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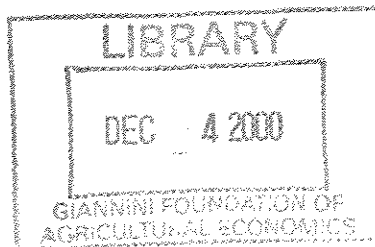
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TIME-CONSISTENT POLICIES

by

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# Time-Consistent Policies

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## Abstract

In many cases the optimal open-loop policy to influence agents who solve dynamic problems is time-inconsistent. We show how to construct a time-consistent open-loop policy rule. We also consider an additional restriction under which the time-consistent open-loop policy is stationary. We use examples to illustrate the properties of these tax rules.

*Keywords:* Time inconsistency, open-loop policies, stationarity

*JEL classification numbers* C72, D83

# 1 Introduction

When non-strategic agents with rational expectations solve dynamic optimization problems, and a government (or some other “leader”) attempts to influence the agents’ decisions, the government’s optimal program is often time-inconsistent. However, the possibility that the optimal program is time-consistent is more general than is widely believed. We extend previous results by developing a simple means of testing whether a given open-loop policy rule, such as a linear income tax, is time-consistent. This approach also identifies the (possibly non-linear) form of the policy that ensures time-consistency, for a wide class of utility and production functions. Alternatively, given a particular functional form for the government’s policy rule, we can select utility and production functions for which that policy rule is time-consistent. The condition for consistency of the tax policies is intuitive: the effect of the tax on the agent’s present discounted value of future utility must be independent of the level of capital (wealth). We then identify a *stationary* time-consistent policy rule. The set of stationary time-consistent policies is a subset of the set of Markov Perfect policies for a particular differential game.

We study an environment in which non-strategic agents with rational expectations make current decisions which affect their future welfare. For example, an individual’s current consumption determines current savings and future wealth, and thus future consumption possibilities. A strategic agent, such as the government, announces a time profile of policies, such as taxes.

Government policy can affect an agent’s decision at a point in time in three ways. First, *past* policies can affect an agent’s current decision indirectly, via the policy’s effect on a state variable. For example, previous taxes affect the individual’s current wealth, which affects current consumption. Second, the *current* policy might directly affect the individual’s current decision. For example, current taxes might affect the current cost of investment. Third, the anticipation of *future* policies can affect the agent’s current decision. For example, future taxes may affect the future value of capital, and thus affect the current consumption and savings decision.

Kydland and Prescott (1977) pointed out that the third channel of causation typically makes the government’s optimal open-loop policy time-inconsistent. They noted (page 476) that if this third channel is absent, i.e. if the agent’s current decision is independent of the government’s announced future policies, then the optimal open-loop policy is time-consistent. This circumstance seems unlikely in models where agents have rational expectations and solve dynamic optimization problems. In Kydland and Prescott’s two-period example, it is obvious

from inspection whether the agent's current decision is independent of the government's future policies. Because this special circumstance seems implausible, and because it would appear that its existence is easy to identify, little attention has been paid to the possibility that the open-loop program is time-consistent.<sup>1</sup>

We use a canonical model in which an agent obtains utility from consumption of a private and a public good. The government finances the public good from taxes and maximizes the utility of the representative agent. The agent takes as given the tax policy and the supply of the public good and maximizes the present discounted stream of utility by choosing consumption (and thus savings). In this model, current consumption depends only on the agent's shadow value of capital. The second channel of causation, in which the current policy directly affects the agent's current control, is absent. Previous taxes affect the current stock of capital, and thus its shadow value, so the first channel of causation is present.

For this model, Xie (1997) derives a restriction on the utility function and the production function that ensure that the current shadow value of capital is independent of future government policies. This restriction eliminates the third channel of causation and renders the government's open-loop policy time-consistent. The restriction appears "reasonable", in the sense that it allows logarithmic utility and constant returns to scale in production. In the continuous time model, unlike the two-period model, the time-consistency of the optimal policy is not obvious. A casual inspection of the necessary conditions for the agent's and the government's problem turns up the usual ingredients that lead us to expect that the open-loop policy is time-inconsistent.

