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Hedges and Trees: Incorporating Fire Risk into Optimal Decisions in Forestry Using a No-Arbitrage Approach

Margaret Insley and Manle Lei

This paper investigates the impact of including the risk of fire in an optimal tree harvesting model at the stand level, assuming timber prices follow a mean-reverting stochastic process. The relevant partial differential equation is derived under different assumptions about hedging the risk of fire. The assumption that fire risk is fully diversifiable is contrasted with the assumption that it can be hedged with another asset. It is conjectured that the risk-neutral probability of fire exceeds the historical probability of fire, which will affect forest land valuation. An empirical example is presented for two different silvicultural regimes.

Key words: fire risk, forest value, hedging, jumps, no-arbitrage, optimal harvesting, Poisson process, real options

Introduction

The valuation of forested land is an ongoing topic of research in the academic literature and is of vital practical concern to those involved in managing timberlands. In the United States a motivation for improving forest land valuation is the significant increase in holdings of private forests by institutional investors. Prior to the 1980s, industrial owners of lumber mills in the United States had typically maintained ownership of forested lands to supply their mills. Recognizing that the value of these lands was not adequately reflected in share prices, a trend emerged whereby integrated forest companies sold off timberland assets to willing institutional investors. These investors were able to manage timberlands more efficiently without the constraint of having to keep any particular mill in operation.[1] Aronow, Binkley, and Washburn (2004) note that changes in timberland property values are not well understood owing, in part, to the lack of a consistent time series of historical data on timberland values.

In the academic literature, the valuation of forested land has been a concern for over 150 years. The work by Faustmann in 1849 introduced the concept of "land expectation value" and the determination of the optimal harvest age assuming the land remained

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Margaret Insley is assistant professor, Department of Economics, University of Waterloo, Waterloo, Ontario; Manle Lei is a former Ph.D. student in the Department of Economics, University of Guelph. This work evolved from Lei's doctoral research at the University of Guelph.

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[1] This shift in the ownership of timberland is discussed in Global Institute of Sustainable Forestry (2002) and Caulfield and Newman (1999).

in forestry over an infinite series of future rotations.[2] Since the publication of Faustmann's paper, much research has been devoted to understanding the classic tree harvesting problem, and its many variations, whereby the opportunity cost of maintaining land in standing timber is balanced against the benefit of allowing the trees to grow for another period. [For a review of this literature, see Newman (2002).]

Over recent decades, one focus of the forest economics literature has been the impact of uncertainty on forest valuation and optimal harvesting.[3] Major sources of uncertainty are volatile prices for timber and production risks such as fire and pests. Although there are numerous papers dealing with either price or production risk alone, fewer studies consider the two jointly. The forestry literature has benefited from developments in finance and real-options theory on the valuation of risky assets, and several papers have adopted a real-options approach to valuing forestry investments (e.g., see Plantinga, 1998). A brief review of different approaches in the literature to optimal harvesting under uncertainty is provided in the following section, with a more thorough review contained in Insley and Rollins (2005).

Much of the existing literature on optimal harvesting under uncertainty estimates stand value using a constant risk-adjusted discount rate. For a stand subject to fire and price risk, this discount rate would be chosen to reflect the return required by the market to induce investors to hold the asset. Dixit and Pindyck (1994) call this the dynamic programming approach. The problem with this approach is that it is generally not correct to assume the discount rate will be constant (Trigeorgis, 1996). In the real-options literature, the preferred approach is to use contingent claims arguments and the assumption that markets are complete enough to permit an investor to hedge the risk of an investment. An assumption of the absence of arbitrage opportunities allows the determination of the fair value of the investment.[4] Insley and Wirjanto (2006) show that depending on key parameter assumptions, the dynamic programming approach with a constant discount rate[5] can give significantly different results than the contingent claims approach. The advantage of the latter approach is that it avoids the need to choose a discount rate and is consistent with modern finance theory.[6]

To apply contingent claims arguments to a tree harvesting problem, it must be decided how to handle the risk of destruction by fire or other catastrophe which causes a sudden jump in the value of the asset. One option is to assume that the risk of destruction is fully diversifiable. This implies the market will not reward an investor for holding an asset subject to fire or other production risk. In this case, when valuing a stand of trees, one would apply normal hedging arguments to eliminate price risk in a hypothetical hedging portfolio and assume the risk of fire, while not eliminated, generates no extra return. The assumption that the risk of a jump in asset value is fully diversifiable has been motivated in the literature by the concern that it is not feasible to hedge a stochastic process characterized by discrete jumps of random size (see Dixit

[2] A translation of Faustmann's original work in German is contained in Faustmann (1995).

[3] Caulfield and Newman (1999) review research on timberland investment risk.

[4] There are now numerous books that introduce the concepts of real options. Dixit and Pindyck (1994) and Trigeorgis (1996) are two of the most accessible.

[5] Here we generally refer to dynamic programming with a constant risk-adjusted discount rate as just the dynamic programming approach. In theory, one could choose a nonconstant discount rate with dynamic programming, but this is less common in practice. This is consistent with the terminology used by Dixit and Pindyck (1994).

[6] Its disadvantage occurs in situations when it is farfetched to assume the risk of an investment can be adequately hedged. There is an emerging literature that addresses this issue, such as Henderson (2006), for example. The issue of incomplete markets is beyond the scope of this paper.

and Pindyck, 1994, p. 120; and Wilmott, 1998, chap. 26). However, recent research has shown that jump processes can be adequately hedged with a reasonable number of hedging assets (see, e.g., Amin, 1993; Meyer, 1998; and D'Halluin, Forsyth, and Labahn, 2004). In addition, assuming the risk of fire is fully diversifiable, while convenient, may not be correct.

With the increased inclusion of timberland in investment portfolios, we conjecture that the ability to hedge production risks in forestry investments is increasing. This raises the question of what impact this may have on the value of timberland. It is possible that the market's perception of the risk of destruction from fire will differ from the historical risk of destruction. This is somewhat analogous to the observation that the premium demanded in bond markets to compensate for the possibility of default is much higher than the historical risk of default. In the finance literature dealing with credit risk, this is called the "credit spread puzzle."

This study extends the work of Insley and Rollins (2005) by including the risk of fire in a model of optimal harvesting over infinite rotations in which timber prices are modeled as a mean-reverting stochastic process. A contingent claims approach is used to derive the appropriate partial differential equation and linear complementarity problem (LCP) which completely specifies the decision problem. Using an empirical example, we compare the impact of fire risk on land value and optimal decisions when prices are stochastic with the case when prices are assumed constant. Two cases are examined—one in which the trees are allowed to regenerate naturally and one in which significant planting and other management expenses are incurred.

We derive three different partial differential equations which could be solved, in conjunction with the LCP, to find the value of a stand of trees. The first uses the dynamic programming approach and assumes that the asset will earn a constant risk-adjusted return. In the second, markets are assumed to be complete enough to allow the investor to hedge both the risks from price volatility and from fire. In the third, it is assumed that we can hedge the risk due to price volatility, while the risk of fire cannot be hedged but is fully diversifiable. For our empirical example, we do not pursue the dynamic programming approach, but rather adopt this third assumption that fire risk is diversifiable. Due to a lack of data on the perceptions of investors about fire risk, we are unable to draw definite conclusions about the impact of assuming the risk of fire can be hedged. However, we are able to observe the significance of any disparity between historical risks of fire and market perceptions.

The remainder of the paper proceeds as follows. The next section presents a review of some of the literature addressing optimal harvesting under uncertainty and, in particular, the previous literature that considers the impact of fire or other production risk. We then detail the different approaches to estimating the value of the stand under price and fire risk. A section devoted to the empirical example is then provided, demonstrating the impact of including the risk of fire in an optimal harvesting problem at the stand level. Concluding remarks are highlighted in the final section.

