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When there are multiple risks threatening the loss of an asset, insurance schemes contingent on one risk alone are incomplete. Two issues concerning such insurance schemes are studied here. The first issue relates to the consequences of incompleteness for the optimal amount of insurance. The second issue relates to the incentive implications of incomplete insurance. We find that, except when individuals have quadratic utility functions, incompleteness has non-trivial effects on the optimal insurance contract. When marginal utility is convex, incompleteness limits the amount of insurance. An increase in the variance of the individual's income (arising from greater uncertainty about the uninsured variable) *decreases* the optimal level of insurance. Also for a class of utility functions which includes some familiar functional forms, increasing risk aversion *reduces* the optimal amount of insurance. As for the incentive implication, the insurance is so weak for decreasing risk averse individuals that they strictly prefer those states of the world where no indemnity is forthcoming. For this reason, the potential moral hazard is least serious for this class of utility functions and the incompleteness of insurance helps to resolve the incentive problem.

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INCOMPLETENESS IN INSURANCE: AN ANALYSIS OF THE MULTIPLICATIVE CASE

I. INTRODUCTION

When there are multiple risks threatening the loss of an asset, insurance schemes contingent on one risk alone are incomplete. In the analysis of such insurance schemes, two issues have received attention in the literature. The first issue relates to the consequences of incompleteness for the optimal amount of insurance. This is studied here by answering two questions. First, how does the existence of a background uninsurable risk modify the optimal amount of insurance and second, how do risk attitudes of individuals influence the optimal insurance? The second issue concerns the incentive implications of incomplete insurance. In this paper, both of these issues are examined for the case of multiplicative risks.

The specification of multiplicative risks is natural in contexts where the assets that are insured against physical loss are also subject to uninsured fluctuations in unit value. Some examples of multiplicative risks are:

(a) when a firm (as in agriculture) faces price and output uncertainty but can obtain insurance against only one risk (e.g., crop insurance).

(b) when a work of art can be insured against loss of theft but not against changes in its market value[1].

(c) when an exporter can obtain insurance against exchange rate risks but not against fluctuations in world market price.

In the complete market case, and in the absence of informational asymmetries, optimal risk sharing between a risk averse individual and risk neutral insurance firm involves equalization of marginal utilities and hence net incomes across the different states. If there is, however, a second source of risk on which insurance is not contingent then the above proposition need not hold and marginal utilities may no longer be equalized.

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[1] This example is due to Turnbull

As a consequence, here it is shown, except when individuals have quadratic utility functions, incompleteness has non-trivial effects on the optimal insurance contract. First, when marginal utility is convex, the existence of an uninsurable risk reduces the amount of insurance. An increase in the variance of the individual's income (arising from greater uncertainty about the uninsured variable) *decreases* the optimal level of insurance. Second, the agent's risk preference and, in particular, the curvature of the marginal utility function is important in determining the optimal level of insurance. For a class of utility functions which includes some familiar functional forms, increasing risk aversion *reduces* the optimal amount of insurance.

As for the incentive implication, we find the optimal insurance to be so limited for decreasing risk averse individuals that they strictly prefer those states of the world where no indemnity is forthcoming to those where they receive indemnities. For this reason, potential moral hazard is least serious for this class of utility functions. In this case the incompleteness (due to the uninsurable risk) of insurance helps to resolve the incentive problem.

Relation to Previous Work

The problem of insuring against a risk x when an uninsurable background risk y is present has been studied by a number of researchers (for a survey see Schlesinger and Doherty). In Doherty and Schlesinger, and in

Mayers and Smith, correlation between the two risks has been shown to be critical in determining the level of optimal insurance. An implication is that if risks are independent, the optimal insurance against risk x is independent of risk y. As will be noted later, this result is a consequence of the assumption of additive risks. If instead, the risks interact multiplicatively, as assumed here, the flavor of the analysis is altered considerably. In the model presented here, although the risks are independent, the optimal insurance is not independent of the uncertainty about the uninsured variable.

The properties of optimal insurance when there are multiple risks has also been studied in terms of the willingness to pay for the removal of a risk x when a risk y is present. In such circumstances, individuals who are more risk averse (in the Arrow-Pratt sense) may not necessarily be willing to pay larger risk premiums (Ross; Kihlstrom, Romer and Williams; Turnbull). In much the same vein, this paper identifies situations when higher risk aversion reduces the optimal amount of insurance. Also for the case of multiplicative risks, Turnbull found the individual's willingness to pay for the removal of a single risk to decrease as a result of an increase in uncertainty about the uninsured variable. This result is proved for decreasing risk averse utility functions. This paper proves, for a larger class of utility functions, that an increase in the riskiness of the distribution of the uninsured variable decreases the optimal level of insurance.

In the complete market case, and in the absence of moral hazard, the optimal contract is such that the insured individual is indifferent between the various income states. If, however, the contract is not contingent on the agent's actions, then such a level of insurance is not optimal because it provides no incentive for the agent to take actions which will reduce the probability of losses. Similarly, in an incomplete insurance context, the optimality of contracts in the presence of moral hazard has been investigated by Imai et.al and Ito and Machina. They consider unemployment insurance schemes where severance payments are made to laid-off workers. But "since severance payments usually do not depend on outcomes at alternative opportunities after layoff, they are considered at best incomplete insurance for layoff" (Ito and Machina). The issue that is investigated is whether the laid off worker could be better off, in an ex-ante sense, than the retained worker. If this were so, it would obviously introduce the moral hazard problem. For a similar question in our model of multiplicative risks, we find the incentive problem to be least serious for individuals with decreasing risk averse utility functions. This is because the background risk reduces the optimal insurance so much that moral hazard problems are considerably moderated.

Plan of Paper

For a fairly general problem, the next section sets out a model of incomplete insurance and derives the condition for optimal insurance. By introducing more structure, we are able to specialize the model to consider the case of additive and multiplicative risks. The results of Doherty and Schlesinger are reviewed for the additive case. The multiplicative case is pursed in Section III, where the effect of background risk on the optimal insurance is completely characterized. In Section IV, the inverse relationship between risk aversion and the level of optimal insurance is demonstrated for a class of utility functions which includes some familiar functional forms.

Sections V and VI consist of extensions to the basic model of Section II.

Since the incompleteness of insurance is intimately tied to the absence of risk markets, Section V considers the effect on the optimal insurance due to the introduction of a market for the uninsurable risk. Section VI allows individual's actions to affect the probability distribution of insurable losses. If the agent's actions are unobservable, the insurance contract cannot be contingent on it. But the optimal contract must take into account the agent's optimal actions in response to the insurance contract (described by the incentive constraint). The analysis considers the situations in which the incentive constraint is likely to be binding. This is shown to depend on the nature of risk preferences and on the agent's disutility towards work. The incentive constraint is least binding for decreasing risk averse individuals. In fact, if the marginal cost to the individual of his actions is small, the incentive constraint for decreasing risk averse individuals may not be binding i.e., moral hazard does not alter the optimal insurance contract. The analysis also reconsiders the effect of background risk on the optimal insurance in the context of moral hazard. Concluding remarks are gathered together in Section VII.

II. A MODEL OF INCOMPLETE INSURANCE

Let w be the value of owning an asset. w depends on the state of the world ω that is realized, where ω is an element of the state space Ω. An insurance contract is a state contingent indemnity schedule $I(\omega)$ and a premium P that is paid in all states.