We generalize Xie's principal result (his Proposition 3), which identifies combinations of utility and production functions under which the open-loop *linear tax* is time-inconsistent. Our procedure shows how to construct a class of (possibly) nonlinear tax rules such that the open-loop policy is time-consistent, *for a wide class of utility and production functions*. Noting the inconsistency of optimal open-loop policies, Kydland and Prescott advocated the use of rules rather than discretion. The problem with this advice is that most rules are time-inconsistent. We are able to identify the class of rules that are time-consistent. Different (non-linear) tax rules are compatible with time-consistency under different combinations of utility and pro-

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<sup>1</sup>The open-loop program is also time consistent if: (i) the agent's decisions have no effect on the government's payoff (Kydland and Prescott, page 476) or if (ii) the government is able to completely control the agent's decisions, and thereby achieve its first best payoff (i.e. the government can achieve the payoff it would have obtained had it directly chosen the agent's control).

duction functions. The time-consistency of policies is not an unusual possibility, but instead depends on the manner in which the government raises revenue. We then obtain an additional restriction that holds for stationary time-consistent policies, and discuss its relation to the set of Markov Perfect policies. Examples illustrate the relation between primitive functions and the set of time-consistent policies, and also the relation among time-consistent, stationary time-consistent, and Markov Perfect policies.

## 2 The model

Our model is standard in the literature on optimal taxation in continuous time (see Xie (1997) and Chamley (1986)). A representative agent chooses a consumption trajectory  $c(t)$  in order to maximize the present discounted value of utility. The agent's wealth (capital stock) is  $k(t)$  which yields the instantaneous output  $f(k(t))$ . The tax rule is  $g(k, t)$ , so after-tax income is  $f(k(t)) - g(k, t)$  and investment is  $\frac{dk}{dt} = f(k(t)) - g(k, t) - c(t)$ . The government pays for the flow of a public good,  $G(t)$ , using taxes, without borrowing, so  $G(t) = g(k, t)$ .<sup>2</sup> The utility of consumption is  $U(c)$  and the utility derived from the public good is  $V(G)$ ; both functions are concave. The agent's optimization problem is

$$\begin{aligned} \max_{\{c(t)\}} \quad & \int_0^{\infty} e^{-\rho t} (U(c(t)) + V(G(t))) dt \\ \text{subject to} \quad & \frac{dk}{dt} = f(k(t)) - g(k, t) - c(t), \quad k_0 \text{ given.} \end{aligned} \tag{1}$$

The agent behaves as if *aggregate* tax collection, and thus the provision of the public good, is given. In view of the (assumed) separability of the instantaneous payoff, we can ignore the term  $V(G(t))$  in studying the agent's control problem. The agent understands that her tax payments may depend on her wealth via  $g(k, t)$ . This function appears in the constraint of problem (1).

We will use the following

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<sup>2</sup>In another interpretation of this model (Long and Shimomura 2000) there are two types of agents: capital owners receive utility  $U(c)$  and individuals whose consumption equals the government transfer (financed by the tax on capital) receive utility  $V(G)$ .

**Definition 1** *The first best outcome is the solution to the following optimization problem:*

$$\begin{aligned} & \max_{\{c(t), G(t)\}} \int_0^{\infty} e^{-\rho t} (U(c(t)) + V(G(t))) dt & (2) \\ & \text{subject to } \frac{dk}{dt} = f(k(t)) - G(t) - c(t), \quad k_0 \text{ given.} \end{aligned}$$

Problem (2) differs from problem (1) in that in the former the government directly chooses the consumption and tax trajectories, whereas in the latter the agent optimizes with respect to only the consumption plan, given the tax policy.

To avoid uninteresting complications caused by corner solutions, we adopt the following:

**Assumption 1** *The levels of consumption and the provision of the public good in the open-loop equilibrium are strictly positive.*

To eliminate a trivial kind of time consistency, we also adopt:

**Assumption 2** *The feasible set of open-loop policy rules does not enable the government to achieve the first best outcome.*

This assumption states that the tax rule does not give the government enough leverage to achieve the first best outcome. If the government were able to achieve the first best outcome using a tax policy, that policy would obviously be time consistent.

Assumption 2 is not innocuous, and therefore requires justification. A tractable model is necessarily simple. In a simple environment it is often easy to find a relatively simple policy that achieves the first best outcome. This success, however, is due to the simplicity of the theoretical model. In the real world most feasible policies do give rise to secondary distortions. In order to describe this feature of the real world using a simple model, we have to tie our hands in some fashion, i.e. to assume that non-distortionary policies are not available.

