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THE PLANTING REAL OPTION IN CASH RENT VALUATION

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After entering into farmland rental contracts in the fall, a tenant farmer has the planting flexibility to choose between corn and soybeans. Failure to account for this switching option will bias estimates of what farmers should pay to rent land. Applying contingent claims analysis methods, this study explicitly derives the real option value function. Comparative statics with respect to the volatilities of underlying state variables and their correlations are derived and discussed. Dynamic hedging deltas in this real option context are also developed. Monte Carlo simulation results show that the average cash rent valuation for the real option approach is 11% higher than that for the conventional net present value (NPV) method. The simulated dynamic hedging deltas are shown to differ from the ones implied by the NPV method.

Key words: cash rent, delta hedging, Monte Carlo simulation, multivariate GARCH, real option, Ricardian rent.

JEL classification: C5, G1, Q1.

Cropland rental rates have adjusted substantially over the period 2005-08 (Edwards and Smith 2007). This has largely been due to price shifts arising from demand for corn as an ethanol plant feedstock. Landlords and tenants have needed to re-evaluate their willingness to pay and accept rents in this new environment. The goal of this paper is to provide a better understanding of willingness to pay for rented cropland.

In the United States, tenant farmers generally rent cropland in the fall to prepare for spring planting. Cash rent is an important feature of midwestern crop production. In Iowa, as an example, about 40% of cropland is rented under cash rental agreements. Our contention is that the fall to spring time gap is important for how cash renters value access to land. The basis cash rent calculation method suggested by farm management textbooks (e.g., Calkins and DiPietre 1983, p. 394; Olson 2004, p. 285; Kay, Edwards, and Duffy 2004, p. 359) is the so-called tenant's residual approach. The method is to derive a residual, or Ricardian rent, for land by deducting operating costs from crop revenue based on estimated yields, prices, and operating expenses. After taking into account planting decisions faced by a farmer who chooses between corn and soybeans, the traditional net present value (NPV) method calculates the present value of expected corn cash flows and also the present value of expected soybeans cash flows. The maximum value of this pair of present values is then used to determine cash rent. The major drawback of the conventional NPV method as applied to cash rent valuation is that it ignores the option to choose what to plant. Thus, it underestimates what farmers should be willing to pay for rental land.

Like other investment decisions, farmer's production intentions with rented land share three distinct features of real options, as described in Dixit and Pindyck (1994). One is irreversibility. Once the crop has been planted, related sunk costs cannot be fully recovered. Another is uncertainty. Profit uncertainty is due to stochastic output, as well as time-varying input and output prices. The third feature is leeway in timing. After entering into a farmland rental

agreement, a tenant farmer has an extensive margin flexibility to “switch” between corn and soybeans for the next crop year. He also has an intensive margin flexibility concerning the level of inputs to apply at planting. And these options mature at the planting time.

The impacts of irreversibility, uncertainty and the choice of timing on investment project decisions and valuation have been widely recognized and applied to various investment problems in agriculture. For example, Musshoff and Hirschauer (2008) apply the dynamic programming and simulation methods to sales contracting decision problems facing German grain farmers. Odening, Mußhoff, and Balmann (2005) calculate investment triggers and option values when accounting for the value of waiting for an investment in hog fattening in Germany. Tzouramani and Mattas (2004) employ the real option approach to better assess investment opportunities when compared with the NPV approach. Luong and Tauer (2006) model Vietnamese coffee growers’ entry and exit decisions as real options. The most relevant application to our work is Marcus and Modest (1984). They applied continuous time option pricing methods to solve a farmer’s optimal production decision problem. Crop futures prices are used as the stochastic state variables that characterize the uncertainty faced by farmers.

The value of a tenant farmer’s potential planting flexibility, which should be reflected in cash rent determination, is largely driven by volatile input and output prices. Failure to account for option values will bias estimates of what farmers should pay to rent land. The literature of farmland cash rent determination is surprisingly limited and the embedded real option component is largely ignored. Kurkalova, Burkart, and Secchi (2004), for example, estimate the cash rental rate as a function of the corn yield in the Upper Mississippi River Basin in 1997. Dhuyvetter and Kastens (2002) use an accounting approach to model cash rents. Lence and Mishra (2003) and Goodwin, Mishra, and Ortalo-Magné (2004) develop regressions of cash rents against crop revenues and government payments in order to understand the role of government interventions. Du, Hennessy, and Edwards (2007) employ a variable profit function Ricardian

rent approach to analyze the determinants of cash rents using Iowa county-level panel data. None of these seeks to model planting time flexibility.

Contrary to the traditional NPV method, in this study, we explicitly derive the value of the switching and input intensity options. Using historical cash and futures prices, as well as experimental production data, we evaluate the option, i.e., flexibility, value by Monte Carlo methods. One contribution of our work to the literature on cash rent will be to identify the importance of the switching option in cash rent determination.

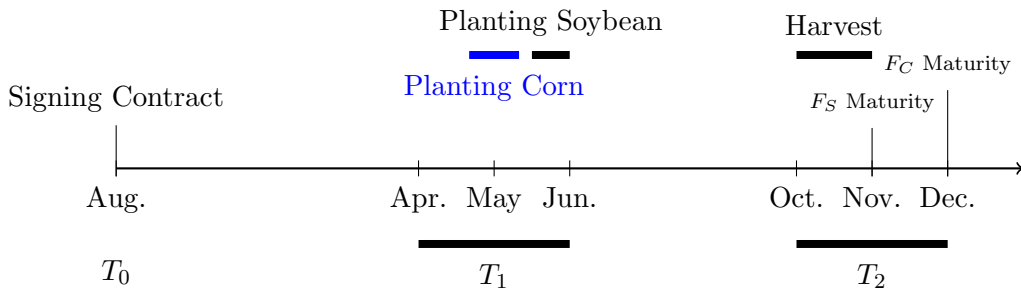
Futures hedging provides tenant farmers with an important instrument for reducing the risk of adverse price changes. There is a considerable body of research on the subject of the optimal futures hedge (e.g., Rolfo 1980; Newbery and Stiglitz 1981; Anderson and Danthine 1983; Moschini and Lapan 1995). Typically, the producer's hedging activities are assumed to start at a known planting decision date. Farmers protect the established long position of a known quantity with an appropriately chosen short position in the corresponding futures contracts. Thus the dynamic hedging strategy represented by delta measures only involves the certain planted crop and its corresponding futures market. In this study, a tenant farmer has the flexibility before planting to switch between intentions to plant corn and soybeans for the next crop year. As far as we know, this has not been allowed for before. In light of this flexibility, the deltas he uses to hedge against input and output price uncertainty should differ from the ones implied by traditional methods. Our second contribution to the literature is to explicitly derive the hedging deltas for underlying random market prices. These delta measures are also quantified by simulation methods.

The paper proceeds as follows. First, a conceptual model of real option valuation is developed. Comparative statics of the real option with respect to volatilities and correlation of underlying state variables are derived and discussed. We also derive the optimal hedging deltas. Second, an empirical Monte Carlo simulation method is described. Estimation and simulation

methods for input and output prices, crop yields, and price bases are presented. The estimation focuses on the option's contribution to cash rent, and also on optimum hedging decisions. The final section concludes with a brief discussion.

Conceptual Model

In Iowa, corn is typically planted between April 20 and May 10 each year. The best planting time for soybeans is from May 15 to June 1. Crops are harvested from October to November of the same year. After signing farmland rental contracts, typically in August the previous year, a tenant farmer makes planting and input choice decisions in April. When making planting decisions, farmers observe and use price information from the futures contracts expiring right after harvest time to formulate harvest price expectations. When deciding what can be paid for rented land, farmers will use futures prices to establish what they may plant, how intensively they will farm, and the value of what they will reap. On the Chicago Board of Trade (CBOT), the December contract for corn and the November contract for soybeans are the first available futures contracts after harvest time. The time line is as follows:



where T_0 is the time when a tenant farmer signs the farmland rental contract, T_1 is the time when the planting and input intensity decisions are made, and T_2 is the harvest time. In addition, time $t \in [T_0, T_2]$ is the continuous time indicator.

NPV vs. Real Option Methods

The traditional NPV approach assumes that a tenant farmer makes the planting decision when agreeing on the cash rent. When corn and soybeans are the crops that may be chosen, a tenant farmer compares expected corn profit, $E_t(\pi_C) = E_t[E_{T_1}(\pi_C)]$, with that of soybeans, $E_t(\pi_S) = E_t[E_{T_1}(\pi_S)]$. $E_t[\cdot]$ denotes the expectation operator conditional on information available at time t under the risk-neutral measure. Expectations $E_{T_1}(\pi_C)$ and $E_{T_1}(\pi_S)$ are expected harvest time corn and soybean profits at planting time T_1 . The present value of $E_t(\pi_C)$ and $E_t(\pi_S)$ are obtained from discounting the expected profits back to the decision-making time t by risk-free rate r . In the standard NPV approach to rent determination, the planting choice is implicitly assumed to have been made with certainty by time t where $t < T_1$.¹ A tenant farmer plants the crop with higher present value of expected profit, which is also the amount of cash rent paid out for landowner and is given as

$$(1) \quad V_1 = e^{-r(T_2-t)} \max \{E_t(\pi_C), E_t(\pi_S)\} \quad (\text{Traditional approach})$$

Contrast this approach with the real option method, in which a tenant farmer is assumed to have the flexibility to switch between corn and soybeans until the planting time. The corresponding cash rent valuation taking into account the real option value is

$$(2) \quad V_2 = e^{-r(T_2-t)} E_t [\max(\pi_C, \pi_S)] \quad (\text{Real option approach})$$

Here, the planting choice is not made until time T_1 . It's readily shown that $V_1 \leq V_2$ is true by

¹The traditional approach also ignores intensive margin planting time flexibility in input use. It identifies an expected profit at T_0 , not allowing for flexibility in waiting for knowledge of $F_{T_1, i}$ ($i = C, S$) to choose input levels for each given crop (Oi 1961).

Jensen's inequality. Also, at maturity, the real option payoff is

$$(3) \quad E_{T_1} \{ \max(\pi_C, \pi_S) \} - \max \{ E_{T_1}(\pi_C), E_{T_1}(\pi_S) \}$$

with the strike price being $\max \{ E_{T_1}(\pi_C), E_{T_1}(\pi_S) \}$. In general, the smaller the difference between corn and soybean expected profits, the higher the real option premium will be. The switching option will have little value if the profit from one crop is almost certain to dominate those from all other crops.

Real Option Valuation

This section considers the production decision of a tenant farmer facing input and output price uncertainty. Applying contingent claim analysis, we derive a closed-form solution for switching option valuation.

Production Decision

First, let's consider the production decision of a tenant farmer. We make the standard assumptions that markets are competitive and frictionless. There are perfectly competitive markets for corn, soybeans, and nitrogen fertilizer. Tenant farmers are price takers who can borrow and lend at the constant riskless rate r . Capital markets are assumed to be open all the time so that a portfolio can be continuously rebalanced. Furthermore, we assume non-stochastic outputs for corn and soybeans in this section.

At planting time T_1 , a tenant farmer is assumed to solve the following expected profit maximization problem when making the production and input choice decision:

$$(4) \quad \max \left\{ \max_{y_C} [F_{T_1,C} y_C - K_C(y_C, W)], \max_{y_S} [F_{T_1,S} y_S - K_S(y_S, W)] \right\}$$

where y_i ($i = C, S$, the same hereafter) are the decision choice variables, denoting expected outputs of corn and soybeans, respectively. The $F_{T_1,i}$'s are expected output prices of harvest time T_2 represented by planting time T_1 prices of futures contracts that mature at harvest time T_2 . To promote concise notation, futures maturity date T_2 has been suppressed. We also simplify by ignoring futures basis in this section. The K_i 's are the corn and soybean production cost function. The input price vector is W .