Literature Review

Early papers dealing with managing forests under price uncertainty include Hool (1966), and Lembersky and Johnson (1975). More recent contributions use insights from the finance literature and real-options theory. Examples of papers that model timber prices as a stochastic differential equation include Morck, Schwartz, and Strangeland (1989); Clarke and Reed (1989); Thomson (1992); and Plantinga (1998). Morck, Schwartz, and Strangeland (1989), and Insley and Wirjanto (2006) employ no-arbitrage arguments and a contingent claims approach to determine the value of a stand of trees, whereas other papers (e.g., Thomson, 1992; and Insley and Rollins, 2005) have estimated expected land value using an exogenously chosen discount rate.

Recent occurrences of very large forest fires in western Canada and the United States, as well as significant infestations of the pine beetle and other pests, have increased awareness of the need to consider production risks in valuation models. Two of the earliest papers to add fire risk to the optimal harvesting problem for an even-aged stand of trees were Martell (1980) and Routledge (1980). Their general result was to show a decline in land value as well as optimal rotation age. Reed (1984) derived the condition for optimal harvesting when the risk of fire or other catastrophe is modeled as a Poisson process. He observed that when fire risk is independent of stand age and causes total destruction of a stand of trees, the effect on the optimal harvesting age is the same as the effect of an increase in the discount rate. Using a stylized tree harvesting example, he found a moderate reduction in the optimal harvest age when fire risk is included.[7] These three papers all assume constant prices and costs.

More recent stand-level models have considered optimal actions by forest managers beyond the choice of the optimal harvest time in response to the risk of destruction by fire or other event. Thorsen and Helles (1998) present a model of optimal harvesting where the risk of destruction depends on stand characteristics and thinning treatments undertaken by the forest manager. In a similar spirit, Amacher, Malik, and Haight (2005) develop a stand-level model that considers actions such as the level and timing of fuel management activities, including planting densities and thinnings to remove surface fuels to reduce the losses once a fire starts. When these fuel management activities are taken into account, and depending on the function for the fire arrival rate, the authors found the optimal rotation age may rise or fall in response to an increase in fire risk. Yoder (2004) also considers optimal actions to control fire, but in his paper the benefits are from the standing forest rather than from periodic timber harvests.

Another stand-level model is the work of Englin, Boxall, and Hauer (2000) who consider the effect of amenity value of the standing forest in a Faustmann-type model in the presence of fire risk. Stollery (2005) examines optimal harvesting in a flammable forest when the benefits of carbon absorption are explicitly modeled.

Several papers (e.g., Reed and Errico, 1985; and Boychuck and Martell, 1996) have addressed the impact of fire risk at a forest level. This allows for multiple stands of different ages and partial harvesting rather than the clear cutting of the Faustmann model.

All of the papers mentioned so far have assumed nonstochastic costs and prices. Reed (1993) tackles the optimal harvesting problem for an old growth stand of trees, assuming the value of the stand follows geometric Brownian motion, given a constant timber volume. The standing timber provides an uncertain stream of amenity benefits modeled as simple geometric Brownian motion. In addition, there is a potential risk of

[7] For example, for a typical one-acre stand of Douglas Fir in British Columbia and a 3% discount rate, the optimal harvest age without fire risk is 70 years, while a fire arrival rate of 0.02 (averaging once in every 50 years) implies an optimal harvest age of 63 years.

catastrophe, modeled as a Poisson process. There are no harvest or planting costs. Reed's model permits an analytic solution and presentation of comparative static results. He solves for an optimal harvest rule, showing that harvesting is optimal if the ratio of the current value of timber to current amenity benefits exceeds a particular critical value. The presence of a time-independent risk of destruction lowers the expected present value of amenity benefits foregone through harvesting and lowers the critical value of the ratio that determines when harvesting is optimal.

Yin and Newman (1996) incorporate stochastic prices and the risk of fire in a forestry problem modeled at the forest level. They use a forest-level profit function, abstracting from the rotational element of the harvesting problem.

Motoh (2004) considers the effect of catastrophic risk on natural resource harvesting using a stylized model in which the value of the resource stock is assumed to follow geometric Brownian motion and is subject to periodic catastrophic events modeled as a Poisson process.[8] The catastrophic event drastically reduces the resource stock. As in Reed (1993), Motoh models stock value as the stochastic variable, rather than considering resource volume and price separately. He reports that the optimal rate of use of the resource is increasing in: the initial stock of the resource, resource stock uncertainty as described by the instantaneous volatility of the diffusion process, and the probability of catastrophe.

A common result in the above papers is that including the risk of destruction in an optimal harvesting model reduces the value of the forest land and reduces the optimal harvest age (when prices are assumed fixed). For papers where price or value are stochastic, the inclusion of risk of destruction reduces a critical value so that the expected harvesting time occurs sooner than when risk is not present. The exception is in Amacher, Malik, and Haight (2005) where the impact of fire risk on the rotation age depends on how the fire arrival rate relates to stand age and the extent to which a stand owner is able to mitigate the effect of fire through various fuel management activities.

The current paper confirms the qualitative results of the previous literature. We use a stand-level model where the only choice is whether or not to harvest the entire stand.[9] It represents an extension of the model developed in Insley and Rollins (2005) which uses a realistic stand-level timber growth curve and assumes timber prices are governed by a stochastic differential equation with a mean-reverting drift term. As such, the model is richer in certain respects than those of previous papers. The problem is defined over infinite rotations, and the opportunity cost of land is determined endogenously. The extension is in incorporating the risk of destruction of the stand and in using no-arbitrage arguments to determine the appropriate partial differential equation. We do not consider the effect of fuel management activities, leaving this issue for future work. However, we do consider the impact of salvaging a portion of the timber from a burned forest.

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[8] Willassen (1998) models forest value as a stochastic differential equation as a means of generalizing the Faustmann formula to stochastic growing forests. Motoh (2004) takes a similar approach but incorporates a Poisson process as well to capture catastrophic risk.

[9] We do not attempt a forest-level model in this paper because allowing for partial harvesting as well as stochastic prices creates many more state variables and adds considerable complexity. Whenever a portion of the forest is harvested, a new age cohort of trees is created (essentially a new stand of trees) which must be tracked.

Table 1. Notation for Key Parameters Used in the Model Specification

Symbol	Definition
P	Price of lumber ($/cubic meter)
η	Speed of mean reversion
$\overline{P}$	Long-run mean of lumber price ($/cubic meter)
σ	Instantaneous standard deviation of the proportionate change in price
z	Standard Wiener process
Q	Volume of wood available for harvest (cubic meters/hectare)
α	Stand age (years)
ϕ	Instantaneous probability of fire
C_h	Harvest and transport cost to the mill ($/cubic meter)
C_m	Management costs of maintaining the stand ($/hectare)
$V(P, t, \alpha)$	Value of the stand of trees under optimal management ($/hectare)
λ_P	Market price of risk with respect to price
λ_F	Market price of risk with respect to fire
r	Risk-free discount rate
S	Salvage value from harvesting trees after a fire ($/hectare)

Note: All $ amounts are given in Canadian dollars.

Different Approaches to Estimating Stand Value

In this section we develop our tree harvesting decision model, contrasting the dynamic programming and contingent claims approaches. For the reader's convenience, a summary of symbols used in the model specification is given in table 1. Assuming an even-aged stand of trees that will be used for commercial forestry, the value of the stand depends on the price of timber P, the age of the stand α, and time t. Denote the value of this asset as $V(P, t, \alpha)$, or just V for ease of notation.