Assumption 2 eliminates some apparently unexceptionable policies, including the poll tax. If the government uses a poll tax ( $g(k, t) = g(t)$ ), then the optimal open-loop tax is first best. In order to demonstrate this claim, we can show that the necessary conditions to the agent's and the government's problems in the open-loop equilibrium are equivalent to the necessary conditions for the first best outcome. The intuition is straightforward. In general, time inconsistency arises because the government's policy causes a secondary distortion. Incurring the secondary distortion in the future is a cost (born by the government in the future) of influencing the agent's

current behavior. The poll tax does not cause a secondary distortion; it is the first best means of raising tax revenue.

Excluding by assumption the simplest policy, which also happens to be time consistent, may seem a bit odd. We explained above the general reason for adopting Assumption 2. As a practical matter it may make sense to ignore the poll tax because of its lack of progressivity, and the associated political problems – as Thatcher discovered. In addition, the poll tax that supports the first best outcome is non-stationary, and therefore is not Markov Perfect. We discuss stationary rules in Section 4.

We also adopt

**Assumption 3** *The government's tax policy is multiplicatively separable:  $g(k, t) = b(k)\tau(t)$  for some functions  $b(k)$  and  $\tau(t)$ .*

A subsequent footnote briefly explains how our major result changes when we drop Assumption 3. This assumption allows us to concentrate on interesting special cases: a linear income tax ( $b(k) = f(k)$ ); a linear capital tax ( $b(k) = k$ ); and a nonlinear income or capital tax ( $b(k) \neq k$ ,  $b(k) \neq f(k)$ ). Assumption 3 permits a poll tax ( $b(k) = \text{a constant}$ ), but as we noted above, Assumption 2 excludes the poll tax.

The function  $b(k)$  typically affects the agent's optimal consumption plan. Fixing the function  $b(k)$  does not restrict the government's ability to raise tax revenue for a given level of  $k$ , because the government is able to choose  $\tau(t)$ .

We use the following

**Definition 2** *Conditional on fixed  $b(k)$ , a tax policy  $b(k)\tau(t)$  is time consistent if and only if the trajectory  $\tau(t)$  that is optimal at time  $t = 0$  (given the state  $k_0$ ) remains optimal at every  $t \geq 0$  along the equilibrium trajectory.*

In other words, if the agent believes that the government will adhere to the announced policy  $b(k)\tau(t)$  and behaves optimally given this belief, then the government has no incentive to deviate from the time dependent part of the policy,  $\tau(t)$ .

The qualifier “conditional on fixed  $b(k)$ ” in Definition 2 means that the policy is “conditionally time-consistent”. All discussions of time consistency (that we know of) implicitly contain this kind of conditionality. For example, Xie finds the time-consistent policy conditional on the use of a linear income tax ( $b(k) = f(k)$ ). Since a major point of our paper is to show that



we can always find a time-consistent policy by the appropriate choice of  $b(k)$ , it is important that we make this conditionality explicit. Hereafter we use the terms “time-consistent” and “conditionally time-consistent” interchangeably, adding the qualifier only for emphasis.

### 3 The time-consistent tax policy

Ignoring the function  $V(G(t))$  (which has no influence on the agent’s optimal consumption decision), we can write the value of the agent’s payoff as the function  $Y(k, t)$ . This function solves the Hamilton-Jacobi-Bellman (H-J-B) equation

$$-Y_t(k, t) = \max_c \{e^{-\rho t} U(c) + Y_k(k, t)(f(k) - b(k)\tau(t) - c)\}, \quad (3)$$

where subscripts denote partial derivatives. We eliminate the effect of discounting by defining the function  $e^{-\rho t} J(k, t) \equiv Y(k, t)$ . Substituting this definition in equation (3) we obtain the current value H-J-B equation

$$\rho J(k, t) = \max_c \{U(c) + J_k(k, t)(f(k) - b(k)\tau(t) - c)\} + J_t(k, t). \quad (4)$$

