Title: AJAE appendix for *Estimability and Identifying the Estimability Status of a System of Restrictions*

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Appendix: Estimability and Identifying the Estimability Status of a System of Restrictions

This appendix contains a necessarily brief review of the concept of estimability and presents a method whereby information from the structure of a lagrangian augmented (LAUG) system of normal equations can be used to identify the estimability status of a set of simultaneous restrictions. (In the following discussion, we use the terms estimable or non-estimable restrictions to refer to restrictions involving estimable or non-estimable contrasts).

Consider the initially unrestricted normal equations:

(A-1) $(X'X)\underline{\beta} = X'\underline{y}$.

If $(X'X)$ is of full rank, $\underline{\hat{\beta}} = (X'X)^{-1} X'\underline{y}$ is a unique solution and the classical results with respect to hypothesis testing and restricted estimation apply. Specifically, all hypotheses with respect to linear functions of the $\underline{\hat{\beta}}$ parameter are testable as are all sets of simultaneous restrictions that are internally consistent.

When $(X'X)$ is singular, the classical results do not apply as system (A-1) has an infinite number of solutions:

(A-2) $\underline{\tilde{\beta}} = (X'X)^{-} X'\underline{y}$

where $(X'X)^{-}$ denotes any one of an infinite number of generalized inverses (g-inverses) of $(X'X)$ such that $(X'X)(X'X)^{-}(X'X) = (X'X)$. Searle provides a description of all such g-inverses of $(X'X)$. Each g-inverse of $(X'X)$ generates a different $\underline{\tilde{\beta}}$ estimate although all such estimates can be shown to generate identical predicted values $\underline{\hat{y}}$ and identical Sum of Square Errors (SSE).

A problem introduced by the singularity of $\left(X'X\right)$ is that some hypotheses on $\underline{\tilde{\beta}}$ are no longer testable. A well-known example in econometrics is the dummy variable problem in which the model is singular if all dummy variables are included in a model with an intercept. Imposing a restriction, i.e., deleting one of the dummy variables, is the usual approach to identifying this model. However, the t-statistic on a given dummy variable is no longer a test of individual dummy variable significance but rather a test of whether the dummy variable statistically differs from the effect of the deleted variable. Identical inferences could have been tested if all dummy variables had been included and the system estimated with a generalized inverse.

The problem with hypothesis testing in a singular system arises when imposing a particular hypothesis testing restriction:

(A-3) $\underline{r}'\underline{\beta} = \delta_r$

There are two possibilities when a single restriction is imposed. If $\underline{r}'$ is in the row space of X or equivalently $\left(X'X\right)$, the function or contrast $\underline{r}'\underline{\beta}$ is termed "estimable" and the SSE_R of the restricted system may exceed the unrestricted SSE_0 for some δ_r. If $\underline{r}'$ is not in the row space of X, the contrast $\underline{r}'\underline{\beta}$ is non-estimable, the restriction $\underline{r}'\underline{\beta} = \delta_r$ is "non-testable" and $SSE_R = SSE_0$ for all possible values of δ_r.

When (A-3) is replaced with a set of consistent restrictions:

(A-4) $R\underline{\beta} = \underline{\delta}_R$

the problem becomes more complex. The additional complexity arises because it is possible that: (1) each restriction in (A-4) is individually testable (the corresponding rows are each estimable with respect to X or (X'X)), (2) each row in R is "system non-estimable" i.e., linearly independent of the rows of $\left(X'X\right)$ and the remaining rows of R, or (3) the contrasts of set of simultaneous restrictions may contain, or be equivalent to, a mixture of estimable and non-estimable contrasts. Situations (1) and (2) are commonly addressed in the literature (Rao 1962; Searle 1997). The problem of mixed estimable and non-estimable contrasts and the corresponding restrictions is not commonly discussed. Since our model involves mixtures of estimable and non-estimable contrasts, we present methods that address the estimable non-estimable restrictions problem.

