Conservation Needs Assessment: Sustainability with Substitution and Biased Technical Change

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Abstract

This study explores the role of rate and biases of technological change in the sustainability of an economy with an exhaustible resource. In order to achieve this goal, the mathematical concept of viability kernel is introduced as a sustainability indicator and necessary conditions for sustainability are completely depicted in terms of technological parameters. The literature has historically assumed substitutability between human capital and natural resources and technological progress that is neutral in terms of relative input productivity and minimum efficient scale of production. Both assumptions have been widely criticized given the patterns of technological change and substitution observed empirically. Hence the theoretical contribution developed here allows calculation of sustainability thresholds in a manner that permits two of the most important drivers of economic behavior, substitution possibilities and biased technical change. As a result, necessary conditions for sustainability are derived in terms of rate and bias of technical change and elasticity of substitution. Results previously derived in the literature are reviewed and comparisons are made with new results derived from more flexible technological specifications. Several important results are found. First, the identification between elasticity of substitution and sustainability breaks down. Second, a rather optimistic result is obtained by which Increasing Returns to Scale sometimes can prevent the economy from extinction even with zero technological progress and positive capital depreciation. Third, input bias of technical change is critical in determining sustainability and further, size-increasing bias of technical change increases the likelihood of an economy to be sustainable in all cases.

At an empirical level, sustainability has occupied a central place in economic policy especially in the last 15 years. Concerns for sustainability have triggered a number of conservation policies around the globe and the trend in recent years has been to devote more attention and funding to conservation programs. In the U.S., natural resource conservation efforts encompass programs in agriculture, Forest Service, Bureau of Land Management, Fish and Wildlife Service, and the divisions of air and water quality of EPA. Funding for water resource programs, recreational services, and pollution control/abatement activities also come under the general rubric of natural resources.

Resources often targeted by these programs are air, land, water and forests/biodiversity. One of the key steps in conservation programs affecting these resources and other exhaustible ones like management of oil, gas and coal reserves is the "conservation needs assessment" which consists of the determination of a level of the resource to be preserved. The **computation** of this threshold **should** depend upon the economy's estimated transformation frontier (describing feasible combinations of sustainable consumption, human-made capital and natural resource capital.) The approximation to the economy's transformation frontier will be directly affected by the equations of motion of state variables and the chosen approximation to the production function. Hence the characteristics of the chosen approximation to the production function are crucial to the estimation of the transformation frontier and therefore to the computation of the resource sustainability threshold. It is obvious that the availability of *substitution possibilities* and the *nature of technical change* will directly affect this type of "conservation needs assessment" and the policies based on it.

According to USDA's National Planning Procedures Handbook, conservation programs try in fact to stay flexible enough so as to adjust to "economic and social changes". The meaning of this is, however, extremely ambiguous. Nothing specific is mentioned in this implementation handbook regarding adjustments of funding or conservation efforts to measures of substitutability or biases of technical change. The Handbook does however mention in Step 2 of the Natural Resource Conservation Service's (NRCS) Planning Process (Determination of Objectives) that "The planner and client should discuss, reach agreement, and document the client's operation in terms of …Production and business goals…"

The Step 8 of this process is also somewhat on the lines of the role of production technology in conservation when it defines the Plan Revision stage as: "Action needed as a result of significant changes in one or more of the conservation systems defined in the conservation plan. This may be caused by changes in land use, changes in technology, changes in the set of practices included in the system, a change in the land units treated by the system, etc."

To sum up USDA and other federal agencies clearly mention the need to adjust conservation plans to current agricultural practices and technologies and it is also acknowledged that they should be flexible enough so as to adapt to technical change. Unfortunately they are not specific in terms of an explicit criterion for incorporating in the analysis technological dimensions such as flexible input substitution and effect of technical change on relative productivities and minimum efficient scale.

If public conservation programs do not consider an economic transformation frontier allowing for flexible capital-resource elasticity of substitution and biased input and size technical change, they may trigger miscalculated resource conservation levels and unreliable sustainability tests, therefore potential misguided conservation policy prescriptions. The calculated sustainability threshold of the natural resource could be "too high", punishing current generations, or "too low", punishing future generations.

To illustrate the empirical importance of these insights take for instance the cases of fossil fuels or forest/biodiversity. A positive rate of neutral technical change implies a "relaxation" of the conservation constraint. However in the case of technical change that is resource-using, the positive technological progress can be partially outweighed by the increase in resource intensity implying that conservation efforts should not be decreased. Examples include observed energy-biased technical change and the related air-bias technical change (pollution). Land and forest/biodiversity-biased technical change has also been extensively documented. Land-using technical change (extensification) is closely linked to forest/biodiversity-using technical change (deforestation). Expansion of modern irrigation techniques has displayed water-saving technical change. And we could keep on listing examples.

A similar argument applies to size-bias of technical change and this is especially relevant to land use. As we will see below, under some parameter configurations, a size-increasing technical change might imply a relaxation of the conservation. More importantly, from a public policy perspective, the findings of this paper open new avenues for policy prescriptions when we show that induced technical change can be used to relax conservation constraints.

Theoretically, most of the accepted definitions of sustainability used as a basis in conservation needs assessment require an approximation to the economy's frontier. In this research we use viability theory, developed by Aubin (1991) and extended to models of growth with exhaustible resources by Martinet and Doyen (2007). Viability models define an ensemble of "viable states", in contrast to undesirable states defined as such by ecological, economic, and/or social constraints. These constraints can be derived from objectives, conservation principles, scientific results of modelling, or precautionary principles, and correspond to limit reference points to be avoided. Viability theory does not attempt to choose any "optimal solution" according to given criteria, but selects "viable evolutions". These evolutions are compatible with the constraints in the sense that they satisfy them at each time and can be delineated by the viability kernel. The lower bound of the viability kernel is used to represent technological possibilities. In the case of *sustainable* consumption the frontier derived yields the maximum level of constant consumption that can be sustained forever given the current level of human-made capital and natural resources. This paper augments the theory by providing more flexible technological specifications of the viability kernel to account for two important economic concepts, flexible substitution and biased technical change. In doing so, it modifies and/or extends the conclusions derived by previous literature, as we will see below, giving different and less pessimistic conservation policy recommendations.

Previous Theoretical Literature on Sustainability and Intertemporal Equity

Dasgupta and Heal (1974) was the first study addressing the question of optimal programmes in an economy with exhaustible resources and utilitarian intertemporal SWFL. Dasgupta and Heal proved under specific set of assumption on preferences (Proposition 8 in their paper) that with exhaustible resources and Cobb-Douglas (CD) technology, the consumption eventually turns down and goes to zero for a finite time, provided productivity of reproducible capital is low enough.

Due to these striking results, "resolutions" to these problems were tackled by several studies. In an economy with a CD production function $y_t = k_t^{\alpha} r_t^{\beta}$ where y is output, k is the stock of human made capital and r is the rate of natural resource used in production, Solow proved that a positive consumption that can be held forever constant exists if $\alpha > \beta$, i.e. share of capital is greater than the share of natural resources. Stiglitz (1974) addressed a second potential resolution to the issue of the existence of a viable path by assuming (within the CD case) exogenous Hicks-neutral technical change. Assuming a utilitarian Social Welfare Function (SWFL) he derives the natural result that if technical progress is high enough (compared to impatience) there exists an optimal non-decreasing level of consumption over time. Furthermore Hartwick (1977) also derives a saving rule (known as the Hartwicks' rule) that prevents the economy from extinction.

After a careful look at the questions addressed in this literature, it is clear that no specific definition of sustainability was offered and no criteria for evaluating policies were developed either. In turn these studies had as their main concern a necessary condition for an economy to be sustainable under any possible definition: future generations enjoy a **positive** level of consumption.

More recent literature has attempted to define sustainability and to develop criteria to evaluate alternative paths of the economy to determine which ones fulfill the conditions to be considered "sustainable". In terms of defining and evaluating paths and as argued before by Pezzey (2002), the literature on weak[1] sustainability and development of an economy with an exhaustible resource can be separated in two strands: sustainability imposed as an aside constraint and the axiomatic approach in which sustainability is a socially desired outcome captured by the Social Welfare Functional in the economy. It is not the purpose of this paper to discuss the merits and drawbacks of each of these[2] but the clarification is necessary to make the theoretical and philosophical choices of the present study more explicit.

This paper goes back to the original literature in the sense that discusses conditions under which an economy with an exhaustible resource is not doomed to extinction in finite time. Specifically it discusses the role of flexible substitution between capital and natural resources and bias of technical change on extinction.

Technological Parsimony versus Flexibility

Dasgupta and Heal (1974), Solow (1974) and Stiglitz (1974) derived the precise range of situations for which a maintainable positive consumption path does not exist. However, by assuming a constant elasticity of substitution[3] (CES) or CD technologies and

[1] Weak sustainability as opposed to strong sustainability where the former assumes some degree of substitutability between natural and human made capital and the latter assumes no substitutability between them.

[2] For this the reader is referred to the extremely enlightening studies by Pezzey (1997) and (2002), Chichilnisky (1996), Heal (2005) and Arrow et.al. (2003)

[3] The CES case was discussed by Stiglitz (1974). In his study he discuses the fact that Hick's neutral technical change is a maintained hypotheses in a Cobb-Douglas specification and derives the result that with increasing returns to scale the technical growth needed to guarantee a constant consumption over time

exogenous, disembodied technical change all of them have neglected the role of input and size bias of technical change on the existence of a path that never converges to zero.

Martinet and Doyen (2007) included endogenous technical change in a CD production function. However all technical change was assumed to be capital-augmenting and hence they derive the un-surprising result that a positive rate of technical change is sufficient for the existence of a forever-positive consumption path. Hence by assuming a CD with capital augmenting technical change they are neglecting once more the role of resource-augmenting technical change, size biased technical change and their interaction with flexibility (substitutability) in the existence problem.

The production function used by these studies can be summarized by $y_t = A(t,k)k_t^{\alpha} r_t^{\beta}$ where y is output, k is the stock of human made capital, r is the rate of natural resource used in production and $A(t,k)$ was specified as e^{gt} by Stiglitz (t is time and g rate of technological progress) and as $k^{1-\alpha+\varepsilon}$ by Martinet and Doyen (ε is rate of capital-augmenting technological progress).

The general argument to model production technology with a CD form is that *within the CES technologies* the only empirically relevant case is presented by the CD case. As discussed by Dasgupta and Heal (1974), the CD case is the only one within the CES family that satisfies the property of essentiality with respect to all of its arguments while at the same time allowing for unbounded average product in both inputs $\lim_{r \to 0} \frac{y}{r} = \infty$ (i.e. average product of natural resource grows without bound as the level of the resource goes to zero). The combination of these two properties is what Dasgupta and Heal (1974) defined as the **essentialness** property that a natural resource was supposed to fulfill for making the problem empirically relevant. Under this last condition the technology allows for unlimited substitution between human made capital and natural resource, such that production (and hence consumption) does not necessarily collapses to 0.

Note that I stressed the fact that CD offers the empirically relevant case, specifically *within the CES family*, but why the literature restrains itself to the CES family is a choice mostly unjustified. One would argue that the main reason is its parsimony compared to other more flexible specifications and its resulting ability to yield analytical solutions to the problem at hand. However, the CD technology has as maintained assumption a fixed (equal to 1) elasticity of substitution between capital and natural resources[4]. Moreover, exogenous disembodied technical change (Dasgupta and Heal, Solow, Stiglitz, Hartwick) or capital-augmenting technical change (Martinet and Doyen) neglect the role of biases and their interaction with substitutability. Both assumptions have been widely criticized given the patterns of technological change and substitution observed empirically[5].

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decreases. Moreover he stresses the fact that in a more general technology $Q = F(\phi(K,R);L)$ an estimate of the substitution elasticity greater than one between K and R also yields existence of an efficient path with a constant consumption and also a resource augmenting technical progress (no matter how small) would yield a positive consumption maintainable forever. Another exception is Dixit et al.

[4] This assumption has been criticized in general and in particular by the "strong" sustainability proponents.

[5] Discussions of bias of estimates of elasticity of substitution towards a value of one (CD) in cross section studies of CES technologies began with Lucas (1969) and Sveikauskas (1974). This issue was reviewed and discussed in a more recent paper by Antras (2004).

If it is in fact the case that technological progress is not neutral in its effects on inputs relative productivities and that some natural resources do not have man-made close substitutes, then a set of questions should be raised. **Is it true that an elasticity of substitution between natural resource and human made capital lower than one is a sufficient condition for unsustainability? Is it true that an elasticity of substitution greater than one is sufficient for the existence of a consumption path that never goes to zero? Can a low rate of technological change be compensated by a higher degree of flexibility (higher elasticity of substitution) in production? Does the scale of production play any role of the sustainability of the economy?** All these can be compacted in one question; how do technological features of the economy affect sustainability and hence the level of the resource that needs to be preserved? This is an important question if one is to adjust conservation policies to changes in production technology.