Because our derivation of the condition for time consistency builds on Xie, we review his argument. The first order condition to equation (4) implies that the optimal consumption at a point in time depends only on the shadow value of capital  $J_k(k, t)$ . Xie uses the Maximum Principle rather than dynamic programming, and denotes the costate variable for capital as  $q$ . Assuming differentiability of the value function, we have  $J_k \equiv q$  (Seierstad and Sydsaeter 1996). We take the following definition from Xie:

**Definition 3** *The costate variable  $q$  is “uncontrollable” if and only if its value at time  $t$  is independent of current and future government actions  $\{\tau(s)\}_{s=t}^{\infty}$ .*

Since consumption at a point in time depends only on the costate variable, consumption is uncontrollable if and only if the costate variable is uncontrollable.

Xie’s Proposition 1 states that the open-loop linear income tax ( $g(k, t) = f(k)\tau(t)$ ) is time consistent only if  $q$  is uncontrollable. The agent’s costate equation (the equation for  $\dot{q}$ ) is a constraint in the government’s problem. Let  $\xi$  be the costate variable, in the government’s problem, associated with the variable  $q$ . If future government policies affect the current value

of  $q$ , then the initial value of  $q$  ( $q_0$ ) is free. This initial value is determined by the government's future policy. In that case, optimality of the government's problem requires  $\xi_0 = 0$ .<sup>3</sup>

The boundary condition  $\xi_0 = 0$  is the source of the inconsistency of the government's open-loop policy. Since  $\dot{\xi}$  is not identically 0 unless the government can completely control the agent's decisions (a possibility excluded by Assumption 2)  $\xi_t \neq 0$  for some  $t$ . If the government were able to revise its future trajectory at  $t$ , it would do so in order to set  $\xi_t = 0$ . The continuation of the policy trajectory chosen at time 0 is therefore not optimal at  $t > 0$ , so the policy trajectory is not time consistent.

If the future values of government policies do not affect the agent's costate variable, then  $q_0$  is not free. In this case,  $\xi_0 \neq 0$  and the government's open-loop policy is time consistent. In other words, time consistency requires that at time  $t$  the agent's shadow value can be a function only of  $k_t$ , not of  $\tau(s)$  for  $s > t$ . We have a slightly more general result. (The Appendix contains proofs that are not included in the text.)

**Lemma 1** *Suppose that Assumptions 1-3 hold. The open-loop tax policy is time consistent if and only if consumption is uncontrollable.*

The lemma implies that the tax policy is time consistent if and only if  $J(k, t)$  is additively separable in  $k$  and  $t$ .

**Proposition 1** *Under Assumptions 1-3, the government's open-loop policy is time-consistent if and only if the agent's value function is additively separable in the state variable and time:  $J(k, t) = W(k) + Z(t)$  for some functions  $W(k)$  and  $Z(t)$ .*

Proposition 1 implies that time consistency of an open-loop policy requires that the function  $b(k)$  is proportional to the reciprocal of the shadow value of capital:

**Corollary 1** *Suppose that the agent's value function is additively separable in the state variable and time. Then*

$$W_k(k)b(k) = \alpha \tag{5}$$

for some constant  $\alpha$ .

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<sup>3</sup>Simaan and Cruz (1973) were among the first to use this boundary condition in Stackelberg differential games. Many applications use this boundary condition. See, for example, Oudiz and Sachs (1984) and Miller and Salmon (1985) for macroeconomic applications. Xie (1997) provides other examples.

**Proof.** From Proposition 1, the optimal consumption rule is a function only of  $k$ . Substituting this optimal rule,  $c = c^*(k)$  into equation (4) and using  $J(k, t) = W(k) + Z(t)$  from Proposition 1, we can write the government's H-J-B equation as

$$\rho(W(k) + Z(t)) = U(c^*(k)) + W_k(k) [f(k) - \tau(t)b(k) - c^*(k)] + Z_t(t). \quad (6)$$

The additive separability of  $J(k, t)$  requires that the right side of equation (6) must also be additively separable for any admissible  $\tau(t)$ . This requirement implies equation (5) with  $\alpha$  equal to a constant. ■

The left side of equation (6) is the present discounted value of future utility, expressed as an annuity with discount rate  $\rho$ . The reduction in the value of this annuity (i.e., the reduction in the value of the agent's program), caused by the tax, is  $W_k(k) [\tau(t)b(k)]$ . Equation (5) means that this effect of the tax is independent of the value of  $k$ . In other words, the agent views a time-consistent tax exactly like a lump sum reduction in the dollar value of future utility, equal to  $\frac{\alpha\tau(t)}{\rho}$ .