Consider the Lagrangian augmented (LAUG) system for the restricted first order conditions:

(A-5) $$\begin{pmatrix} \left(X'X\right) & R' \\ R & 0 \end{pmatrix}\begin{pmatrix} \underline{\beta} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} X'\underline{y} \\ \underline{\delta}_R \end{pmatrix}$$

or

(A-6) $$L \qquad \underline{\alpha} \qquad = \qquad \underline{g}$$

where $\underline{\lambda}$ is a vector of lagrangian multipliers. A solution to the above system can be written as:

(A-7) $$\begin{pmatrix} \underline{\tilde{\beta}} \\ \underline{\tilde{\lambda}} \end{pmatrix} = \begin{pmatrix} \left(X'X\right) & R' \\ R & 0 \end{pmatrix}^{-} \begin{pmatrix} X'\underline{y} \\ \underline{\delta}_R \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}\begin{pmatrix} X'\underline{y} \\ \underline{\delta}_R \end{pmatrix}$$

or

(A-8) $\underline{\tilde{\alpha}} = L^{-} \underline{g} = G \underline{g}$

when $\underline{\tilde{\alpha}}$ denotes “a” solution to the restricted normal equations and G is a generalized inverse of the augmented matrix L.

If R contains a sufficient number of system non-estimable restrictions, L is nonsingular and $G = L^{-1}$. If L is singular, there are an infinite number of G and $\underline{\tilde{\alpha}}$ in (A-7) and (A-8) that minimize the SSE_R and generate identical predicted values $\underline{\tilde{y}}$ and SSE_R. Results available from authors prove that if R is of full row rank, G_{22} is symmetric and invariant for all G-Inverses of L[1]. It can also be shown that:

(A-9) $\underline{\tilde{\lambda}} = G_{22}\left(\underline{\delta}_R - \underline{\delta}_0\right)$

and

(A-10) $SSE_R - SSE_0 = -\left(\underline{\delta}_R - \underline{\delta}_0\right)' G_{22}\left(\underline{\delta}_R - \underline{\delta}_0\right)$

where $\underline{\tilde{\delta}}_0 = R\,\underline{\tilde{\beta}}_0$ and $\underline{\tilde{\beta}}_0$ is any solution to the original unrestricted system. Using (A-10), a number of results can be derived including: (i) the rank of G_{22} is equivalent to the dimension of a basis set for the estimable restriction space implied by R, (ii) the eigenvectors of G_{22} can be used to derive an equivalent[2] set of restrictions R^* in which the first (rank of (G_{22})) restrictions are system estimable and the remaining restrictions are non-system-estimable, and (iii) the G_{22}^* associated with R^* is diagonal with the first (rank of (G_{22})) elements of G_{22}^* equal to the non-zero eigen-values of the G_{22}. These results imply that if any rows (and corresponding columns) in G_{22} are zero, the corresponding contrasts in R are system non-estimable and the lagrangians will remain zero for all δ_R associated with the corresponding rows. It is also important to note that the converse is not true, i.e. $\underline{\tilde{\lambda}} \neq \underline{0}$ does not imply that the entire set of corresponding rows in R are system estimable. This result is demonstrated in the following example.

In the following, we apply the above results and demonstrate that: (a) simultaneously imposing the T with-in year (and individually non-testable) restrictions $\sum_u a_u \Delta_{u,t} = 0$ results in one implicitly non-testable and T-1 implicitly testable restrictions and (b) simultaneously adding U-1 of the across year restrictions $\sum_t \Delta_{u,t} = 0$ enables the identification of the system but does not affect the SSE as the across-year restrictions are jointly non-testable. The latter result implies that the right-hand-sides of the across-year summing restrictions are arbitrary with $\sum_u \Delta_{u\,t} = \delta_u$ generating the same SSE for all δ_u.

Tables A1 and A2 presents the yield vectors and the X, $(X'\hat{\Sigma}^{-1}X)$, $X'\hat{\Sigma}^{-1}\underline{y}$, and R matrices from the main text's example wheat farm. In the main body's table 2, farm yields for HONEST and SWITCH are the reported yields for the main text's example (table 1) of returns to yield switching for a wheat farm[3]. The reported yields for HAIL are a construct with all yields (except $y_{1,10}$ = 0) set equal to these from HONEST. Below we refer to HONEST, SWITCH, AND HAIL as if the data came from three different "farms" that were being subjected to yield switching tests. Since each "farm's" reported yields are from the same county and years, each "farm" has an identical design and Σ matrix. For this example, we assume that each "farm" has equal acreage in each unit resulting in an identical R matrix (table A1) for each farm. The first four rows in R are the within year $\sum_u a_u \Delta_{u,t} = 0$ restrictions. The fifth row of R is the across year $\sum_t \Delta_{1,t} = 0$ restriction.