As said before the theoretical contribution developed here allows calculation of sustainability thresholds taking under consideration substitution possibilities and biased technical change and thus it offers answers to these set of questions.

Alternatives to Cobb Douglas Specification

As I discussed before the "empirically relevant case" seems to be depicted in the literature as the family of functional forms that satisfy what has been called essentialness of inputs in production in the sense of Dasgupta and Heal (1974). This concept of essentiality denotes those resources that are needed in production but with an unbounded potential production. Furthermore a wide variety of functional forms more flexible than CES or not belonging to the CES family at all satisfy these two properties as well.

Due to the potential problems with constant elasticity of substitution and neutral technical change it is the goal of this paper to use a more flexible, "empirically relevant" functional form to approximate the production technology and discuss the generality of common results in this literature and their robustness to changes in the technological specification.

The rest of the paper is organized as follows. In the next section, the model of the economy is presented and the relevant parameters describing the technology are identified and derived. Then, the maintained hypotheses of the CD specification are expressed in terms of those parameters and sustainability conditions (expressed in terms of the same parameters) derived previously in the literature are reviewed. We then proceed to develop the selection mechanism of the more flexible functional forms used for the purposes of this study (i.e. Transcendental and Generalized Quadratic production functions). We will prove how the use of these new forms:

- Change the effect of the parameters on the existence and shape of the viability kernel and hence on sufficient conditions for sustainability and;
- Add the effect of new parameters (previously neglected by CD's maintained hypotheses) on sustainability. The introduction of non neutral technical change and the results here derived are then contrasted with previous results that assumed size neutral technical change and Hicks neutral or capital augmenting technical change. This will in turn allow us to show how conservation policies should be adjusted to the biases of technical change along with flexible substitution. These are two aspects of technology

whose empirical relevance has been widely accepted and theoretically developed, but never introduced and discussed in this literature.

After this we will proceed to extend the method to exhaustible resources and finally we summarize and conclude.

The Model

The economy is subject to the following standard dynamics:

$$\dot{S}(t) = -r(t) \qquad (1)$$

$$\dot{K}(t) = f(K(t), r(t)) - c(t) - \lambda K(t) \qquad (2)$$

where:

$K(t)$ is the stock of human-made capital

$S(t)$ is the level of natural resources

$r(t)$ is the rate of extraction of natural resources

$f(.,.)$ is the production function

$c(t)$ is consumption and λ is the depreciation rate

Dots above variables denote time derivatives

This economy is nothing but a replication of what has come to be known as Solow-Heal-Dasgupta's economy. The evolution of the exhaustible natural resources depends negatively and in a one to one relation upon the rate of extraction. Zero extraction costs are assumed. The evolution of the stock of human made capital of net investment is the difference between output, consumption and depreciation.

Viability Approach

To discuss the issue of the conditions under which an economy with exhaustible natural resources is doomed to extinction allowing for biased technical change and flexible elasticity of I will use the concept, which will be made precise soon, of viability kernel to characterize sustainable paths of control and state variables. From here on I will follow the same clear definition and description offered by Martinet and Doyen 2007: "The viability kernel is the set of initial resources and capital levels from which it is possible to define acceptable regimes of exploitations and consumption paths satisfying all of the constraints throughout time".

The viability approach (Aubin 1991) offers an alternative approach to sustainability. It allows us to address the issue of viable[6] intertemporal use of natural resources by defining a set of state and control variables paths which satisfy a vector of constraints describing the sustainability of the economy. These constraints are called the viability constraints and are determined exogenously. Second, the constraints include a guaranteed consumption level and a guaranteed level of an exhaustible resource. Based on these constraints, conditions are derived such that the constraints are fulfilled at any time.

[6] By viable the literature means those in which exhaustion of natural resources is not achieved in finite time and hence consumption does not collapse to zero in finite time.

This approach is subject to two main critiques. One is the fact that the constraints are set exogenously and they do not respond to an optimality criterion. Three reasons counterbalance this apparent drawback. First, by setting the guaranteed level of natural resource to zero, this method will allow us to depict the maximum level of sustainable consumption with the resource going asymptotically towards zero but not hitting zero in finite time. In this case the resulting level of consumption can be linked to the optimal level of consumption yielded by an optimum maxi-min path (Martinet 2007). Second, this approach yields a relationship between sustainable consumption and minimum resource level. This allows the researcher to derive explicit conditions describing the trade off between conservation of natural resources and sustainable consumption and hence it reflects the intertemporal trade-off facing society for different levels of the resource[7]. Third in the case of implementation of the precautionary principle by a government which sets a guaranteed minimum level of a natural resource (strong sustainability), this approach allows for the impact of that conservation policy in terms of sustainable consumption.

In the economy described by (1) and (2), no assumptions are included regarding preferences because in the viability approach the existence of constant consumption path is analyzed for all feasible (not necessarily optimal) consumption paths. Consequently, the sufficient conditions for existence of a feasible constant consumption over time will be depicted in terms of technology parameters only.

The viability or sustainability constraints are:

$$0 \le r(t) \qquad (3) \qquad\qquad 0 = S_b(t) \qquad (4)$$
$$S_b < S(t) \qquad (5) \qquad\qquad 0 < f(K(t), r(t)) - c(t) \qquad (6)$$
$$0 \le K(t) \qquad (7) \qquad\qquad 0 < c_b < c(t) \qquad (8)$$

Where the variables are as defined in equations (1) and (2) and S_b and c_b are arbitrarily chosen values.

Now that we have the set of viability constraints, a more technical definition of the viability kernel is offered and it is also taken form Martinet and Doyen (2007):

$$Viab(f, c_b, S_b) = \begin{Bmatrix} (S_0, K_0) \mid \text{there exists decisions } (c(.), r(.)) \text{ and states } (S(.), K(.)) \\ \text{starting from } (S_0, K_0) \text{ satisying conditons } (3)-(8) \text{ for any time } t \in \Re^+ \end{Bmatrix}$$

So, in short, the viability kernel is the set of initial values of the capital stock and natural resource stock such that the economy can maintain forever a consumption level greater or equal to c_b. Hence there exists a positive level of indefinitely maintainable consumption which is the same existence problem first asked by Solow (1974), Hartwick and Dixit et.al. (1980) with a Rawlsian SWFL and Dasgupta and Heal (1974) and Stiglitz (1974) with a utilitarian SWFL. The results derived in this literature were recovered by Martinet and Doyen (2007) using the mathematical concept of viability kernel as a

[7] This is not an option in the optimality approach since there is no way to rationalize such a constraint and hence a relationship depicting the trade-off is not derived.

sustainability indicator. We propose to exploit this approach to the case of more general technologies.

This is a one sided sustainability check in the sense that if the existence conditions do not hold for a given economy then the path followed by this economy is NOT sustainable. However if conditions hold, then we can not assert that the economy is in fact sustainable. This becomes clear as soon as it is noted that positive consumption of future generations is only a necessary but not a sufficient condition for sustainability.

This papers offers, through the use of viability theory, a more general, analytical way to check a necessary condition for non-conflict. This is, it finds technological conditions that, if violated, there is no SWFL whose maximization yields a program consistent with *any* sustainability constraints.

There is by now a rich theoretical and empirical literature regarding viability theory. The theoretical foundations of viability theory in general and its natural extension to economic theory can be found in Aubin (1991) and (1997) respectively. Moreover, there is a growing literature applying this approach to economic problems and sustainable use of natural resources. An illustrative but very incomplete list of applications is: Bonneuil (1994), Bene, Doyen, and Gabay (2001), Aubin (2003), Doyen and Bene (2003), and once again, Martinet and Doyen (2007) and Martinet (2007).

Viability Kernel with a General Form

Hereby we will present the procedure used to derive sufficient conditions for non emptiness of the viability kernel (indefinite sustainability of consumption level c_b) and explain the analytical derivation of the boundary of the kernel when possible.

Suppose we have a general approximation to the economy's technology $f(K(t), r(t))$. As explained by Aubin, Doyen, Martinet and others given a viable current state, the relevant viable controls ensure that the velocities $\dot{K}, \dot{S}$ are tangent or inward to the viability kernel. It turns out that viable decisions at the boundary (consumption constant and equal to c_b) are reduced to those corresponding to the Hartwick's rule $c_b = f(K(t), r(t)) - rf_r(K(t), r(t))$. Hence the optimal extraction rate r is implicitly defined in this expression. I will describe it as $r = G(k, c_b) \quad (B)$.

In this context the viability kernel can be shown to be the epigraph of a function $V(k)$. The epigraph is a set defined in the space of the natural resource S in the following way:
$Epi(V) = \{(K, S), V(K) \leq S\}$

Let us define a function H(.) which is the Hamiltonian expressed as:
$H(V'(K), K, r, c) = V'(K)(f(K, r) + c_b) + r$

Then the function V(k) solves the following Hamilton-Jacobi-Bellman equation:
$$\min_{(r,c)\in C(K,S)} H(V'(K), K, r, c) = 0$$
where: $C(K, S) = \{(r, c): c_b < c < f(K(t), r(t)) \text{ and } 0 \leq \text{r}\}$

The First Order Condition is defined as:

$$\Rightarrow V'(K)(f_r(k,r))+1=0$$

$$\Rightarrow V'(K)=-\frac{1}{f_r(k,r)} \qquad (A)$$

Combining (A) and (B), integrating both sides, taking limits and rearranging yields:

$$V(c_b)=\int_{c_b}^{\infty}\frac{1}{f_1(G(k,c_b),k)}+S_b$$

Therefore all is left to derive is the conditions under which the integral in the first term converges (i.e. it does not equal ∞). For this we can use one of the many existing tests of convergence. We will see when we work with specific functional forms that in all of them the so called comparison test will be enough to derive the convergence (and hence Non-Emptiness) conditions. In conclusion we can not say much without imposing more specific structures on the functional forms used to approximate the existing technology.

Parametric Description of Technology $f(K(t),r(t))$

We would like to draw conclusions on the role of some particular technological features (more specifically, their parametric representations) on the sustainability of this economy, i.e. we are interested in their role in the determination of viable intertemporal paths. Within the framework of viability theory this is captured by the parametric restrictions under which the viability kernel is non-empty.

The above mentioned technological features are basically parameters representing slopes and curvatures of the production function with respect to the vector of inputs and parameters representing the curvature of level curves on the one hand and parameters representing the nature of technological change on the other.

Following Perrin (1998), these technological features can be categorized in two broad sets: the set of features describing the basic technology and the set describing the nature of the technological change. The first set is composed by: input elasticity of production (not explicitly included in Perrin[8]), returns to scale, slope of elasticity of scale, elasticity of substitution and expansion path. The second set is composed by: rate of technological change, size bias of technological change and input bias of technological change.

*No Technological Change*_ $f(K(t),r(t))$

- Input i's Elasticity of Production (IiEP). The elasticity of production of an input it is just defined as the percentage change in output after a one percent change in the level of input as it is expressed by: $\frac{d\ln f(x)}{d\ln x_i}$
- Elasticity of Scale (ES). It is the percentage change in output after a *proportional* change in the level of *all* inputs and it is computed as: $\left.\frac{d\ln(f(\lambda K(t),\lambda r(t)))}{d\ln\lambda}\right|_{\lambda=1}$

[8] This measure is not needed for the purposes of that study and in any case is captured by the other parameters calculated.

- Slope of Elasticity of Scale (SES). This is a curvature measure since it shows the change in the slope of the production function for the case of a simultaneous proportional change in all inputs $\left.\frac{d\left[\frac{d\ln\left(f\left(\lambda K(t),\lambda r(t)\right)\right)}{d\ln\lambda}\right]}{d\lambda}\right|_{\lambda=1}$
- (Morishima)[9] Elasticity of Substitution (MES). It is a measure of the slope of the isoquant or expressed in terms of price, the percentage change in optimal input ratio as a result of a change in relative prices. It is expressed as: $\sigma_{kr}^{M}=\frac{f_r}{x_k}\frac{F_{kr}}{F}-\frac{f_k}{x_r}\frac{F_{kr}}{F}$ where F is the bordered Hessian determinant and F_{ij} is the cofactor associated with f_{ij}. σ_{kr}^{M} >0 means the inputs are substitutes and σ_{kr}^{M} <0 means they are complements.
- Expansion Path (EP). It is defined as the change in the Marginal Rate of Transformation after a change in the level of output. It is expressed as: $\frac{d\left(f_k/f_r\right)}{dy}$

For these parameters to be computable we need the production function to fulfill some regularity conditions, in particular we need it to be continuous and twice continuously differentiable. Continuity is guaranteed by assuming that f(x) is a function (single valued) defined in the positive orthant of the real line and is finite. Moreover, the input requirement set V(y), is closed and non-empty for all y>0.