Let I(.) partition Ω into Ω_G and Ω_B such that $\Omega_G = \{\omega \in \Omega\colon I(\omega) = 0\}$ and $\Omega_B = \{\omega \in \Omega\colon I(\omega) = k)$ where k is any arbitrary positive constant. We can think of Ω_G as the no-accident state of the world (the 'good' state) and of Ω_B as the accident state of the world (the 'bad' state). If Ω contains at least three elements, a description of the state of the world in terms of Ω_G and Ω_B is incomplete since Ω_G or Ω_B or both contain at least two elements.

Let π be an element of the probability vector Π on Ω where Π is exogenous to the individual seeking insurance[2]. We suppose the insurance firm is risk neutral and offers actuarially fair insurance, i.e., $P = \Sigma_{\omega\in\Omega}\Big[\pi(\omega)I(\omega)\Big] = I\Sigma_{\omega\in\Omega_B}\Big[\pi(\omega)\Big] \equiv \gamma I$ where $\gamma = \Sigma_{\omega\in\Omega_B}\Big[\pi(\omega)\Big]$. From the set of actuarially fair contracts the optimal insurance is found by maximizing the expectation of an increasing and strictly concave von Neumann-Morgenstern utility function.

$$(1) \quad \underset{I}{\text{Max}}\ \eta(I) = \Sigma_{\omega\in\Omega_G}\Big[\pi(\omega)U(w(\omega) - \gamma I)\Big] + \Sigma_{\omega\in\Omega_B}\Big[\pi(\omega)U(w(\omega) + (1 - \gamma)I)\Big]$$

If an interior solution exists, it satisfies

$$\eta'(I) = (1 - \gamma)(\Sigma_{\Omega_B}\pi(\omega)U'(w(\omega) + (1 - \gamma)I) - \gamma\Sigma_{\Omega_G}\pi(\omega)U'(w(\omega) - \gamma I) = 0$$

or

$$(2)\ (1 - \gamma)\gamma\Big[\Sigma_{\Omega_B}(\pi(\omega)/\gamma)U'(w(\omega) + (1 - \gamma)I) - \Sigma_{\Omega_G}(\pi(\omega)/(1-\gamma))U'(w(\omega) - \gamma I)\Big] = 0$$

For $\omega \in \Omega_B$, $\pi(\omega)/\gamma = \text{Prob}(\omega|\omega \in \Omega_B)$ and

for $\omega \in \Omega_G$, $\pi(\omega)/(1-\gamma) = \text{Prob}(\omega|\omega \in \Omega_G)$.

Rewriting (2) in terms of the conditional probabilities, the optimal I satisfies,

$$(3) \quad \eta'(I) = (1 - \gamma)\gamma\Big[E[U'(w(\omega) + (1 - \gamma)I)|\Omega_B] - E[U'(w(\omega) - \gamma I)|\Omega_G]\Big] = 0$$

The optimal insurance equates the expected marginal utility across the accident and no-accident states of nature. While (3) is the basic optimality condition of incomplete insurance schemes, different cases arise depending upon assumptions about Ω_G, Ω_B and Π.

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[2] The exogeniety assumption is relaxed in Section VI

To introduce more structure, suppose that the randomness in w the value of owning the asset is induced by randomness in two variables x and y. Then the state of the world is described by the pair (x,y). Assume two point distributions for x and y, with outcomes X_1, X_2 and Y_1, Y_2 respectively. Also let an insurance scheme be contingent on x but not on y and identify the no-accident state as X_1 and the accident state as X_2, i.e., $X_1 > X_2$. Or more formally, $\Omega_G = \{(X_1,Y_1), (X_1,Y_2)\}$ and $\Omega_B = \{(X_2,Y_1), (X_2,Y_2)\}$. Then (3) becomes

$$(4) \quad \eta'(I) = (1-\gamma)\gamma\Big[E[U'(w(x,y) + (1-\gamma)I)|X_2] - E[U'(w(x,y) - \gamma I)|X_1]\Big] = 0$$

The Additive Case

The interaction of x and y or more generally the manner in which w depends on x and y matters in the analysis of incomplete insurance schemes. One specification, analyzed by Doherty and Schlesinger is when $w = x + y$. In this case,

$$(5) \quad \eta'(I) = (1-\gamma)\gamma\Big[E[U'(X_2 + y + (1-\gamma)I)|X_2] - E[U'(X_1 + y - \gamma I)|X_1]\Big]$$

If no background risk were present, i.e., if y was non-random, it would be optimal to insure fully against the risk of X_2, i.e., $I = X_1 - X_2$. The optimal insurance in the presence of background risk is greater than, equal to or less than full coverage depending upon whether $\eta'(I)$ evaluated at $X_1 - X_2$ is greater than, equal to or less than zero[3].

If $I = X_1 - X_2$, notice that

$$(6) \quad X_2 + Y_i + (1-\gamma)I = X_2 + Y_i + X_1 - X_2 - \gamma I = X_1 + Y_i - \gamma I, \ldots . i = 1,2$$

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[3] The strict concavity of the utility function guarantees the strict concavity of $\eta'(I)$.

Using (6),

$$\eta'(I)\Big|_{I=X_1-X_2} = (1-\gamma)\gamma\Big[E[U'(X_1 + y - \gamma I)|X_2] - E[U'(X_1 + y - \gamma I)|X_1]\Big]$$

$$\eta'(I)\Big|_{I=X_1-X_2} = (1-\gamma)\gamma\sum_i\Big[\text{Prob}\ (Y_i|X_2) - \text{Prob}(Y_i|X_1)\Big]\Big[U'(X_1 + Y_i - \gamma I)\Big]$$

Since $\text{Prob}(Y_1|X_2) - \text{Prob}(Y_1|X_1) = \text{Prob}(Y_2|X_1) - \text{Prob}(Y_2|X_2)$,

$$\eta'(I)\Big|_{I=X_1-X_2}$$

$$= (1-\gamma)\gamma\Big[\text{Prob}(Y_1|X_2) - \text{Prob}(Y_1|X_1)\Big]\Big[(U'(X_1+Y_1-\gamma I) - U'(X_1+Y_2-\gamma I))\Big]$$

Without loss of generality, suppose $Y_1 > Y_2$. Then,

$$\eta'(I)\Big|_{I=X_1-X_2} \underset{<}{\geq} 0 \text{ as } \text{Prob}(Y_1|X_2) - \text{Prob}(Y_1|X_1) \underset{>}{\leq} 0.$$

If x and y are positively (negatively) correlated, the background risk increases (decreases) the optimal insurance compared to the no-background risk situation. If x and y are independent random variables, the randomness in y does not affect the optimal amount of insurance.

The Multiplicative Case

We now turn to the case of independent multiplicative risks, i.e., when $w = xy$.

By the independence of x and y, the first order condition (4) becomes

$$(7)\quad \eta'(I) = (1-\gamma)\gamma\Big[E^y[U'(w(X_2,y) + (1-\gamma)I)] - E^y[U'(w(X_1,y) - \gamma I)]\Big] = 0$$

where the superscript on the expectations operator denotes that the expectations are with respect to the distribution of y.

For the exposition it is convenient to regard x and y as output and price respectively and to consider an insurance scheme contingent on output[4]. It is also a description of the economic setting in agriculture where crop insurance schemes (contingent on output) are rendered incomplete by price risks. Then it is natural to extend the discussion to consider the existence of a futures market, i.e., a market for the uninsurable risk.