For analytical convenience, we assume that the cost function for soybean production follows the output homogeneous and input price separable form:

$$(5) \quad K_S(y_S, W) = y_S^{\phi_S} k_S(W)$$

where ϕ_S is the elasticity of scale parameter. The expected profit for soybean at planting time takes the form

$$(6) \quad \pi_S(F_{T_1,S}, W) = \varphi_S F_{T_1,S}^{\delta_S}$$

where $\varphi_S = (1 - \frac{1}{\phi_S}) [\phi_S k_S(W)]^{-\frac{1}{\phi_S-1}}$ and $\delta_S = \frac{\phi_S}{\phi_S-1}$. By the property of profit function convexity, $\delta_S > 1$.²

For corn production, nitrogen fertilizer is the second most expensive input after farmland cost. Natural gas is the primary raw material in producing ammonia for nitrogen fertilizer. The volatile natural gas price largely affects nitrogen's production capacity and price. For simplicity, we assume an actively traded futures or forward market for nitrogen fertilizer. We also assume that all nitrogen fertilizer is applied at planting. A Cobb-Douglas cost function $K_C(y_C, W)$ is

²Observe that inserting the time T_0 futures price into π_S , rather than the time T_1 price, will generally lead to an understatement of profit. This is due to an application of Jensen's inequality since π_S is convex in prices and $F_{T_1,S}$ is random from the viewpoint of T_0 . So the traditional approach is likely to undervalue Ricardian rent for more than one reason.

also assumed for corn production:

$$(7) \quad K_C(y_C, W) = y_C^{\phi_C} F_{T_1, N}^{\lambda} k_C(W)$$

where ϕ_C is the elasticity of scale parameter, $F_{T_1, N}$ is the planting time price of an assumed nitrogen futures or forward contract that matures at planting time T_1 , and λ is the demand elasticity of nitrogen fertilizer.³ The planting time expected profit function for corn is ⁴

$$(8) \quad \pi_C(F_{T_1, C}, F_{T_1, N}, W) = \varphi_C F_{T_1, C}^{\delta_C} F_{T_1, N}^{\delta_N}$$

where $\varphi_C = (1 - \frac{1}{\phi_C}) [\phi_C k_C(W)]^{-\frac{1}{\phi_C - 1}}$, $\delta_C = \frac{\phi_C}{\phi_C - 1}$, and $\delta_N = -\frac{\lambda}{\phi_C - 1}$. By the property of profit function convexity, $\delta_C > 1$.

These are the inputs that enter equations (1) and (2).

Valuation of the Crops

This is the environment in which we identify the value of the crops as V^C and V^S . Given expected profit functions for crop production at planting time T_1 , the crop present values at any time t before planting can be derived using the contingent claim analysis methods developed in Black and Scholes (1973), Merton (1973, 1977), and Dixit and Pindyck (1994). Given a dynamically complete market for a contingent claim on the profits from the crop and using futures contract markets on the commodities, a tenant farmer may form a hedged portfolio to eliminate systemic risk and earn the risk-free rate of return instantaneously.

Assume that expected corn and soybean prices at harvest time T_2 , $F_{t, C}$, $F_{t, S}$, and nitrogen

³While corn requires nitrogen for adequate growth and grain production, soybean generally receives little or no nitrogen.

⁴Bear in mind that $F_{T_1, C}$ and $F_{T_1, S}$ are planting time T_1 prices of harvest time T_2 maturity contracts. The nitrogen contract used has planting maturity, and not harvest maturity, as planting maturity is what is needed for hedging.

fertilizer price at planting time T_1 , $F_{t,N}$, follow geometric Brownian motions as

$$(9) \quad \frac{dF_{t,i}}{F_{t,i}} = \mu_{F_{t,i}} dt + \sigma_{F_{t,i}} dz_i \quad i = C, S, N.$$

over $t \in [T_0, T_1]$ where $\mu_{F_{t,i}}$ is the instantaneous expected rate of return, $\sigma_{F_{t,i}}^2$ is the volatility of the expected prices, and dz_i follows a Wiener process. In addition, ρ_{CS} , ρ_{CN} , and ρ_{SN} denote the instantaneous correlation between the Wiener processes dz_C and dz_S , dz_C and dz_N , dz_S and dz_N , respectively.

In the case of soybean, $F_{t,S}$ is considered to be the only stochastic state variable determining the value of soybean profit, $V_t^S(F_{t,S})$, at time t . By applying contingent claim analysis methods, it is shown in the appendix that the value function $V_t^S(\cdot)$ is

$$(10) \quad V_t^S(F_{t,S}) = \varphi_S(F_{t,S})^{\delta_S} \exp \left\{ \left[\frac{1}{2} \sigma_{F_{t,S}}^2 \delta_S (\delta_S - 1) - r \right] (T_1 - t) \right\}$$

The expected corn and nitrogen fertilizer prices, $F_{t,C}$ and $F_{t,N}$, are considered to be the stochastic state variables driving the changes of corn value. Following Marcus and Modest (1984), it is shown in the appendix that the value function $V_t^C(\cdot)$ is as follows:

$$(11) \quad \begin{aligned} V_t^C(F_{t,C}, F_{t,N}) = & \varphi_C F_{t,C}^{\delta_C} F_{t,N}^{\delta_N} \exp \left\{ \left[\frac{1}{2} \delta_C (\delta_C - 1) \sigma_{F_{t,C}}^2 \right. \right. \\ & \left. \left. + \frac{1}{2} \delta_N (\delta_N - 1) \sigma_{F_{t,N}}^2 + \delta_C \delta_N \sigma_{F_{t,C}} \sigma_{F_{t,N}} \rho_{CN} - r \right] (T_1 - t) \right\} \end{aligned}$$

Notice that

a) the time t present value of crop V_t^i ($i = C, S$) increases with higher expected output prices

$$F_{t,i} \text{ since } \frac{\partial V_t^i}{\partial F_{t,i}} = \frac{\delta_i V_t^i}{F_{t,i}} > 0;$$

b) the time t present value of corn V_t^C decreases with nitrogen fertilizer futures price $F_{t,N}$ for

$$\delta_N < 0 \text{ since } \frac{\partial V_t^C}{\partial F_{t,N}} = \frac{\delta_N V_t^C}{F_{t,N}} < 0;$$

- c) the time t value of the soybean crop goes up with an increase in the volatility of soybean's futures price $\sigma_{F_{t,S}}^2$, as implied by $\frac{\partial V_t^S}{\partial \sigma_{F_{t,S}}^2} = \frac{1}{2}\delta_S(\delta_S - 1)V_t^S(T_1 - t) > 0$;
- d) higher correlation between corn and nitrogen fertilizer prices ρ_{CN} reduces the value of corn for $\delta_C\delta_N < 0$ as $\frac{\partial V_t^C}{\partial \rho_{CN}} = \delta_C\delta_N\sigma_{F_{t,C}}\sigma_{F_{t,N}}V_t^C(T_1 - t) < 0$.

Value of the Switching Option

The option of choosing between corn and soybeans is equivalent to an option to exchange one risky asset for another. Values of the crops are assumed to be the two assets to be exchanged. Following Margrabe (1978), in the appendix the value of the switching option is shown as

$$\begin{aligned}
\Pi(V^C, V^S, t) &= V_t^C\Phi(d_1) - V_t^S\Phi(d_2) \geq 0 \\
d_1 &= \frac{\ln(V_t^C/V_t^S) + \frac{1}{2}\sigma_V^2(T_1 - t)}{\sigma_V\sqrt{T_1 - t}}; \quad d_2 = d_1 - \sigma_V\sqrt{T_1 - t} \\
\sigma_V^2 &= \delta_C^2\sigma_{F_{t,C}}^2 + \delta_S^2\sigma_{F_{t,S}}^2 + \delta_N^2\sigma_{F_{t,N}}^2 \\
&\quad + 2\delta_C\delta_N\sigma_{F_{t,C}}\sigma_{F_{t,N}}\rho_{CN} - 2\delta_C\delta_S\sigma_{F_{t,C}}\sigma_{F_{t,S}}\rho_{CS} - 2\delta_S\delta_N\sigma_{F_{t,S}}\sigma_{F_{t,N}}\rho_{SN} \\
(12) \quad &= \begin{pmatrix} \delta_C & \delta_S & -\delta_N \end{pmatrix} \begin{pmatrix} \sigma_{F_{t,C}}^2 & \sigma_{F_{t,C},F_{t,S}} & \sigma_{F_{t,C},F_{t,N}} \\ \sigma_{F_{t,C},F_{t,S}} & \sigma_{F_{t,S}}^2 & \sigma_{F_{t,S},F_{t,N}} \\ \sigma_{F_{t,C},F_{t,N}} & \sigma_{F_{t,S},F_{t,N}} & \sigma_{F_{t,N}}^2 \end{pmatrix} \begin{pmatrix} \delta_C \\ \delta_S \\ -\delta_N \end{pmatrix} \geq 0
\end{aligned}$$

where $\sigma_{F_{t,C},F_{t,S}} = \sigma_{F_{t,C}}\sigma_{F_{t,S}}\rho_{CS}$ is the covariance between $F_{t,C}$ and $F_{t,S}$, and $\sigma_{F_{t,C},F_{t,N}}$ as well as $\sigma_{F_{t,S},F_{t,N}}$ are similarly defined. The cdf of standard normal distribution is $\Phi(\cdot)$.

Comparative Statics of the Switching Option

The comparative statics of the switching option with respect to volatilities of the underlying price variables, also called the option Vegas, measure how much the option price would change

when the volatility of the underlying state variable changes. They can be derived as follows

i. effect of a change in $\sigma_{F_{t,C}}$:

$$(13) \quad \frac{\partial \Pi}{\partial \sigma_{F_{t,C}}} = V_t^C \Phi(d_1) \overbrace{[\delta_C(\delta_C - 1)\sigma_{F_{t,C}} + \delta_C \delta_N \sigma_{F_{t,N}} \rho_{CN}]}^A (T_1 - t) \\ + V_t^C \phi(d_1) \underbrace{(\delta_C^2 \sigma_{F_{t,C}} + \delta_C \delta_N \sigma_{F_{t,N}} \rho_{CN} - \delta_C \delta_S \sigma_{F_{t,S}} \rho_{CS})}_{B} \frac{\sqrt{T_1 - t}}{\sigma_V}$$

where $\phi(\cdot)$ is the pdf of standard normal distribution. Note that if $\delta_C \approx 1$ and $\rho_{CN} \approx 0$, then term A in (13) is approximately 0. But term B could be negative if $\delta_C \sigma_{F_{t,C}} < \delta_S \sigma_{F_{t,S}} \rho_{CS}$, so that the whole expression can have negative value.

ii. effect of a change in $\sigma_{F_{t,S}}$:

$$\frac{\partial \Pi}{\partial \sigma_{F_{t,S}}} = V_t^S \Phi(d_2) \overbrace{[\delta_S(\delta_S - 1)\sigma_{F_{t,S}}]}^{A'} (T_1 - t) \\ + V_t^S \phi(d_2) \underbrace{(\delta_S^2 \sigma_{F_{t,S}} - \delta_S \delta_N \sigma_{F_{t,N}} \rho_{SN} - \delta_S \delta_C \sigma_{F_{t,S}} \rho_{CS})}_{B'} \frac{\sqrt{T_1 - t}}{\sigma_V}$$

A sufficient condition for a positive overall effect is $\delta_S \sigma_{F_{t,S}} > \delta_N \sigma_{F_{t,N}} \rho_{SN} + \delta_C \sigma_{F_{t,S}} \rho_{CS}$ given that $\delta_S > 1$. If $\delta_S \approx 1$, then $A' \approx 0$, but B' could be negative. So a negative overall effect cannot be precluded.

iii. effect of a change in $\sigma_{F_{t,N}}$:

$$\frac{\partial \Pi}{\partial \sigma_{F_{t,N}}} = V_t^C \Phi(d_1) \overbrace{[\delta_N(\delta_N - 1)\sigma_{F_{t,N}} + \delta_C \delta_N \sigma_{F_{t,C}} \rho_{CN}]}^{A''} (T_1 - t) \\ + V_t^C \phi(d_1) \underbrace{(\delta_N^2 \sigma_{F_{t,N}} - \delta_C \delta_N \sigma_{F_{t,C}} \rho_{CN} - \delta_S \delta_N \sigma_{F_{t,S}} \rho_{SN})}_{B''} \frac{\sqrt{T_1 - t}}{\sigma_V}$$

Also note that if $\delta_N \approx 1$ and $\rho_{CN} \approx 0$, then $A'' \approx 0$, but B'' could be negative if $\delta_N \sigma_{F_{t,N}} < \delta_N \sigma_{F_{t,S}} \rho_{SN}$.

The results indicate that the effects of changes in the volatility of the state variables, $\sigma_{F_{t,C}}$, $\sigma_{F_{t,S}}$, and $\sigma_{F_{t,N}}$, on the option value are, in general, ambiguous. The standard result that an increase in the volatility of the underlying state variable increases the option value doesn't hold here. The sign of the Vega depends on the correlation between the underlying state variables, as well as the relative magnitudes of their volatilities.