Technological Parameters for Cobb Douglas Specification

We will derive in this section a technological description of the Cobb Douglas technology focusing upon its restrictive nature for the purposes of sustainability analysis. We will do this by showing its maintained hypothesis the their implications.

Parametric Description

The CD specification is in fact very restrictive in terms of maintained hypotheses regarding these technological features. Specifically, for the first set of parameters identified in the previous section it yields the following:

- Input Elasticity of Production: $\frac{d\ln f(.,.)}{d\ln k}=\alpha$ and $\frac{d\ln f(.,.)}{d\ln r}=\beta$
- Elasticity of Scale: $\alpha+\beta$
- Slope of Elasticity of Scale: 0
- Elasticity of substitution: constant and equal to 1

[9] We use Morishima and not Allen for several properties that the former has over the later. For a detailed technical comparison see Blackorby and Rusell (1989).

- Expansion path: linear (i.e. assumes homotheticity). The optimal input ratio $\left(\frac{r^*}{k^*}=\frac{p_k}{p_r}\frac{\beta}{\alpha}\right)$ is constant.

Note that there is a very close relationship between input elasticity of production and elasticity of scale. Indeed any increase in the input elasticities of production will cause an increase in the elasticity of scale

Next we will review the main results in growth with exhaustible resources regarding sustainability conditions under four cases: Zero Technical Progress and Zero Capital Depreciation, Zero Technical Progress and Positive Capital Depreciation, Positive Technical Progress and Zero Capital Depreciation and Positive Technical Progress and Positive Capital Depreciation

We will summarize the results in the literature and the ones derived here in several propositions. This has the purpose of ordering and summarizing the main lessons of this study and drawing a parallel with previous, more restrictive, results. In addition, stating the results in propositions allows us to relegate proofs to the Appendix and focus the analysis in the intuitive interpretations of the findings.

Sustainability Results For Cobb Douglas Specification Zero Technical Progress (ZTP) and Zero Capital Depreciation (ZCD)

PROPOSITION 1. For a CES technology an elasticity of substitution greater than one is a sufficient condition for existence of an indefinitely-maintainable positive level of consumption since the resource is inessential. For proof see Dasgupta and Heal (1974), section 1.4 based on results 1.22a-1.22c

PROPOSITION 1'. If the resource is strongly essential for the technology in the sense that $r_b(f)>0$ then the economy is unsustainable, i.e. $Viab(f,c_b,S_b)=\varnothing$. For proof see proof of proposition 1 in Martinet and Doyen (2007), Appendix A2

Proposition 1' is in a sense a generalization of Proposition 1 since it is not restricted to CES specification. This is further shown in their study when the case of a linear additively separable production function is discussed.

PROPOSITION 2. For a CES technology an elasticity of substitution lower than one is a sufficient condition for non-existence of an indefinitely-maintainable positive level of consumption since the resource's average product is unbounded. For proof see Dasgupta and Heal (1974), section 1.4 based on results 1.20a-1.20c

PROPOSITION 2'. If the resource is non-essential for the technology, i.e. there is a capital level K^+ with $r_b(f)=0=r_b(f,K^+)$ then the economy is sustainable in the sense that the viability kernel is not empty or $Viab(f,c_b,S_b)\neq\varnothing$. For proof see proof of proposition 2 in Martinet and Doyen (2007), Appendix A2.

As in Proposition 1', Proposition 2' is a generalization of Proposition 2 since it is not restricted to CES specification. Martinet and Doyen discuss the case of a Leontief for the sake of illustration of the point.

The generalizations in propositions 1' and 2' are important since they focus on the true conditions required from the technology: essentiality and unbounded average product. One consequence of assuming CES technology is that there is a monotonic relation between the value of the elasticity of substitution and essentialness of an input as defined by Dasgupta and Heal (1974) that might create a misleading general identification of these two concepts. I will go back to this discussion in Section V.

More specifically, within the CES family an elasticity of substitution greater than one implies that the inputs are not strictly essential, an elasticity of substitution equal to one imply both strictly essential inputs and unbounded average product (Dasgupta and Heal's essentiality) and an elasticity of substitution below one implies strict essentiality but with bounded average product. I will soon show that the monotonic relation between elasticity of substitution and essentialness is specific to the CES family and does not extend to more flexible technologies.

We now turn to results derived in the literature for the particular case of a CD specification.

PROPOSITION 3. In an economy with an exhaustible resource and a CD technology; an elasticity of output with respect to reproducible capital (α) lower than an elasticity of output with respect to exhaustible resource (β) is a sufficient condition for unsustainability (i.e. non existence of an indefinitely-maintainable positive level of consumption). For proof see Solow (1974) Appendix B.

This result was generalized first by Stiglitz (1974):

PROPOSITION 3' (Proposition 5b in Stiglitz):

"A necessary and sufficient condition for a constant level of consumption with no technical change and no growth is that the share of natural resources (α_3) be less than the share of capital (α_1)."

A generalization was also provided in the context of Viability Theory by Martinet and Doyen who prove (under the assumption $\beta < 1$) that in fact $\alpha \leq \beta$ is a necessary and sufficient for unsustainability and hence by definition of unsustainability (complement of sustainability), $\alpha > \beta$ is a necessary and sufficient condition for sustainability.

Sustainability Conditions and Viability Theory

PROPOSITION 3''. In an economy with an exhaustible resource, a CD technology and $\beta < 1$, an elasticity of output with respect to reproducible capital higher than the elasticity of output with respect to exhaustible resource is a necessary and sufficient condition for sustainability (i.e. non emptiness of the Viability Kernel). For proof see proof of proposition 3 in Martinet and Doyen (2007) Appendix A2. In fact:

$$Viab(f, c_b, S_b) = \begin{cases} \varnothing & \text{if } \alpha \leq \beta \\ \{(S, K) \text{ such that } S \geq V(K, c_b, S_b)\} & \text{if } \alpha > \beta \end{cases}$$

where V is a function defined by

$$V(K, c_b, S_b) = \frac{1}{\alpha - \beta}\left(\frac{c_b}{1-\beta}\right)^{\frac{1-\beta}{\beta}} K^{\frac{(\beta-\alpha)}{\beta}} + S_b$$

Implications of Proposition 3'' in Terms of Technological Parameters

- Input Elasticity of Production: $\frac{d \ln f(.,.)}{d \ln k} = \alpha > \frac{d \ln f(.,.)}{d \ln r} = \beta$. For the economy to be sustainable, capital elasticity of output has to be greater than resource elasticity of output.
- Elasticity of Scale $(\alpha + \beta)$: no restrictions are imposed on this parameter.
- Slope of Elasticity of Scale: 0
- Elasticity of substitution: constant and equal to 1
- Expansion path $\left(\frac{r^*}{k^*} = \frac{p_k}{p_r}\frac{\beta}{\alpha}\right)$: no restrictions are imposed on this parameter.

Sustainability Results For Cobb Douglas Specification Zero Technical Progress (ZTP) and Positive Capital Depreciation (ZCD)

PROPOSITION 4. In an economy with an exhaustible resource and a CD technology, an elasticity of output with respect to reproducible capital lower than one $(\alpha < 1)$ and positive capital depreciation $(\lambda > 0)$ imply unsustainability (i.e. non existence of an indefinitely-maintainable positive level of consumption). For proof see Solow (1974).

In sum, an economy without technological progress and with capital depreciation is doomed to extinction.

Sustainability Results For Cobb Douglas Specification Positive Technical Progress (PTP) and Zero Capital Depreciation (ZCD)

As pointed out by Stiglitz (1974): "There are at least three economic forces offsetting the limitations imposed by natural resources: technical change, the substitution of man-made factors of production (capital) for natural resources, and returns to scale. This study is an attempt to determine more precisely under what conditions a sustainable level of per capita consumption is feasible…"

Consequently, Stiglitz derives the following result presented there as Proposition 4':

PROPOSITION 4'**.** "If the rate of population growth is positive, necessary and sufficient condition for sustaining a constant level of consumption per capita is that the ratio of the rate of technical change, γ, to the rate of population growth must be greater than or equal to the share of natural resources."

Although this case is one of PTP and ZCD it considers positive population growth and it is not thus directly comparable to the results here. However recalling the inclusion of technological progress in previous studies is needed for the purpose of illustrating the nature of the technological change considered: CD technology with Hicks-neutral and size-neutral technical change. Stiglitz shows however consciousness of the great limitation that this entails[10].

Sustainability Results For Cobb Douglas Specification Positive Technical Progress (PTP) and Positive Capital Depreciation (PCD)

Since the purpose of this section is to review previous results, we will consider here the especial case of capital augmenting endogenous technical progress. This is the type of technical change considered by Martinet and Doyen for which they derive necessary conditions for sustainability.

Sustainability Conditions

I will summarize what has been done in Proposition 5 below. This proposition was stated and proved (also as Proposition 5) in Martinet and Doyen (2007).

PROPOSITION 5. If there is a positive capital depreciation term $(\lambda > 0)$ and if the production function has the form $f(A,K,r) = A(k)K^{\alpha} r^{\beta}$ with $\beta < \alpha$, $\beta < 1$ and $A(K) = K^{1-\alpha+\varepsilon}$ with $\varepsilon > 0$, then the viability kernel is not empty. For proof see proof of proposition 5 in Martinet and Doyen (2007), Appendix A2.

Implications of Proposition 5 in Terms of Technological Parameters

- Input Elasticity of Production: $\frac{d \ln f(.,.)}{d \ln k} = \alpha > \frac{d \ln f(.,.)}{d \ln r} = \beta$. For the economy to be sustainable, capital elasticity of output has to be greater than resource elasticity of output.
- Elasticity of Scale $(1+\varepsilon+\beta)$: Increasing Returns to Scale are imposed.
- Slope of Elasticity of Scale: 0
- Elasticity of substitution: constant and equal to 1
- Expansion path $\left(\frac{r^*}{k^*} = \frac{p_k}{p_r}\frac{\beta}{\alpha}\right)$: no restrictions are imposed on this parameter.

Summary So Far

The results of the literature on sustainability of an economy with an exhaustible natural resource were summarized and compacted in eight propositions. In addition, five parameters describing the technology of the economy were proposed and derived for the case of a CD specification and the implications of the eight propositions were re-

[10] A passage of his paper is cited at the beginning of Section IV to illustrate this.

expressed in terms of these parameters. This was done for neater comparison with results that will soon be derived for more flexible technologies.

Selection of Flexible Approximations

It was already noted by Stiglitz (1974) that in a Cobb-Douglas production function, if technological change is exogenous, we need not distinguish between labor, capital and resource augmenting technical progress. Moreover an elasticity of substitution greater than one yields a sustainable economy and increasing returns to scale, make the required rate of technical progress necessary to offset the effects of the decreased input of natural resources smaller.

Therefore the introduction of a more flexible technology in the analysis is not just a matter of completeness but the result of a search for more specific and realistic sustainability conditions. The use of functional forms other than CD would either enrich the analysis on the role of technological parameters in sustainable development or change some existing insights or both. Therefore, the goal of using different specifications than CD is to use more flexible functional forms that will allow us to analyze the effect of technological parameters on sustainability (non emptiness of viability kernel) under a different (less restrictive) set of maintained hypotheses. Whatever the functional chosen though, it must satisfy the two properties for empirical relevance (in the Dasgupta-Heal sense) and as mentioned before a wide variety of functional forms more flexible than CES or not belonging to the CES family at all, satisfy them.

To choose among these functional forms we first need to define a set from where to decide which to work with. For that we need first to put together a pool of functionals widely used in empirical work. From this pool we will proceed to check which of these satisfy the conditions needed for empirical relevance (essentiality and $\lim_{r\to 0}\frac{y}{r} = \lim_{r\to 0} AP_r = \infty$). Finally from the subset of forms satisfying those two properties we will use the ones nesting CD either by nesting CES or directly CD. Thus these functionals fulfill a total of three key properties: essentiality, unbounded average product and flexibility.

To build a complete list of available functional forms we will rely on a very complete survey by Griffin et.al. that summarized functional forms used in empirical work and discussed their properties. There's another prominent survey in chapter 4 in Fuss et.al. but this is (in terms of functional forms revised) a subset of Griffin's. The final list of functional forms discussed and analyzed there is displayed in their Table 1.

As we can see from the table in Appendix 1[11] there are 7 functional forms nesting the CD. Three of them nest CD directly[12] (Generalized CD, Transcendental and Translog), one is the CES, there is one that nests CD indirectly through CES[13] (Generalized

[11] Regarding the generalized power production function I will consider a special case first proposed by de Janvry (1972) in which $y = \alpha k^{\alpha_1} r^{\alpha_2+\beta_2 k} \exp(\gamma_1 k)$. Expressed in terms of the table in Appendix 1 this implies $f_k(r,k) = \alpha_1$, $f_r(r,k) = \alpha_2 + \beta_2 k$ and $g(k,r) = \gamma_1 k$.