The price of timber is assumed to follow a known stochastic process:

$$dP = a(P,t)dt + b(P,t)dz, \tag{1}$$

where $a(P,t)$ and $b(P,t)$ represent known functions, and dz is the increment of a Wiener process. Note that $a(P, t)$ is called the expected instantaneous drift rate since $E[dP] = a(P, t)dt$, and $b^2(P, t)$ is denoted the instantaneous variance rate since $\text{Var}[dP] = b^2(P,t)dz$. Following Insley and Rollins (2005), we adopt a mean-reverting price process of the form:

$$dP = \eta(\overline{P} - P)dt + \sigma P dz, \tag{2}$$

which implies $a(P,t) \equiv \eta(\overline{P} - P)$, and $b(P,t) \equiv \sigma P$.

The logic for this choice is that a stochastic process whereby the price level tends to revert to a long-run mean over time should provide a better description of the price path of commodities such as lumber or oil or copper [see Schwartz (1997) for support of this view]. This long-run mean would reflect production costs and the cost of substitutes.

Other authors have appealed to efficient markets to argue that commodity prices would follow some sort of random walk. However, McGough, Plantinga, and Provencher (2004) show that an efficient market can produce mean-reverting prices. Unfortunately, it is difficult to make definitive conclusions based on historical data as to whether mean reversion or a random walk is more likely the true data-generating process. Further discussion of this issue as well as statistical tests on the price data used in this paper are reported in Insley and Rollins (2005).

The volume of timber (Q) on the stand at any time is a deterministic function of stand age (α):

$$Q = g(\alpha). \tag{3}$$

However, stand age is a stochastic variable depending on the time of the last harvest (t_h), which depends on P, and on the occurrence of fire. If no fire has occurred, the stand age can be computed as:

$$\alpha = t - t_h. \tag{4}$$

If a fire occurs, the stand age will jump suddenly to zero. We assume for simplicity that fire will always consume the entire stand, although some timber may be salvaged. The risk of fire is specified as a Poisson process, which we denote by q, where

$$dq = \begin{cases} 0 \text{ with probability } 1 - \phi dt, \\ 1 \text{ with probability } \phi dt. \end{cases} \tag{5}$$

In equation (5), ϕ, the average arrival rate, represents the probability of fire over the infinitesimal interval dt. It follows that:

$$d\alpha = dt - \alpha dq. \tag{6}$$

Recall that a Poisson probability density function for a discrete random variable Z may be written as:

$$f(Z) = \begin{cases} \dfrac{\phi w^Z e^{-\phi w}}{Z!}, & Z = 0, 1, 2, \ldots; \\ 0 \text{ otherwise.} \end{cases} \tag{7}$$

Here, Z represents the number of events (in this case the occurrence of fire) in a fixed interval of time of length w; ϕw represents the average number of occurrences in the interval w. As noted above, the probability of a fire occurring in the infinitesimal interval dt is ϕdt. Using equation (7), the probability that no fire occurs ($Z = 0$) in the interval $(0, T)$ is $e^{-\phi T}$. With a Poisson process, the probability of one event in an interval is independent of events in other non-overlapping intervals. Hence, the probability that the first fire occurs in the interval $(T, T+dT)$ is equal to the probability of no fire in $(0, T)$ times the probability that a fire occurs in dT: $\phi dTe^{-\phi T}$. It follows that the expected time until the first fire occurs is:[10]

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[10] This is shown in Dixit and Pindyck (1994, p. 170). For a review of the Poisson distribution, see Hogg and Craig (1970).

(8) $$E(T) = \int_0^\infty \phi T e^{-\phi T} dT = \frac{1}{\phi}.$$

In deciding whether or not to harvest, the stand owner will consider the benefits from harvesting immediately versus delaying until the next period. If harvesting occurs today, the owner will receive the net revenue from harvesting and selling the timber, i.e., $(P - C_h)Q(\alpha)$, where C_h refers to harvesting costs per cubic meter, as well as the value of the bare land, $V(P, t, 0)$. If, instead, harvesting is delayed and the stand is allowed to grow for another period, the stand owner will experience a capital gain or loss due to the change in wood volume, change in price, and the possible occurrence of fire. In addition, the stand owner may have to incur management costs, denoted C_m in \$/ha, to maintain the stand. If a fire does occur, there may be costs of clearing and revenue from salvaged timber. S denotes the value of salvaged timber net of harvesting costs per hectare. In this paper, we assume S is some constant proportion of the value of the wood if it had been harvested immediately before the fire: $S = \gamma(P - C_h)Q(\alpha)$ for some parameter γ, which we set at either 0 or 0.5 for illustrative purposes.

The Dynamic Programming Approach

Assuming neither the risks from price changes nor fire can be hedged, then the value of the stand of trees must be estimated as an expected value with respect to price and the physical risk of fire. Hence, the dynamic programming approach is sometimes called the expected value approach. Using Ito's lemma adjusted for our Poisson process (see Dixit and Pindyck, 1994, p. 86), dV can be expressed as:

(9) $$dV = \mu V dt + sV dz - C_m dt + \left[V(P, t, 0) - V + S\right]dq,$$

where

(10) $$\mu = \left[V_t + a(P,t)V_P + V_\alpha + \tfrac{1}{2}b^2(P,t)V_{PP}\right]\frac{1}{V},$$

$$s = \frac{b(P,t)}{V}V_P.$$

Note that:

(11) $$V_P \equiv \frac{\partial V}{\partial P}; \quad V_{PP} \equiv \frac{\partial^2 V}{\partial P^2}; \quad V_t \equiv \frac{\partial V}{\partial t}; \quad V_\alpha \equiv \frac{\partial V}{\partial \alpha}.$$

We assume that in order to hold this asset, an investor would require a risk-adjusted return of ρ. This implies:

(12) $$E[dV] = \rho V,$$

where $E[dV]$ refers to the expectation over dP and dq. Substitute for μ and s in equation (9), take the expectation, and substitute the result into equation (12) to obtain:

(13) $$\rho V = V_t + V_\alpha + \tfrac{1}{2}b^2(P,t)V_{PP} + a(P,t)V_P + \phi\left[V(P,t,0) - V + S\right] - C_m.$$

Equation (13) is a partial differential equation which describes the value of the forested land, V, while it is optimal to refrain from harvesting. The required return on the asset in order for an investor to hold it willingly is given on the left-hand side as ρV. On the right-hand side we see the sources of return from holding the standing timber. These include how the value of the land changes with time (V_t), with price (V_P and V_{PP}), and with stand age (V_α). In addition, the expected loss from a fire is given by $\phi[V(P, t, 0) - V + S]$, which is the probability of fire multiplied by the loss from fire. Note that after a fire, the investor would lose the value V but would still have the value of the bare land $V(P, t, 0)$ plus any salvage value S.

Collecting terms in V and rearranging, equation (13) can be rewritten as:

$$V_t + \tfrac{1}{2}b^2(P,t)V_{PP} + a(P,t)V_P + V_\alpha + \phi\left[V(P,t,0) + S\right] - (\rho + \phi)V - C_m = 0. \tag{14}$$

As observed from equation (14), including the possibility of fire in the model has the effect of adding a risk premium (ϕ) to the risk-adjusted discount rate (ρ). This is consistent with results reported by Reed (1984).

The difficulty with using the expected value (or dynamic programming approach) is in determining the appropriate value for ρ. As discussed by Insley and Wirjanto (2006), we would not expect ρ to be constant as it will depend on the ratio V_P/V. We therefore focus on the contingent claims approaches detailed below.