In addition, the partial derivative of the switching option with respect to correlation between underlying price variables can be derived as:

iv. effect of a change in ρ_{CS} :

$$(14) \quad \frac{\partial \Pi}{\partial \rho_{CS}} = -V_t^C \phi(d_1) \delta_C \delta_S \sigma_{F_{t,C}} \sigma_{F_{t,S}} \frac{\sqrt{T_1 - t}}{\sigma_V} < 0$$

Intuitively, given price volatility $\sigma_{F_{t,C}}$ and $\sigma_{F_{t,S}}$, a higher correlation between corn and soybean prices makes them move up and down more closely. A tenant farmer is less likely to change crop choice and thus the switching option has less value to him.

v. effect of a change in ρ_{CN} :

$$\frac{\partial \Pi}{\partial \rho_{CN}} = V_t^C \delta_C \delta_N \sigma_{F_{t,C}} \sigma_{F_{t,N}} \left[\Phi(d_1)(T_1 - t) + \phi(d_1) \sqrt{T_1 - t} / \sigma_V \right] < 0 \quad \text{as} \quad \delta_C > 0 > \delta_N.$$

Given corn and nitrogen price volatilities, $\sigma_{F_{t,C}}$ and $\sigma_{F_{t,N}}$, a higher correlation between the input and output prices, ρ_{CN} , leads to more stabilized value of corn and in turn reduces the value of the option to exchange the crops.

vi. effect of a change in ρ_{SN} :

$$\frac{\partial \Pi}{\partial \rho_{SN}} = -V_t^C \phi(d_1) \delta_S \delta_N \sigma_{F_{t,S}} \sigma_{F_{t,N}} \sqrt{T_1 - t} / \sigma_V > 0 \quad \text{as} \quad \delta_S > 0 > \delta_N.$$

In this case, the value of σ_V^2 in (12) increases with the correlation between soybeans and nitrogen fertilizer, ρ_{SN} , i.e., the underlying volatility of the exchange option increases with the correlation. Hence, an increase in ρ_{SN} leads to a higher option value. These results indicate that under certain parameter conditions, a change in the correlation between vibration of state variables unambiguously changes the value of the switching option.

Optimal Hedging Strategy

A tenant farmer has the option to choose between corn and soybeans after signing a farmland rental contract in August. This flexibility will also affect his hedging decision. Taking into account the real option value, a tenant farmer hedges the value

$$(15) \quad V_t = e^{-r(T_2-t)} E_t [\max(V_{T_1}^C, V_{T_1}^S)] = V_t^C \Phi(d_1) - V_t^S \Phi(d_2) + V_t^S = V_t^C \Phi(d_1) + V_t^S [1 - \Phi(d_2)]$$

where d_1 and d_2 are as defined in (12). Note that both $\Phi(d_1)$ and $[1 - \Phi(d_2)]$ are in the range of $[0, 1]$, and they add up to a number greater than one because $\Phi(d_2) < \Phi(d_1)$. The “excess probability” characterizes the switching option as it establishes that the value under flexibility exceeds the average of values absent flexibility.

To get an idea of the hedging scheme that can be used in this real option context, we derive the dynamic delta measure of the underlying assets, corn, soybeans, and nitrogen fertilizer

prices. The deltas with respect to corn, soybeans, and nitrogen prices are

$$\begin{aligned}
 (16) \quad \Delta_C &= \Phi(d_1)\delta_C V_t^C / F_{t,C} \\
 \Delta_S &= [1 - \Phi(d_2)]\delta_S V_t^S / F_{t,S} \\
 \Delta_N &= \Phi(d_1)\delta_N V_t^C / F_{t,N}
 \end{aligned}$$

The above measures indicate that using delta hedging of the value of V_t involves maintaining the position of Δ_C in corn futures contracts, Δ_S in soybean futures contracts, and Δ_N in assumed nitrogen fertilizer futures contracts. Since the value he is hedging for is a weighted average of corn and soybean crop values, the optimal hedging strategy is different from the one with a certain crop decision at the contract signing data T_0 .

Empirical Model

Using monthly average corn and soybean futures prices, local cash prices, and crop production data collected from controlled experiments, we apply Monte Carlo methods to value the real option. Income uncertainty faced by a tenant farmer comes from four sets of random variables. These are output prices, input prices, crop yields, and price bases. A fundamental feature of these random variables is that they are correlated. For example, the corn price is correlated with the soybean price, the nitrogen fertilizer price, corn and soybean yields, and also corn and soybean price bases. All prices, yields, and bases are treated as random variables in our simulation. As indicated in table 1, we explicitly model the correlation between corn, soybean, and nitrogen fertilizer prices in a trivariate Student- t distribution. Corn and soybean price bases are modeled as bivariate normal variables. Because of computational difficulties, we don't take into account the correlations between output prices and crop yields.

Under the assumption of risk neutrality, the Monte Carlo method involves evaluating the cash rent implied by the NPV and real option approaches at planting time T_1 , then discounting back at the risk-free rate as given by V_1 in (1) and V_2 in (2). The corn and soybeans profits are assumed to be

$$(17) \quad \pi_{T_1,C} = (F_{T_1,C} + B_{T_1,C})y_{T_1,C} - F_{T_1,N}N - K_C \quad \pi_{T_1,S} = (F_{T_1,S} + B_{T_1,S})y_{T_1,S} - K_S$$

where $F_{T_1,i}$ ($i = C, S$) denotes futures prices at the planting time T_1 , and $F_{T_1,N}$ denotes price of nitrogen fertilizer at time T_1 . Amount K_C is the corn production cost excluding the cost of nitrogen fertilizer, K_S is soybean production cost, and local corn and soybean bases are $B_{T_1,i}$ ($i = C, S$). Symbol N denotes the quantity of nitrogen fertilizer input and $y_{T_1,i}$ ($i = C, S$) are expected yields of corn and soybeans, respectively.

The Monte Carlo simulation consists of the following steps:

- a) Based on estimated parameters, simulate sample paths of the underlying state variables, i.e., generate n price trajectories of $F_{T_1,C}$, $F_{T_1,S}$, and $F_{T_1,N}$; generate corresponding local bases $B_{T_1,C}$ and $B_{T_1,S}$, and corn and soybean yields forecasts $y_{T_1,C}$ and $y_{T_1,S}$ at planting time T_1 .
- b) Applying Iowa annual crop production budget data (Duffy and Smith 1995-2005) for K_C and K_S and generated quantities in step (a), get n terminal corn and soybean profits, $\pi_{T_1,C}^i, \pi_{T_1,S}^i$, $i \in \{1, 2, \dots, n\}$;
- c) Take average of the discounted final option values under the NPV and real option approaches to obtain an estimate of these values at time t as

$$(18) \quad \begin{aligned} \hat{V}_1(t) &= e^{-r_t(T_2-t)} \max \left\{ \frac{1}{n} \sum_{i=1}^n (\pi_{T_1,C}^i), \frac{1}{n} \sum_{i=1}^n (\pi_{T_1,S}^i) \right\} \\ \hat{V}_2(t) &= e^{-r_t(T_2-t)} \frac{1}{n} \sum_{i=1}^n \max (\pi_{T_1,C}^i, \pi_{T_1,S}^i) \end{aligned}$$

As shown in table 2, in step (a) price series $F_{T_1,C}$, $F_{T_1,S}$, and $F_{T_1,N}$ are random draws from multivariate $t(\hat{\Sigma}, \hat{\nu})$, where $\hat{\Sigma}$ and $\hat{\nu}$ are the variance-covariance matrix and degree of freedom parameters of the distribution. These are estimated from the multivariate Generalized Autoregressive Conditional Heteroskedastic (MGARCH) model. The typical multivariate normality assumption is considered to be unrealistic (Hong 1988) because the kurtosis of most financial asset returns is larger than 3. They have more extreme values than those implied by the normal distribution. For this reason, the multivariate Student- t distribution is a natural alternative in which as the degree of freedom parameter ν tends to zero, the tails of the density become thicker.⁵

The expected corn and soybean yields are obtained from ordinary least squares (OLS) estimates of experimental data. Yield random variations are re-sampled from regression residuals by the bootstrap method. Corn and soybean bases at harvest time are drawn from a bivariate normal distribution with mean \bar{B} and variance-covariance matrix $\hat{\Sigma}_B$, where the parameters are obtained from historical bases data. The estimation and simulation methods for each of these parameters is now discussed in greater detail.

Price Simulation

Corn and soybean futures prices are highly correlated. Using daily settlement prices of the nearby futures contracts traded at the Chicago Board of Trade (CBOT), Malliaris and Urrutia (1996) confirm the strong, statistically significant, long-term interdependence between corn and soybean futures prices. And their relationship has gone through changes over time because of changing market conditions, technology innovation, and agricultural policy adjustments (Lin and Riley 1998).

⁵Although skewness has been found in some distributions of financial data, designing an appropriate multivariate distribution for consistent estimation while allowing for skewness and kurtosis properties is still a challenge.

Another correlation that needs to be accounted for is that between nitrogen and corn prices. Adequate nitrogen fertilizer is critical for profitable corn production. In 2006, nitrogen fertilizer accounted for 56.6% of the 21.3 million tons of chemical fertilizer nutrients applied by U.S. farmers (Huang 2007). The nitrogen fertilizer price directly relates to the price of natural gas (methane). During recent years, the correlation between corn and nitrogen fertilizer has strengthened because more corn is being used as feedstock in corn-based ethanol production. Further expansion of ethanol production as an alternative energy source will likely strengthen the price linkage between corn and other energy prices including natural gas. To take into account the dynamics of correlation structure between corn, soybeans, and nitrogen fertilizer, we employ a MGARCH model. This model simultaneously estimates the mean and conditional variance-covariance matrix of corn, soybeans, and nitrogen fertilizer prices, and we jointly draw the sample paths of prices from the estimated multivariate distribution.

Among several different MGARCH formulations proposed in the literature, the VEC and the BEKK parameterization are the two most popular ones. The acronym BEKK refers to the unpublished work by Baba et al. (1990). The VEC parameterization is quite flexible, as it allows every element of the conditional variance-covariance matrix to be a function of every element of the lagged conditional variance-covariance matrix as well as products of lagged residuals. But it cannot guarantee that the variance-covariance matrix is positive semi-definite, and it requires a large number of estimated parameters. In a trivariate system, a total of 78 parameters need to be estimated. By contrast, the structure of the BEKK model guarantees that the estimated variance-covariance matrix is positive semi-definite, and it considerably reduces the number of parameters to be estimated. In addition, it allows for time-varying correlation among variables. In this study, the BEKK model proposed in Engle and Kroner (1995) is employed.

Consider a (3×1) vector stochastic process $\{\mathbf{R}_t\}$, where \mathbf{R}_t are period-to-period changes

in the logarithm of corn and soybean futures prices, and nitrogen fertilizer price is given as

$\mathbf{R}_t = (r_{t,C} \ r_{t,S} \ r_{t,N})' = \left(\ln \frac{F_{t,C}}{F_{t-1,C}} \quad \ln \frac{F_{t,S}}{F_{t-1,S}} \quad \ln \frac{F_{t,N}}{F_{t-1,N}} \right)'$. With a vector of parameters $\boldsymbol{\theta}$, the model of \mathbf{R}_t is written as

$$(19) \quad 100\mathbf{R}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t \quad \boldsymbol{\varepsilon}_t = \mathbf{H}_t^{1/2}(\boldsymbol{\theta})\mathbf{z}_t$$

where $\boldsymbol{\mu}_t = (\mu_C \ \mu_S \ \mu_N)'$ is the conditional mean vector and $\mathbf{H}_t^{1/2}(\boldsymbol{\theta})$ is a (3×3) positive semi-definite matrix. Furthermore, we assume that (3×1) random vector \mathbf{z}_t has the following first two moments: $E(\mathbf{z}_t) = \mathbf{0}$, $var(\mathbf{z}_t) = \mathbf{I}_3$, where \mathbf{I}_3 denotes the (3×3) identity matrix.

The BEKK(1,1) parameterization is given as

$$(20) \quad \mathbf{H}_{t+1} = \mathbf{C}'\mathbf{C} + \mathbf{A}'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{A} + \mathbf{B}'\mathbf{H}_t\mathbf{B}$$

The individual elements of the \mathbf{C} , \mathbf{A} , and \mathbf{B} matrices are

$$\mathbf{C} = \begin{pmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Therefore, the parameter vector $\boldsymbol{\theta}$ includes the estimated matrices elements of \mathbf{C} , \mathbf{A} , and \mathbf{B} with twenty-four parameters to be estimated.