[12] By "directly" I mean they do not nest a functional form that in turn nests CD.

[13] This means that there is no way to obtain CD without first passing through CES.

Quadratic), one that nets CD indirectly through transcendental and one that nests CD both directly and also indirectly through CES (Generalized Box-Cox).

From this set of functionals there is a subset of three that satisfy strict essentiality of all inputs and, under certain conditions, unbounded natural resource average product (check for satisfaction of properties is done in Appendix 1):

- Generalized Quadratic (it satisfies properties in the two inputs case)

$$y = \left[\beta_{kr}\, k^{\delta\gamma} r^{\delta(1-\gamma)}\right]^{\frac{\gamma}{\delta}}$$

- Transcendental

$$y = \alpha k^{\beta_k} r^{\beta_r} \exp(\delta_k k) \exp(\delta_r r)$$

- Generalized Power (special case from de Janvry 1972)

$$y = \alpha k^{\alpha_1} r^{\alpha_2 + \beta_2 k} \exp(\gamma_1 k)$$

Analytical Results

A commonly used argument for the use of a CD production function in sustainability theory is that it offers the possibility of deriving analytical results. The idea that analytical results provide a level of generality that numerical results do not, seems to be widely accepted in economics.

As it turns out, among the functional forms fulfilling all three criteria defined above, the generalized quadratic yields analytical solutions as it is, the transcendental yields analytical solutions when $\delta_r = 0$ [14] and hence $y = \alpha k^{\beta_k} r^{\beta_r} \exp(\delta_k k)$ and finally the special case of Generalized Power does not yield analytical results. The first two cases will then be used in the next section for the specification of the production function to assess sustainability conditions.

We will proceed now to the analysis of sustainability conditions with flexible specifications, their implications in terms of technological parameters and how are these compared to previous results in the literature. The goal of these following sections are to identify the new role (on non-emptiness and shape of the viability kernel) of parameters already discussed before but now in the context of more flexible approximations and in addition determine the role of new parameters, previously neglected, on necessary conditions for sustainability

Technological Parameters and Sustainability Results for Cobb Douglas Specification with ZTP and ZCD

Previous Results in the literature and their parametric implications were reflected in Proposition 3'', Section III.

[14] This implies that productivity of natural resource is constant and elasticity of scale is independent of the level of natural resource as we will show below.

Technological Parameters and Sustainability Results for Generalized Quadratic Specification with ZTP and ZCD

Parametric Description

The Generalized Quadratic specification is less restrictive than the CD in terms of maintained hypotheses regarding the technological parameters defined in Section II. Specifically it yields the following:

- Input Elasticity of Production: $\frac{d\ln f(.,.)}{d\ln k} = \gamma^2$ and $\frac{d\ln f(.,.)}{d\ln r} = \gamma(1-\gamma)$
- Elasticity of Scale: γ
- Slope of Elasticity of Scale: 0
- $\sigma^M = \dfrac{\left(\dfrac{\gamma^3-\gamma^4}{\gamma^{10}-3\gamma^9+6\gamma^8-7\gamma^7+3\gamma^6-\gamma^4+2\gamma^3-\gamma^2}\right)}{\left(\dfrac{k-r}{r^2k}\right)}$ so it's neither constant nor necessarily equal to one.
- Expansion path: linear (i.e. assumes homotheticity). The optimal input ratio $\left(\frac{r^*}{k^*} = \frac{p_k}{p_r}\frac{(1-\gamma)}{\gamma}\right)$ is constant.

Note once again the relationships among the different parameters. An increase in γ increases both capital and resource elasticities of production and it also increases returns to scale. Moreover, increasing returns to scale (γ>1) would imply a negative resource elasticity of production. Also, by looking at the expansion path, an increase in γ (i.e. an increase in elasticity of scale and capital elasticity of production) reduces the ratio of resource to capital.

Sustainability conditions

I will define sustainability conditions by using the viability kernel as indicator of sustainability. This means that the conditions for an economy with an exhaustible resource to be sustainable are equivalent to the conditions for non-emptiness of the viability kernel (NEC). I will now derive in proposition 6 those NEC and depict the viability kernel for the GQ technology.

PROPOSITION 6. Consider an economy with an exhaustible natural resource and a Generalized Quadratic technology $y = \left[\beta_{kr} k^{\delta\gamma} r^{\delta(1-\gamma)}\right]^{\frac{\gamma}{\delta}}$. Then the viability kernel depends on parameter γ as follows:

$$Viab(f, c_b, S_b) = \begin{cases} \{(S,K) \text{ such that } S \geq V(K, c_b, S_b)\} & \text{if } 0 < \gamma \leq 1, \beta_{kr} > 0 \\ \varnothing & \text{otherwise} \end{cases}$$

where V is a function defined by

$$V(K, c_b, S_b) = (1-\gamma)\gamma\beta_{kr}^{\frac{1}{1+\gamma}} \left(\frac{c_b}{\gamma}\right)^{\frac{(1-\gamma)\gamma-1}{(1-\gamma)\gamma}} K^{\frac{2-\gamma}{1-\gamma}} + S_b$$

See proof in Appendix 2.

From now on I will refer to $0 < \gamma \leq 1, \beta_{kr} > 0$ as NEC (Non-Emptiness Conditions). In addition EC (essentialness condition) requires $\gamma < 1$.

Implications in Terms of Technological Parameters of EC and NEC

- Input Elasticity of Production: $0 < \gamma^2 < 1$ and $0 < \gamma(1-\gamma) \leq \frac{1}{2}$. Moreover $\frac{d \ln f(.,.)}{d \ln k} > \frac{d \ln f(.,.)}{d \ln r}$ $\forall \gamma > 0.707106$, Therefore in the range $0 < \gamma < 0.707106$, the elasticity of output with respect to reproducible capital γ^2 is lower than the elasticity of output with respect to exhaustible resource $\gamma(1-\gamma)$ **and yet the economy is not un-sustainable** (i.e. the viability kernel is non empty). This shows that the conditions required in propositions 3 and 4 DO NOT extend to the more flexible case offered by a Generalized Quadratic technology.
- Since Elasticity of Scale is γ, the NEC implies non increasing returns to scale. Moreover the addition of EC implies **decreasing returns to scale** for the economy to be sustainable. This is in fact more restrictive than NEC in the CD case since there the two conditions $(\alpha > \beta \text{ and } \beta < 1)$ do not restrict returns to scale to be increasing, constant or decreasing. This shouldn't be surprising once one notes that decreasing returns to scale is equivalent to non negative resource elasticity of production. Hence this condition should be interpreted as a lower bound on resource productivity instead of an upper bound on returns to scale.
- NEC and EC do not affect Slope of Elasticity of Scale since this is 0 always.

- The constant part of σ^M is not defined for $\gamma = 1$, however EC requires $\gamma < 1$ and hence the elasticity is defined in the range satisfying NEC and EC. Just to illustrate the strange behavior of the elasticity of substitution under NEC and EC one can check that for example, for a value of γ=0.77143, σ^M equals 2,127,923 and for another, very close, value of γ such as 0.77142 we find a value of σ^M of -64,667.9. So NEC and EC do not really restrict the range of σ^M.

o EC and NEC make the optimal ratio (constant along the expansion path) $0 < \left(\frac{r^*}{k^*}\right) < \infty$. Additionally note that this ratio depends negatively upon γ.

In conclusion, sustainability conditions in the generalized quadratic specification with essential inputs are less restrictive than CES in terms of elasticity of substitution required for empirical relevance. Moreover sustainability conditions are more restrictive in terms of elasticity of scale but less restrictive than the CD conditions in terms of output elasticity of inputs, slope of elasticity of scale and the evolution of optimal ratio along the expansion path.

Technological Parameters and Sustainability Results for Transcendental Specification with ZTP and ZCD

It turns out that, as it stands, this form does not yield closed-form analytical results. However we will use a nested case consisting of $\delta_r = 0$ which means that resource productivity is entirely captured by (although different to) β_r.

Parametric Description

o Input Elasticity of Production: $\frac{d \ln f(.,.)}{d \ln k} = \beta_k + \delta_k k$ and $\frac{d \ln f(.,.)}{d \ln r} = \beta_r$

o Elasticity of Scale: $\left.\frac{d \ln f(\lambda x)}{d \ln \lambda}\right|_{\lambda=1} = \beta_k + \beta_r + \delta_k k$

o Slope of Elasticity of Scale: $\delta_k k$

o $\sigma^M = \left(\frac{\left(\beta_k k^{-1} + \delta_k\right)^2}{\left(\beta_k k^{-1} + \delta_k\right)^2 + \beta_k \beta_r}\right)\left(\frac{k-r}{r\,k}\right)$ so it's neither constant nor necessarily equal to one.

o Expansion path: it can be shown that the optimal input ratio can be expressed implicitly as $\left(\frac{r^*}{k^*} = \left[\frac{p_k}{p_r} - \frac{\delta_k}{\beta_r} r^*\right]\frac{\beta_r}{\beta_k}\right)$ where $r^* = \frac{y}{\alpha k^{\beta_k} \exp(\delta_k k)}$ and hence is not linear.

It is worth noting that this approximation is rich enough so as to allow for some degree of independence among parameters; an increase in β_r affects the resource elasticity of production and the elasticity of scale but not the capital elasticity of production. On the other hand an increase in $\beta_k + \delta_k k$ increases capital elasticity of production and elasticity of scale but does not affect the resource elasticity of production.

There is also a relationship between elasticities of production and elasticity of substitution, in fact we can show that $\frac{\partial \sigma^M}{\partial \beta_r} < 0 \Leftrightarrow k > r$ and $\frac{\partial \sigma^M}{\partial \beta_k} < 0 \Leftrightarrow \left(k\delta_k > \beta_k\right)$ and $k > r$.

Sustainability conditions

I will now derive in proposition 7 the conditions for non emptiness of the viability kernel (NEC) for a transcendental technology and depict its viability kernel.

PROPOSITION 7. Consider an economy with an exhaustible natural resource and a transcendental technology $y = \alpha k^{\beta_k} r^{\beta_r} \exp(\delta_k k)$. Then the viability kernel depends on parameters β_k, β_r and δ_k as follows:

$$Viab(f, c_b, S_b) = \begin{cases} \{(S,K) \text{ such that } S \geq V(K, c_b, S_b)\} & \text{if } \beta_k > \beta_r, \delta_k > 0 \\ \varnothing & \text{otherwise} \end{cases}$$

where V is a function defined by

$$V(K, c_b, S_b) = \int_k -\left(\frac{c_b}{\alpha(1-\beta_r)}\right)^{\frac{1-\beta r}{\beta r}} K^{-\frac{\beta k}{\beta r}} \exp\left(-\frac{\delta_k}{\beta_r} k\right) dk + S_b$$

Proof proceeds as illustration in Appendix 2.
Then the EC requires $\beta_r < 1$ and the NEC require $\beta_k > \beta_r$ and $\delta_k > 0$

Implications in Terms of Technological Parameters of EC and NEC

- Input Elasticity of Production: EC and NEC yield $(\beta_k + \delta_k k) > 0\ \forall$k and $\beta_r < \beta_k + \delta_k k$. Hence in the case of the transcendental technology an elasticity of output with respect to reproducible capital greater than the elasticity of output with respect to exhaustible resource **is required as in the CD case for the economy to be sustainable** (i.e. non empty viability kernel). In this case then the conditions required in propositions 3 and 4 do extend to the more flexible case offered by a Transcendental technology.

- Since Elasticity of Scale is $\beta_k + \beta_r + \delta_k k$, the EC and NEC allow for increasing returns to scale $(\beta_k + \beta_r > 1 - \delta_k k)$, constant returns to scale $(\beta_k + \beta_r = 1 - \delta_k k)$ or decreasing returns to scale $(\beta_k + \beta_r = 1 - \delta_k k)$. Moreover there are increasing returns to scale $\forall k > \frac{1-(\beta_k + \beta_r)}{\delta_k}$. Therefore this is in fact equally restrictive than ES and NEC in the CD case since neither of both pose restrictions on returns to scale.

Comparing restrictions imposed by generalized quadratic and transcendental there seems to be a trade off between restrictions on inputs elasticities of output and restrictions in returns to scale. More specifically, for the economy to be sustainable in the empirical relevant case if there is no restriction on returns to scale then a restriction on input elasticity of output exists (transcendental) or vice versa (generalized quadratic).

- Since the slope of elasticity of scale is $\delta_k k$, NEC and EC guarantee a positive value of this parameter.