The Contingent Claims Approach: Hedging Price and Fire Risk

We derive a partial differential equation followed by V assuming that asset markets are rich enough to allow us to hedge the risks due to price volatility and fire. We now denote the value of our stand of trees by V_1. We require one asset that depends on price (and not fire), which we denote as $V_2(P, t, \alpha)$ (or just V_2), and another asset whose value depends on the risk of fire alone, which we denote $V_3(P, t, \alpha)$ (or just V_3). V_2 and V_3 might represent shares or a portfolio of shares in forestry firms or ownership in an investment fund specializing in timberlands. We construct a portfolio with n_1 of V_1, n_2 of V_2, and n_3 of V_3. The value of our hedging portfolio, π, is then:

$$\pi = n_1V_1 + n_2V_2 + n_3V_3. \tag{15}$$

Over the interval dt, holding n_1, n_2, and n_3 constant, the change in the value of the portfolio is:

$$d\pi = n_1dV_1 + n_2dV_2 + n_3dV_3. \tag{16}$$

Using Ito's lemma, we can express dV_j, $j = 1, 2, 3$, as follows:

$$\begin{aligned} dV_1 &= \mu_1V_1dt + s_1V_1dz - C_{m1}dt + \left[V_1(P,t,0) - V_1(P,t,\alpha) + S_1\right]dq, \\ dV_2 &= \mu_2V_2dt + s_2V_2dz - C_{m2}dt, \\ dV_3 &= \mu_3V_3dt - C_{m3}dt + \left[V_3(P,t,0) - V_3(P,t,\alpha) + S_3\right]dq, \end{aligned} \tag{17}$$

where μ_j and s_j are defined as:

$$\mu_j = \left[(V_j)_t + a(P,t)(V_j)_P + (V_j)_\alpha + \tfrac{1}{2}b^2(P,t)(V_j)_{PP}\right]\frac{1}{V_j}, \quad j = 1, 2; \tag{18}$$

$$s_j = \frac{b(P,t)}{V_j}(V_j)_P, \quad j = 1, 2;$$

$$\mu_3 = \left[(V_3)_t + (V_3)_\alpha\right]\frac{1}{V_3}.$$

C_{mj} and S_j refer, respectively, to management costs and salvage value for asset j. For simplicity, we assume these elements are zero for the two hedging assets, V_2 and V_3. Note that V_2 depends only on price risk, and V_3 depends only on the risk of fire.[11] To eliminate risk in our hedging portfolio, as is reflected in dz and dq, we choose n_1, n_2, and n_3 so that the following equations are satisfied:

$$n_1 s_1 V_1 + n_2 s_2 V_2 = 0 \tag{19}$$

and

$$n_1\left[V_1(P,t,0) - V_1 + S\right] + n_3\left[V_3(P,t,0) - V_3\right] = 0. \tag{20}$$

Equation (19) is required to get rid of the dz terms in equation (17), and equation (20) is required to eliminate the terms with dq in equation (17). With no risk, this portfolio must earn the risk-free return over dt:

$$d\pi = r\pi dt. \tag{21}$$

Substituting from equations (15) and (16) for π and $d\pi$ in equation (21) and combining the result with equations (19) and (20), we get a system of three equations and three unknowns:

$$\begin{bmatrix} s_1 V_1 & s_2 V_2 & 0 \\ V_1(P,t,0) - V_1 + S_1 & 0 & V_3(t,0) - V_3 \\ V_1 * (\mu_1 - r) - C_m & V_2 * (\mu_2 - r) & V_3 * (\mu_3 - r) \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{22}$$

For a nontrivial solution, the determinant to the 3 × 3 matrix in equation (22) must vanish, which will occur if the rows of the matrix are linearly dependent. In other words, there must be two parameters, λ_P and λ_F, such that the following will hold:

$$(\mu_1 - r)V_1 - C_m = \lambda_P s_1 V_1 - \lambda_F\left[V_1(P,t,0) - V_1 + S\right], \tag{23}$$

$$(\mu_2 - r)V_2 = \lambda_P s_2 V_2, \tag{24}$$

$$(\mu_3 - r)V_3 = -\lambda_F\left[V_3(t,0) - V_3\right]. \tag{25}$$

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[11] This assumption is for convenience. The same fundamental partial differential equation [equation (28)] is obtained if each of the hedging assets are assumed to depend on both fire and price risk.

Solving for λ_P and λ_F in equations (24) and (25) gives:

(26)
$$\lambda_P = \frac{\mu_2 - r}{s_2}$$

and

(27)
$$\lambda_F = \frac{\mu_3 - r}{\left[V_3(t, 0) - V_3\right]/V_3},$$

where λ_P is known as the market price of P-risk and λ_F as the market price of fire risk.[12]

As can be seen from equations (26) and (27), the market price of risk for either price or fire reflects the extra return over the risk-free rate per unit of variability. That variability is measured by s_2 for price, which is the instantaneous standard deviation for $V_2(P, t)$. For fire risk, the variability is measured by $-[V_3(t, 0) - V_3]/V_3$, which is the proportionate loss in asset value if a fire occurs. In theory, both λ_P and λ_F could be estimated using historical data on log prices and timberland sales. This issue is discussed in appendix B.

Substituting for μ and s from equation (18) into equation (23) (and dropping the subscript for $j = 1$), we obtain the partial differential equation (PDE) that holds while it is optimal to refrain from harvesting the trees:

(28)
$$V_t + \tfrac{1}{2}b^2(P,t)V_{PP} + \left[a(P,t) - \lambda_P b(P,t)\right]V_P - (r + \lambda_F)V + V_\alpha + \lambda_F\left[V(P,t,0) + S\right] - C_m = 0.$$

Equation (28) is the partial differential equation which must be satisfied by the value of the stand of trees when it is not optimal to harvest and assuming that both the risk of fire and the risk due to price volatility can be hedged. This may be contrasted with equation (14) which was derived assuming these risks could not be hedged, but that the asset in question (V) would earn a risk-adjusted return of ρ. Comparing these two equations, we observe that in equation (28) the risk-free rate (r) has replaced the risk-adjusted discount rate (ρ), and λ_F has replaced the probability of fire (ϕ). In addition, the term associated with V_P is $[a(P, t) - \lambda_P b(P, t)]$, rather than just $a(P, t)$. This demonstrates the principle of risk-neutral valuation, which is discussed in the subsection below.

Alternatively, it may be assumed that it is possible to hedge price risk but not fire risk. With stochastic changes in V due to fire risk uncorrelated with changes in the market portfolio, an investor would not be rewarded with extra return for taking on fire risk. Under these assumptions, the stand of trees in question could be valued by creating a hypothetical portfolio of V_1 and a hedging asset V_2, whereby price risk is eliminated. This portfolio should earn the risk-free return. Using similar steps to those used to derive equation (28), the following partial differential equation can be derived which describes V:

(29)
$$V_t + \tfrac{1}{2}b^2(P,t)V_{PP} + rPV_P - (r + \phi)V(P,t,\alpha) + V_\alpha + \phi\left[V(P,t,0) + S\right] - C_m = 0.$$

[12] Note that we use a negative sign before λ_F in equation (25). The determinant vanishes whether this sign is negative or positive. Using a negative sign allows us to define λ_F as a positive number, which makes sense intuitively. As will be seen in equation (28), λ_F is added to r as a risk premium.

This is the same as equation (28) except that the true probability of fire (ϕ) appears instead of λ_F. Recall that in equation (14) the true probability of fire was added as a risk premium to the risk-adjusted discount rate. Here we observe that the probability of fire is added to the risk-free rate, due to the assumption that we are able to hedge the risk from price volatility.