We use monthly average prices for the corn December futures contract and soybean November futures contract as traded on the CBOT. We use the producer price index (PPI) for nitrogen fertilizer. Monthly PPI data are obtained from the website of the Bureau of Labor Statistics, U.S. Department of Labor over the time frame January 1976 to August 2005. To eliminate the maturity effect, which refers to increasing futures price volatility as the contract approaches

expiration, we switch over to the next maturing futures contracts in the expiration month. This means that data from the maturity month for each contract are not used in our estimation. The three original price data series are shown in figure 1. The original data series are transformed to the return series by evaluating period-to-period changes in the logarithm of prices and then multiplying by 100. Figure 2 shows the resulting time series without multiplication. Table 3 presents the summary of descriptive statistics of the three transformed price series, $100(r_{t,C} \ r_{t,S} \ r_{t,N})$. Sample mean, median, maximum, minimum, standard deviation, skewness, kurtosis, unit root tests and ARCH effect test statistics and corresponding P values are also reported in table 3.

We applied the Augmented Dickey-Fuller (ADF) test (Greene 2003, p. 643) and KPSS test (Kwiatkowski et al. 1992) for the presence of a unit root on all three return series. The null hypothesis of the ADF test is that the series contains a unit root, and the alternative is that the series is level stationary. The KPSS test, in contrast to the ADF test, takes stationarity as the null hypothesis. We include both a constant and lag one first difference term for the tests. The lags of the first difference are included to account for possible serial autocorrelation. From the results in table 3, the null hypothesis of the ADF test is rejected at the 1% significance level. In the KPSS test, we fail to reject the null hypothesis for all three series. The results indicate that all three return series are stationary.

The LM test for the existence of autoregressive conditional heteroskedasticity suggested by Engle (1982) shows that all three have significant ARCH effects. The distributional properties of the three data series generally appear non-normal as indicated by sample skewness and kurtosis values. In addition, compared with the multivariate normal distribution assumption of \mathbf{z}_t , the multivariate Student- t distribution assumption provides a better fit of data with lower Akaike information criterion (AIC) and Bayesian information criterion (BIC) values. So we choose multivariate Student- t as the conditional distribution for the error term in the BEKK

model.

The estimation results of the MGARCH model with BEKK parameterization and degree of freedom parameter $\hat{\nu}$ for the Student- t distribution are reported in table 4. Figure 3 presents the in-sample dynamic conditional correlations among the three return series implied by the estimated conditional variance-covariance matrix of BEKK model. Figure 3 shows the evolution of the conditional correlation among three time series. On average, the correlation between corn and soybeans, corn and nitrogen, and soybeans and nitrogen are about 0.46, 0.05 and 0.02, respectively.

The price simulation method used in this study is an extension to the multivariate setting of Myers and Hanson (1993). One-step-ahead forecasts of variance-covariance matrices are recursively generated by the BEKK model, given as

$$(21) \quad \hat{\mathbf{H}}_{t+1} = \hat{\mathbf{C}}'\hat{\mathbf{C}} + \hat{\mathbf{A}}'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\hat{\mathbf{A}} + \hat{\mathbf{B}}'\mathbf{H}_t\hat{\mathbf{B}}$$

where $\hat{\mathbf{C}}$, $\hat{\mathbf{A}}$, and $\hat{\mathbf{B}}$ are estimated model parameters, $\boldsymbol{\varepsilon}_t$ is the error from the last observation in the sample used to estimate the MGARCH model, and \mathbf{H}_t is the time t sample variance-covariance matrix. Thus, the log futures price of corn, soybean, and nitrogen fertilizer prices at time $t + 1$, represented by a (3×1) vector $\hat{\mathbf{f}}_{t+1}$, are calculated as

$$(22) \quad \hat{\mathbf{f}}_{t+1} = \mathbf{f}_t + \sqrt{\frac{\hat{\nu} - 2}{\hat{\nu}}} \hat{\mathbf{h}}_{t+1} \mathbf{e}_{t+1}$$

where \mathbf{e}_{t+1} is a random draw from standardized multivariate t -distribution with estimated degree of freedom parameter $\hat{\nu}$, and $\hat{\mathbf{h}}_{t+1}$ is the Cholesky factorization (Greene 2003, p. 832) of the estimated $\hat{\mathbf{H}}_{t+1}$ matrix.

Similarly, two or more steps-ahead forecasts of variance-covariance matrices and log prices $\hat{\mathbf{f}}_{t+i}$ ($i \geq 2, 3, \dots, m$) at time $t + i$, where m is the number of periods between t and T_1 , are

simulated as

$$\begin{aligned}
\hat{H}_{t+i} &= \hat{C}'\hat{C} + \hat{A}'\varepsilon_{t+i-1}\varepsilon'_{t+i-1}\hat{A} + \hat{B}'H_{t+i-1}\hat{B} \\
(23) \quad \hat{f}_{t+i} &= f_{t+i-1} + \sqrt{\frac{\hat{\nu}-2}{\hat{\nu}}}\hat{h}_{t+i}e_{t+i}
\end{aligned}$$

in which the variables and parameters are defined similar to (22).

To evaluate the switching option in our case, T_0 is August the prior year, the option maturity time T_1 is April of the crop year, and thus the number of periods for maturity time, m , is equal to 7. The process represented by (21), (22), and (23) results in one realization of the terminal log prices \widehat{f}_T^i , $i \in 1, 2, \dots, n$. The total number of simulations is n , which is chosen to be 5,000 in this study. The terminal log prices are then converted to one realization of terminal prices of $F_{T_1,C}^i$, $F_{T_1,S}^i$, and $F_{T_1,N}^i$, by taking the antilog. Under the risk-neutral measure, the average terminal futures prices of corn and soybean should not have drift. In order to correct for the drifts in simulated price series, following Myers and Hanson (1993), we multiply the individual simulated corn (soybean) prices by the initial futures price at T_0 , and then divide the results by the average of the simulated corn (soybean) price.

Yield and Basis Simulation

Corn and soybean yields can differ by soil quality, climate, and many other natural factors. In this study, local controlled experimental production data enable us to model corn yield as a function of time and the input quantity of nitrogen fertilizer, and also soybean yield as a function of time only. The appropriate distribution for yield variation is still subject to debate. The Beta distribution is popular because of the common view that crop yield distributions can be skewed (Nelson and Preckel 1989). Just and Weninger (1999) find empirical support for normally distributed crop yields, while Ker and Goodwin (2000) prefer non-parametric yield

density estimation. Ker and Coble (2003) point out that sufficient yield data are lacking to accept or reject various reasonable parametric distribution models. Without exploring this issue further here, we apply nonparametric density estimation methods for corn and soybean yields and resample the variation from the residuals of OLS regression.

In this study, crop production data are from controlled experiments conducted at Iowa State University’s Research and Demonstration Farm located in Floyd County, Iowa, from 1979 to 2003 (Mallarino, Ortiz-Torres, and Pecinovsky 2004). The data are collected under 5 rotations, $\langle C \rangle$, $\langle CS \rangle$, $\langle CCS \rangle$, $\langle CCCS \rangle$, and $\langle S \rangle$, where $\langle CCCS \rangle$ is to be read as the corn-corn-corn-soybeans rotation. Four nitrogen application levels, 0 lb/ac, 80 lb/ac, 160 lb/ac, and 240 lb/ac were applied. Each combination of rotation and nitrogen level are replicated three times in a year. To make corn and soybean yields comparable for our simulation purposes, we choose the second corn yield data in $\langle CCS \rangle$ rotation and the soybean yield data in $\langle CS \rangle$ rotation for estimation. In doing so, both corn and soybeans under estimation are planted after a soybeans-corn crop sequence and the corresponding yields data should be comparable.

The independent variables included in the OLS regression of corn and soybean yields are chosen by AIC and BIC model selection criteria. The regression results are shown in table 5. All explanatory variables are significant at the 5% level except that the time variable, year, is only marginally significant in the case of corn.⁶ The Lagrange Multiplier (LM) test for heteroskedasticity is applied on the OLS residuals and the test statistics for corn and soybeans are 6 and 2.87, respectively. The P values of 0.20 for corn and 0.24 for soybeans indicate no sign of heteroskedasticity in either regression equation.

Figure 4 illustrates the nonparametric kernel density estimations of corn and soybean yield residuals after OLS. We resample yield variations using the bootstrapping method (Efron 1979),

⁶Note that when nitrogen fertilizer levels exceed 128 lb/ac, the marginal effect of time is positive. Typical nitrogen rates are 160-200 lb/ac in northern Iowa (Iowa State University Extension 2004). This suggests that technical change has been biased in favor of nitrogen use.

i.e., drawing yield variation randomly with replacement from the set of OLS regression residuals. These random draws are then added to mean yield forecasting of OLS regression.

In order to match with the crop yields specifically estimated from experimental data for Floyd County, we also simulate local crop price bases using statistics from historical data for that county. The basis is computed by subtracting the futures price from the monthly average cash price received by farmers. Historical local cash prices are represented by monthly average (November for corn and October for soybeans) cash prices quoted in North Central Iowa from 1985 to 2005. The data are reported in the “Daily Historical Grain Report” by the Iowa Department of Agriculture and Land Stewardship. For historical futures prices, we use settlement prices on November 3rd (October 3rd) or nearest trading date for corn (soybeans) each year.

The simulated price basis of corn and soybeans are jointly simulated as follows:

$$(24) \quad \hat{\mathbf{B}} = \bar{\mathbf{B}} + \hat{\Sigma}_{\mathbf{B}}^{1/2} \times \mathbf{e}$$

where $\bar{\mathbf{B}}$ is a (2×1) vector of historical average bases over the sampling data frame and $\bar{\mathbf{B}} = \begin{pmatrix} \bar{B}_C \\ \bar{B}_S \end{pmatrix} = \begin{pmatrix} 0.30 \\ 0.38 \end{pmatrix}$ are used in our simulation; $\hat{\Sigma}_{\mathbf{B}} = \begin{pmatrix} \hat{\sigma}_{B_C}^2 & \hat{\sigma}_{B_C B_S} \\ \hat{\sigma}_{B_C B_S} & \hat{\sigma}_{B_S}^2 \end{pmatrix} = \begin{pmatrix} 0.0099 & 0.0039 \\ 0.0039 & 0.0202 \end{pmatrix}$ is the variance-covariance matrix estimated from the historical bases data; and \mathbf{e} is a random draw from the standard bivariate normal distribution. Finally, the realizations of crop cash prices are obtained by summing up the generated crop futures prices and price bases. Given simulated crop cash prices, and nitrogen fertilizer price and crop yields, the optimal quantity of fertilizer input is obtained from the corn profit maximization problem. The functional forms estimated in the OLS regression of corn yield shown in table 5 are also used to solve for the optimal input quantity.

Simulation Results

Given simulated input, output cash prices, and yield realization, we get expected revenues from corn and soybeans at planting time T_1 . The corn and soybean profits are then obtained by subtracting expected crop production costs from simulated revenues. Iowa annual crop production cost budget data (Duffy and Smith 1995-2005) are used for approximation of the production costs excluding cash rent costs. Then $\hat{V}_1(t)$ and $\hat{V}_2(t)$ in (18) are calculated. To quantify the value of the real option embedded in cash rent valuation, we define $\widehat{\% \Pi} = \left(\frac{\hat{V}_2(t)}{\hat{V}_1(t)} - 1 \right) \times 100\%$ as the relative percentage of the switching option value in terms of $\hat{V}_1(t)$, where $\hat{V}_1(t)$ is the amount of cash rent determined by the traditional NPV method. The simulated cash rents evaluated by the NPV and real option methods and the relative real option value from 1995 to 2005 are presented in table 6.

From the simulation results, the average cash rent evaluated by the real option approach is about 11% higher than that of the traditional NPV method. As corn and soybean profits get closer, i.e., corn is planted in about 50% of all simulation draws, the option value increases with a maximum value of 18.46% in 2003. When the profit of one crop dominates the other, the switching option is not as valuable. This was the case in 2000 and 2005 where in each case the option premium was less than 5% for our simulation context.