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[4] We could just as well have considered price insurance. The analysis equally applies to the choice of optimal hedge under production uncertainty.

The uncertain elements are assumed to have independent probability distributions of the following form

$$(8) \qquad q = \begin{cases} Q_1 & \text{with probability } \gamma \\ Q_2 & \text{with probability } (1-\gamma) \end{cases}$$

and

$$(9) \qquad p = \begin{cases} P_1 & \text{with probability } \lambda \\ P_2 & \text{with probability } (1-\lambda) \end{cases}$$

where q is output and p is output price. Suppose $Q_1 > Q_2$ and $P_1 > P_2$. The value of output is a random variable w distributed as:

$$(10) \qquad w = \begin{cases} W_1 = P_1Q_1 & \text{with probability } (1-\gamma)\lambda \\ W_2 = P_2Q_1 & \text{with probability } (1-\gamma)(1-\lambda) \\ W_3 = P_1Q_2 & \text{with probability } \gamma\lambda \\ W_4 = P_2Q_2 & \text{with probability } \pi(1-\gamma) \end{cases}$$

W can be ordered in one of two ways. Either $W_1 > W_2 > W_3 > W_4$ or $W_1 > W_3 > W_2 > W_4$.

Denoting as r, the revenue with output insurance, we have

$$(11) \qquad r = \begin{cases} R_1 = P_1Q_1 - \gamma I & \text{with probability } (1-\gamma)\lambda \\ R_2 = P_2Q_1 - \gamma I & \text{with probability } (1-\gamma)(1-\lambda) \\ R_3 = P_1Q_2 + (1-\gamma)I & \text{with probability } \gamma\lambda \\ R_4 = P_2Q_2 + (1-\gamma)I & \text{with probability } \gamma(1-\lambda) \end{cases}$$

From (7), the optimal output insurance satisfies[5]

$$E^P[U'(pQ_2 + (1 - \gamma)I)] = E^P[U'(pQ_1 - \gamma I)] \tag{12}$$

An immediate consequence is the following.

Proposition 1 Let I^* be the optimal level of insurance. Then $R_1(I^*) > R_3(I^*) > R_4(I^*) > R_2(I^*)$ or equivalently $P_1(Q_1 - Q_2) > I^* > P_2(Q_1 - Q_2)$.

Proof From (10) note that $R_3 > R_4$ for all I. So what needs to be shown is $R_1(I^*) > R_3(I^*)$ and $R_4(I^*) > R_2(I^*)$.

From the first order condition, I^* satisfies

$$\lambda U'(R_3^*) + (1-\lambda)U'(R_4^*) = \lambda U'(R_1^*) + (1-\lambda)U'(R_2^*) \tag{13}$$

where R_j^* denotes $R_j(I^*)$ for $j = 1,..,4$.

Now let $\eta_1(I) = U'(R_3) - U'(R_1)$ and $\eta_2(I) = U'(R_2) - U'(R_4)$. Substituting and rearranging terms, (12) becomes

$$\lambda\eta_2(I^*) - (1-\lambda)\eta_2(I^*) = 0 \tag{14}$$

Clearly, $\eta_1(I^*)$ and $\eta_2(I^*)$ must both be of the same sign. Suppose they are both negative.

$$\eta_1 < 0 \Rightarrow R_3^* > R_1^* \Rightarrow P_1Q_2 + (1-\gamma)I^* > P_1Q_1 - \gamma I^*$$

$$\Rightarrow I^* > P_1(Q_1-Q_2) \tag{15}$$

$$\eta_2 < 0 \Rightarrow R_2^* > R_4^* \Rightarrow P_2Q_1 - \gamma I^* > P_2Q_2 + (1-\gamma)I^*$$

$$\Rightarrow P_2(Q_1 - Q_2) > I^* \tag{16}$$

Combining the two inequalities, we get $P_2 > P_1$ which is not possible. For a similar reason η_1 and η_2 cannot both be zero; so both of them have to be positive. This means $R_3^* < R_1^*$ and $R_2^* < R_4^*$. Since $R_3^* > R_4^*$, we obtain the ordering $R_1^* > R_3^* > R_4^* > R_2^*$. Notice also that the inequalities in (15) and (16) are reversed and so we obtain upper and lower bounds on the optimal amount of insurance , i.e., $P_1(Q_1 - Q_2) > I^* > P_2(Q_1 - Q_2)$.

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[5] The optimal output insurance is strictly positive because $\eta'(I)\big|_{I=0} = \gamma(1-\gamma)\left\{E^P[U'(pQ_2)] - E^P[U'(pQ_1)]\right\} > 0$ by the strict concavity of U and from the fact $pQ_1 > pQ_2$ for all p.

The complete ordering of the R_j's is a direct consequence of risk averse behavior and the first order condition (12). Recall, that in the absence of insurance, we know that either $W_1 > W_2 > W_3 > W_4$ or $W_1 > W_3 > W_2 > W_4$. In either case the worst income state was W_4 when both price and output are low. With insurance, however, the ordering changes in a significant way; the worst income state is R_2, when price is low but output is high. In this state, premium payments have to be made, even though the individual suffers losses due to low prices. For the individual seeking output insurance, its incomplete nature creates a difficult trade-off between output and price risks. While output risks are clearly reduced, the individual is worse off in the low price-high output state R_2. Further the fact that R_2 decreases with greater purchase of insurance, suggests that I^* cannot be "too high". The argument is made more precise in the following propositions.

III. THE EFFECT OF PRICE RISK

Proposition 2 If p^e is expected output price, then

(i) $I^* < p^e(Q_1 - Q_2)$ if $U'''(.) > 0$

(ii) $I^* = p^e(Q_1 - Q_2)$ if $U'''(.) = 0$

(iii) $I^* > p^e(Q_1 - Q_2)$ if $U'''(.) < 0$

Proof We will consider the case when U''' is strictly positive. It is straightforward to alter the reasoning for the cases when U''' is zero or negative. The proof consists in showing $\eta'(I)$ to be negative for all $I \geq$

$p^e(Q_1 - Q_2)$[6].

Let $r_G(p)$ denote the random income in the high output states and $r_B(p)$ the random income in low output states. So

$$(17) \qquad r_G(p) = pQ_1 - \gamma I = \begin{cases} R_1 = P_1Q_1 - \gamma I & \text{with probability } \lambda \\ R_2 = P_2Q_1 - \gamma I & \text{with probability } (1-\lambda) \end{cases}$$

$$(18) \qquad r_B(p) = pQ_2 + (1-\gamma)I = \begin{cases} R_3 = P_1Q_2 + (1-\gamma)I & \text{with probability } \lambda \\ R_4 = P_2Q_2 + (1-\gamma)I & \text{with probability } (1-\lambda) \end{cases}$$

Then $\eta'(I)$ can be written more compactly as

$$\eta'(I) = \gamma(1-\gamma) \left\{E^pU'(r_G(p)) - E^pU'(r_B(p))\right\}$$

The sign of $\eta'(I)$ depends on the difference in expected marginal utilities between low and high output states. Now,

$r_G(p) - r_B(p) = pQ_1 - \gamma I - pQ_2 - (1 - \gamma)I = p(Q_1 - Q_2) - I.$

So $E^pU'(r_G(p)) = E^pU'(r_B(p) + p(Q_1 - Q_2) - I)$

Let $\nu(p) = p(Q_1 - Q_2) - p^e(Q_1 - Q_2)$

$\nu(P_1) > 0$, $\nu(P_2) < 0$ and $E^p\nu(p) = 0$. Then

$E^pU'(r_G) = E^pU'(r_B + \nu + p^e(Q_1 - Q_2) - I)$

If $I \geq p^e(Q_1 - Q_2)$, $E^pU'(r_G) \geq E^pU'(r_B + \nu) > EU'(r_B)$ where the second inequality follows from the convexity of the marginal utility ($U''' > 0$) and from the observation that $(r_B + \nu)$ is a mean preserving spread of r_G Therefore, the optimal insurance is less than the expected value of output loss.

