oil only		167
Other fuel only		14
Electricity and fuel oil		108
Electricity and other		196
Fuel oil and other		386
Electricity, fuel oil and other		442
TOTAL		1410
	mean	S.D.
shares:		
electricity	.381	.397
fuel oil	.379	.379
other fuels	.240	.269
log prices:		
electricity	.873	.207
fuel oil	70.7	5.62
other fuels	2.48	.061
firm characteristics:		
Java = 1	.867	.339
Urban = 1	.784	.411
Start-up year	65.5	11.1

TABLE 2

Parameter Estimates of the Translog Energy Cost Function[ab]

	Maximum likelihood estimates	Asymptotic t-ratios
γ_{10}	-1.3532	14.68
γ_{11}	0.2613	6.37
γ_{12}	0.2886	8.52
γ_{13}	-0.0521	-12.86
γ_{20}	-4.7240	-1.86
γ_{21}	-0.1554	-1.69
γ_{22}	-0.1691	-1.59
γ_{23}	0.0236	4.88
α_{11}	-1.1361	-19.02
α_{12}	0.6071	19.28
α_{22}	-0.2486	-14.97
σ_{11}	0.1940	11.35
σ_{22}	1.0324	133.29
σ_{12}	-0.1371	-8.25

log likelihood -2251.77

[a] For parameters γ_{ij}: $i = 1$ (electricity), $i = 2$ (fuel oil), $j = 0$ (intercept), $j = 1$ (Java), $j = 2$ (urban), $j = 3$ (start-up year). For parameters α_{ij}: $i, j = 1$ (electricity), $i, j = 2$ (fuel oil), $i, j = 3$ (other fuels).

[b] Other parameters are easily derived from the homogeneity and symmetry conditions.

TABLE 3

Partial Fuel Price Elasticities

Fuels: \ Prices:	electricity	fuel oil	other fuels
electricity	-3.60	1.97	1.63
fuel oil	1.98	-1.28	-0.70
other fuels	2.59	-1.12	-1.47

9. Conclusions.

In this paper, we have considered the specification and estimation of models of consumers and producer demand with kink points. These kink points can arise from binding non-negativity constraints, quantity rationing, block prices or production quotas. The models specified recognize that observed demands are the result of optimal choice. The basic structures can be either a specific utility function or indirect utility function for consumer demand analysis, and a production function or profit function for production analysis.

Our analysis unifies the direct and dual approaches in consumption and production economics with kink points. The presence of kink points divides the demand schedule or production schedule into different regimes. Switching conditions, which determine the occurrence probabilities of different demand regimes, are provided. Our approach utilizes the concept of virtual prices originated in the quantity rationing literature. The virtual prices transform binding quantities into non-binding quantities and provide a rigorous justification for structural change in the observed demand functions across regimes. The comparison of virtual prices with market prices is sufficient to determine regime occurrences. Such comparisons are intuitively appealing as the virtual prices are actually reservation or shadow prices.

As an application of our approach, we have estimated a three input translog energy cost function. As some of these three fuels are close substitutes, non-negativity constraints are often binding. The empirical results are appealing and computation was inexpensive. Elsewhere, we estimate a demand system for five aggregated commodities using a sample of 767 households from a budget survey for Indonesia (Pitt and Lee, 1983). The computational cost for estimating that system with more than 50 parameters in which there were at most two non-consumed goods for each household was still quite moderate. However, because the econometric model is highly non-linear and multivariate in nature, computational difficulty and cost may increase rapidly with the number of non-consumed commodities.

FOOTNOTES

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(1) On the other hand, commodities such as electricity often have decreasing block pricing which creates a concave budget set. Concave budget sets create special problems and will be considered elsewhere.

(2) Survey sampling errors, such as reporting errors, may alone be sufficient to introduce zero quantities in observed samples. Pure measurement error problems will not be considered in this paper. Deaton and Irish (1982) suggest a relatively simple model of demand with reporting errors.

(3) Two textbook examples on increasing block input prices in production can be found in Henderson and Quandt (1980). One of the examples is on discontinuous labor contract for which the firm has to pay higher wage rates for overtime labor.

(4) Our analysis can be generalized in a straightforward manner to incorporate quantity rationing with a fixed amount of quantity. This case is the main concern of the studies of Deaton (1981) and Blundell and Walker (1982).

(5) Browning (1983) has shown that the unconditional cost function can theoretically be recovered from a conditional cost function. The necessary conditions for the conditional cost function are also sufficient for the recovery of the unconditional cost function when the rationed quantities are positive. Our approach starts with the unconditional functions. Identification in this paper refers to parameter identification given functional forms for the uncondi-

tional functions.

(6) One can also specify other distributions if they are of interest. Normality is attractive because of its additive property.

(7) It is necessary to specify $\Sigma_{i=1}^{K} \varepsilon_i = 0$, since, for the homogenous case $\Sigma_{i=1}^{K} \beta_{ij} = 0$ so that $D = -1$ in the share equations (4.11) and the sum of the shares is unity.

(8) The category "other fuel" includes diesel oils, gas oils and kerosene. The three categories of fuels delimited in this study comprised 86% of the value of energy used by firms in 1978.

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