- The constant part of σ^M is always positive under EC and NEC. However these conditions do not impose any restrictions on σ^M in terms of being lower, equal or higher to one. Therefore, as in the generalized quadratic case, there is no identification between elasticity of substitution and essentiality i.e. **essentiality of inputs is consistent with values of σ^M above, equal or below 1.** We could say, however, that since $\frac{\partial \sigma^M}{\partial \beta_r} < 0 \Leftrightarrow k > r$ and EC and NEC are imposing an upper bound on β_r (i.e. $\beta_r < \beta_k + \delta_k k$) then EC and NEC tend in fact to require a higher elasticity of substitution, although is not restrictive enough so as to require a certain level of substitutability.
- Finally EC and NEC do not impose any restriction on the expansion path.
-

In conclusion, sustainability conditions in the transcendental specification with essential inputs are less restrictive than CES in terms of elasticity of substitution required for empirical relevance. Further, sustainability conditions are equally restrictive in terms of input elasticity of output and elasticity of scale but less restrictive than the CD conditions in terms slope of elasticity of scale and the evolution of optimal ratio along the expansion path.

Going back to the discussion of identifying elasticity of substitution with essentiality in the Dasgupta-Heal sense we can see here that EC does not pose significant restrictions on the value of σ^M whatsoever. So there is no identification between elasticity of substitution and essentiality. In fact essentiality of inputs is consistent with values of σ^M above, equal or below 1.

Technological Parameters and Sustainability Results for Cobb-Douglas Specification with ZTP and PCD

Previous Results in the literature were reflected in Proposition 4, Section III. Now we will reconsider the case of positive capital depreciation and zero technical change with more flexible technologies and check the generality of the result in Proposition 4.

Technological Parameters and Sustainability Results for Generalized Quadratic Specification with ZTP and PCD

Sustainability Conditions

As it turns out the existence of a positive rate of capital depreciation does not change the NEC in the case of a GQ technology which means that the economy can be sustainable even under ZTP and PCD and an elasticity of production with respect to capital less than one. This result will be stated in the next proposition.

PROPOSITION 8. Consider an economy with an exhaustible natural resource and a Generalized Quadratic technology $y = \left[\beta_{kr} k^{\delta\gamma} r^{\delta(1-\gamma)}\right]^{\frac{\gamma}{\delta}}$. Then the viability kernel depends on parameters β_{kr} and γ as follows:

$$Viab(f,c_b,S_b)=\begin{cases}\{(S,K)\text{ such that } S\geq V(K,c_b,S_b)\} & \text{if } \gamma>1, \beta_{kr}>0\\ \varnothing & \text{otherwise}\end{cases}$$

where V is a function defined by

$$V(K,c_b,S_b)=\int_k (1-\gamma)\gamma\beta_{kr}^{\frac{1}{1+\gamma}}\left(\frac{1+\lambda}{\gamma}\right)^{\frac{(1-\gamma)\gamma-1}{(1-\gamma)\gamma}} K^{-\frac{1}{\gamma}}dk+S_b$$

Proof proceeds as illustration in Appendix 2.

Implications in Terms of Technological Parameters of EC and NEC

We have a very interesting case here. Note that NEC is inconsistent with EC[15] and hence the economy is actually doomed *for the empirically relevant case.* This means that even though there is a configuration of parameters for which the economy can sustain a positive level of consumption forever, this configuration makes the resource non essential and hence this case is not relevant for our purposes. The economy is then unsustainable.

Technological Parameters and Sustainability Results for Transcendental Specification with ZTC and PCD

Sustainability conditions

As it turns out the use of more flexible functional forms like the transcendental still leave room for a sustainable economy even under ZTP and PCD. This result will be stated in the next two propositions.

PROPOSITION 9**.** Consider an economy with an exhaustible natural resource, a positive rate of capital depreciation $(\lambda>0)$ and a transcendental technology $y=\alpha k^{\beta_k}r^{\beta_r}\exp(\delta_k k)$. Then the viability kernel depends on parameters β_k, β_r and δ_k as follows:

$$Viab(f,c_b,S_b)=\begin{cases}\{(S,K)\text{ such that } S\geq V(K,c_b,S_b)\}\text{ if}\left[\begin{array}{l}\beta_k+\beta_r>1\\ \text{and } \delta_k>0\end{array}\right.\\ \varnothing \qquad\qquad\qquad\qquad \text{otherwise}\end{cases}$$

where V is a function defined by

$$V(K,c_b,S_b)=-\int_k\left(\frac{\frac{c_b}{k}+\lambda}{\alpha(1-\beta_r)}\right)^{\frac{1-\beta r}{\beta r}} K^{\frac{1-(\beta r+\beta k)}{\beta r}} e^{\left(-\frac{\delta_k}{\beta_r}k\right)}dk+S_b$$

Proof proceeds as illustration in Appendix 2.

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[15] If $\gamma>1$ then $y(k,0)=\infty$ and the resource is not strictly essential.

Implications in Terms of Technological Parameters of EC and NEC

- Input Elasticity of Production: EC and NEC yield $\left(\beta_k + \delta_k k\right) > 0\ \forall$k and $\beta_r > 1 - \beta_k$. Note that the new NEC do not change the former with respect to the case with ZCD but they change the latter and they do so by relaxing a constraint. This is in fact surprising; an elasticity of output with respect to reproducible capital greater than the elasticity of output with respect to exhaustible resource **is NOT required for the economy to be sustainable** (i.e. non empty viability kernel). However, we will see that this relaxation is so because the new conditions will impose more structure upon other parameters (i.e. elasticity of scale and slope of elasticity of scale).
- Since Elasticity of Scale is $\beta_r + \beta_k + \delta_k k$, the EC and NEC imply strictly increasing returns to scale. This is in fact the first case in which sustainability conditions require increasing returns to scale. Once again (as in the comparison between transcendental and generalized quadratic with ZCD and ZTP) an apparent trade off between restrictions on inputs elasticities of output and restrictions in returns to scale seems to have emerged. More specifically, for the economy to be sustainable in the empirical relevant case with a transcendental technology, ZTP and PCD, if there is no restriction on inputs elasticities of output then a restriction on returns to scale needs to be imposed.
- Since the slope of elasticity of scale is $\delta_k k$, NEC and EC guarantee a positive value of this parameter.
- The constant part of σ^M is not generally affected by the new EC and NEC. Therefore, there is no identification between elasticity of substitution and essentiality i.e. **essentiality of inputs is consistent with values of σ^M above, equal or below 1.**
- Finally the new EC and NEC do not impose any restriction on the optimal ratio.

Proposition 9 offers both a surprising and an also (as the case of a GQ) somewhat optimistic result. Even without any kind of technological progress and positive capital depreciation the economy IS NOT doomed and an indefinitely-maintainable positive level of consumption exists.

Positive Technical Progress (PTP)

In this section we ask the same questions but this time we will do so by including technical change. One implication of considering technical change in the analysis and allowing for biased technical change is that in addition to the five parameters describing the technology, we will consider three other parameters describing the nature of technological change (Perrin 1998). The second set of parameters is composed by:

- Rate of technological change: $\frac{d\ln y}{dt}$. Where t does not represent time but technology.

The relevance of this clarification comes from the fact that we will model technical change as being endogenous and hence the t will not represent time but the value of parameters affecting input effectiveness.

- Size bias of technological change. The first reference to this concept was discussed in Perrin (1998). There the author describes the dual analogous of the concept I will derive here. To define this parameter in primal space I will make use of two well known concepts in production theory (Chambers 1988): Marginal Ray Product (MRP) and Average Ray Product (ARP). The MRP is the slope of $f(\lambda x)$ for a given x and it is expressed as MRP=$\frac{\partial f(\lambda x)}{\partial \lambda}$ and ARP=$\frac{f(\lambda x)}{\lambda}$. The MRP and ARP are the primal to marginal cost and average cost respectively and hence following the expression in Perrin we can define the size bias of technological change as $\frac{d \ln MRP}{dt} - \frac{d \ln ARP}{dt}$. Moreover, since the elasticity of scale can be expressed as the ratio of the MRP to ARP we can interpret this measure as the percentage change in elasticity of scale due to technological change.

- Input bias of technological change. Since it is my intention to avoid expressions depending on prices and, in that way, derive sustainability conditions completely described by technological parameters, I will express the input bias of technological change as the percentage change in TRS due to a change in technology: $\frac{d \ln f_k(.)}{dt} - \frac{d \ln f_r(.)}{dt}$

Choice of Technical Change Modeling

The type of technical change included here is factor augmenting endogenous technical change as opposed to disembodied, exogenous, factor augmenting technical change. In particular for the purposes of this study we will introduce a factor specific augmenting technical change: $A(x_i) = x_i^{\gamma}$. This term enters the production function multiplicatively as we will show below. The endogenous technical change is flexible in the sense that it allows for non-neutral technical change and it also allows, under certain conditions that we will discuss below, for the decomposition between changes affecting just a subset of technological parameters.

Finally another reason to use endogenous technical change is that in the context of viability theory there is a very important difference between endogenous and exogenous technical change. The endogenous version then yields a bound for the viability kernel that is autonomous in time. On the other hand the exogenous version yields a time-dependent bound of the viability kernel which technically means that we are calculating what in the context of viability theory is called a “viability tube” which sometimes requires more structure on the topological space to be worked out.

Positive Technical Progress (PTP) and Zero Capital Depreciation (ZCD)

Technological Parameters for Cobb Douglas Specification

$\left[f(k(t), r(t)) = A(k) k^{\alpha} A(r) r^{\beta} \text{ where } A(k) = k^{\varepsilon}, A(r) = r^{\gamma} \right]$

The addition of technical change changes the first set of parameters. The parameters for the CD specification with endogenous technical change are:

- Input Elasticity of Production: $\frac{d\ln f(.,.)}{d\ln k}=\alpha+\varepsilon$ and $\frac{d\ln f(.,.)}{d\ln r}=\beta+\gamma$
- Elasticity of Scale: $\alpha+\varepsilon+\beta+\gamma$
- Slope of Elasticity of Scale: 0
- Elasticity of substitution: constant and equal to 1
- Expansion path: linear (i.e. assumes homotheticity). The optimal input ratio is depicted by $\left(\frac{r^*}{k^*}=\frac{p_k}{p_r}\frac{\beta+\gamma}{\alpha+\varepsilon}\right)$.

The second set of parameters describing the nature of technological change need to be derived. Specifically, for the second set of parameters the CD yields the following:

- Rate of technological change (RTC): $\frac{d\ln y}{dt}=(\ln k)d\varepsilon+(\ln r)d\gamma$. Then, if we start at $\varepsilon=\gamma=0$, then $d\varepsilon=\varepsilon$ and $d\gamma=\gamma$ and we can re-express this as $RTC=(\ln k)\varepsilon+(\ln r)\gamma$.
- Size bias of technological change (SBTC). It can be shown that the technological change is neutral in terms of size.
- Input bias of technological change (IBTC). The input bias of technical change is resource saving (using) if $IBTC=\frac{d\varepsilon}{(\alpha+\varepsilon)}-\frac{d\gamma}{(\beta+\gamma)}>(<)0$. Therefore technical change is resource saving (using) $\Leftrightarrow\frac{d\varepsilon}{(\alpha+\varepsilon)}>(<)\frac{d\gamma}{(\beta+\gamma)}$. Then, if we start at $\varepsilon=\gamma=0$, then $d\varepsilon=\varepsilon$ and $d\gamma=\gamma$ and we can re-express this as $IBTC>(<)0\Leftrightarrow\varepsilon>(<)\frac{\alpha}{\beta}\gamma$.

Sustainability conditions for Cobb Douglas Specification

As it turns out the use of more flexible functional forms like the transcendental still leaves room for a sustainable economy even under ZTP and PCD. This result will be stated in the next two propositions.

PROPOSITION 10**.** Consider an economy with an exhaustible natural resource, no capital depreciation $(\lambda=0)$ and a CD technology with endogenous capital augmenting technical change $y=(k^{\varepsilon}k^{\alpha})(r^{\gamma}r^{\beta})$ where $\varepsilon,\gamma>0$. Then the viability kernel depends on parameters as follows:

$$Viab(f,c_b,S_b)=\begin{cases}\{(S,K)\text{ such that } S\geq V(K,c_b,S_b)\} & \text{if } (\alpha+\varepsilon)>(\beta+\gamma)\\ \varnothing & \text{otherwise}\end{cases}$$

where V is a function defined by

$$V(K,c_b,S_b)=(\beta+\gamma-\alpha-\varepsilon)\left(\frac{1}{1-(\beta+\gamma)}\right)^{\frac{1-(\beta+\gamma)}{\beta+\gamma}}\left(\frac{c_b}{1-(\beta+\gamma)}\right)^{\frac{1-(\alpha+\varepsilon)}{\beta+\alpha}}+S_b$$

The conditions and the function V(.) are the same as in Propositio n 5 for $\varepsilon=\gamma=0$

Proof proceeds as illustration in Appendix 2.