---

[6] This is enough since η is strictly concave in I.
$\eta''(I) = \gamma(1-\gamma)(\lambda\eta_1'(I) - (1-\lambda)\eta_2'(I))$ where $\eta_1'(I) = U''(R_3)(1-\gamma) + U''(R_1)\gamma < 0$ and $\eta_2'(I) = -\gamma U''(R_2) - (1-\gamma)U''(R_4) > 0$.

The result for the complete market case is the following.

Proposition 3 If output price is certain at the mean ($P_1 = P_2 = p^e$), then $I^* = p^e(Q_1 - Q_2)$.

Proof $P_1 = P_2 \Rightarrow R_1 = R_2$ and $R_3 = R_4$. The first order conditions reduce to $U'(R_1) = U'(R_3)$ or $R_1 = R_3$ which implies $I^* = p^e(q_1 - q_2)$.

If we refer to $p^e(Q_1 - Q_2)$ as the complete insurance, then proposition 2 compares the optimal level of incomplete insurance to the complete insurance. Under the reasonable assumption of a positive U'''[7], incompleteness reduces the optimal insurance.

The next proposition is concerned with the "marginal" impact of price uncertainty i.e. the effect of making a given distribution "slightly more risky". Following Rothschild and Stiglitz a mean preserving increase in price risk is represented by a decrease in price which leaves the mean unchanged. Since $p^e = \lambda P_1 + (1 - \lambda)P_2$, $dP_1/dP_2 = -(1 - \lambda)/\lambda$.

Proposition 4 For the class of utility functions with a positive (zero, negative) third derivative everywhere, an increase in price risk reduces (leaves unchanged, increases) the optimal level of insurance.

The proof is in appendix A.

IV. THE EFFECT OF RISK AVERSION

If U_1 and U_2 are two utility functions, U_1 is said to be globally more risk averse than U_2 if $-U_1''(x)/U_1'(x) \geq -U_2''(x)/U_2'(x)$ for all x. The following theorem, proved by Pratt, is useful for later results.

Theorem (Pratt) The following conditions are equivalent

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[7] The subset of the class of concave utility functions which satisfies $U''' > 0$ includes all constant and decreasing absolute risk aversion utility functions.

(a) $-U_1''(x)/U_1'(x) \geq -U_2''(x)/U_2'(x)$ for all x [and > for at least one x]

(b) $\frac{U_1(y) - U_1(x)}{U_1(w) - U_1(v)} < \frac{U_2(y) - U_2(x)}{U_2(w) - U_2(v)}$ for all v, w, x, y with $v < w \leq x < y$

Before we state the proposition, a little notation is helpful. Let $V_j(.) \equiv -U_j'(.)$, $A_{uj} = -(U_j''/U_j')$ and $A_{vj} = -(V_j''/V_j')$ for j = 1,2, i.e., A_{uj} and A_{vj} are the risk aversion functions with respect to the utility functions U_j and V_j respectively. Also let I_1^* and I_2^* be optimal levels of insurance for individuals 1 and 2 respectively. The proposition below makes use of a condition about the change in the curvature of the marginal utility as risk aversion increases. The condition is

(CMU) $A_{u1}(x) \geq A_{u2}(x)$ for all x [and > for at least one x]

$\Rightarrow A_{v1}(x) \geq A_{v2}(x)$ for all x [and > for at least one x]

From Pratt's work, we know that when $A_{u1} \geq A_{u2}$, U_1 is a concave transformation of U_2. The CMU condition says that increasing risk aversion also results in a concavifying transformation of the V (or -U') function. This property is exhibited by the constant absolute risk aversion (CARA) and the constant relative risk aversion (CRRA) utility functions. This is verified in appendix B.

Proposition 5 For utility functions satisfying the CMU condition, an increase in risk aversion reduces the optimal amount of insurance i.e.,

$A_{u1}(x) \geq A_{u2}(x)$ for all x [and > for at least one x] $\Rightarrow I_1^* < I_2^*$.[8]

Proof Let η_j denote the expected utility of the jth individual where

$\eta_j(I) = (1-\gamma)\lambda U_j(R_1) + (1-\gamma)(1-\lambda)U_j(R_2) + \gamma\lambda U_j(R_3) + \gamma(1-\lambda)U_j(R_4)$

---

[8] Note that if $P_1 = P_2 = p$, i.e., a situation of complete insurance, then as shown in Proposition 3, the optimal insurance is $I^* = p(q_1 - q_2)$ which is independent of the agent's risk attitudes.

Since $U_j'' < 0$, $\eta_j''(.) < 0$ for all I. To prove $I_1^* < I_2^*$, it is then enough to show $\eta_2'(.)\big|_{I=I_1^*} > 0$.

I_1^* satisfies

$$\eta_1'(I_1^*) = \gamma(1-\gamma)\{\lambda(U_1'(R_3(I_1^*)) - U_1'(R_1(I_1^*))) + (1-\lambda)(U_1'(R_4(I_1^*) - U_1'(R_2(I_1^*))))\} = 0$$

$$\Rightarrow (19) \qquad \frac{U_1'(R_3(I_1^*)) - U_1'(R_1(I_1^*))}{U_1'(R_2(I_1^*)) - U_1'(R_4(I_1^*))} = (1-\lambda)/\lambda$$

or $\dfrac{V_1(R_1(I_1^*)) - V_1(R_3(I_1^*))}{V_1(R_4(I_1^*)) - V_1(R_2(I_1^*))} = (1-\lambda)/\lambda$ since $V_j = -U_j'$, $j = 1,2$.

By the CMU condition $A_{v1} \geq A_{v2}$. Applying Pratt's theorem to the V function,

$$(20) \qquad \frac{V_1(R_1(I_1^*)) - V_1(R_3(I_1^*))}{V_1(R_4(I_1^*)) - V_1(R_2(I_1^*))} < \frac{V_2(R_1(I_1^*)) - V_2(R_3(I_1^*))}{V_2(R_4(I_1^*)) - V_2(R_2(I_1^*))}$$

since, by Proposition 1, $R_1(I_1^*) > R_3(I_1^*) > R_4(I_1^*) > R_2(I_1^*)$.

(19) and (20) imply

$$(21) \qquad \frac{U_2'(R_3(I_1^*)) - U_2'(R_1(I_1^*))}{U_2'(R_2(I_1^*)) - U_2'(R_4(I_1^*))} > (1-\lambda)/\lambda$$

or

$$(22) \qquad \lambda(U_2'(R_3(I_1^*)) - U_2'(R_1(I_1^*))) + (1-\lambda)(U_2'(R_4(I_1^*) - U_2'(R_2(I_1^*))))\} > 0$$

Therefore, $\eta_2''(I_1^*) = \gamma(1-\gamma)\{\lambda(U_2'(R_3(I_1^*)) - U_2'(R_1(I_1^*)))$

$$+ (1-\lambda)(U_2'(R_4(I_1^*) - U_2'(R_2(I_1^*))))\} > 0 \ \square$$

V. THE INTRODUCTION OF A FUTURES MARKET

The characterization of insurance schemes as complete or incomplete is intimately tied to the presence or absence of the appropriate risk markets. The incomplete aspects of crop/output insurance arise largely out of the absence of the markets for price risk. But, of course, markets for price risk (e.g., forward and futures markets) are available for some commodities. This section examines the relation between crop insurance schemes and futures markets[9]. This issue is also of policy interest because crop insurance schemes are generally sponsored by the government, while, futures markets are privately organized.