The augmented lagrangian matrix for the system is singular when only the within year restrictions are imposed but becomes invertible with the addition of the fifth or across-year restriction. Table A2 presents the partitioned inverse of the augmented lagrangian matrix L and the $X'\Sigma^{-1}\underline{y}$, $\underline{\tilde{\beta}}$, and $\underline{\tilde{\lambda}}$ vectors for each of the three "farms." The lower right section of table A2 also presents the eigenvalues of the G_{22} matrix that will be used in discussing the estimability status of the restriction set R.[4]

The L^{-1} in table A2 has been partitioned in a manner corresponding to the following partition of the lagrangian augmented matrix:

$$\text{(A-11)} \quad \begin{pmatrix} X'\hat{\Sigma}^{-1}X & R_1' & R_2' \\ R_1 & 0 & 0 \\ R_2 & 0 & 0 \end{pmatrix}$$

when R_1 denotes the farm within-year restrictions and R_2 denotes the fifth or across-year restriction. With this particular structure, the partition of L^{-1} associated with:

$$\text{(A-12)} \quad \begin{pmatrix} X'\hat{\Sigma}^{-1}X & R_2 \\ R_1 & 0 \end{pmatrix}$$

is equivalent to the Moore-Penrose G-Inverse of L_1, our original system without the across-year summing restrictions.

An advantage of this example is that the results presented in table A2 can be used to demonstrate properties of both the singular L_1 system and the larger (and invertible) L system. With both systems there are (2T+U) or 10 parameters and only (2T) or 8 observations. The rank of the 10 x 10 ($X'\hat{\Sigma}^{-1}X$) matrix is eight (2T) indicating that ($X'\hat{\Sigma}^{-1}X$) is rank deficient of order 2 (U). Searle and Rao demonstrate that obtaining an invertible augmented system requires the identification of a set of system non-estimable restrictions equal in number to the rank deficiency. In our example, we need two such restrictions.

Each of the four within-year summing restrictions in system L_1 can be shown to be individually non-estimable (i.e., not in the row space of ($X'\hat{\Sigma}^{-1}X$)). However, when the four restrictions are imposed simultaneously, any three rows in R_1 can be expressed as a linear combination of the rows of ($X'\hat{\Sigma}^{-1}X$) and the remaining row in R_1. Hence, when the four R_1 restrictions are imposed simultaneously, the resulting L_1 system involves only one implicitly non-estimable and three implicitly estimable restrictions. Since the system contains only one implicitly non-estimable restriction, the rank deficiency of the L_1 system is only decreased from U to U-1 (or one in our example).

The estimability status of L_1 is readily identified by examining the eigenvalues of its G_{22}^1 matrix (the upper left portion of the G_{22} matrix in table A2). Table A2 indicates that the 4x4 G_{22}^1 matrix's eigenvalues are (-.0083, -.0070, -.0052, 0) implying that G_{22}^1 has rank three. These results imply that restriction set R_1 is equivalent to a restriction set with three system estimable and one system non-estimable restriction. Details for recovering an equivalent restriction set are available from the authors.

We wish to make a final comment with respect to system L_1. For this example, all solutions to the L_1 system are unchanged from those associated with L found in the corresponding partitions of the larger system solution. For system L_1, the four resulting lagrangians multipliers are non-zero. The four non-zero lagrangians imply that individually changing any of the restriction right-hand-side values from zero will influence the SSE. However, four non-zero lagrangian multipliers do not imply that the system has four implicitly estimable restrictions. A non-zero lagrangian only implies that the corresponding restriction can be written as a linear combination of at least one estimable restriction and other possibly non-estimable restrictions. It is possible that linear combinations of one system estimable and three system non-estimable restrictions can generate four non-zero lagrangians. The key point here is that the G_{22} matrix can be used to identify the estimability status of the system. While the lagrangian multipliers are closely related to the G_{22} matrix (see expression (A-9) above) the G_{22} matrix is the more fundamental of the two.