Furthermore, to value the embedded real option in cash rent for 2008, we fix the nitrogen fertilizer price as of August 2007 at \$0.46/lb, corn production costs at \$191.47, and soybean production costs at \$146.06 from the 2007 Iowa crop production budget (Duffy and Smith 2007). The corn price is assumed to vary from \$3 to \$6 and the soybean price is assumed to be in the range of \$6 to \$15. The simulation result is shown in figure 5, which is a three-dimensional ridge diagram summarizing the variation of relative real option value with changes in corn and soybean prices. The average relative real option value is 8.55%, with ranges from 0% to 23.15%. The typical pattern for the switching option value is that the option tends to

become more valuable as it gets closer to the money, i.e., the corn and soybean profits are similar to each other. This is also the time when there is really a planting decision for a tenant farmer to make. Figure 5, panel b, is the corresponding contour plot of the ridge diagram. It represents level sets of the ridge diagram surface by plotting constant option value slices on corn and soybean price axes. The contour plot shows that the highest real option values are achieved along a line in the middle. Also the option iso-value curves are almost straight lines.

Fixing the corn futures price as of August 2007 at \$4.2/bu and soybeans futures price at \$8.6/bu, we simulate the effect of correlation changes on the real option value for 2008. The results are presented in table 7. In the base scenario, about 11% of the cash rent paid out by a tenant farmer is estimated to be the embedded real option component. The results of other cases indicate that the switching option value decreases with the increase of the correlation between corn and soybeans, as well as that between corn and nitrogen fertilizer, but increases with higher correlation between soybeans and nitrogen fertilizer. The sensitivities are small and confirm our comparative statics results in the conceptual model.

To simulate the hedging strategy in the context of the real option framework, we apply the standard finite differencing approach to compute the deltas with respect to corn and soybean prices (Jäckel 2002, p. 140). In this approach, a base value of the cash rent including the real option component, $\hat{V}_2(T_0)$, is determined by an initial simulation. Then we re-simulate the cash rent with an increase in the price of corn (soybeans). The delta measure for that price is the difference between two simulated values divided by the price increase, shown as

$$\hat{\Delta}_i = \frac{\partial \hat{V}_2(T_0)}{\partial F_{T_1,i}} \approx \frac{\hat{V}_2(F_{T_1,i} + \Delta F_{T_1,i}) - \hat{V}_2(F_{T_1,i})}{\Delta F_{T_1,i}} \quad i = C, S.$$

Assuming that the corn price varies from \$3 to \$6, the soybean price is in the range of \$6 to \$15, and the upshifts of corn and soybeans price are chosen to be one twentieth of the corresponding

price ranges. To reduce the simulation variances, the deltas are averaged over 100 simulations. The resulting deltas for each price grid point for corn and soybeans are shown in panel a of figures 6 and 7, respectively.

The deltas for corn are in the range of $[0, 176.09]$ with an average of 94.41, and the deltas for soybeans are from 0 to 53.99 with an average of 23.94. It is clear that the deltas for both corn and soybeans vary continuously and form continuous planes over the price ranges. The deltas for corn and soybeans implied by the conventional NPV methods are shown in panel b of figures 6 and 7, respectively. Under the NPV assumption, a tenant farmer only hedges against a single price risk since it is assumed that he makes the planting decision when signing the rental contract. As indicated in panel b of figures 6 and 7, the hedging deltas for corn and soybeans jump from zero to a constant level after the points where expected profit of corn equals that of soybeans.

Discussion

Iowa state-level crop planting data collected by the National Agricultural Statistical Service (NASS) show that from 1995 to 2007, the annual change in crop planting acreage is about 1-3%. The biggest change, at 7%, happened in 2007 because of strong demand for ethanol production. The planting acreage has largely remained at 55% for corn and 45% for soybeans. Although it may not be reasonable to make inference on an individual farmer's choice based only on aggregate data, these facts may still indicate that the real option value may not be as big as our simulation results suggest.

Farmers' planting choice is limited by rotation effects, plant diseases and other technology limitations. Rotation benefits for the plant environment have been known for a long time. Continuous corn is known to suffer potential yield losses when compared with rotated corn. Furthermore, corn root worm is primarily a problem for continuous corn. There are also eco-

nomie benefits from the use of rotations. For example, rotations provide a farmer with revenue diversification. In addition, a fixed amount of machinery and limited planting time make farmers disposed to planting half corn and half soybeans. But the profit maximization assumption and the model in this study provide a solid starting point to further understand cash rent valuation as well as farmers' crop planting and hedging decisions.

Conclusion

After entering into a rental agreement in the fall, a tenant farmer has the flexibility to switch between corn and soybeans for the next crop year. This planting flexibility can be treated as a real option, which should be reflected in cash rent paid out to the landowner. Without taking into account this real option component, conventional NPV methods underestimate what farmers should pay for rental land. Furthermore, the dynamic hedging strategy applied by a tenant farmer is different if he takes planting flexibility into consideration. Applying contingent claims analysis, this study explicitly derives the value function of the real option. Comparative statics with respect to the volatilities of underlying state variables and their correlations are derived and discussed. Dynamic hedging deltas in the context of the real option are also developed.

Based on the estimation results, we simulate the switching option value by Monte Carlo methods. The results show that on average, the cash rent valuation by the real option approach is about 11% higher than that by the traditional NPV method. The option value becomes higher as corn and soybean profits become closer to each other. Planting flexibility is worth little if profit from one crop looks as if it will dominate the others. The simulation results of comparative statics confirms our theoretical derivation. The simulated dynamic hedging deltas are shown to be different from the ones implied by the NPV method.

Our research suggests the need for future research in the following areas. First, explicitly modeling the correlation between crop yield and price would allow more realistic simulations of real option valuation. A negative correlation between corn price and yield has long been recognized in Iowa (Babcock and Hennessy 1996), which means that natural movements in price and yield stabilize farmers' incomes to a greater extent than in other areas; i.e., the natural hedge is more effective. Taking this correlation into account would improve the accuracy of our empirical analysis. Second, to reduce the variation of Monte Carlo simulation results and thus to reduce the number of simulations required for a given level of accuracy, especially in case of delta estimates, variance reduction technologies could be applied, such as antithetic sampling, importance sampling, and moment matching. (see Jäckel 2002, Ch. 10 for a comprehensive treatment).

Third, in this study, geometric Brownian motion is assumed and used extensively to describe the dynamic processes of the underlying state variables and the switching option, which is convenient as a starting point. Other stochastic processes, such as mean reverting processes, may be explored and tested to better describe the real world decision problem (e.g., Insley 2002; Insley and Rollins 2005). Finally, the existence and magnitude of the real option in the urban land development context has been tested and investigated (e.g., Cunningham 2006, 2007). Following the lines of an empirical real option pricing model introduced by Quigg (1993), the existence of a real option component in cash rental rates data can be tested empirically using local cash rents data (e.g., Edwards and Smith 2007). The data may be decomposed to statistically test for the real option premium for appropriate years when the profits from corn and soybeans are comparable.

References

- Anderson, R., and J. Danthine. 1983. "Hedger Diversity in Futures Markets." *Economic Journal* 93:370–389.
- Baba, Y., R.F. Engle, D. Draft, and K. Kroner. 1990. "Multivariate Simultaneous Generalized ARCH." Unpublished manuscript, University of California, San Diego.
- Babcock, B., and D. Hennessy. 1996. "Input Demand Under Yield and Revenue Insurance." *American Journal of Agricultural Economics* 81:637–54.
- Black, F., and M. Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *The Journal of Political Economy* 81:637–654.
- Calkins, P., and D. DiPietre. 1983. *Farm Business Management: Successful Decisions in a Changing Environment*. New York: Macmillan Publishing.
- Cunningham, C. 2007. "Growth Controls, Real Options, and Land Development." *The Review of Economics and Statistics* 89:343–358.
- . 2006. "House Price Uncertainty, Timing of Development, and Vacant Land Prices: Evidence for Real Options in Seattle." *Journal of Urban Economics* 59:1–31.
- Dhuyvetter, K., and T. Kastens. 2002. "Landowner vs. Tenant: Why Are Land Rents So High?" Paper presented at the Risk and Profitability Conference, Manhattan KS, 15-16 August.
- Dixit, A., and R. Pindyck. 1994. *Investment Under Uncertainty*. Princeton, New Jersey: Princeton University Press.
- Du, X., D.A. Hennessy, and W. Edwards. 2007. "Determinants of Iowa Cropland Cash Rental Rates: Testing Ricardian Rent Theory." CARD Working Paper 07-WP 454, Center for Agricultural and Rural Development, Iowa State University.

- Duffy, M., and D. Smith. 1995-2008. "Estimated Costs of Crop Production in Iowa." Iowa State University Extension Publication FM-1712.
- Edwards, W., and D. Smith. 2007. "Cash Rental Rates for Iowa 2007 Survey." Iowa State University Extension Publication FM1851.
- Efron, B. 1979. "Bootstrap Methods: Another Look At The Jackknife." *The Annals of Statistics* 7:1–26.
- Engle, R. 1982. "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of the U.K. Inflation." *Econometrica* 50:987–1008.
- Engle, R., and K. Kroner. 1995. "Multivariate Simultaneous Generalized ARCH." *Econometric Review* 11:122–150.
- Goodwin, B.K., A.K. Mishra, and F. Ortalo-Magné. 2004. "Landowners' Riches: The Distribution of Agricultural Subsidies." Working Paper, School of Business, University of Wisconsin, Madison.
- Greene, W.H. 2003. *Econometric Analysis*. Upper Saddle River, NJ: Prentice Hall.
- Hong, C. 1988. "Options, Volatilities and The Hedge Strategy." Unpublished PhD. Dissertation, University of San Diego, Department of Economics.
- Huang, W. 2007. "Tight Supply and Strong Demand May Raise U.S. Nitrogen Fertilizer Prices." *Amber Waves* 5:7.
- Hull, J. 2002. *Options, Futures, and Other Derivatives (5th Ed.)*. Upper Saddle River, NJ: Prentice Hall.
- Insley, M. 2002. "A Real Options Approach to the Valuation of a Forestry Investment." *Journal of Environmental Economics and Management* 44:471–92.

- Insley, M., and K. Rollins. 2005. "On Solving The Multirotational Timber Harvesting Problem With Stochastic Prices: A Linear Complementarity Formulation." *American Journal of Agricultural Economics* 87:735–55.
- Iowa State University Extension. 2004. "Managing Manure Nutrients for Crop Production." Extension Publication PM1811.
- Jäckel, P. 2002. *Monte Carlo Methods in Finance*. New York: John Wiley & Sons.
- Just, R., and Q. Weninger. 1999. "Are Crop Yields Normally Distributed?" *American Journal of Agricultural Economics* 81:287–304.
- Kay, R.D., W.M. Edwards, and P. Duffy. 2004. *Farm Management*. Boston: McGraw-Hill.
- Ker, A., and K. Coble. 2003. "Modeling Conditional Yield Densities." *American Journal of Agricultural Economics* 85:291–304.
- Ker, A., and B. Goodwin. 2000. "Nonparametric Estimation of Crop Insurance Rates Revisited." *American Journal of Agricultural Economics* 83:463–478.
- Kurkalova, L.A., C. Burkart, and S. Secchi. 2004. "Cropland Cash Rental Rates in the Upper Mississippi River Basin." Technical Report, Center for Agricultural and Rural Development, Iowa State University.
- Kwiatkowski, D., P. Phillips, P. Schmidt, and Y. Shin. 1992. "Testing the Null Hypothesis of Stationarity Against the Alternative of a Unit Root." *Journal of Econometrics* 54:159–178.
- Lence, S., and A. Mishra. 2003. "The Impacts of Different Farm Programs on Cash Rents." *American Journal of Agricultural Economics* 85:753–761.