To introduce a futures market, consider a two period model where a farmer makes hedging decisions (and crop insurance decisions if insurance is available) at time 1. The futures contracts are for the duration of one period - i.e., if the farmer buys a contract at time 1 he agrees to deliver the specified quantity of the commodity at time 2. At time 2, output is realized and the uncertainty about the spot price is resolved. If the futures market is unbiased (as is the assumption here), p^f the futures price is equal to p^e, the expected spot price[10]. Let f denote the farmer's position in the futures market.

In this section, it is convenient for the purpose of exposition to specify the insurance to be of the form where $I = \rho(q_1 - q_2)$ where ρ is the co insurance parameter. Since the size of loss is fixed, a specification of the above kind is no restriction on I; choosing the optimal ρ is equivalent to choosing the optimal I. Following crop insurance practice in the U.S.[11], ρ

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[9] The distinction between futures and forward markets is ignored here.

[10] In other words, we are considering a pure hedger. Since $p^f = Ep^e$, the farmer who has a position in the forward market cannot expect to make any speculative profits. The unbiasedness assumption is therefore a condition for fair insurance and in this case the futures position exists only because of hedging considerations. In general, the futures position consists of hedging and speculative components. In a mean-variance context, the futures position can be quite easily be decomposed into its components (see Anderson and Danthine or Newberry and Stiglitz, Ch 13). The empirical evidence on the existence of bias in futures markets is mixed but unbiasedness is usually a reasonable assumption for markets with active trading. See Peck for a collection of papers on this subject.

[11] For a description see the report by the General Accounting Office, 1984.

can also be regarded as the "price-election" which is the price at which the insurance company compensates the farmer for a unit loss of the commodity.

The farmer's revenue for a price election of ρ and a hedge of f is distributed as

$$(22)\quad r = \begin{cases} R_1 = P_1Q_1 - \gamma\rho(Q_1-Q_2) + (p^e - P_1)f & \text{with probability } (1-\gamma)\lambda \\ R_2 = P_2Q_1 - \gamma\rho(Q_1-Q_2) + (p^e - P_2)f & \text{with probability } (1-\gamma)(1-\lambda) \\ R_3 = P_1Q_2 + (1-\gamma)\rho(Q_1-Q_2) + (p^e - P_1)f & \text{with probability } \gamma\lambda \\ R_4 = P_2Q_2 + (1-\pi)\rho(Q_1-Q_2) + (p^e - P_2)f & \text{with probability } (1-\lambda)\gamma \end{cases}$$

The farmer's problem is

$$\underset{\rho,f}{\text{Max}}\ \eta(\rho,f) \equiv (1-\gamma)\lambda U(R_1) + (1-\gamma)(1-\lambda)U(R_2) + \gamma\lambda U(R_3) + \gamma(1-\lambda)U(R_4)$$

It is easy to show that the optimal hedge and insurance are strictly positive. Therefore, they satisfy the first order conditions

$$(23)\quad \eta_\rho = (1-\gamma)\gamma(Q_1-Q_2)[\lambda U'(R_3) + (1-\lambda)U'(R_4) - \lambda U'(R_1) - (1-\lambda)U'(R_2)] = 0$$

and $$\eta_f = \lambda[(1-\gamma)U'(R_1) + \gamma U'(R_3)](p^e - P_1) + (1-\lambda)[(1-\gamma)U'(R_2) + \gamma U'(R_4)](p^e - P_2) = 0$$

Substituting for $p^e = \lambda P_1 + (1-\lambda)P_2$

(24)

$$\eta_f = \lambda(1-\lambda)(P_1 - P_2)[(1-\gamma)U'(R_1) + \gamma U'(R_3) - (1-\gamma)U'(R_2) - \gamma U'(R_4)] = 0$$

(23) and (24) can be expressed more compactly in terms of the conditional distributions of r. As defined earlier, $r_G(p)$ and $r_B(p)$ are the random income in the high output ("good") and low output ("bad") states.

$$r|Q_1 \equiv r_G(p) = \begin{cases} R_1 \text{ with probability } \lambda \\ R_2 \text{ with probability } (1-\lambda) \end{cases} \quad \text{and}$$

$$r|Q_2 \equiv r_B(p) = \begin{cases} R_3 \text{ with probability } \lambda \\ R_4 \text{ with probability } (1-\lambda) \end{cases}$$

Similarly, the distribution of revenue conditional on price is

$$r|P_1 \equiv r_H(q) = \begin{cases} R_1 \text{ with probability } (1-\gamma) \\ R_3 \text{ with probability } \gamma \end{cases} \quad \text{and}$$

$$r|P_2 \equiv r_L(p) = \begin{cases} R_2 \text{ with probability } (1-\gamma) \\ R_4 \text{ with probability } \gamma \end{cases}$$

$r_H(q)$ and $r_L(p)$ are the random income in the high and low price states respectively.

Then the first order conditions become

$$(25) \qquad \eta_\rho = (1-\gamma)\gamma(Q_1 - Q_2)[E^pU'(r_B(p)) - E^pU'(r_G(p))] = 0$$

$$(26) \qquad \eta_f = \lambda(1-\lambda)(P_1 - P_2)[E^qU'(r_L(q)) - E^qU'(r_H(q))] = 0$$

where the superscript indicates the random variable over which expectations are taken. The optimal insurance equates the expected marginal utility across the good and bad output states while the optimal hedge equates the expected marginal utility across the high and low price states[12].

Proposition 6: Let ρ^* and f^* be the optimal insurance and forward position. Then $Q_1 > f^* > Q_2$ and $P_1 > \rho^* > P_2$.

For a proof please see appendix A.

The question that is of interest is whether the existence of hedging opportunities affects the optimal insurance. To be more concrete, we wish to compare the optimal insurance solved from (25) and (26) with the optimal insurance when there are no futures markets. This is best achieved by considering a model where the farmer's choice of the hedge is constrained i.e.,

$$\max_{(\rho,f)} \eta(\rho,f) \equiv (1-\gamma)\lambda U(R_1) + (1-\gamma)(1-\lambda)U(R_2) + \gamma\lambda U(R_3) + \gamma(1-\lambda)U(R_4)$$

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[12] Note that the optimality conditions do not equate the marginal utility across all states of income. This means that some risk markets are still absent. This is not surprising since the relevant state of the world, for which contingent claims must exist to complete markets, is crop revenue, i.e., $w = p \times q$.