The results from the full system with the added across-year summing restriction are also presented in table A2. The G_{22} matrix is in the lower right corner of the L^{-1} matrix. For this representation of the restrictions, we note that the fifth row and column of G_{22} are zero vectors and the fifth lagrangian multiplier is zero. Since $SSE_R - SSE_0 = -\left(\underline{\delta}_R - \underline{\delta}_0\right)' G_{22}\left(\underline{\delta}_R - \underline{\delta}_0\right)$ and $\underline{\tilde{\lambda}} = G_{22}\left(\underline{\delta}_R - \underline{\delta}_0\right)$, it is immediately obvious that the fifth element of $\underline{\delta}_R$ can be set to any value without affecting the $SSE_R - SSE_0$ or

$\underset{\sim}{\tilde{\lambda}}_5$. This is demonstrated in the third section in the bottom half of the table where we set $\sum_{t=7}^{10} \Delta_{1,t} = \delta_5 = 5$ for the first farm. The numerical example demonstrates that changing δ_5 in $\sum_{t=7}^{10} \Delta_{1,t} = \delta_5$ does not affect the SSE of the switching model as farm HONEST's SSE-SW equals 3.49 for both $\delta_5 = 0$ and $\delta_5 = 5$.

Changing the δ_5 value from 0 to 5 changes the individual parameter estimates but it does not change the estimability status of the system or that of the parameters. Only those hypotheses that were testable under system L_1 are testable within system L. However, there is one additional complication that must be considered when selecting identification restrictions such as $\underline{r}'_5\underline{\beta} = \delta_5$. If additional hypotheses testing restrictions are to be imposed upon the identified system L, the additional restrictions must be consistent with any previously imposed restrictions. Since we wish to test additional restrictions that $\Delta_{u,t}$ are all simultaneously equal to zero, we choose identifiability restrictions of the form $\underline{r}'_5\underline{\beta} = 0$.

Endnotes

[1] If R is of less than full row rank, G_{22} is no longer unique. However, the following results remain applicable for the Moore-Penrose generalized inverse of L.

[2] Equivalent in the sense that the solutions $\underline{\tilde{\beta}}$ and $\underline{\tilde{\lambda}}$ remain unchanged by the change in restriction representation.

[3] We have shortened the number of years to four for the example.

[4] All solutions, inverses, Moore Penrose generalized inverses, and eigenvalues reported in table A2 and discussed in this appendix were generated in Microsoft Excel© using the Simetar addin.

TABLE A1: DATA AND DESIGN MATRICES FOR THE EXAMPLE WHEAT FARM

DESIGN MATRIX

UNIT-YR	γ_1	γ_2	$\Delta_{1,7}$	$\Delta_{1,8}$	$\Delta_{1,9}$	$\Delta_{1,10}$	$\Delta_{2,7}$	$\Delta_{2,8}$	$\Delta_{2,9}$	$\Delta_{2,10}$
1-7	29.0	0	1	0	0	0	0	0	0	0
1-8	22.0	0	0	1	0	0	0	0	0	0
1-9	19.0	0	0	0	1	0	0	0	0	0
1-10	40.0	0	0	0	0	1	0	0	0	0
2-7	0	29.0	0	0	0	0	1	0	0	0
2-8	0	22.0	0	0	0	0	0	1	0	0
2-9	0	19.0	0	0	0	0	0	0	1	0
2-10	0	40.0	0	0	0	0	0	0	0	1

FARM YIELDS

HONEST	SWITCH	HAIL
44.0	11.0	44.0
15.0	30.0	15.0
26.0	46.0	26.0
68.0	17.0	0.0
32.0	65.0	32.0
25.0	5.0	25.0
27.0	7.0	27.0
47.0	98.0	47.0