The Historical Probability of Fire versus the Risk-Neutral Probability

Under the principle of risk-neutral valuation, a risky asset is valued assuming that investors require no extra return for bearing risk. When done correctly, risk-neutral valuation gives the appropriate value for the asset even when investors are risk averse. As shown by Cox, Ingersoll, and Ross (1985), "the equilibrium price of a claim is given by its expected discounted value, with discounting done at the risk-free rate, when the expectation is taken with respect to a risk-adjusted process for wealth and the state variables" (p. 380). This is accomplished by reducing the expected growth rate of each of the stochastic underlying variables by an appropriate factor risk premium. In our case, the instantaneous drift rate, $a(P, t)$, is reduced by the risk premium $\lambda_P b(P, t)$. The term $[a(P, t) - \lambda_P b(P, t)]$ is often referred to as the risk-neutral drift rate. If we assume the stochastic underlying variable (P) exhibits this risk-neutral drift rate, and we use the risk-free interest rate in our key partial differential equation [equation (28)], we will obtain the correct asset value for the "real world" where investors are risk averse and the true drift rate is $a(P, t)$.

Similarly, if we were able to hedge the risk of jumps, we could use risk-neutral valuation by finding the expected value of the asset with respect to a risk-adjusted probability of fire. In our case, this risk-adjusted probability of fire is just λ_F, which, as seen in equation (28), is the appropriate probability of fire to use to calculate the asset's value either if fire risk can be hedged or if investors are indeed risk neutral. Hence, λ_F may be called the risk-neutral probability of fire. The assumption of whether the risk of losses due to fire can be hedged or whether it is a diversifiable risk may have an important impact on V depending on the relative size of ϕ versus λ_F. For intuition about this, we turn to the literature on credit risk for corporate bonds.

In the finance literature, a popular way of modeling credit risk is to assume that the risk of default on corporate bonds follows a Poisson jump process.[13] It is possible to estimate the market's perception of the probability of default from bond prices and compare this with the actual historical risk of corporate defaults. It has been found that the risk of default calculated from bond prices is much larger than the historical risk of default. As a consequence, the spread of the yield on corporate bonds over the risk-free rate is greater than is justified by historical default rates (Amato and Remolona, 2003). This is called the "credit spread puzzle."

As an example, Hull (2006) compares the default intensity calculated from bond prices to the historical default intensity for bond yields published by Merrill Lynch averaged over a seven-year period from 1996. The difference in estimated probabilities of default ranges from 0.63% for the highest quality bonds to 4.40% for lower quality bonds. This translates into a small but significant excess return for corporate bonds above what can be accounted for by expected default rates based on historical data. Hull

[13] See Wilmott (2006, chap. 40) and Hull (2006, chap. 20) for discussions of modeling credit risk.

notes some reasons for the difference between the risk of default implied by bond prices and actual default rates. These include the low liquidity of some corporate bonds, the possibility that bond traders may be allowing for future depression scenarios, the fact that bonds do not default independently of each other giving rise to systematic risk, and finally the fact that bond returns are highly skewed in the downward direction which makes it difficult to diversify risks.

There is a parallel between the modeling of bond default risk and the risk of destruction of a forest by some catastrophe. We might expect $\lambda_F > \phi$ for some of the same reasons cited to explain the credit risk puzzle—i.e., lack of liquidity and difficulty in diversifying risks. In our empirical example, we estimate stand value for various different fire probabilities. This allows us to see the sensitivity of land value to this parameter. A discussion of the estimation of both λ_F and λ_P is provided in appendix B.

The Linear Complementarity Problem (LCP)

Equations (13), (28), and (29) are partial differential equations that will hold when it is optimal not to harvest the stand of trees (the continuation region). To specify the complete harvesting problem, including when it is optimal to harvest the stand, we formulate a linear complementarity problem (LCP) as in Insley and Rollins (2005).[14]

We set up the LCP under the assumption that we can hedge price risk, but not fire risk. The LCP for the other cases can be expressed in a parallel fashion. Let T denote the terminal time when the option to harvest expires. Let τ be defined as time remaining in the option's life, i.e., $\tau \equiv T - t$. Rearranging equation (29) and substituting τ for t, an expression, HV, is defined as follows:

$$(30)\qquad HV \equiv rV - \Big[\tfrac{1}{2}\sigma^2 P^2 V_{PP} + \big[a(P,\tau) - \lambda_P b(P,\tau)\big]V_P - C_m + V_\alpha - V_\tau + \phi\big[V(P,\tau,0) - V + S\big]\Big],$$

where rV represents the opportunity cost of not harvesting the stand. The expression within large square brackets represents the return over the infinitesimal time interval $d\tau$ due to the time change, price change, and the expected loss due to fire. This latter component is given by $\phi[V(P,\tau,0) - V + S]$ and includes the loss in V for the stand prior to the fire which is offset by the value of the bare land $V(P,\tau,0)$ and any salvage value S.

The LCP is given as:

$$(31a)\qquad HV \ge 0,$$

$$(31b)\qquad V(P,\tau,\alpha) - \big[(P - C_h)Q(\alpha) + V(P,\tau,0)\big] \ge 0,$$

$$(31c)\qquad HV\Big[V(P,\tau,\alpha) - \big[(P - C_h)Q(\alpha) + V(P,\tau,0)\big]\Big] = 0.$$

[14] The optimal harvesting problem we have posed is an optimal stopping problem. Jaillet, Lamberton, and Lapeyre (1990) have shown that the option value determined by the linear complementarity problem is the solution to the full optimal stopping problem under certain conditions, which hold for our problem. Wilmott, Dewynne, and Howison (1993) and Tavella (2002) provide further discussion on the linear complementarity problem. Note also, with a constant price assumption, it can be shown that the LCP is consistent with the decision rule developed by Reed (1984).

Table 2. Silviculture Costs Under a Basic Regime

Item	Cost ($/ha)	Age at Which Cost Incurred	Item	Cost ($/ha)	Age at Which Cost Incurred
Site Preparation	$200	1	First Tending	$120	5
Nursery Stock	$360	1	Monitoring	$10	35
Planting	$360	2			

The LCP specifies the optimal actions of a forest owner seeking to maximize the net present value of a stand of trees. With appropriate boundary conditions, the LCP is solved numerically which involves discretizing the relevant partial differential equation including a penalty term that enforces the American constraint [equation (31b)].[15] We are left with a series of nonlinear algebraic equations which must be solved iteratively. We use a fully implicit finite difference method, as detailed by Insley and Rollins (2005). The boundary conditions are the same as those in Insley and Rollins and are described in appendix A.

Empirical Example

This section demonstrates the impact of including the risk of fire in an optimal harvesting problem at the stand level. We use data for a hypothetical stand of Jack Pine trees in Ontario's boreal forest. Ontario's forests are largely publicly owned, and private companies acquire harvesting rights through long-term lease agreements. The perspective of this analysis is the forest owner, i.e., the public, so we ignore taxes and stumpage payments made by firms to the government. All dollar amounts given in the paper are expressed in Canadian dollars.

Volume and silviculture cost data were kindly provided by Tembec, Inc. The estimated volumes used for our example reflect "extensive" and "basic" levels of forestry management. The extensive regime involves essentially no management, as trees are allowed to regenerate naturally and nothing is spent on silvicultural treatments. Under basic management, $1,040 per hectare are spent within the first five years on site preparation, planting, and tending. These costs are detailed in table 2. Note that in the Canadian context, these basic silviculture expenses are mandated by government regulation for certain stands.