The previous results lead to the following necessary and sufficient condition for time consistency<sup>4</sup>

**Proposition 2** *Under Assumptions 1-3, the government's open-loop policy is time-consistent if and only if*

$$U' \left( \left[ \frac{\rho - f_k(k)}{b_k(k)} \right] b(k) + f(k) \right) b(k) = \alpha. \quad (7)$$

The proof of this proposition shows that when  $b(k)$  satisfies equation (7), the agent's consumption rule is

$$c^*(k) = \left( \frac{\rho - f_k(k)}{b_k(k)} \right) b(k) + f(k). \quad (8)$$

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<sup>4</sup>In stating this proposition – and also Proposition 3 – we assume that the necessary conditions to the control problem provide a solution to the problem. Given our assumption that  $U(c)$  is concave, the necessary conditions are sufficient if  $f(k) - \tau(t)b(k)$  is concave. For specific examples we can confirm that this function is concave, but in general we cannot do so. Of course, the solution to the necessary conditions may provide the solution to the agent's optimal control problem even if  $f(k) - \tau(t)b(k)$  is not concave.

We refer to equation (7) as the *consistency constraint*, since the government's optimal program is time-consistent if and only if it holds (subject to our assumptions).<sup>5</sup> Proposition 2 extends Xie's Proposition 3 in two respects. First, it shows that the possibility of time consistency is very general. Given primitive functions  $U$  and  $f$ , we can construct  $b$  to obtain a time-consistent tax. Xie restricts  $b(k) \equiv f(k)$ , i.e. he assumes that the government must use a linear income tax. Second, our Proposition 2 is a necessary and sufficient condition, rather than only a sufficient condition.

Given the utility and production functions, the consistency constraint is an ordinary differential equation (ODE). The solution to this ODE depends on two parameters,  $\alpha$  and a constant of integration that determines the boundary condition to equation (7). Denote this constant of integration by  $\gamma$ . The set of time-consistent rules is the two-parameter family of functions  $b(k; \alpha, \gamma)$ ; when there is no ambiguity, we suppress the arguments  $\alpha$  and  $\gamma$ .

Alternatively, we can start with a particular type of tax rule and find primitive functions ( $U$  and  $f$ ) such that the tax rule is time-consistent. For each tax rule we obtain the combination of production and utility functions that satisfy the consistency condition.

## 4 Stationarity

If the function  $b(k)$  satisfies the time consistency constraint, then for values of  $k$  along the *optimal trajectory* the government has no incentive to revise the optimal time dependent component of the tax  $\tau(t)$  announced at time 0. If for some reason the state  $k$  departs from the equilibrium path, the government might want to change the original open-loop policy  $\tau(t)$ . In that case, the optimal open-loop policy is not subgame perfect.

In this section we obtain an additional constraint on the functions  $U$ ,  $V$ ,  $f$ , and  $b$ ; if the functions satisfy this constraint, then the time dependent component of the government's optimal tax

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<sup>5</sup>We mentioned above that Assumption 3 is unnecessarily restrictive. It is easy to show that additive separability of the value function only requires  $g(k, t) = x(k) + b(k)\tau(t)$  for some function  $x(k)$  (rather than  $g(k, t) = b(k)\tau(t)$  as Assumption 3 maintains). With this more general functional form, we can repeat the steps used to derive equation (7) to obtain the more general consistency constraint

$$U' \left( \frac{b(\rho - (f_k + x_k))}{b_k} + f + x \right) b(k) = \alpha.$$

This constraint involves the two functions  $x(k)$  and  $b(k)$ .

policy,  $\tau(t)$ , is a constant: the policy is stationary. We then show that if  $b$  is a time-consistent stationary policy and if  $c = c^*(k)$ , (given by (8)), then the pair of functions  $b(k)$  and  $c^*(k)$  are a Markov Perfect equilibrium to a differential game.