subject to $f \leq \hat{f}$

If $\hat{f} = 0$, the situation corresponds to the absence of futures markets. By the Kuhn-Tucker theorem, there exists a $\mu \geq 0$ such that

$$(27) \qquad E^{p}U'(r_B(p)) - E^{p}U'(r_G(p)) = 0$$

$$(28) \qquad \lambda(1 - \lambda)(P_1 - P_2)[E^{q}U'(r_L(q)) - E^{q}U'(r_H(q))] = \mu$$

$$(29) \qquad \mu(f - \hat{f}) = 0$$

$$(30) \qquad \mu \geq 0$$

Since we want to investigate the effect of the opening up of a futures market, suppose that $\hat{f}$ is small enough (less than the unconstrained optimum f^{*}) to be a binding constraint i.e., $\mu > 0$. Let ρ_f^{*} and μ_f^{*} denote the constrained solution to (27)-(30). From (28),

$$E^{q}U'(r_L(q)) - E^{q}U'(r_H(q)) = \mu/[\lambda(1 - \lambda)(P_1 - P_2)] > 0.$$

Because of insufficient hedging, the expected marginal utility in the low price state remains higher than the expected marginal utility in the high price states. The optimal crop insurance, however, equates the expected marginal utility across the high and low output states. Consequently, from the argument in Proposition 6,

Proposition 7: $R_1^{*} > R_3^{*}$ and $R_4^{*} > R_2^{*}$

Suppose $\hat{f}$ were to be increased i.e., the constraint is made less binding. What is the effect on ρ_f^{*} ? In the appendix (proof of Proposition 8 below) it is shown that ρ_f^{*} responds positively to an increase in $\hat{f}$. Since the case of no futures markets corresponds to an extreme constraint, the existence of futures markets leads to an increase in the optimal output insurance.

Proposition 8 The existence of futures markets increases (decreases, leaves unchanged) the optimal insurance if the third derivative of the utility function is positive (negative or zero).

Since the problem of choosing the optimal hedge under production uncertainty is conceptually equivalent to the problem of choosing the optimal price election under price risk, a corollary to Proposition 8 would be that the farmers choose larger hedge positions in the presence of crop insurance[13].

VI. INCENTIVE IMPLICATIONS

The analysis so far has abstracted from any considerations of the farmer's actions which may affect the probability distribution of output. If insurance contracts are contingent on the farmer's actions as well as output, then the optimal contract will once again equalize the expected marginal utilities across the high and the low states of output and the results of the earlier sections will go through unaltered. If, however, the insurer cannot observe the actions of the farmer, the contracts remain contingent on output alone and the optimal contract will have to be consistent with the farmer's choice of action described by the *incentive constraint.* This section studies how the introduction of the incentive constraint alters the optimal amount of insurance. This also necessitates a restatement of the effect of background risk.

Let z represent the farmer's choice of action. To be concrete, we could consider z as the input or the effort used to produce the output q The probability distribution of output depends on the chosen action,

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[13] For a treatment of hedging decisions under output uncertainty, see Losq

$$q = \begin{cases} Q_1 \text{ with probability } (1 - \gamma(z)) \\ Q_2 \text{ with probability } \gamma(z) \end{cases}$$

The probability of crop failure will be assumed to decline with greater effort, i.e., $\gamma'(z) \leq 0$. Also the farmer's utility is taken to be separable in income r and effort z, i.e., total utility is $U(r) - C(z)$ where U is increasing, concave and thrice differentiable and C(.) is the cost of taking action z. It is assumed that the farmer dislikes working harder, i.e., $C'(z) > 0$.

The optimal contract is found by solving the following program

$$(31) \quad \max_{I} \lambda((1-\gamma(z)U(R_1) + \gamma(z)U(R_3)) + (1-\lambda)((1-\gamma(z))U(R_2) + \gamma(z)U(R_4)) - C(z)$$

subject to

$$(32) \qquad \gamma'(z)\{\lambda U(R_3) + (1-\lambda)U(R_4) - \lambda U(R_1) - (1-\lambda)U(R_2)\} - C'(z) = 0$$

where the R_i's are defined in (11)[14] and (32) is the first order condition for maximizing the farmer's expected utility with respect to z. (32), which is the incentive constraint[15] ensures the consistency of the optimal contract with the action chosen by the farmer. (32) can be rewritten as

$$(33) \qquad E^{p}U(r_B(p)) - E^{p}U(r_G(p)) = C'(z)/-\gamma'(z)$$

Since the RHS of (33) is non-negative, the optimal contract, in order to preserve incentives, is such that the expected utility in the low output states is greater than the expected utility in the high output states.

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[14] Notice that the fair insurance condition is already embedded in the objective function.

[15] More generally, the incentive constraint is $z \in \operatorname{argmax}_{z'} EU(r) - C(z')$. This is equivalent to (29) if $EU(r) - C(z)$ is strictly concave in z which is guaranteed by the convexity of γ and C, i.e., $\gamma''(z) > 0$ and $C''(z) > 0$. See Rogerson.

Let I^* be the solution to the unconstrained problem (i.e., the optimal insurance without the incentive constraint) and I^{**} be the solution to the constrained problem. Then it is easy to see that $I^* \geq I^{**}$. The inequality is strict if I^* does not satisfy the incentive constraint. The following result is helpful in assessing the circumstances in which I^* violates the constraint.

Proposition 9 The unconstrained solution I^* has the following property

(i) $E^pU(r_G(p)) > E^pU(r_B(p))$ for decreasing absolute risk aversion utility functions

(ii) $E^pU(r_G(p)) = E^pU(r_B(p))$ for constant absolute risk aversion utility functions

(iii) $E^pU(r_G(p)) < E^pU(r_B(p))$ for increasing absolute risk aversion utility functions

where the utilities are evaluated at $I = I^*$

Proof: See Appendix A.

Thus, if $I = I^*$, the incentive constraint is violated whenever the utility function exhibits constant or increasing risk aversion. In such instances, the optimal incentive compatible insurance (I^{**}) will be smaller than I^*. In the case of decreasing risk aversion, the nature of constraint depends upon the marginal cost of actions C'(z). For small C'(z), the constraint will not be violated and the optimal insurance will still be that which equates the expected marginal utilities across the two output states. In general, the incentive problem is least serious for decreasing risk averse individuals simply because their insurance coverage is already curtailed by price risk[16].

We now turn to the issue of the effect of price risk on the optimal incentive compatible insurance. If the incentive constraint is not binding, previous results are unaltered. So suppose now that the incentive constraint is binding. The first order condition for maximizing (31) subject to (33) is

$$\gamma(1 - \gamma)(E^PU'(r_B) - E^PU'(r_G)) - \mu(\gamma E^PU'(r_G) + (1 - \gamma)E^PU'(r_B)) = 0$$

where μ is the Lagrange multiplier or

$$\gamma(1 - \gamma)(E^PU'(r_B) - E^PU'(r_G)) = \mu(\gamma E^PU'(r_G) + (1 - \gamma)E^PU'(r_B))$$

When the incentive constraint is binding, the optimal insurance does not equate the expected marginal utilities. Instead, to preserve incentives, the optimal insurance I^{**} is less than I^* and consequently $E^PU'(r_B) > E^PU'(r_G)$.

Proposition 10 $I^{**} < P_1(Q_1 - Q_2)$

Proof In Proposition 1 it was shown that $I^* < P_1(Q_1 - Q_2)$. Since $I^* \geq I^{**}$, the result follows. Note the above result could also be rewritten as $R_1^* > R_3^*$.

Proposition 11 Suppose the optimal amount of insurance is at least as great as value of the crop loss in the low price state. Then, if $U''' > 0$, increasing price risk reduces the optimal amount of insurance. If the optimal insurance is smaller than the value of the crop loss in the low price state, the effects of increasing price risk are indeterminate.