Note that under the assumption imposed so far $\varepsilon > \beta + \gamma - \alpha$ means that the necessary condition for sustainability can be achieved even for a negative capital augmenting technical change since α can be greater than $\beta + \gamma$. It is also noticeable that the condition $\alpha > \beta$ is neither sufficient (as it was for the case of ZTP and ZCP) nor necessary to achieve the non-emptiness condition. If ε or γ or both are positive then α could be greater (lower) than β but still $\alpha > (<) \beta + \gamma - \varepsilon$ and hence the economy would be unsustainable (non-unsustainable).

Implications in Terms of Technological Parameters of EC and NEC

- Input Elasticity of Production: $\alpha + \varepsilon > \beta + \gamma$. This means that the productivity of capital (capital elasticity of production) must be higher than the productivity of natural resource for the economy not to be doomed.
- Elasticity of Scale: $\alpha + \varepsilon + \beta + \gamma$. NEC do not impose restrictions on this since EC requires $\beta + \gamma < 1$ so the expression above can be less, equal or greater than 1.
- Slope of Elasticity of Scale: 0
- Elasticity of substitution: constant and equal to 1
- Expansion path: $\frac{r^*}{k^*} < 1$ if $p_k = p_r$. It limits the use of the resource to be lower than that of capital.

For the second set of parameters:

- Rate of technological change (RTC): No restrictions are imposed.
- Size bias of technological change (SBTC). Technological change is neutral in terms of size.
- Input bias of technological change (IBTC). EC and NEC impose $\varepsilon > \frac{\beta + \gamma}{\alpha}$ and since the input bias of technical change is resource saving (using) if $IBTC > (<) 0 \Leftrightarrow \varepsilon > (<) \frac{\alpha}{\beta}\gamma$ then by imposing a lower bound on ε sustainability points towards a resource-saving technical change. In fact, it is worth noting that sustainability requires resource-saving technical change whenever $\alpha^2 < \frac{\beta^2}{\gamma} + 1$ which is basically always expected to be true since in the opposite case we would observe $\alpha > 1$ which would be strangely high in a CD technology like the one here. Hence we should conclude that **sustainability requires resource-saving technical change for a wide range of parametric values** which intuitively means that if capital productivity is relatively low then technical change should increase that productivity (should be capital using, or equivalently, resource saving) to “compensate” that fact and reduce the intensity in the use of natural resources.

Technological Parameters for Transcendental Specification $y = \alpha k^{\beta_k + \varepsilon} r^{\beta_r + \gamma} \exp\left(\delta_k k^{1+\varepsilon}\right)$

The first set of parameters can now be expressed as:

- Input Elasticity of Production: $\frac{d \ln f(.,.)}{d \ln k} = \beta_k + \varepsilon + \delta_k (1+\varepsilon) k^{1+\varepsilon}$ and $\frac{d \ln f(.,.)}{d \ln r} = \beta_r + \gamma$
- Elasticity of Scale: $\left.\frac{d \ln f(\lambda x)}{d \ln \lambda}\right|_{\lambda=1} = \beta_k + \varepsilon + \beta_r + \gamma + (1+\varepsilon)\delta_k k^{1+\varepsilon}$
- Slope of Elasticity of Scale: $(1+\varepsilon)\delta_k k^{1+\varepsilon}$
- $\sigma^M = \left(\frac{\left(\beta_k k^{-1} + \delta_k\right)^2}{\left(\beta_k k^{-1} + \delta_k\right)^2 + \beta_k (1+\gamma)} \right) \left(\frac{k - r}{r k} \right)$ so it's neither constant nor necessarily equal to one.
- Expansion path: it can be shown that the optimal input ratio is $\left(\frac{r^*}{k^*} = \left[\frac{p_k}{p_r} - \frac{\delta_k}{1+\gamma} r^* \right] \frac{(1+\gamma)}{\beta_k} \right)$ where $r^* = \frac{y}{\alpha k^{\beta_k} \exp(\delta_k k)}$ and hence is not linear.

The second set of parameters describing the nature of technological change need to be derived. Specifically, for the second set of parameters the CD yields the following:

- Rate of technological change (RTC): $d \ln y = \ln r d\gamma + \ln k \left(1 + \delta_K k^{1+\varepsilon}\right) d\varepsilon$
- Size bias of technological change (SBTC): $SBTC = \left[\frac{1}{\beta_k + \beta_r + \delta_k k^{1+\varepsilon}} \right] \left[(1 + 2\delta_k k)\varepsilon + \gamma \right] - \left[\frac{1}{\beta_k + \beta_r - 1} \right] (\varepsilon + \gamma)$
- Input bias of technological change (IBTC): $\frac{d \ln f_k(.)}{dt} - \frac{d \ln f_r(.)}{dt} = \frac{k^{-1} + \delta_k k^{\varepsilon} + \delta_k (1+\varepsilon)(\ln k) k^{\varepsilon}}{(\beta_k + \varepsilon) k^{-1} + \delta_k (1+\varepsilon) k^{\varepsilon}} d\varepsilon - \frac{1}{\beta_r + \gamma} d\gamma$

Sustainability Conditions for Transcendental Specification

As it turns out the use of more flexible functional forms like the transcendental still leave room for a sustainable economy even under ZTP and PCD. This result will be stated in the next propositions.

PROPOSITION 11. Consider an economy with an exhaustible natural resource, no capital depreciation $(d = 0)$ and a transcendental approximation to production technology with endogenous capital augmenting technical change $y = \alpha k^{\beta_k + \varepsilon} r^{\beta_r + \gamma} \exp\left(\delta_k k^{1+\varepsilon}\right)$. Then the viability kernel depends on technological parameters as follows:

$$Viab\left(f, c_b, S_b\right) = \begin{cases} \left\{\left(S, K\right) \text{such that } S \geq V\left(K, c_b, S_b\right)\right\} & \text{if } \beta_k + \varepsilon > \dfrac{\beta_r + \gamma}{1-\left(\beta_r + \gamma\right)} \\ \varnothing & \text{otherwise} \end{cases}$$

where V is a function defined by

$$V\left(K, c_b, S_b\right) = \int_k \frac{1}{\beta_r + \gamma}\left(\frac{c_b}{\gamma}\right)^{\frac{1-(\beta_r+\gamma)}{\beta_r+\gamma}} k^{-\frac{(\beta_k+\varepsilon)(1-(\beta_r+\gamma))}{\beta_r}} \exp\left(-\delta_k\left(\frac{1-\left(\beta_r+\gamma\right)}{\beta_r+\gamma}\right)k^{1+\varepsilon}\right)dk + S_b$$

The conditions and the function V(.) are the same as in Propositio n 9 for $\varepsilon = 0$
Proof proceeds as illustration in Appendix 2.

Implications in Terms of Technological Parameters of EC and NEC

The first set of parameters can now be expressed as:

- Input Elasticity of Production: $\beta_k + \varepsilon + \delta_k\left(1+\varepsilon\right)k^{1+\varepsilon}$ and $\beta_r + \gamma$. Since NEC implies $\beta_r + \gamma < \dfrac{\beta_k + \varepsilon}{1+\beta_k+\varepsilon}$ then $\beta_r + \gamma < \beta_k + \varepsilon$ which under $\delta_k > 0$ and $\varepsilon > -1$ implies $\beta_r + \gamma < \beta_k + \varepsilon + \delta_k\left(1+\varepsilon\right)k^{1+\varepsilon}$, then sustainability requires a capital elasticity of output greater than the resource elasticity of output. In addition EC puts an upper bound on resource elasticity of production $\left(\beta_r + \gamma\right) < 1$
- Elasticity of Scale: by imposing a lower bound on ε, NEC impose a lower bound on the elasticity of scale.
- Slope of Elasticity of Scale: once again NEC impose a lower bound on $\left(1+\varepsilon\right)\delta_k k^{1+\varepsilon}$.
- σ^M it's neither constant nor necessarily equal to one. It is worth noting that given the fact that σ^M depends negatively upon γ, by imposing an upper bound on γ, NEC imposes a lower bound on σ^M, although this might not be sufficient for both to be substitutes if k is high enough relative to r.
- The Expansion path is not necessarily restricted although it tends to decrease with NEC.

The implications for the second set of parameters are:

- No constraints are imposed on the rate of technological change (RTC).
- Size bias of technological change (SBTC). Since size increasing technical change implies $\gamma < \left[\left[\dfrac{\beta_k + \beta_r - 1}{1+\delta_k k^{1+\varepsilon}}\right]2\delta_k k - 1\right]\varepsilon$ and NEC also impose an upper bound for γ, then sustainability is favored by a size increasing technical change. However neither a size increasing nor a size decreasing technical change is required for sustainability.
- Input bias of technological change (IBTC). Just as before resource-saving technical change imposes an upper bound on γ and thus even though NEC also imposes an upper bound on γ, sustainability do not directly require resource saving technical change.

Positive Technical Progress (PTP) and Positive Capital Depreciation (PCD)

Technological Parameters for Cobb Douglas Specification $f(k,r)=\left(k^{\varepsilon}k^{\alpha}\right)\left(r^{\gamma}r^{\beta}\right)$
The parameters are exactly the same as in the case with ZCD and CD technology.

Sustainability Conditions for Cobb Douglas Specification

PROPOSITION 12. Consider an economy with an exhaustible natural resource, positive capital depreciation $(d>0)$ and a CD technology with endogenous factor augmenting technical change $y=\left(k^{\varepsilon}k^{\alpha}\right)\left(r^{\gamma}r^{\beta}\right)$ where $\varepsilon,\gamma>0$. Then the viability kernel depends on parameters as follows:

$$Viab\left(f,c_b,S_b\right)=\begin{cases}\left\{\left(S,K\right)\text{such that } S\geq V\left(K,c_b,S_b\right)\right\} & \text{if } \left(\alpha+\varepsilon\right)>\left(\beta+\gamma\right)+1 \\ & \text{and } \left(\beta+\gamma\right)<1 \\ \varnothing & \text{otherwise}\end{cases}$$

where V is a function defined by

$$V\left(K,c_b,S_b\right)=\left(\beta+\gamma-\alpha-\varepsilon\right)\left(\frac{1-d}{1-\left(\beta+\gamma\right)}\right)^{\frac{1-(\beta+\gamma)}{\beta+\alpha}}c_b^{\frac{1-(\alpha+\varepsilon)}{\beta+\gamma}}+S_b$$

Proof proceeds as illustration in Appendix 2.

Implications in Terms of Technological Parameters of EC and NEC

The addition of technical change changes the first set of parameters. The parameters for the CD specification with endogenous technical change are:

- Input Elasticity of Production. Here NEC imply that $\frac{d\ln f(.,.)}{d\ln k}=\alpha+\varepsilon>\left(\beta+\gamma\right)+1=1+\frac{d\ln f(.,.)}{d\ln r}$ hence sustainability requires that capital elasticity of output (including technical change) is greater than resource elasticity of output.
- Elasticity of Scale. NEC imply $\alpha+\varepsilon+\beta+\gamma>1$ and hence sustainability requires increasing returns to scale.
- Slope of Elasticity of Scale: 0
- Elasticity of substitution: constant and equal to 1
- Expansion path. NEC imposes $\frac{r^*}{k^*}<1$ for $p_k=p_r$. Hence sustainability requires under same prices that more capital is used in production than natural resource.

The implications for the second set of parameters are:

- Rate of technological change (RTC). No restrictions are imposed.
- Size bias of technological change (SBTC) is assumed to be neutral always.

○ Input bias of technological change (IBTC). EC and NEC impose $\varepsilon > \frac{\beta+\gamma}{\alpha}$ and since the input bias of technical change is resource saving (using) if $IBTC > (<) 0 \Leftrightarrow \varepsilon > (<) \frac{\alpha}{\beta}\gamma$ then by imposing a lower bound on ε sustainability points towards a resource-saving technical change. In fact, it is worth noting that sustainability requires resource-saving technical change whenever $\alpha < \frac{1+\beta+\gamma}{\beta+\gamma}\beta$ which is, once again expected to be true most of the time. Hence we should conclude that **sustainability requires resource-saving technical change for a wide range of parametric values.**
○ Input bias of technological change (IBTC). Sustainability does not impose restrictions on bias.

Technological Parameters for Transcendental Specification

The parameters are exactly the same as in the case with ZCD and CD technology.

Sustainability Conditions for Transcendental Specification

As it turns out the use of more flexible functional forms like the transcendental still leave room for a sustainable economy even under ZTP and PCD. This result will be stated in the next two propositions.