#### 4.1 The stationarity constraint

We adopt the following:

**Assumption 4** *The initial value of the state variable,  $k_0$ , is not equal to its steady state value.*

In general, the solution to the government's control problem is not a constant value of  $\tau$ . The following proposition provides a restriction involving the primitive functions  $U$ ,  $V$ , and  $f$  and the tax policy  $b$  that is necessary and sufficient for the optimal  $\tau$  to be a constant.

**Proposition 3** *Suppose that Assumption 4 holds, that  $b(k)$  satisfies equation (7) (so that the tax is time-consistent) and that  $c = c^*(k)$  satisfies equation (8) (so that the agent behaves optimally). The government takes  $b(k)$  and  $c^*(k)$  as given and chooses the time dependent component of the tax  $\tau(t)$ , in order to solve*

$$\max_{\{\tau(t)\}} \int_0^{\infty} e^{-\rho t} (U(c^*(k)) + V(\tau b(k))) dt \quad (9)$$

subject to  $\frac{dk}{dt} = f(k(t)) - \tau(t)b(k) - c^*(k)$ ,  $k_0$  given.

*The optimal trajectory  $\tau^*(t)$  is a constant  $\tau$  if and only if there exists a  $\tau$  such that the following equation holds identically in  $k$ :*

$$(\rho + \eta) V' = (f' + \eta) U' - (\rho - f' + \tau b') \tau b V'' \quad (10)$$

where

$$\eta = - \left( \frac{U'' b'}{U'' b} + f' \right). \quad (11)$$

Hereafter, we refer to equation (10) as the stationarity constraint.

We showed above that given the primitive functions  $U$  and  $f$ , the set of time-consistent policy rules  $b(k)$  is a two-parameter family of functions that depend on  $\alpha$  and  $\gamma$ . If the government is required to use a time-consistent policy, then at time 0 it is able to choose  $\alpha$  and  $\gamma$  and the open-loop trajectory  $\tau(t)$  to maximize its payoff. If we also impose the requirement that the policy is stationary, then the policy rule  $b(k)$  must satisfy equation (10). In general, there is no guarantee that there exist functions  $b(k)$  and  $c^*(k)$  that satisfy equations (7), (8), and (10). The next section uses examples to show that such functions exist in some cases.

It might appear that when imposing the stationarity constraint we obtain an additional degree of freedom, the parameter  $\tau$ . That is, it might appear that in selecting a stationary time-consistent policy the government is able to choose three parameters,  $\alpha, \gamma$  and  $\tau$ . This interpretation is incorrect; without loss of generality, we can normalize  $\tau = 1$ . The government has only two free parameters, as the following corollary shows.

**Corollary 2** *If the government is required to choose a stationary time-consistent policy, it must select a function  $b(k)$  that satisfies equation (7) and equation (10) evaluated at  $\tau = 1$ . Thus, the government has two free parameters,  $\alpha$  and  $\gamma$ .*

## 4.2 An interpretation of the stationary time-consistent equilibrium

The requirement of Markov Perfection eliminates plans that are contingent on state variables that have no direct bearing on the payoff. For our autonomous problem, time does not directly affect the agents' payoff functions. Thus, a Markov Perfect equilibrium must be stationary. Here we show that the stationary time-consistent equilibrium is a Markov Perfect equilibrium.

If  $b(k)$  satisfies the time-consistency constraint, then  $c^*(k)$  is the private agent's best response to the tax policy. If  $b(k)$  satisfies both the time consistency and the stationarity constraint, then (in view of the normalization  $\tau = 1$ )  $b(k)$  is the government's best response to  $c^*(k)$ . It is worth emphasizing that in this equilibrium the government takes the function  $c^*(k)$  as given: the government does not recognize that  $c^*(k)$  depends on  $b(k)$ .