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[16] Imai, Geanakoplos and Ito report a result opposite to ours. They consider unemployment insurance schemes where severance payments are made to laid-off workers. But "since severance payments usually do not depend on outcomes at alternative opportunities after layoff they are considered at best incomplete insurance for layoff" (Ito). The issue that is investigated is whether the laid off worker could be better off, in an ex-ante sense, than the retained worker. This is indeed the case for individuals with decreasing risk averse utility functions. Individuals with constant risk aversion utility functions are indifferent between the two states while increasing risk averse individuals prefer to be retained. So, the incentive problem is most serious for decreasing risk averse individuals. The difference between their model and ours lies in the assumption about the uninsured variable. The uncertainty in the rehiring wage, in the Imai et.al model, affects only the marginal benefit of insurance (the expected marginal utility if the worker is laid off) and not the marginal cost (the marginal utility of the *sure* wage net of premium payments). See Ito and Machina, and Ito for further variants of the problem.

Proof The first order condition for optimal insurance is

$$\gamma(1 - \gamma)(E^PU'(r_B) - E^PU'(r_G)) - \mu(\gamma E^PU'(r_G) + (1 - \gamma)E^PU'(r_B)) = 0 \tag{34}$$

Concavity of the utility function guarantees the satisfaction of second order conditions. So the effect of price risk on I^{**} depends on how the LHS of (34) responds to changes in price risk.

Consider the case when $I^* > P_2(Q_1 - Q_2)$. Then $R_4^* > R_2^*$. Combining with Proposition 10, we have $R_1^* > R_3^* > R_4^* > R_2^*$. This inequality and the positivity of U''' is sufficient for increases in price risk to negatively affect the first term of (34) (see Proposition 4). On the other hand, a mean preserving increase in price risk increases $(\gamma E^PU'(r_G) + (1 - \gamma)E^PU'(r_B))$ which is a linear combination of two convex functions. Therefore, the sum effect of an increase in price risk is to decrease the the terms on the LHS of (34), and hence the optimal amount of insurance is reduced.

If $I^* < P_2(Q_1 - Q_2)$, then it is not possible to sign the response of the first term in the LHS of (34) to an increase in price risk. Therefore, the net effect is indeterminate.

VII. SUMMARY AND CONCLUDING REMARKS

This paper has explored some properties of incomplete insurance schemes. The basic idea motivating all insurance schemes is that the insured individual pays premiums in "good" times in return for protection during "bad" times. This idea is, however, weakened by an output insurance scheme which does not make fine enough distinctions between the various income states. The optimality condition requires the equalization of the expected marginal utility across the output states of nature. This reflects the fact that the incomplete insurance scheme transfers income from the high output to the low output states. The costs of this transfer are borne in the high output states which includes the state of the world when output is high but price is low. Insurance is, therefore, less attractive to individuals who dislike the prospect of a cash drain (due to premium payments) at a time of a loss in value of the output.

If the actions of farmers are not observable, optimal insurance contracts have to be consistent with the incentive constraint. For decreasing risk averse individuals, even without the incentive constraint, the insurance is so weak that they prefer the high output states to the low output states i.e., $E^P U(r_G(p)) > E^P U(r_B(p))$. For this reason, the potential moral hazard associated with an insurance contract that equalizes expected marginal utilities, is least serious for this class of utility functions and the incompleteness due to price risk helps resolve the incentive problem.

In general, moral hazard considerations may lead to a decline in the amount of optimal insurance. If the resulting insurance coverage is so limited as to be less than the value of output loss in the low price state, the effect of increasing price risk is ambiguous. However, if insurance is at "significant" levels, i.e., greater than $P_2(Q_1 - Q_2)$, increasing price risk will reduce the optimal amount of insurance.

The results of this paper extend to all insurance schemes which are incomplete due to multiplicative risks. The implications of these results will depend on the specific context in which multiplicative risks arise. In this chapter, by way of example, we have highlighted the incompleteness in crop insurance caused by price risk. In the U.S., crop insurance is an element of agricultural policy and recent policies have emphasized federal crop insurance as the most appropriate way for taxpayers to share farmers' risks (Todd). The rationale for increasing government's financial commitment to crop insurance rests on the presumed benefits to risk reduction. But, as we have argued here, the incompleteness of crop insurance is a factor which limits the transfer of risk from the agricultural sector to the government. In this connection, there have been suggestions that the crop insurance program be transformed into a scheme insuring crop revenue/income rather than crop output (Farm Income Insurance Task Force, Offut). Apart from the daunting complexity of administering such a scheme, the effects on private risk markets (futures, options) must also be considered (Petzel). However, as our results indicate, the same objective might be achieved by exploiting the complementarity between crop insurance and hedging activity. As a policy option, it might also be more feasible for the government agencies selling insurance to work with the futures trading organizations and the managers of the grain elevators in order to jointly market crop insurance and forward contracts.

For the study of incomplete insurance schemes, the implications of our analysis are principally to draw attention to the important role of assumptions about utility functions and about the interaction between multiple risks. Since the first order condition involves expected marginal utilities, the impact of the background risk on the choice of optimal insurance will depend on the third derivative of the utility function. In models with mean-variance or quadratic utility functions, the effects of background risk will wash out unless it is correlated with the insurable risk. As regards the interaction of risks, the appropriate specification of risks will have to be guided by the context. In this chapter, we contrasted the additive case with the multiplicative specification. Many real world incomplete insurance schemes may occur in settings which satisfy neither alternatives[17]. For this reason, it might be worthwhile in future investigations, to consider more general structures capable of accommodating a variety of cases.

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[17] The unemployment insurance scheme considered by Imai, Geanakoplos and Ito is an example

APPENDIX A

Proof of Proposition 4:

Since $\eta''(I) < 0$, $\partial I^*/\partial P_2$ is of the same sign as $\partial\eta'(I)/\partial P_2$ evaluated at I^*.

$$\partial\eta'(I)/\partial P_2\big|_{I^*} = \gamma(1-\gamma)\Big\{\lambda(U''(R_3^*)Q_2 - U''(R_1^*)Q_1)dP_1/dP_2 - (1-\lambda)(U''(R_2^*)Q_1 - U''(R_4^*)Q_2)\Big\}$$

Substituting for dP_1/dP_2

$$\begin{aligned}\partial\eta'(I)/\partial P_2\big|_{I^*} &= \gamma(1-\gamma)(1-\lambda)\Big\{U''(R_1^*)Q_1 - U''(R_3^*)(Q_1 - (Q_1 - Q_2)) \\ &\quad + U''(R_4^*)(Q_1 - (Q_1 - Q_2)) - U''(R_2^*)Q_1\Big\}. \\ &= \gamma(1-\gamma)(1-\lambda)\Big\{(U''(R_1^*) - U''(R_3^*))Q_1 \\ &\quad + (U''(R_4^*) - U''(R_2^*))Q_1 + (U''(R_3^*) - U''(R_4^*))(Q_1 - Q_2)\Big\}\end{aligned}$$

which is strictly positive (zero, strictly negative) because $R_1^* > R_3^* > R_4^* > R_2^*$ and $U''' > (=, <)\ 0$.

Proof of Proposition 6:

Let R_j^* denote R_j evaluated at the optimal insurance and hedge level.