Volumes, estimated by product, are shown in figure 1 for the extensive and basic regimes.[16] It is interesting to note the different products that make up the net merchantable volume of wood. SPF refers to spruce-pine-fir logs. SPF1 and SPF2 are defined as being greater than 12 centimeters at the small end, SPF3 is less than 12 centimeters, and "other" refers to other less valuable species (poplar and birch). SPF1 receives the

[15] The American constraint is a restriction on American-type options, defined as an option which can be exercised at any time before the expiry date. The constraint restricts the value of the option to being no less than the payout. If the value of an American option drifted below the payout for an instant, it would immediately be exercised. This contrasts with a European option which can only be exercised on the expiry date. Prior to the expiry date, the value of a European option could conceivably drop below the payout. The option to harvest a stand of trees is typically modeled as an American option in that harvesting can occur at any time.

[16] The yield curves were estimated by Margaret Penner of Forest Analysis, Ltd., Huntsville, Ontario, for Tembec, Inc., and are available from the authors on request.

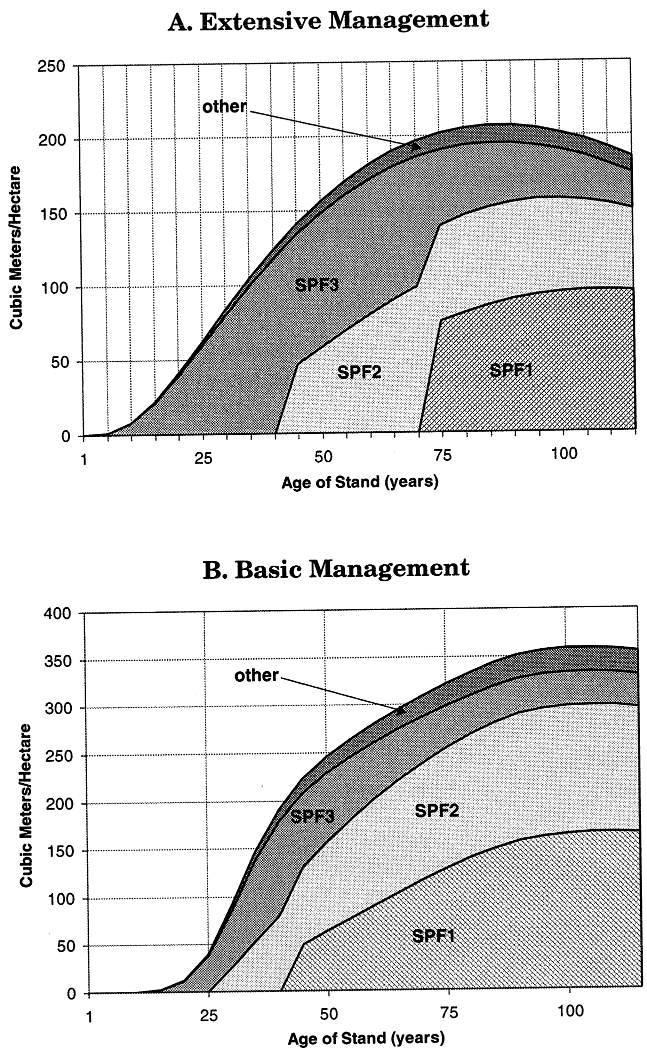

Figure 1. Volumes by product for hypothetical Jack Pine stands in Ontario's boreal forest under extensive and basic management

highest price and, in an extensively managed stand, does not appear until after 70 years. In a stand with basic management, SPF1 appears much earlier (at around age 40) in response to the silvicultural treatments.

The parameters of the price process [equation (2)] are adopted from Insley and Wirjanto (2006). Their paper used weekly data from January 1980 to June 2005 for the price of softwood lumber delivered to Toronto and applied ordinary least squares to a discrete approximation of the continuous-time stochastic process.[17] Parameter estimates are given in table 3.[18] We also use the estimate of the market price of price risk (λ_P) from

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[17] Insley and Wirjanto derived their data from *Madison's Canadian Lumber Reporter*.

[18] Note that the estimated value for $\overline{P}$ was \$230 per cubic meter in Toronto. This had to be translated into a price at the millgate. Since the mean price of \$230 was close to the Toronto price for 2003, we adopted our estimated 2003 millgate price of \$60 per cubic meter for SPF1 logs as $\overline{P}$ at the millgate.

Table 3. Parameter Estimates for Price Process and Market Price of Risk

Parameter	Estimate
Speed of mean reversion, η	0.80
Long-run equilibrium price, $\overline{P}$ ($/$m^3$)	$60
Volatility of price, σ	0.27
Market price of risk for price, λ_P	0.01

Note: The relevant price process is $dP = \eta(\overline{P} - P)dt + \sigma P dz$.

Table 4. Assumed Values for Log Prices and Cost of Delivering Logs to the Mill

Description	Cost ($/cu. meter)
Harvest and transportation cost	$47
Price of SPF1	$60
Price of SPF2	$55
Price of SPF3	$30
Price of poplar/birch	$20

Insley and Wirjanto which is based on the approach of Hull (2006). The λ_P value is reported in table 3, and details on the estimation approach are provided in appendix B.

Assumptions for harvesting costs and log prices are given in table 4. These prices are considered representative for 2003 prices at the millgate in Ontario's boreal forest. Average delivered wood costs to the mill for 2003 are reported as $55 per cubic meter in a recent Ontario government report (Ontario Ministry of Natural Resources, 2005). From this is subtracted $8 per cubic meter as an average stumpage charge in 2003, giving $47 per cubic meter.[19] It will be noted the lower valued items (SPF3 and poplar/birch) are harvested at a loss. These items must be harvested according to Ontario government regulation. Because the price for poplar/birch is at roadside, there is no transportation cost to the mill.

We examine a range of fire arrival rates in our example. According to the Ontario Ministry of Natural Resources (2001), in the boreal forest, estimates of fire return intervals vary considerably from 20 to 300 years. The risk of fire is the reciprocal of the fire return interval, so this implies ϕ ranges from 0.003 to 0.05. For this paper, we consider ϕ values from 0 to 0.06.

We have not made any attempt to estimate the risk-neutral probability of fire (λ_F). However, through observing the impact of different levels of ϕ, we can observe the impact of a risk-neutral probability of fire that exceeds the true probability of fire.

Empirical Results

Using the parameters described above, we solve the optimal harvesting model to determine the value of the stand at the beginning of the rotation when the land is bare ($\alpha = 0$) under both extensive and basic silvicultural regimes, assuming zero salvage value. We also solve for critical harvesting prices for various stand ages. For a stand of a particular age, if the current price equals or surpasses the critical price, then the stand should be harvested. For comparison, we calculate the value of an extensively managed stand using a Faustmann-type model with a constant price of $P = \overline{P}$(= $60) and a range of values for the probability of fire. Faustmann values for the basic regime are not reported, as these values are negative for all levels of ϕ when $P = \overline{P}$.

The results are presented in table 5 and figure 2.[20] The estimates of land value are found to be much larger under the assumption that log prices are stochastic, implying

[19] This consists of $35 per cubic meter for harvesting and $12 per cubic meter for transportation. Average stumpage charges are available from the Canadian Council of Forest Ministers (2007).

[20] Richardson extrapolation was employed to increase the accuracy of the results. [See Wilmott (1998) for an explanation of Richardson extrapolation.] The numerical results are considered accurate to within approximately $5 per hectare.

Table 5. Land Values for Different Fire Arrival Rates

	Faustmann Land Value[a] ($/ha)	Bare Land Value, Stochastic Price ($/ha)	
Fire Arrival Rate (ϕ)	Extensive Regime	Extensive Regime	Basic Regime
0.00	60	663	1,630
0.005	44	534	1,365
0.01	32	432	1,145
0.02	16	285	808
0.03	7	190	572
0.04	2	126	406
0.05	0	85	288
0.06	0	57	205

[a] Faustmann values for the basic regime are not reported as these values are negative for all levels of ϕ when $P = \bar{P}$.