- Lin, W., and P. Riley. 1998. "Rethinking the Soybeans-to-Corn Price Ratio: Is It Still a Good Indicator for Planting Decisions." *Feed Yearbook* Economic Research Service, USDA, pp. 22-33.
- Luong, Q., and L. Tauer. 2006. "A Real Options Analysis of Coffee Planting in Vietnam." *Agricultural Economics* 35:49–57.
- Mallarino, A., E. Ortiz-Torres, and K. Pecinovsky. 2004. "Effects of Crop Rotation and Nitrogen Fertilization on Crop Production." Iowa State University Agronomy Extension Annual Report.
- Malliaris, A., and J. Urrutia. 1996. "Linkage Between Agricultural Commodity Futures Contracts." *Journal of Futures Market* 16:595–609.
- Marcus, A., and D. Modest. 1984. "Futures Markets and Production Decisions." *The Journal of Political Economy* 92:409–426.
- Margrabe, W. 1978. "The Value of an Option to Exchange One Asset for Another." *The Journal of Finance* 33:177–186.
- Merton, R. 1977. "On the Pricing of Contingent Claims and The Modigliani-Miller Theorem." *Journal of Financial Economics* 5:241–249.
- . 1973. "Theory of Rational Option Pricing." *The Bell Journal of Economics and Management Science* 4:141–183.
- Moschini, G., and H. Lapan. 1995. "The Hedging Role of Options and Futures Under Joint Price, Basis, and Production Risk." *International Economic Review* 36:1025–1049.
- Musshoff, O., and N. Hirschauer. 2008. "Investment Planning Under Uncertainty and Flexibil-

- ity: The Case of a Purchasable Sales Contract.” *The Australian Journal of Agricultural and Resource Economics* 52:17–36.
- Myers, R., and S. Hanson. 1993. “Pricing Commodity Options When The Underlying Futures Price Exhibits Time-Varying Volatility.” *American Journal of Agricultural Economics* 75:121–130.
- Nelson, C., and P. Preckel. 1989. “The Conditional Beta Distribution as a Stochastic Production Function.” *American Journal of Agricultural Economics* 71:370–78.
- Newbery, D., and J. Stiglitz. 1981. *The Theory of Commodity Price Stabilization*. Oxford: Clarendon Press.
- Odening, M., O. Mußhoff, and A. Balmann. 2005. “Investment Decisions in Hog Finishing: An Application of The Real Options Approach.” *Agricultural Economics* 32:47–60.
- Oi, W. 1961. “The Desirability of Price Instability Under Perfect Competition.” *Econometrica* 29:58–64.
- Olson, K. 2004. *Farm Management: Principles and Strategies*. Ames, IA: Blackwell Publishing.
- Quigg, L. 1993. “Empirical Testing of Real Option-Pricing Model.” *The Journal of Finance* 48:621–640.
- Rolfo, J. 1980. “Optimal Hedging Under Price and Quantity Uncertainty: The Case of a Cocoa Producer.” *The Journal of Political Economy* 88:100–116.
- Tzouramani, L., and K. Mattas. 2004. “Employing Real Options methodology in Agricultural Investments: The Case of Greenhouse Construction.” *Applied Economics Letter* 11:355–359.

Table 1. Correlation Structure in Empirical Model

	$F_{T_1,C}$	$F_{T_1,S}$	$F_{T_1,N}$	$B_{T_1,C}$	$B_{T_1,S}$	$y_{T_1,C}$	$y_{T_1,S}$
$F_{T_1,C}$	*	*	*	-	-	-	-
$F_{T_1,S}$	*	*	*	-	-	-	-
$F_{T_1,N}$	*	*	*	-	-	-	-
$B_{T_1,C}$	-	-	-	*	*	-	-
$B_{T_1,S}$	-	-	-	*	*	-	-
$y_{T_1,C}$	-	-	-	-	-	*	*
$y_{T_1,S}$	-	-	-	-	-	*	*

Note: * indicates that the random variation or correlation is explicitly modeled; – indicates that the correlation is ignored.

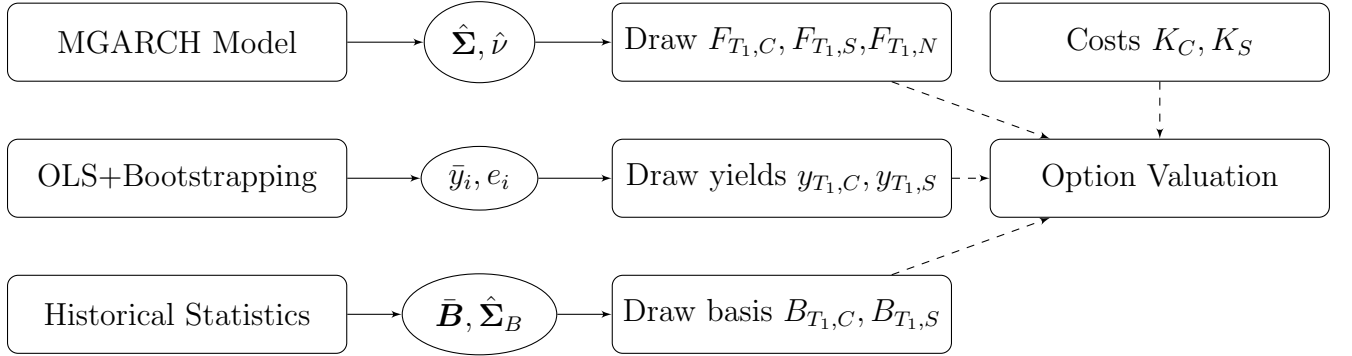
Table 2. Schematic for Simulation Methods

Table 3. Summary Statistics of Period-to-Period Changes in Log Prices

	Corn	Soybean	Nitrogen Fertilizer
Mean	-0.044	0.066	0.258
Median	0.167	0.229	-0.037
Maximum	16.498	23.094	21.969
Minimum	-18.344	-22.800	-9.631
Std. dev.	4.032	5.505	3.116
Skewness	-0.406	0.154	0.943
Kurtosis	3.403	2.694	7.841
ADF	-7.713 (< 0.001)	-8.864 (< 0.001)	-7.325 (< 0.001)
KPSS	0.030 (> 0.10)	0.042 (> 0.10)	0.063 (> 0.10)
LM test	21.578(0.023)	16.648 (0.034)	25.925 (0.011)

Note: For ADF, KPSS, and LM tests, the first number is the test statistic and the second is the corresponding P value.

**Table 4. Multivariate GARCH BEKK
Model Estimation Results**

	Estimates	Std. Error	t Value	$P(> t)$
μ_1	0.090	0.188	0.481	0.631
μ_2	0.149	0.285	0.493	0.622
μ_3	0.118	0.135	0.879	0.380
c_{11}	0.871	0.518	1.681	0.094
c_{21}	-3.418	0.940	-3.638	< 0.001
c_{31}	-0.698	0.695	-1.004	0.316
c_{22}	0.103	4.693	0.022	0.983
c_{32}	0.544	25.822	0.021	0.983
c_{33}	0.585	23.685	0.025	0.980
a_{11}	0.151	0.070	2.162	0.031
a_{21}	-0.119	0.131	-0.909	0.363
a_{31}	-0.025	0.072	-0.352	0.725
a_{12}	0.041	0.053	0.764	0.445
a_{22}	0.301	0.096	3.142	0.002
a_{32}	0.030	0.052	0.574	0.566
a_{13}	-0.046	0.091	-0.507	0.612
a_{23}	0.271	0.163	1.660	0.098
a_{33}	0.503	0.095	5.292	< 0.001
ν	4.346	0.754	5.764	
AIC	5792.105			
BIC	5900.524			

Table 5. OLS Regression Results for Corn and Soybean Yields

	Estimates	Std. Error	t Value	$P(> t)$
Corn Yield				
Nitrogen	0.656	0.115	5.710	< 0.001
Year	-1.149	0.606	-1.900	0.061
(Nitrogen) ²	-0.002	0.0004	-4.45	< 0.001
Nitrogen \times Year	0.009	0.004	2.400	0.018
Intercept	69.185	9.377	7.380	< 0.001
R^2	0.6368			
Soybean Yield				
Year	2.206	0.515	4.280	< 0.001
(Year) ²	-0.053	0.019	-2.770	0.007
Intercept	28.324	2.910	9.730	< 0.001
R^2	0.3473			

Note: When nitrogen fertilizer levels exceed 128 lb/ac, the marginal effect of time is positive. Typical nitrogen rates are 160-200 lb/ac in northern Iowa.

Table 6. Simulation Results for Real Option Value, 1995-2005

Year	August Prices				Production Cost			Profit		Option Value			
	Corn	Soybeans	Nitrogen	Corn	Soybeans	Mean	Mean	Corn	Soybeans	% of Draws	Traditional	Real	% Real
						Corn Profit	Soybeans Profit	Corn		is Grown	Value	Option Value	Option Value
1995	2.5198	6.0653	0.1787	102.87	78.54	174.26	208.86	33.06			208.86	233.76	11.92
1996	2.9383	7.7245	0.1784	94.06	87.21	243.58	286.13	32.58			286.13	315.77	10.62
1997	2.6908	6.2836	0.1904	106.66	89.65	195.95	207.61	42.42			207.61	243.61	17.34
1998	2.7556	5.3357	0.1571	113.80	95.33	207.25	157.86	65.24			207.25	226.83	9.45
1999	2.4450	4.7744	0.1319	114.60	93.73	267.23	129.18	61.10			267.23	287.51	7.59
2000	2.6458	4.6714	0.1787	101.65	92.68	203.78	126.98	76.96			203.78	213.92	4.97
2001	2.3818	4.5888	0.1806	114.09	92.20	156.15	122.13	61.44			156.15	176.36	12.94
2002	2.3188	5.4384	0.1560	112.94	94.04	143.23	163.14	36.86			163.14	191.17	17.18
2003	2.4569	5.5388	0.2140	114.19	95.56	150.03	165.69	39.52			165.69	196.27	18.46
2004	2.8733	5.8519	0.2507	119.57	94.04	215.69	117.49	59.80			215.69	242.73	12.53
2005	2.2872	6.3751	0.2774	136.77	102.95	90.46	201.28	11.50			201.28	208.42	3.55

Notes: Mean corn profit: $\frac{1}{n} \sum_{i=1}^n (\pi_{T_1,C}^i)$; mean soybean profit: $\frac{1}{n} \sum_{i=1}^n (\pi_{T_1,S}^i)$; % of draws corn is grown: $\%(\pi_{T_1,C}^i > \pi_{T_1,S}^i)$;
traditional value: $\hat{V}_1(t)$; real option value: $\hat{V}_2(t)$; % real option value: $\widehat{\%}\Pi = \left(\frac{\hat{V}_2(t)}{\hat{V}_1(t)} - 1 \right) \times 100\%$.

Table 7. Simulation Results for Changing Correlations, 2008

Scenario	Mean Corn Profit	Mean Soybeans Profit	% of Draws Corn is Grown	Traditional Value	Real Option Value	% Real Option Value
Base Case	314.04	235.39	63.84	314.04	347.95	10.81
$\rho_{CS} = 0$	315.30	236.42	63.74	315.30	352.69	11.86
10% increase in ρ_{CS}	313.99	235.17	63.82	313.99	347.23	10.58
20% increase in ρ_{CS}	313.78	236.89	64.14	313.78	346.68	10.48
$\rho_{CS} = 0.80$	312.55	237.80	64.72	312.55	344.62	10.26
$\rho_{CN} = 0$	314.69	237.21	63.64	314.69	349.00	10.90
10% increase in ρ_{CN}	312.14	237.81	64.36	312.14	345.59	10.72
20% increase in ρ_{CN}	314.84	235.77	64.82	314.84	348.38	10.65
$\rho_{CN} = 0.80$	314.00	235.69	64.54	314.00	346.10	10.22
$\rho_{SN} = 0$	314.64	236.92	64.98	314.64	347.85	10.55
10% increase in ρ_{SN}	314.03	238.34	63.58	314.03	348.68	11.03
20% increase in ρ_{SN}	311.87	236.25	62.58	311.87	346.30	11.04
$\rho_{SN} = 0.80$	313.36	237.45	63.64	313.36	348.61	11.25

Note: In the base case scenario, $F_{T_1,C} = \$4.2/\text{bu}$, $F_{T_1,S} = \$8.6/\text{bu}$, $F_{T_1,N} = \$0.46/\text{lb}$, $K_C = \$191.47$, $K_S = \$146.06$. In addition, over the forecasting period, September in year t to April of year $t + 1$, average forecasting of $\rho_{CS} = 0.51$, $\rho_{CN} = 0.095$, $\rho_{SN} = 0.024$.