PROPOSITION 13. Consider an economy with an exhaustible natural resource, positive capital depreciation $(d > 0)$ and a transcendental approximation to production technology with endogenous factor augmenting technical change $y = \alpha k^{\beta_k+\varepsilon} r^{\beta_r+\gamma} \exp\left(\delta_k k^{1+\varepsilon}\right)$. Then the viability kernel depends on technological parameters as follows:

$$Viab\left(f, c_b, S_b\right) = \begin{cases} \left\{\left(S, K\right) \text{such that } \mathrm{S} \geq \mathrm{V}\left(K, c_b, S_b\right)\right\} & \text{if } \beta_{\mathrm{k}} + \varepsilon > \dfrac{1}{1-\left(\beta_r+\gamma\right)} \\ \varnothing & \text{otherwise} \end{cases}$$

where V is a function defined by

$$\mathrm{V}\left(K, c_b, S_b\right) = \int_k \frac{1}{\left(\beta_r+\gamma\right)} \left(\frac{1-d}{\gamma}\right)^{\frac{1-(\beta_r+\gamma)}{(\beta_r+\gamma)}} k^{\frac{[1-(\beta_r+\gamma)][1-(\beta_k+\varepsilon)]}{\beta_r+\gamma}} \exp\left(-\delta_k \left(\frac{1-\left(\beta_r+\gamma\right)}{\beta_r+\gamma}\right) k^{1+\varepsilon}\right) dk + S_b$$

The conditions and the function V(.) are the same as in Propositio n 9 for $\varepsilon = 0$
Proof proceeds as illustration in Appendix 2.

Comparing the result here with the result under no depreciation we can see that depreciation imposes further constraints on technical change to achieve sustainability since $\frac{1}{1-(\beta_r+\gamma)} > \frac{\beta_r+\gamma}{1-(\beta_r+\gamma)}$ whenever $\beta_r+\gamma<1$ which in turn holds true by EC in appendix 1.

Implications in Terms of Technological Parameters of EC and NEC

- Input Elasticity of Production: $\beta_k+\varepsilon+\delta_k(1+\varepsilon)k^{1+\varepsilon}$ and $\beta_r+\gamma$. NEC implies $\beta_k+\varepsilon > \frac{1}{1-(\beta_r+\gamma)}$ which implies a capital elasticity of substitution greater than the resource elasticity of substitution for $(\beta_r+\gamma)<1$ which holds true by EC. Therefore by incorporating positive capital depreciation in the transcendental approximation we are back to the result with CD and zero capital depreciation regarding elasticities of production.
- Elasticity of Scale: by plugging the lower bound of ε on the expression for the elasticity of scale we can show that $\left.\frac{d\ln f(\lambda x)}{d\ln\lambda}\right|_{\lambda=1} = \frac{1+\overbrace{[1-(\beta_r+\gamma)][\beta_r+\gamma+(1+\varepsilon)\delta_k k^{1+\varepsilon}]}^{POSITIVE}}{1-(\beta_r+\gamma)} > 1\ \forall(\beta_r+\gamma)<1$ which is true by EC. Hence NEC and EC jointly impose increasing returns to scale.
- Slope of Elasticity of Scale: once again NEC impose a lower bound on $(1+\varepsilon)\delta_k k^{1+\varepsilon}$.
- σ^M it's neither constant nor necessarily equal to one.
- No restrictions are imposed upon the expansion path.

The implications for the second set of parameters are:

- NEC do not impose restrictions on the rate of technological change (RTC).
- Size bias of technological change (SBTC). Since size increasing technical change implies a lower bound on $\beta_k : \frac{\gamma}{\varepsilon}\frac{1+\delta_k k^{1+\varepsilon}}{2\delta_k k}+1-\beta_r<\beta_k$ and NEC also impose a lower bound on β_k, then sustainability is favored by a size increasing technical change. However neither a size increasing nor a size decreasing technical change is required for sustainability.
- Input bias of technological change (IBTC). Just as before resource-saving technical change imposes a lower bound on β_k and thus even though NEC also imposes an upper bound on β_k, sustainability do not directly require resource saving technical change.

Extension of the Method to Exhaustible Resources

The three previous sections were based on the so called Solow-Dasgupta-Heal-Stiglitz economy modeled by:

$$\dot{S}(t) = -r(t) \qquad (1)$$

$$\dot{K}(t) = f(K(t), r(t)) - c(t) - \lambda K(t) \qquad (2)$$

Of course sustainability is an issue in this economy because we have an input that is essential and also exhaustible. The method derived and applied in this paper to finding necessary conditions for sustainability was also based on these two features of resources. A natural question is; how would these conditions be affected if we extended the analysis to renewable resources? Would there still be necessary parametric conditions for sustainability at all? Or the mere existence of a positive rate of natural reposition would be enough for sustainability?

We will proceed to answer these questions by extending the method above to renewable resources. We will do so by changing equation (1) ($\dot{S}(t) = -r(t)$) in the model above to consider the positive rate of natural reposition. We will use two different alternatives to (1) both have the desirable feature of keeping the analysis simple while at the same time useful for illustrating the effects of non-exhaustibility on sustainability.

In the first alternative, the evolution of the stock of resource is determined by the rate of consumption but outweighed (at least partially) by a fixed rate of reposition δ:

$$\dot{S}(t) = \delta - r(t) \qquad (1')$$

In the second alternative, the evolution of the stock of resource is determined by the rate of consumption outweighed (at least partially) by a flexible rate of reposition. Specifically the rate of reposition is a linear function of the resource consumption rate $r(t)\delta$:

$$\dot{S}(t) = r(t)\delta - r(t) = r(t)(\delta - 1) \qquad (1'')$$

where: δ is the natural rate of reposition of the natural resource.

Fixed Rate of Reposition

The equation of motion for the natural resource is $(1')$. To analyze this economy we will introduce, as in Martinet and Doyen (2007) the extraction indicator denoted by $r_b(f, K)$ and defined by:

$$r_b(f,k) = \inf(r \geq 0 : f(k,r) \geq c_b)$$

Based on this we also introduce the concept of the minimal extraction indicator $r_b(f)$ defined by:

$$r_b(f) = \inf_{k \geq 0} r_b(f,k)$$

Following the same steps of the proofs they offer there ***for the case of exhaustible resources***, it can be shown that if $r_b(f)>0$[16], an economy is unsustainable or doomed to extinction (Viability Kernel is empty). Moreover, if there is a capital level k* such that $r_b(f)=0=r_b(f,k^*)$[17] then the Viability Kernel is non-empty and the economy is not doomed to extinction. Finally if $r_b(f)=0<r_b(f,k^*)$, the economy *might not be unsustainable* for a precise range of parameters and this range is captured by NEC conditions as developed above.

[16] Martinet and Doyen call this type of inputs strongly essential.
[17] Martinet and Doyen call this type of inputs non-essential.

Since the purpose of this section is illustration rather than a complete development of conditions like the three sections above; we will limit our analysis to the CD technology ($y = k^{\alpha} r^{\beta}$). From this expression we can show that $r_b(f,k) = k^{-\frac{\alpha}{\beta}} c_b^{\frac{1}{\beta}} > 0$ and $r_b(f) = 0$. Then, as explained before, the viability kernel will not be empty for a precise configuration of parameters.

The statement above rests upon the case of exhaustible resources and the lesson learned is: for non emptiness of viability kernel, if resources are exhaustible then inputs must not be strongly essential in the sense of Martinet and Doyen. How does this statement change when resources are renewable?

For the case of fixed rate of reposition, solving the differential equation by simple integration yields:

$S(t) = S_0 - [r_b(f) - \delta]\mathrm{t}$.

Hence the extraction of the natural resource does not even reduces the stock in the long run if $r_b(f) < \delta$ since in this case $\lim_{t \to \infty} S(t) = \infty$. More importantly, even in the case of $r_b(f) > 0$ the economy may still be sustainable as long as $r_b(f) < \delta$. However if $r_b(f) \geq \delta$, we are once again in the situation in which viability kernel may not be empty but for a precise parameter configuration and we need to find that particular configuration.

The way we do this is exactly the same as in the previous sections. We find conditions under which with the given technology and equation of motion, the viability kernel is non-empty. It turns out that this particular change in the equation of motion does not affect the Hamilton-Jacobi-Bellman equation from which the kernel is derived and hence the NEC condition still is $\alpha > \beta$ and the shape of the viability kernel remains also unchanged.

Variable Rate of Reposition

The technological part of the discussion below still applies. It is left for us to check what changes, if any, occur in the NEC conditions after the change in the equation of motion. With our new equation of motion $(1'')$, the H-J-B equation changes and the shape of the viability kernel does in fact change. NEC, however, remains unchanged ($\alpha > \beta$). The new lower bound of the viability kernel is:

$$\mathrm{V}^{\mathrm{R}}(K, c_b, S_b) = \frac{1}{\alpha - \beta}\left(\frac{c_b}{1-\beta}\right)^{\frac{1-\beta}{\beta}} K^{\frac{(\beta-\alpha)}{\beta}} + S_b$$

where the superscript R stands for renewable.

If we equate the lower bound of the natural resource to zero (i.e. S_b=0) then we can express the new lower bound as a proportion of the lower bound for exhaustible resources discussed in PROPOSITION 3'':

$\mathrm{V}^{\mathrm{R}}(K, c_b, S_b) = (\delta - 1)\mathrm{V}(K, c_b, S_b)$ which implies a reduction in the lower bound (i.e. a relaxation of the sustainability constraint) for $\delta < 2$ which is a very plausible value. This makes perfect intuitive sense: after the introduction of a positive rate of natural resource reposition we would expect a relaxation of the sustainability constraint.

Summary and Conclusions

This study was structured in the following way: assumptions regarding technology specification and type of technical change were undertaken. Based on those, conditions for essentialness -EC- (i.e. empirical relevance) and non-emptiness –NEC- of the viability kernel (and hence existence of positive indefinitely sustainable consumption) were derived and the implications of those conditions on technological parameters were discussed. For the sake of intuitive clarity we will gather and summarize here our findings and their implications -based upon EC and NEC- in terms of the questions that this paper posed at the beginning and the answers to those questions provided by different approximations to production technology, i.e. CD and more flexible approximations for the different combinations of technical change and capital depreciation.

Zero Technical Change and Zero Capital Depreciation

1. Is it true that an elasticity of substitution between natural resource and human made capital lower than one is a sufficient condition for unsustainability?

Cobb Douglas technology

Yes. In this case we are back in the CES technology and the conventional wisdom (exposed in Proposition 2) states that assuming a constant elasticity of substitution, a value of the elasticity of substitution lower than one is a sufficient condition for non-existence of an indefinitely-maintainable positive level of consumption since the resource's average product is unbounded.

Generalized Quadratic (Proposition 6) **and Transcendental** (Proposition 7)

No. In neither of both there is an identification between elasticity of substitution and essentiality i.e. essentiality of inputs is consistent with values of σ^M above, equal or below 1.

2. Is it true that an elasticity of substitution greater than one is sufficient for the existence of a consumption path that never goes to zero?

Cobb Douglas

Yes, the conventional wisdom (exposed in Proposition 1) states that for a CES technology an elasticity of substitution greater than one is a sufficient condition for existence of an indefinitely-maintainable positive level of consumption since the resource is inessential.

Generalized quadratic and Transcendental

No. Once again in neither of both, there is identification between elasticity of substitution and essentiality.

3. Does the production elasticity of capital have to be higher than the production elasticity of natural resource for the economy no to be doomed?

Cobb Douglas (Proposition 3)

Yes, it does.

Generalized quadratic (P6)

No, a positive level of consumption can be held forever constant even under resource production elasticity greater than capital production elasticity.

Transcendental (P7)

Yes, an elasticity of output with respect to reproducible capital greater than the elasticity of output with respect to exhaustible resource is required as in the CD case for the economy to be sustainable (i.e. non-empty viability kernel). In this case then the conditions required in propositions 3 and 4 do extend to the more flexible case offered by a Transcendental technology.

4. Does the scale of production play any role on the sustainability of the economy?

Cobb Douglas (Proposition 3)

No. Sustainability does not require increasing or decreasing returns to scale.

Generalized Quadratic (P6)

Yes, NEC implies non increasing returns to scale which by contrapositive means that if technology displays increasing returns to scale then the economy is unsustainable. This shouldn't be surprising once one notes that decreasing returns to scale is equivalent to non-negative resource elasticity of production. Hence this condition should really be interpreted as a lower bound on resource productivity instead of an upper bound on returns to scale.

Transcendental (P7)

Not necessarily; EC and NEC allow for increasing, constant or decreasing returns to scale.

Zero Technical Change and Positive Capital Depreciation

1. Is it true that an elasticity of substitution between natural resource and human made capital lower than one is a sufficient condition for unsustainability?

Cobb Douglas (P4)

Yes, we would be back in the CES case.

Generalized Quadratic (P8)

Yes, we would be back in the CES case.

Transcendental (P9)

No, since sustainability is consistent with values of σ^M above, equal or below 1.

2. Is it true that an elasticity of substitution greater than one is sufficient for the existence of a consumption path that never goes to zero?

Cobb Douglas

Yes, we would be back in the CES case.

Generalized Quadratic

Yes, we would be back in the CES case.

Transcendental

No, since essentiality of inputs is consistent with values of σ^M above, equal or below 1.