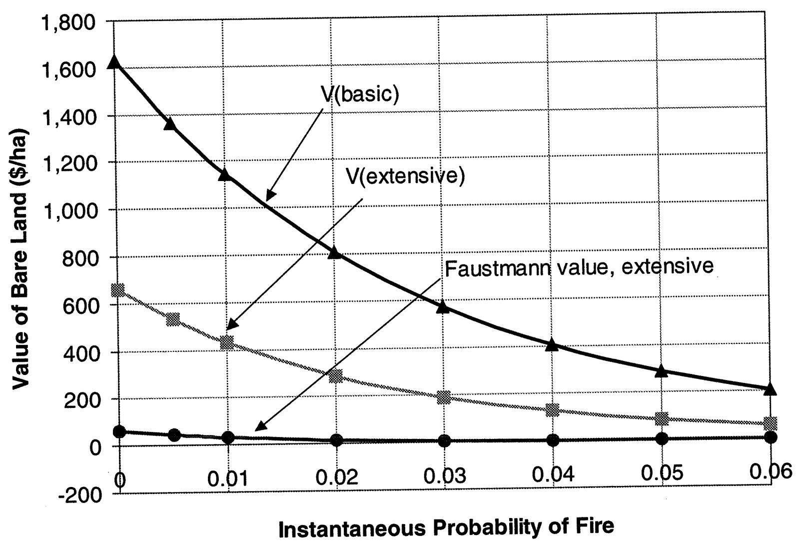


Figure 2. Value of land at the beginning of the first rotation for Faustmann (constant price) and stochastic price cases for various fire arrival rates

that incorrectly ignoring stochasticity can have a very large effect on the estimated land value. Our model assumes the stand owner is free to react optimally to changing lumber prices. The owner can therefore increase the value of the stand by delaying harvesting when prices are low and harvesting immediately to take advantage of any price spikes. This is a standard real-options result. To the extent the forest owner is unable to freely choose the harvesting date due to regulation, weather, or some other factor, the bare land values calculated using the real-options approach will overestimate the true stand value.

The relatively low values for the Faustmann result are a reflection of the high costs of harvesting and delivering wood to the mill in Ontario's boreal forests, combined with the assumption that the price at the time of harvest will equal the long-run average of $60 per cubic meter. We would need to assume a significantly higher price in the Faustmann model to achieve a land value anywhere close to that obtained under the stochastic price assumption. For example, at a price of $90 per cubic meter, the Faustmann result for the extensive regime (with no fire risk) is around $660 per hectare, which is close to the land value with a stochastic price. For the basic regime, a price of around $112 per cubic meter is required to obtain a value that is close to the bare land value under stochastic prices.

We also observe from table 5 that land value falls sharply as the risk of fire increases. First consider the extensively managed stand. When price is stochastic, land value for the extensive regime at $\phi = 0$ is about four times the value at $\phi = 0.04$. Using the Faustmann model with price fixed at the long-run mean, land value at $\phi = 0$ is 30 times the value at $\phi = 0.04$. Because of the low land values under the Faustmann rule when $P = \overline{P}$, increases in the probability of fire have a larger relative effect under the Faustmann rule and the estimated value of land reaches zero sooner as the fire arrival rate is increased.

Comparing the basic and extensive regimes, at $\phi = 0$, the basic land value is more than double the extensive land value (table 5). Although silviculture costs are much higher under basic management, this is offset by the earlier appearance and greater magnitude of the most valuable wood products, SPF1 and SPF2. The absolute difference in value between the two regimes is observed to fall as the fire arrival rate increases. Specifically, under the basic regime, every time the stand is destroyed by fire (or harvesting), large planting and management expenses (relative to the extensive regime) must be incurred.

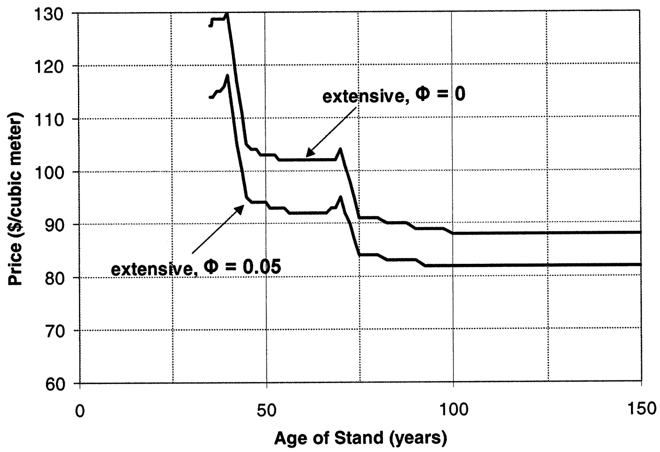
Also of interest is the timing of the harvesting decision. For the Faustmann case, the optimal harvest age in an extensively managed stand is 85 years for $\phi = 0$, falling to 75 years for $\phi = 0.04$. The long rotation ages reflect the fact that the highest value product does not begin to appear until age 70.

For the stochastic price cases, we are interested in critical harvest prices, and these are plotted in figure 3 for basic and extensive regimes and for $\phi = 0$ and $\phi = 0.05$. First, we observe that prior to about age 75, the critical prices for the extensive cases are higher than for the basic cases. The valuable products take longer to appear in the extensive case, so it is reasonable that we expect to wait longer before harvesting. In addition, the critical harvesting price for $\phi = 0$ is higher than for $\phi = 0.05$ in both extensive and basic cases, indicating the stand will likely be harvested earlier when fire risk is positive. This is consistent with results found by other authors surveyed in the literature. At age 60 for both the extensive and basic regimes, the critical harvesting price with $\phi = 0$ exceeds the value with $\phi = 0.05$ by $8 to $\$10/m^3$.

Overall, critical prices decline with stand age, reflecting falling tree growth rates in a maturing stand. However, as shown in figure 3, at several points critical prices jump up with stand age. This follows from using specific yield curves for different product types, rather than a homogeneous net merchantable volume. An increase in critical price occurs when a new higher valued product makes an appearance, making it worthwhile delaying the harvest to give the more valuable product time to grow—e.g., under the extensive regime, critical prices increase at age 40 when SPF2 makes an appearance and again at age 70 when SPF1 appears.

Even if a decision maker is using a Faustmann decision rule, we would expect her to update her price forecast as time passes. For example, if at the time the stand is planted

A. Extensive Management



B. Basic Management

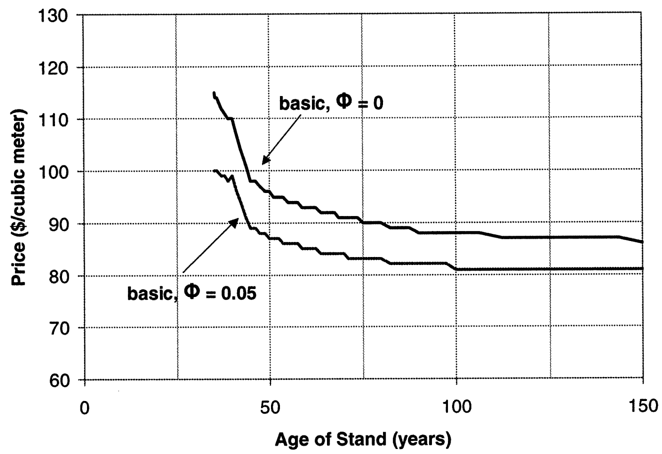


Figure 3. Critical harvesting prices, extensive and basic management (ϕ = 0 and ϕ = 0.05)

the Faustmann rule says that the optimal harvest date is 70 years of age, it would not be surprising to see the stand harvested earlier if price rises above what had been expected, or later if prices fall below previous expectations. Compared to Faustmann, the real-options approach gives a larger estimate of today's land value because it explicitly takes account of the ability of managers to react optimally to price changes in the future. Additionally, the real-options approach accounts for price volatility, so that an increase in the variance of price would typically result in a higher critical price and a later harvest date (Insley and Rollins, 2005). This is because a higher variance means there is more likelihood that a very high price might be reached in the future, which makes it desirable to wait a bit longer for harvesting. The impact of volatility on stand value and optimal harvest timing is ignored in the Faustmann rule.