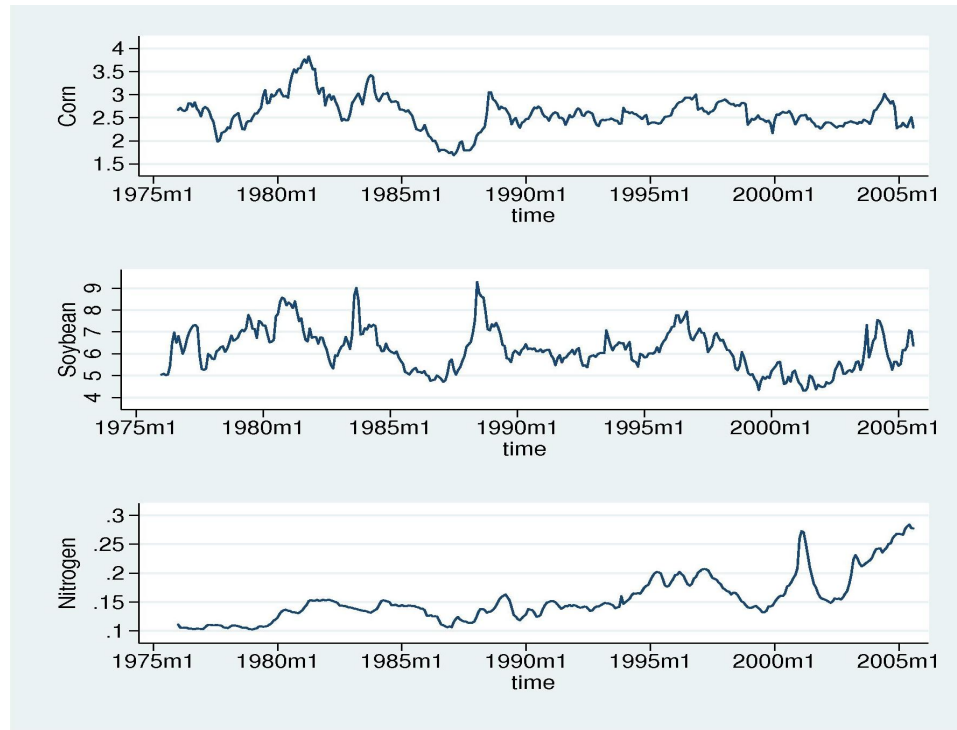


Figure 1. Original Price Series, Jan. 1976-Aug. 2005

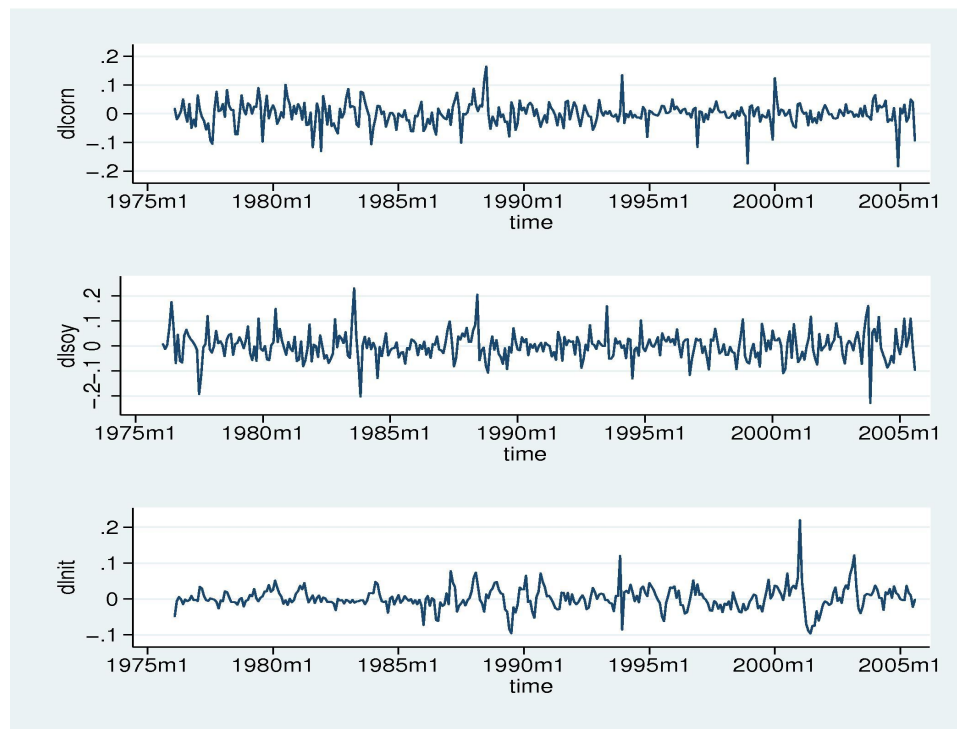


Figure 2. Transformed Price Series

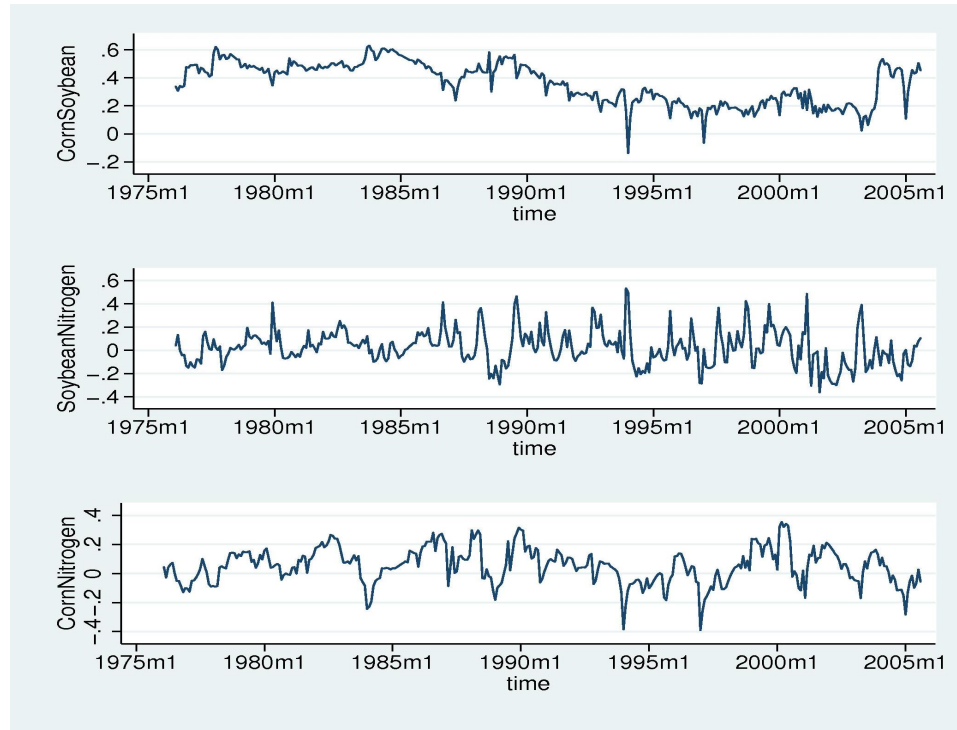


Figure 3. Implied Dynamic Conditional Correlations

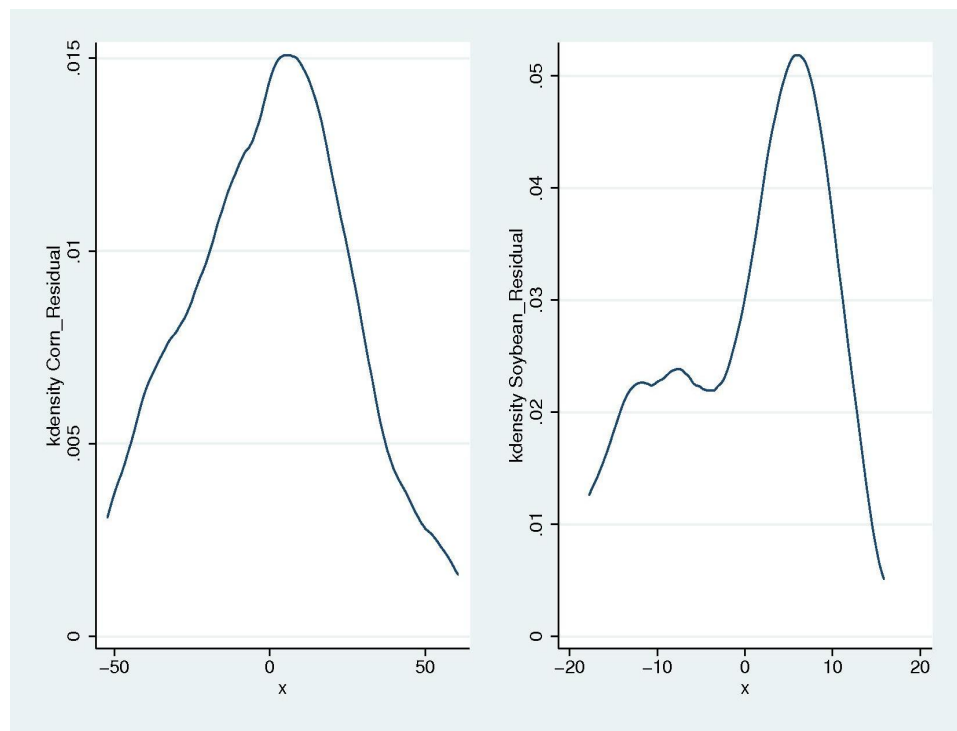


Figure 4. Kernel Density Estimation of OLS Residuals

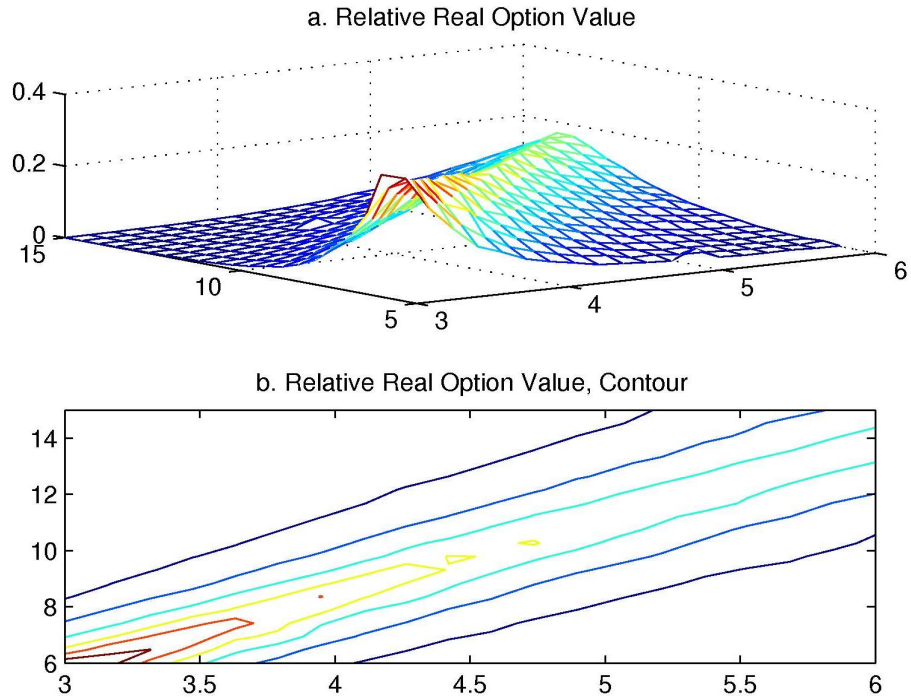


Figure 5. Simulated Relative Real Option Value (a) and Contour (b), 2008

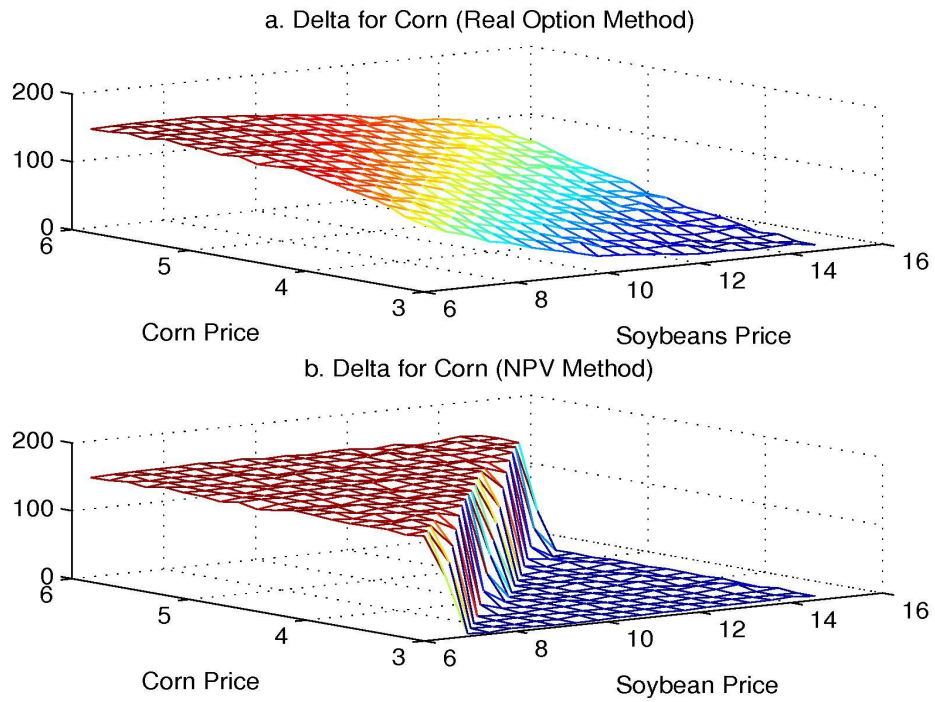


Figure 6. Delta of Corn Implied by the Real Option (a) and NPV (b) Methods

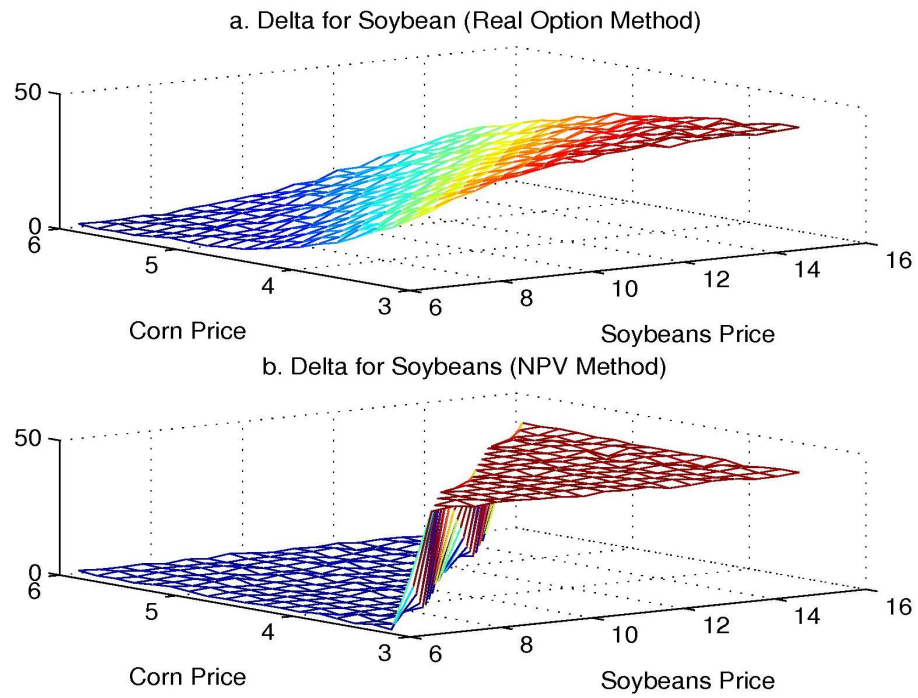


Figure 7. Delta of Soybeans Implied by the Real Option (a) and NPV (b) Methods

Appendix

Derivation of Equation (10)

By Itô's lemma, the value function $V_t^S(\cdot)$ follows the following stochastic process:

$$dV_t^S = \left(\frac{\partial V_t^S}{\partial F_{t,S}} \mu_{F_{t,S}} F_{t,S} + \frac{\partial V_t^S}{\partial t} + \frac{1}{2} \frac{\partial^2 V_t^S}{\partial F_{t,S}^2} \sigma_{F_{t,S}}^2 F_{t,S}^2 \right) dt + \frac{\partial V_t^S}{\partial F_{t,S}} \sigma_{F_{t,S}} F_{t,S} dz_S$$

It satisfies the Black differential equation (Hull 2002, p. 298):

$$\frac{\partial V_t^S}{\partial t} + \frac{1}{2} \frac{\partial^2 V_t^S}{\partial F_{t,S}^2} \sigma_{F_{t,S}}^2 F_{t,S}^2 = r V_t^S$$

with the only non-trivial boundary condition as $V_{T_1}^S(F_{T_1,S}) = \varphi_S F_{T_1,S}^{\delta_S}$. The unique value function $V_t^S(\cdot)$ satisfying the above partial differential equation and the boundary condition is given as equation (10). In addition, the value of soybeans profit follows an Itô process as

$$\frac{dV_t^S}{V_t^S} = \alpha_S dt + \sigma_{V_S} dz_{V_S}$$

where $\alpha_S = \delta_S \mu_{F_{t,S}} + r$ and $\sigma_{V_S} dz_{V_S} = \delta_S \sigma_{F_{t,S}} dz_S$.

Derivation of Equation (11)

Applying Itô's lemma to the two state variables, $F_{t,C}$ and $F_{t,N}$, we get the dynamics of the corn value as:

$$\begin{aligned}
dV_t^C &= \frac{\partial V_t^C}{\partial F_{t,C}} dF_{t,C} + \frac{\partial V_t^C}{\partial F_{t,N}} dF_{t,N} + \frac{\partial V_t^C}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V_t^C}{\partial F_{t,C}^2} (dF_{t,C})^2 + \frac{1}{2} \frac{\partial^2 V_t^C}{\partial F_{t,N}^2} (dF_{t,N})^2 \\
&\quad + \frac{\partial^2 V_t^C}{\partial F_{t,C} \partial F_{t,N}} dF_{t,C} dF_{t,N} \\
&= \left(\frac{\partial V_t^C}{\partial F_{t,C}} \mu_{F_{t,C}} F_{t,C} + \frac{\partial V_t^C}{\partial F_{t,N}} \mu_{F_{t,N}} F_{t,N} + \frac{\partial V_t^C}{\partial t} + \frac{1}{2} \frac{\partial^2 V_t^C}{\partial F_{t,C}^2} \sigma_{F_{t,C}}^2 F_{t,C}^2 + \frac{1}{2} \frac{\partial^2 V_t^C}{\partial F_{t,N}^2} \sigma_{F_{t,N}}^2 F_{t,N}^2 + \right. \\
&\quad \left. \frac{\partial^2 V_t^C}{\partial F_{t,C} \partial F_{t,N}} F_{t,C} F_{t,N} \sigma_{F_{t,C}} \sigma_{F_{t,N}} \rho_{CN} \right) dt + \frac{\partial V_t^C}{\partial F_{t,C}} F_{t,C} \sigma_{F_{t,C}} dz_C + \frac{\partial V_t^C}{\partial F_{t,N}} F_{t,N} \sigma_{F_{t,N}} dz_N
\end{aligned}$$

Following Marcus and Modest (1984), the hedging portfolio includes (1) the claim to the farmer's corn profit, (2) $\frac{\partial V_t^C}{\partial F_{t,C}}$ short positions in the corn futures contracts, (3) $\frac{\partial V_t^C}{\partial F_{t,N}}$ short positions in the assumed nitrogen fertilizer futures contracts, and (4) borrowing the amount of V_t^C at the risk free rate r . By design, the return on this portfolio is instantaneously riskless, which implies that the valuation function $V_t^C(\cdot)$ satisfies the partial differential equation

$$\frac{\partial V_t^C}{\partial t} + \frac{1}{2} \frac{\partial^2 V_t^C}{\partial F_{t,C}^2} \sigma_{F_{t,C}}^2 F_{t,C}^2 + \frac{1}{2} \frac{\partial^2 V_t^C}{\partial F_{t,N}^2} \sigma_{F_{t,N}}^2 F_{t,N}^2 + \frac{\partial^2 V_t^C}{\partial F_{t,C} \partial F_{t,N}} F_{t,C} F_{t,N} \sigma_{F_{t,C}} \sigma_{F_{t,N}} \rho_{CN} - r V_t^C = 0$$