3. Does the production elasticity of capital have to be higher than the production elasticity of natural resource for the economy no to be doomed?

Cobb Douglas

Yes, in fact sustainability requires even more than this form the elasticity of output with respect to capital, it requires $(\alpha > 1)$

Generalized Quadratic

Yes, but that is not enough for the economy to survive. In fact, the economy is doomed *for the empirically relevant case.*

Transcendental

An elasticity of output with respect to reproducible capital greater than the elasticity of output with respect to exhaustible resource is NOT required for the economy to be sustainable (i.e. non empty viability kernel). However, this relaxation is so because the new conditions impose more structure upon other parameters (i.e. elasticity of scale and slope of elasticity of scale).

4. Does the scale of production play any role on the sustainability of the economy?

Cobb Douglas

Yes, increasing returns to scale are required for the economy to be sustainable.

Generalized Quadratic

No, because the economy is doomed anyway.

Transcendental

EC and NEC imply strictly increasing returns to scale.

Positive Technical Change and Zero Capital Depreciation

1. Is it true that an elasticity of substitution between natural resource and human made capital lower than one is a sufficient condition for unsustainability?

Cobb Douglas (P10)

Yes, we would be back in the CES case.

Transcendental (P11)

No, since sustainability is consistent with values of σ^M above, equal or below 1 as long as technical change increases relative productivity of capital (resource-saving).

2. Is it true that an elasticity of substitution greater than one is sufficient for the existence of a consumption path that never goes to zero?

Cobb Douglas

Yes, we would be back in the CES case.

Transcendental

No it is not, by the same arguments in the previous question.

3. Does the production elasticity of capital have to be higher than the production elasticity of natural resource for the economy no to be doomed?

Cobb Douglas

Yes.

Transcendental

Yes

4. Can a low rate of technological change be compensated by a higher degree of flexibility (higher elasticity of substitution) in production?

Cobb Douglas

Obviously not since the elasticity of substitution is always 1.

Transcendental

Yes, a low enough value of γ would achieve both NEC conditions and also a high value of elasticity of substitution even with a low rate of technical change.

5. Does the scale of production play any role on the sustainability of the economy?

Cobb Douglas

No, nothing is required in terms of scale.

Transcendental

Not necessarily but by imposing a lower bound on ε, NEC impose a lower bound on the elasticity of scale.

6. Features of technological change

Cobb Douglas

No restrictions in the Rate of technological change (RTC) are imposed. Size bias of technological change (SBTC) is assumed away and sustainability requires resource-saving technical change for a wide range of parametric values.

Transcendental

No restrictions in the Rate of technological change (RTC) are imposed. Moreover sustainability is favored by a size increasing technical change, although size increasing technical change is neither necessary nor sufficient for sustainability. Regarding the input bias of technological change (IBTC), just as before resource-saving technical change imposes an upper bound on γ and thus even though NEC also imposes an upper bound on γ, sustainability do not directly require resource saving technical change.

Positive Technical Change and Positive Capital Depreciation

1. Is it true that an elasticity of substitution between natural resource and human made capital lower than one is a sufficient condition for unsustainability?

Cobb Douglas (P12)

Yes, we would be back in the CES case.

Transcendental (P13)

A higher elasticity of substitution favors sustainability but is neither necessary nor sufficient.

2. Is it true that an elasticity of substitution greater than one is sufficient for the existence of a consumption path that never goes to zero?

Cobb Douglas

Yes, we would be back in the CES case.

Transcendental

A higher elasticity of substitution favors sustainability but is neither necessary nor sufficient.

3. Does the production elasticity of capital have to be higher than the production elasticity of natural resource for the economy no to be doomed?

Cobb Douglas

Yes, for the economy to be sustainable, capital elasticity of output has to be greater than resource elasticity of output.

Transcendental

Yes. Therefore by incorporating positive capital depreciation in the transcendental approximation we are back to the result with CD and zero capital depreciation regarding elasticities of production.

4. Can a low rate of technological change be compensated by a higher degree of flexibility (higher elasticity of substitution) in production?

Cobb Douglas

Obviously not since the elasticity of substitution is always 1.

Transcendental

Yes, a low enough value of γ would achieve both NEC conditions and also a high value of elasticity of substitution even with a low rate of technical change.

5. Does the scale of production play any role on the sustainability of the economy?

Cobb Douglas

NEC imply $\alpha + \varepsilon + \beta + \gamma > 1$ and hence sustainability requires increasing returns to scale.

Transcendental

The transcendental technology also requires IRS for sustainability.

6. Features of technological change

Cobb Douglas

No restrictions in the Rate of technological change (RTC) are imposed. Size bias of technological change (SBTC) is assumed away and sustainability requires resource-saving technical change for a wide range of parametric values.

Transcendental

No restrictions in the Rate of technological change (RTC) are imposed. Moreover sustainability is favored by a size increasing technical change, although size increasing technical change is neither necessary nor sufficient for sustainability. Sustainability is also favored by a resource-saving technical change.

The **main lesson** we draw from this paper and we would like to convey to the reader is that by using more flexible approximations to the economy's production frontier we confirm our hypothesis that the conventional wisdom on necessary conditions for sustainability might be based on "too stringent" maintained hypotheses on parameters and result in pessimistic conclusions. More specifically, we learned from here that by considering flexible substitution and potentially biased technical change at the same time (all of these neglected by CD), the economy might actually display "compensations" that allow it to keep fulfilling sustainability conditions. Indeed, a low productivity of capital relative to natural resources can be compensated by a higher substitutability between them and/or by a resource-saving technical change preventing the economy from extinction at a finite time. Additionally a high enough productivity of capital relative to resources or a high enough resource-saving technical change can compensate for a low level of substitutability between capital and natural resources.
Finally this has important empirical implications. If conservation needs assessment conducted by federal agencies neglects this type of compensations then conservationist policies might be too stringent punishing current generations by limiting in excess the use of natural resources.

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Appendix 1

Check of essentiality and unbounded natural resource average product. The result of this check will be a set of essentiality conditions (EC), which are parametric restrictions ensuring essentiality in the Dasgupta and Heal sense

Transcendental:
The version of the transcendental production function considered here is of the form $y = \alpha k^{\beta_k} r^{\beta_r} \exp(\delta_k k)\exp(\delta_r r)$

- Essentiality

$$y(0,r) = \alpha 0 r^{\beta_r} \overbrace{\exp(\delta_k 0)}^{1} \exp(\delta_r r) = 0 = \alpha k^{\beta_k} 0 \exp(\delta_k k) 1 = y(k,0)$$

- Unbounded Natural Resource Average Product

$$AP_r = \frac{y}{r} = \alpha k^{\beta_k} r^{\beta_r - 1} \exp(\delta_k k)\exp(\delta_r r) \Rightarrow \lim_{r\to 0} AP_r = \infty \ \forall \beta_r < 1$$

Then $\text{EC}: \beta_r < 1$

Transcendental With Endogenous Technical Change:

$$y = \alpha k^{\beta_k + \varepsilon} r^{\beta_r + \gamma} \exp(\delta_k k^{1+\varepsilon})$$

- Essentiality

$$y(0,r) = \alpha 0 r^{\beta_r} \overbrace{\exp(\delta_k 0)}^{1} = 0 = \alpha k^{\beta_k + \varepsilon} 0 \exp(\delta_k k^{1+\varepsilon}) = y(k,0)$$

- Unbounded Natural Resource Average Product

$$AP_r = \frac{y}{r} = \alpha k^{\beta_k + \varepsilon} r^{\beta_r + \gamma - 1} \exp(\delta_k k^{1+\varepsilon}) \Rightarrow \lim_{r\to 0} AP_r = \infty \ \forall (\beta_r + \gamma) < 1$$

Then $\text{EC}: (\beta_r + \gamma) < 1$

Cobb Douglas with Endogenous Technical Change $y = (k^{\alpha+\varepsilon})(r^{\beta+\gamma})$:

- Essentiality

$$y(0,r) = y = (0)(r^{\beta+\gamma}) = 0 = (k^{\alpha+\varepsilon})(0) = y(k,0)$$

- Unbounded Natural Resource Average Product

$$AP_r = \frac{y}{r} = \left(k^{\alpha+\varepsilon}\right)\left(r^{\beta+\gamma-1}\right) \Rightarrow \lim_{r\to 0} AP_r = \infty \forall(\beta+\gamma) < 1$$

Then $\mathrm{EC}: \beta+\gamma<1$

Generalized Quadratic:

- Essentiality

$$y(0,r) = \left[\beta_{kr} 0^{\delta\gamma} r^{\delta(1-\gamma)}\right]^{\frac{\gamma}{\delta}} = 0 = \left[\beta_{kr} k^{\delta\gamma} 0^{\delta(1-\gamma)}\right]^{\frac{\gamma}{\delta}} = y(k,0)\ \forall \delta,\gamma \neq 0$$

- Unbounded Natural Resource Average Product

$$AP_r = \frac{y}{r} = \left[\beta_{kr} k^{\delta\gamma}\right]^{\frac{\gamma}{\delta}} r^{\gamma(1-\gamma)} \Rightarrow \lim_{r\to 0} AP_r = \infty \forall \gamma < 1$$

Then $\mathrm{EC}: \gamma < 1$ and $\gamma \neq 0$

Generalized Power:

The special case of the Generalized Power considered here (de Janvry 1972) takes the form $y = \alpha k^{\alpha_1} r^{\alpha_2+\beta_2 k} \exp(\gamma_1 k)$

- Essentiality

$$y(0,r) = \alpha 0 r^{\alpha_2+0} \overbrace{\exp(\gamma_1 0)}^{1} = 0 = \alpha k^{\alpha_1} 0 \exp(\gamma_1 k) = y(k,0)$$

- Unbounded Natural Resource Average Product

$$AP_r = \frac{y}{r} = \alpha k^{\alpha_1} r^{\alpha_2+\beta_2 k-1} \exp(\gamma_1 k) \Rightarrow \lim_{r\to 0} AP_r = \begin{cases} \infty & \forall(\alpha_2+\beta_2 k) < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathrm{EC}: (\alpha_2+\beta_2 k) < 1$

Appendix 2: Derivation of conditions for non-emptiness of the viability kernel

We will proceed to derive sustainability conditions for Proposition 6. The rest will follow in exactly the same manner but with expressions corresponding to each technological case: $y = \left[\beta_{kr} k^{\gamma} r^{\gamma(1-\gamma)}\right]$

The Hartwick's rule is expressed in general as $c_b = f(K(t), r(t)) - r f_r(K(t), r(t))$. In this case then $c_b = \beta_{kr} k^{\gamma} r_b^{\gamma(1-\gamma)}\left[1-\gamma(1-\gamma)\right] \Rightarrow r_b = \left\{\frac{c_b}{\left[1-\gamma(1-\gamma)\right]}\right\}^{\frac{1}{\gamma(1-\gamma)}} k^{-\frac{1}{(1-\gamma)}} \beta_{kr}^{-\frac{1}{\gamma(1-\gamma)}}$ (B).

This is the resource consumption rate that solves the H-J-B equation:

$$H(V'(K), K, r, c) = V'(K)\left(\beta_{kr} k^{\gamma} r_b^{\gamma(1-\gamma)} + c_b\right) + r_b$$

Then the function V(k) solves the following Hamilton-Jacobi-Bellman equation:

$$\min_{(r,c)\in C(K,S)} H(V'(K), K, r, c) = 0 \quad \text{where } C(K,S) = \left\{\begin{matrix}(r,c): c_b < c < f(K(t), r(t)) \\ \text{and } 0 \leq \mathrm{r}\end{matrix}\right\}$$

The FOC is defined as:

$$\Rightarrow V'(K)(f_r(k,r))+1=0 \Rightarrow V'(K)=-\frac{k^{-\gamma}r_b^{\,1-\gamma(1-\gamma)}}{\gamma(1-\gamma)\beta_{kr}} \qquad (A)$$

Combining (A) and (B), integrating both sides, taking limits and rearranging yields:

$$V(c_b)=\frac{\left\{\frac{c_b}{[1-\gamma(1-\gamma)]}\right\}^{\frac{1-\gamma(1-\gamma)}{\gamma(1-\gamma)}}\beta_{kr}^{-\frac{1-\gamma(1-\gamma)}{\gamma(1-\gamma)}}-1}{\gamma(1-\gamma)}\int_{c_b}^{\infty}k^{-\frac{1}{(1-\gamma)}}dk+S_b$$ According to the comparison test, the integral in the first term of the RHS converges if and only if $-\frac{1}{(1-\gamma)}$<0 which implies $\gamma<1$ (NEC).

In this context the viability kernel can be shown to be the epigraph of a function $V(k)$. The epigraph is a set defined in the space of the natural resource S in the following way:

$Epi(V)=\{(K,S),V(K)\leq S\}$