(20) implies $(1 - \gamma)[U'(R_2^*) - U'(R_1^*)] = \gamma[U'(R_3^*) - U'(R_4^*)]$

$\Rightarrow R_2^* \overset{<}{\underset{>}{=}} R_1^*$ as $R_3^* \overset{<}{\underset{>}{=}} R_4^*$

Now $R_2^* - R_1^* = (P_2 - P_1)Q_1 + (P_1 - P_2)f^* = (P_1 - P_2)(f^* - Q_1)$ and

$R_3^* - R_4^* = (P_1 - P_2)(Q_2 - f^*)$

If $R_2^* > R_1^*$ and $R_3^* > R_4^*$, that would imply $f^* > Q_1$ and $Q_2 > f^*$ which contradicts that Q_2 is the smaller output. Similarly, $R_2^* = R_1^*$ and $R_3^* = R_4^*$ is a contradiction. Hence $R_1^* > R_2^*$ and $R_4^* > R_3^*$ and therefore $Q_1 > f^* > Q_2$.

(24) implies

$\lambda[U'(R_3^*) - U'(R_1^*)] = (1 - \lambda)[U'(R_2^*) - U'(R_4^*)]$

$\Rightarrow R_3^* \overset{<}{\underset{>}{=}} R_1^*$ as $R_2^* \overset{<}{\underset{>}{=}} R_4^*$.

Now $R_3^* - R_1^* = P_1Q_2 + (1 - \gamma)\rho^*(Q_1 - Q_2) + f^*(p^e - P_1)$

$$- P_1Q_1 + \gamma\rho^*(Q_1-Q_2) - f^*(p^e- P_1)$$

$$= (\rho^* - P_1)(Q_1-Q_2)$$ and similarly

$$R_2^* - R_4^* = (P_2 - \rho^*)(Q_1-Q_2)$$

If $R_3^* > R_4^*$ and $R_2^* > R_4^*$ that would imply $\rho^* > P_1$ and $P_2 > \rho^*$ which contradicts P_1 being the higher price. Similarly $R_3^* = R_1^*$ and $R_2^* = R_4^*$ is a contradiction. Hence $R_3^* < R_1^*$ and $R_2^* < R_4^*$ and therefore, $P_1 > \rho^* > P_2$.

Proof of Proposition 8:

The strategy is to apply the implicit function theorem to equations (25) and (26) to discover the response of the endogenous variables ρ_f^* and μ^* to a change in the exogenous variable $\hat{f}$.

The general statement of the implicit function theorem is as follows. X is a vector of choice variables, b is vector of parameters, and G is a differential map such that $G(X^o;b) = 0$ and the matrix $\partial G(X^o;b)/\partial X = D_xG$ is non singular. Then one can solve for a differentiable function $x^*(b)$ such that $G(x^*(b),b) = 0$ holds as an identity. Further $D_bx^* = -[D_xG(X^o;b)]^{-1}[D_bG(X^o;b)]$

In the analysis here,

$X = \begin{bmatrix} \rho_f^* \\ \mu^* \end{bmatrix}$, $G = \begin{bmatrix} L_\rho \\ L_f \end{bmatrix}$, and $b = \hat{f}$, where L is the Lagrangian function of the constrained maximization problem i.e., $L(\rho,f,\mu) = \eta(\rho,f) + \mu(\hat{f} - f)$.

Therefore,

$$\begin{bmatrix} \partial\rho_f^*/\partial\hat{f} \\ \partial\mu^*/\partial\hat{f} \end{bmatrix} = \frac{-1}{|D_xG|} \begin{bmatrix} L_{f\mu} & -L_{\rho\mu} \\ -L_{f\rho} & L_{\rho\rho} \end{bmatrix} \begin{bmatrix} L_{\rho f} \\ L_{ff} \end{bmatrix}$$

where $|D_xG| = L_{\rho\rho}L_{f\mu} - L_{\rho\mu}L_{f\rho} = L_{\rho\rho}L_{f\mu}$ since $L_{\rho\mu} = 0$. Further $L_{f\mu} = - 1$ and $L_{\rho\rho} = \eta_{\rho\rho}$ is also negative due to the concavity of the utility function. Hence $|D_xG|$ is strictly positive.

$\partial\mu^*/\partial\hat{f} = \frac{-1}{|D_xG|}[L_{\rho\rho}L_{ff} - (L_{\rho f})^2] < 0$ because the numerator is positive by the second order sufficient condition.

$$\partial\rho_f^*/\partial\hat{f} = \frac{-1}{|D_xG|}[L_{f\mu}L_{\rho f}]$$

$L_{f\mu} = -1$ and $|D_xG|$ is positive. So the sign of $\partial\rho_f^*/\partial\hat{f}$ depends on the sign of $L_{\rho f}$ which is equal to $\eta_{\rho f}$.

$$\eta_{\rho f} = (1-\gamma)\gamma(Q_1-Q_2)[\lambda U''(R_3^*)(p^e - P_1) + (1-\lambda)U''(R_4^*)(p^e - P_2)$$
$$- \lambda U''(R_1^*)(p^e - P_1) - (1-\lambda)U''(R_2^*)(p^e - P_2)]$$
$$= (1-\gamma)\gamma(Q_1-Q_2)[\lambda(U''(R_1^*) - U''(R_3^*))(P_1 - p^e)$$
$$+ (1-\lambda)(U''(R_4^*) - U''(R_2^*))(p^e - P_2)]$$

$\gtreqless 0$ as $U''' \gtreqless 0$ because $R_1^* > R_3^*$ and $R_4^* > R_2^*$ (Proposition 7).

Proof of Proposition 9:

Define the inverse of the marginal utility function by m: $m \equiv (U')^{-1}$. Also define ν as the composite function of u and m: $\nu = U \circ m$. Denoting by α a value of marginal utility of income the following relation holds

$$\nu(\alpha) = U(m(\alpha))$$

Lemma: $\nu''(\alpha)$ is greater than, equal to, or less than zero depending on whether the utility function exhibits decreasing, constant or increasing absolute risk aversion.

Proof of lemma: See Imai, Geanakoplos and Ito

From the first order condition (to the unconstrained maximum) $EU'(r_G(p)) = EU'(r_B(p))$. From Proposition 1, we also know $U'(R_2^*) > U'(R_4^*) > U'(R_3^*) < U'(R_1^*)$. At the optimum, $U'(r_G)$ is a mean preserving spread of $U'(r_B)$. But if absolute risk aversion is decreasing, ν is a strictly convex function of marginal utility (from lemma). Therefore $E\nu(U'(r_G)) = EU(r_G) > EU(r_B) = E\nu(U'(r_B))$ The proof is similar for other cases.

APPENDIX B

$A_{uj}'(x) = (U_j''/U_j')^2 - U_j'''/U_j'$. Dividing throughout by A_{uj}, the equation becomes $A_{uj}'(x)/A_{uj} = A_{uj} - A_{vj}$ or

$A_{uj} = A_{vj} + (A_{uj}'(x)/A_{uj})$.

Consider the CARA utility function. Since $A_{uj}' = 0$, $A_{uj} = A_{vj}$ and the CMU is verified. If the utility function is of the CRRA type with k as the constant relative risk aversion parameter, $A_{uj} = k_j/x$ and $A_{uj}' = -k_j/x^2$. Therefore, $A_{u1} \geq A_{u2}$ implies $A_{v1} - (k_1/x^2)/(k_1/x) \geq A_{v2} - (k_2/x^2)/(k_2/x)$ or $A_{v1} \geq A_{v2}$ which verifies the CMU condition.

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