X' SIGMA-INV X

γ_1	γ_2	$\Delta_{1,7}$	$\Delta_{1,8}$	$\Delta_{1,9}$	$\Delta_{1,10}$	$\Delta_{2,7}$	$\Delta_{2,8}$	$\Delta_{2,9}$	$\Delta_{2,10}$
47.6786	0.0000	0.3867	0.3729	0.1919	0.6154	0.0000	0.0000	0.0000	0.0000
0.0000	47.6786	0.0000	0.0000	0.0000	0.0000	0.3867	0.3729	0.1919	0.6154
0.3867	0.0000	0.0133	0	0	0	0.0000	0	0	0
0.3729	0.0000	0	0.0169	0	0	0	0.0000	0	0
0.1919	0.0000	0	0	0.0101	0	0	0	0.0000	0
0.6154	0.0000	0	0	0	0.0154	0	0	0	0.0000
0.0000	0.3867	0.0000	0	0	0	0.0133	0	0	0
0.0000	0.3729	0	0.0000	0	0	0	0.0169	0	0
0.0000	0.1919	0	0	0.0000	0	0	0	0.0101	0
0.0000	0.6154	0	0	0	0.0000	0	0	0	0.0154

X' SIGMA-INV Y

HONEST	SWITCH	HAIL
69.4426	34.7296	27.5965
55.8003	88.6489	55.8003
0.5867	0.1467	0.5867
0.2542	0.5085	0.2542
0.2626	0.4646	0.2626
1.0462	0.2615	0.0000
0.4267	0.8667	0.4267
0.4237	0.0847	0.4237
0.2727	0.0707	0.2727
0.7231	1.5077	0.7231

R MATRIX

RESTRICTIONS	γ_1	γ_2	$\Delta_{1,7}$	$\Delta_{1,8}$	$\Delta_{1,9}$	$\Delta_{1,10}$	$\Delta_{2,7}$	$\Delta_{2,8}$	$\Delta_{2,9}$	$\Delta_{2,10}$
	0	0	1	0	0	0	1	0	0	0
WITHIN-YR	0	0	0	1	0	0	0	1	0	0
	0	0	0	0	1	0	0	0	1	0
	0	0	0	0	0	1	0	0	0	1
ACROSS-YR	0	0	1	1	1	1	0	0	0	0

TABLE A2: LAUG-INVERSE MATRIX AND SOLUTIONS FOR THE EXAMPLE WHEAT FARM

LAUG-INVERSE MATRIX

	G11									G12				
γ_1	γ_2	$\Delta_{1,7}$	$\Delta_{1,8}$	$\Delta_{1,9}$	$\Delta_{1,10}$	$\Delta_{2,7}$	$\Delta_{2,8}$	$\Delta_{2,9}$	$\Delta_{2,10}$	λ_1	λ_2	λ_3	λ_4	λ_5
0.0228	-0.0018	-0.0162	-0.0027	0.2160	-0.1971	0.0162	0.0027	-0.2160	0.1971	0.0005	0.0006	0.0025	-0.0019	-0.0091
-0.0018	0.0228	0.0162	0.0027	-0.2160	0.1971	-0.0162	-0.0027	0.2160	-0.1971	-0.0086	-0.0085	-0.0066	-0.0110	0.0091
-0.0162	0.0162	28.0834	-7.4209	-12.7422	-7.9202	-28.0834	7.4209	12.7422	7.9202	0.3682	-0.1318	-0.1318	-0.1318	0.2636
-0.0027	0.0027	-7.4209	23.6600	-9.8482	-6.3909	7.4209	-23.6600	9.8482	6.3909	-0.1000	0.4000	-0.1000	-0.1000	0.2000
0.2160	-0.2160	-12.7422	-9.8482	36.8454	-14.2550	12.7422	9.8482	-36.8454	14.2550	-0.0864	-0.0864	0.4136	-0.0864	0.1727
-0.1971	0.1971	-7.9202	-6.3909	-14.2550	28.5661	7.9202	6.3909	14.2550	-28.5661	-0.1818	-0.1818	-0.1818	0.3182	0.3636
0.0162	-0.0162	-28.0834	7.4209	12.7422	7.9202	28.0834	-7.4209	-12.7422	-7.9202	0.6318	0.1318	0.1318	0.1318	-0.2636
0.0027	-0.0027	7.4209	-23.6600	9.8482	6.3909	-7.4209	23.6600	-9.8482	-6.3909	0.1000	0.6000	0.1000	0.1000	-0.2000
-0.2160	0.2160	12.7422	9.8482	-36.8454	14.2550	-12.7422	-9.8482	36.8454	-14.2550	0.0864	0.0864	0.5864	0.0864	-0.1727
0.1971	-0.1971	7.9202	6.3909	14.2550	-28.5661	-7.9202	-6.3909	-14.2550	28.5661	0.1818	0.1818	0.1818	0.6818	-0.3636
0.0005	-0.0086	0.3682	-0.1000	-0.0864	-0.1818	0.6318	0.1000	0.0864	0.1818	-0.0051	0.0015	0.0008	0.0025	0.0000
0.0006	-0.0085	-0.1318	0.4000	-0.0864	-0.1818	0.1318	0.6000	0.0864	0.1818	0.0015	-0.0070	0.0008	0.0024	0.0000
0.0025	-0.0066	-0.1318	-0.1000	0.4136	-0.1818	0.1318	0.1000	0.5864	0.1818	0.0008	0.0008	-0.0047	0.0012	0.0000
-0.0019	-0.0110	-0.1318	-0.1000	-0.0864	0.3182	0.1318	0.1000	0.0864	0.6818	0.0025	0.0024	0.0012	-0.0037	0.0000
-0.0091	0.0091	0.2636	0.2000	0.1727	0.3636	-0.2636	-0.2000	-0.1727	-0.3636	0.0000	0.0000	0.0000	0.0000	0.0000
	G21									G22				