with the boundary condition $V_{T_1}^C(F_{T_1,C}, F_{T_1,N}) = \varphi_C F_{T_1,C}^{\delta_C} F_{T_1,N}^{\delta_N}$. The value function satisfying the above partial differential equation and the boundary condition is as equation (11).

The value of corn profit follows a geometric Brownian motion as

$$\frac{dV_t^C}{V_t^C} = \alpha_C dt + \sigma_{V_C} dz_{V_C}$$

where $\alpha_C = \delta_C \mu_{F_{t,C}} + \delta_N \mu_{F_{t,N}} + r$, $\sigma_{V_C} dz_{V_C} = \delta_C \sigma_{F_{t,C}} dz_C + \delta_N \sigma_{F_{t,N}} dz_N$.

Derivation of Equation (12)

We assume that the option value function $\Pi(\cdot)$ is linear homogeneous in V_t^C and V_t^S . Now let V_t^S be the numéraire with price of unity and define the price of V_t^C as $V_t = V_t^C/V_t^S$. Given (10) and (11), the relative value also follows geometric Brownian motion. Applying Itô's lemma, the dynamics of V_t are given by

$$\frac{dV_t}{V_t} = \mu_V dt + \sigma_V dz_V$$

where $\mu_V = \alpha_C - \alpha_S + \delta_S^2 \sigma_{F_{t,S}}^2 - \delta_C \delta_S \sigma_{F_{t,C}} \sigma_{F_{t,S}} \rho_{CS} - \delta_S \delta_N \sigma_{F_{t,S}} \sigma_{F_{t,N}} \rho_{SN}$, and $\sigma_V dz_V = \delta_C \sigma_{F_{t,C}} dz_C + \delta_N \sigma_{F_{t,N}} dz_N - \delta_S \sigma_{F_{t,S}} dz_S$.

Now, the option to switch between corn and soybeans is a call option on the value of corn, with exercise price equal to unity and interest rate equal to zero. Applying the Black-Scholes formula on this special case, the value of the switching option is given as equation (12).

Derivation of Equation (13)

$$\begin{aligned} \frac{\partial \Pi}{\partial \sigma_{F_{t,C}}} &= \frac{\partial V_t^C}{\partial \sigma_{F_{t,C}}} \Phi(d_1) + V_t^C \frac{\partial \Phi(d_1)}{\partial d_1} \frac{\partial d_1}{\partial \sigma_{F_{t,C}}} - \frac{\partial V_t^S}{\partial \sigma_{F_{t,C}}} \Phi(d_2) - V_t^S \frac{\partial \Phi(d_2)}{\partial d_2} \frac{\partial d_2}{\partial \sigma_{F_{t,C}}} \\ &= \frac{\partial V_t^C}{\partial \sigma_{F_{t,C}}} \Phi(d_1) + V_t^C \phi(d_1) \frac{\partial d_1}{\partial \sigma_{F_{t,C}}} - \frac{\partial V_t^S}{\partial \sigma_{F_{t,C}}} \Phi(d_2) - V_t^S \phi(d_2) \left(\frac{\partial d_1}{\partial \sigma_{F_{t,C}}} - \frac{\partial \sigma_V}{\partial \sigma_{F_{t,C}}} \sqrt{T_1 - t} \right) \\ &= \frac{\partial V_t^C}{\partial \sigma_{F_{t,C}}} \Phi(d_1) + V_t^S \phi(d_2) \frac{\partial \sigma_V}{\partial \sigma_{F_{t,C}}} \sqrt{T_1 - t} \end{aligned}$$

Equation (13) follows. The third equality holds because we have (a) $V_t^C \phi(d_1) = V_t^S \phi(d_2)$; (b) $\frac{\partial V_t^S}{\partial \sigma_{F_{t,C}}} = 0$.

Derivation of Equation (14)

$$\begin{aligned}
\frac{\partial \Pi}{\partial \rho_{CS}} &= \frac{\partial V_t^C}{\partial \rho_{CS}} \Phi(d_1) + V_t^C \phi(d_1) \frac{\partial d_1}{\partial \rho_{CS}} - \frac{\partial V_t^S}{\partial \rho_{CS}} \Phi(d_2) - V_t^S \phi(d_2) \frac{\partial d_2}{\partial \rho_{CS}} \\
&= V_t^C \phi(d_1) \frac{\partial \sigma_V}{\partial \rho_{CS}} \sqrt{T_1 - t}
\end{aligned}$$

Equation (14) follows. The second equality holds because we have (a) $V_t^C \phi(d_1) = V_t^S \phi(d_2)$; (b)

$$\frac{\partial V_t^C}{\partial \rho_{CS}} = \frac{\partial V_t^S}{\partial \rho_{CS}} = 0.$$

Derivation of Equation (16)

$$\begin{aligned}
\Delta_C &= \frac{\partial V_{T_0}^{OP}}{\partial F_{t,C}} \\
&= \Phi(d_1) \frac{\partial V_t^C}{\partial F_{t,C}} + \phi(d_1) V_t^C \frac{\partial d_1}{\partial F_{t,C}} + [1 - \Phi(d_2)] \frac{\partial V_t^S}{\partial F_{t,C}} - \phi(d_2) V_t^S \frac{\partial d_2}{\partial F_{t,C}} \\
&= \Phi(d_1) \frac{\partial V_t^C}{\partial F_{t,C}} = \Phi(d_1) \delta_C V_t^C / F_{t,C}
\end{aligned}$$