As noted earlier, we do not have an estimate for the risk-neutral probability of fire. However, if it exceeds the actual probability of fire by a significant degree, it will affect the valuation of the stand and optimal harvest time. This finding is made clear by our observations of the impact of an increase in ϕ. The likelihood that actual timberland returns will reflect this unknown risk-neutral probability of fire is something to keep in mind in any analysis of returns to timberlands.

In the results so far presented, we have assumed zero salvage value after a fire. In North America, salvage logging is increasingly being considered as a means of reducing the economic losses due to fire. It has been highly controversial in the United States, where there are fears it may delay forest recovery and change the ecology of an area.

Accordingly, we examined the effect on our results of assuming 50% of the volumes are salvageable once the stand is at least 45 years old. The value of the amount salvaged will depend on timber prices after the fire has occurred. It was found that the value of the bare land stayed basically the same for the extensive regime, and increased from $4 to $10 per hectare for the basic regime. The ability to salvage a portion of merchantable timber increased the critical price at which the stand should be harvested by $1 to $3 for both regimes. The presence of salvageable timber does not have a large effect on the timing of the harvest in our example.

Concluding Remarks

We have demonstrated the inclusion of catastrophic risk in a stand-level model of optimal tree harvesting under stochastic prices which exhibit mean reversion in the drift term. Our findings reveal that as the probability of fire increases the critical harvesting price falls; consequently, the stand would be expected to be harvested sooner with a larger probability of fire. In our empirical example, the probability of fire had a larger impact on the value of a stand where significant silvicultural expenses are mandated, compared to one left to regenerate naturally.

We compared land values calculated using a Faustmann rule with price set at the long-run expected value. In our example, Faustmann values were found to be considerably lower than those calculated in the real-options model with stochastic prices. The risk of fire had a much larger relative effect in the Faustmann model because of its very low land values.

The conventional way to treat the presence of a Poisson jump process is to assume it is a diversifiable risk and hence commands no return above the risk-free rate. We demonstrated the alternative assumption that markets are complete enough so that fire risk can be hedged. Under this assumption, we must be concerned about the probability of fire in the risk-neutral world compared to estimated physical probability of fire. Based on the credit risk literature on the likelihood of bond default, we conjecture that the risk-neutral probability of fire may exceed the estimated physical or historical probability of fire. While we do not currently have adequate data to explore this further, we can observe the effect of a higher risk-neutral probability of fire by observing the sensitivity of land value and critical harvesting prices to changes in the probability of fire. We have found in our empirical example that land value is very sensitive to the assumed probability of fire. It follows that exploring the estimation of the risk-neutral probability of fire would increase our ability to evaluate potential investments in forest land. This is one avenue for future research. Another would be to investigate hedging in a model where fire size is itself a random variable.

This paper confirms the results of previous literature in terms of the qualitative impact of fire risk on land value and expected harvesting dates. However, by demonstrating the inclusion of catastrophic risk in a real-options model of optimal harvesting, new insight has been gained. In particular, our results show that catastrophic risk should not be considered in isolation, but rather should be examined in conjunction with price volatility, which is the other major risk faced by forest owners and investors.

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Appendix A: Boundary Conditions

The boundary conditions for solving the linear complementarity problem [text equation (31)] are given as follows:

- **As $P \to 0$,** we observe from text equation (2) no special boundary conditions are needed to prevent negative prices.
- **As $P \to \infty$,** we follow Wilmott (1998) and set $V_{PP} = 0$. This implies that for very large prices the value of the option is approximately linear with price.
- **As $\alpha \to 0$,** we require no boundary condition since the PDE is first-order hyperbolic in the α direction, with outgoing characteristic in the negative α direction.
- **As $\alpha \to \infty$,** we assume $V_\alpha \to 0$. This means that as stand age gets very large, the value of the option to harvest (V) does not change with α. In essence, we are presuming the wood volume in the stand has reached some sort of steady state.
- **Terminal condition.** As T gets large, it is assumed that $V = 0$. T is made large enough that this assumption has a negligible effect on V today.

Appendix B: Estimating the Market Price of Risk

We postulate a hypothetical contract that depends linearly on the underlying stochastic variable P. Using the Capital Asset Pricing Model, we assume the expected return of this hypothetical asset (μ^*) is described as:

$$\text{(A1)} \qquad \mu^* = r + [E(r_m) - r]\beta,$$

where r_m refers to the return on the market portfolio and β is a parameter. For any asset dependent on one stochastic variable, P, we know text equation (26) will hold such that $\mu^* = r + \lambda_P s$. Combining these two specifications for μ^*, λ_P can be expressed as:

$$\text{(A2)} \qquad \lambda_P = \frac{[E(r_m) - r]\beta}{s}.$$

We need estimates for the components of equation (A2). Assume the value of our hypothetical asset, V^*, depends linearly on P: $V^* = g \times P$ for some constant g. From the definition of s in text equation (18), we know that $s = \sigma = 0.27$. The estimated value for β is determined by an OLS regression of $\mu^* - r$ on $r_m - r$. Historical values for μ^* are the percentage change in P. The estimated β is 0.09, as detailed in Insley and Wirjanto (2006). We assume a real risk-free interest rate of 3% and a real return to the market portfolio of 6%. Using these parameters, the market price of risk is estimated as:

$$\text{(A3)} \qquad \lambda = \frac{[E(r_m) - r]\beta}{\sigma} = \frac{(0.06 - 0.03)0.09}{0.27} = 0.01.$$

Note that a more satisfactory way of estimating the market price of price risk would be to make use of futures prices. However, lumber futures contracts are only available for dates up to a year, and it is questionable whether these would be helpful for our harvesting problem with harvesting intervals upwards of 50 years.

While we did not attempt to estimate the market price of fire risk, λ_F, it could in theory be estimated from a historical time series of timberland prices. This would involve a calibration exercise in which data on timberland values would be used in the full model to back out implied values for λ_F, given an estimate of λ_P. The data on timberland value would also need to contain information on location of the land so that the average historical risk of fire could also be determined. At the moment, time-series data on timberland values are lacking.[21] Although it would be interesting to consider a cross-section of data on timberland prices in regions subject to different fire risk, this task is beyond the scope of the current paper.[22] Without an estimate of λ_F, we can at the very least observe the sensitivity of our results to this factor.

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[21] Aronow, Binkley, and Washburn (2004) discuss some of the problems associated with one available time series—the National Council of Real Estate Investment Fiduciaries (NCREIF) Timberland Property Index. These include the fact that values are based largely on appraisals rather than market transactions, all properties are not revalued each quarter, the sample of properties in the index changes from quarter to quarter, and timber inventory changes on each property over time. The authors describe in detail their approach to addressing these shortcomings in the data.

[22] Before undertaking a calibration exercise, further enhancements to the model used in this paper would be desirable, one of which would be to add the price of pulp as an additional stochastic factor.