GLS NORMAL EQUATION RHS VALUES AND RESULTS WITH RESTRICTIONS

X' SIGMA-INV Y				BETAHAT VECTORS		
HONEST	SWITCH	HAIL		HONEST	SWITCH	HAIL
69.4426	34.7296	27.5965	γ_1	1.4134	0.9711	0.6655
55.8003	88.6489	55.8003	γ_2	1.2134	1.6166	1.0837
0.5867	0.1467	0.5867	$\Delta_{1,7}$	3.1000	-17.6409	12.0636
0.2542	0.5085	0.2542	$\Delta_{1,8}$	-7.2000	19.6000	-0.4000
0.2626	0.4646	0.2626	$\Delta_{1,9}$	-2.4000	25.6318	3.4727
1.0462	0.2615	0.0000	$\Delta_{1,10}$	6.5000	-27.5909	-15.1364
0.4267	0.8667	0.4267	$\Delta_{2,7}$	-3.1000	17.6409	-12.0636
0.4237	0.0847	0.4237	$\Delta_{2,8}$	7.2000	-19.6000	0.4000
0.2727	0.0707	0.2727	$\Delta_{2,9}$	2.4000	-25.6318	-3.4727
0.7231	1.5077	0.7231	$\Delta_{2,10}$	-6.5000	27.5909	15.1364
0.0000	0.0000	0.0000	λ_1	-0.0012	0.0064	0.1685
0.0000	0.0000	0.0000	λ_2	-0.1508	-0.1858	0.0129
0.0000	0.0000	0.0000	λ_3	0.0156	0.0194	0.0998
0.0000	0.0000	0.0000	λ_4	0.0764	0.0884	-0.1767
0.0000	0.0000	0.0000	λ_5	0.0000	0.0000	0.0000
			SSE_0	**3.49**	**5.17**	**10.31**

EFFECT OF CHANGING RHS RESTRCTION 5

X' SIGMA-INV Y		BETAHAT
HONEST		HONEST
69.443	GAM1	1.3680
55.800	GAM2	1.2589
0.587	D1-7	4.4182
0.254	D1-8	-6.2000
0.263	D1-9	-1.5364
1.046	D1-10	8.3182
0.427	D2-7	-4.4182
0.424	D2-8	6.2000
0.273	D2-9	1.5364
0.723	D2-10	-8.3182
0.000	LAMDA1	-0.0012
0.000	LAMDA2	-0.1508
0.000	LAMDA3	0.0156
0.000	LAMDA4	0.0764
5.000	LAMDA5	0.0000
	SSE_0	**3.49**

G22 RANK TEST

G22
EIGVALS
-0.0083
-0.0070
-0.0052
0.0000
0.0000