More formally, we have

**Proposition 4** *Suppose that a tax function  $b(k)$  satisfies equations (7) and (10) and that the consumption function  $c^*(k)$  is given by equation (8). In this case, the functions  $b(k)$  and  $c^*(k)$  satisfy the necessary conditions for a Markov Perfect equilibrium to the game described by the*

following pair of control problems

$$\begin{aligned} & \max_{\{c(t)\}} \int_0^{\infty} e^{-\rho t} (U(c(t))) dt & (12) \\ \text{subject to} & \quad \frac{dk}{dt} = f(k(t)) - b(k) - c(t), \quad k_0 \text{ given.} \end{aligned}$$

$$\begin{aligned} & \max_{\{b(t)\}} \int_0^{\infty} e^{-\rho t} (U(c^*(k) + V(b))) dt & (13) \\ \text{subject to} & \quad \frac{dk}{dt} = f(k(t)) - b(t) - c^*(k), \quad k_0 \text{ given.} \end{aligned}$$

As we noted in a previous footnote, our results use only the necessary conditions for the various control problems. If the solution to these necessary conditions actually solves the control problem, then Proposition 4 has the following implication: The set of stationary time-consistent tax policies  $b(k)$  and the corresponding set of consumption functions  $c^*(k)$  (i.e., those that satisfy equation (8)) are a subset of the set of Markov Perfect equilibria to the game in which the government takes the consumption rule as given, and the agent take the tax rule as given.

The intuition for Proposition 4 is straightforward. We already know from Proposition 2 that  $c^*(k)$  is the best response to a time-consistent tax rule  $b(k)$ . In addition, it is rather obvious that when the government takes  $c^*(k)$  as given, the two control problems (9) and (13) are equivalent. The government is able to achieve exactly the same set of outcomes by choosing  $\tau$  when it holds  $b(k)$  (and thus  $c^*(k)$ ) fixed (as it does in problem (9)), and when it chooses  $b(k)$  holding  $c^*(k)$  fixed (as it does in problem (13)). If equation (10) holds, then for problem (9) the optimal  $\tau^o(t) \equiv 1$ , so  $b(k)$  is the best response to  $c^*(k)$ .

Proposition 4 does *not* state that equations (7), (8), and (10) are equivalent to the necessary conditions to the differential game above. In particular, the proposition does not state that the necessary conditions to the differential game imply equations (7), (8), and (10). The time consistency condition requires that  $c^*(k)$  is the best response to  $\tau(t)b(k)$  for all functions  $\tau(t)$ . This requirement is obviously stronger than the Markov Perfect equilibrium condition that  $c^*(k)$  is the best response to  $b(k)$ . Thus, we expect that the set of stationary time-consistent equilibria is a strict subset of the set of Markov Perfect equilibria. Our example in the next section supports this conjecture.

In both the consistent equilibrium studied in Section 3, and in the subgame perfect (stationary time-consistent) equilibrium in this section, the agent recognizes that the *level* of taxes depends on  $k$ . The agent can influence the evolution of  $k$ . The fact that the agent knows that it can affect the level of taxes (*as distinct from the tax rule*) does not contradict the assumption that the agent is non-atomic. The agent takes the provision of the public good as given, whereas the government recognizes that taxes determine the provision of the public good. This difference reflects the fact that the agent is non-atomic and the government is strategic.<sup>6</sup>

The differential game given above and the government's problem under the restriction that it use a stationary time-consistent policy have different interpretations. The differential game does not provide a criterion for selecting the "most plausible" Markov Perfect equilibria. In describing the government's problem under the time-consistent stationary restrictions, on the other hand, we explicitly recognize that the government is the "leader". It may be reasonable to assume that leadership gives the government the ability to commit to a particular policy from the set of stationary time-consistent policies.

Of course, if the government is able to select the equilibrium from among the set of stationary time-consistent policies, its optimal choice will, in general, depend on the initial condition  $k_0$ . Therefore, if the government were able to choose a different stationary time-consistent policy at a later date (for  $k(t) \neq k_0$ ), and in the process also change the agent's consumption rule, it would typically want to do so. In other words, leadership requires some degree of commitment ability.<sup>7</sup>

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<sup>6</sup>Tsutsui and Mino (1990) study a similar differential game, except that in their setting agents are symmetric in every sense. Karp (1996) uses the same methods to study a differential game between a strategic agent (a durable goods monopoly) and a continuum of non-strategic buyers. There typically exists a continuum of differentiable Markov Perfect equilibria to differential games of this sort, because of the "incomplete transversality condition". Remark I in the Appendix sketches the derivation of the ODE's for  $c^*(k)$  and  $b(k)$  implied by the necessary conditions to the pair of control problems (12) and (13). The incomplete transversality condition means that the necessary conditions do not provide a boundary condition for these ODE's, leading to non-uniqueness of the Markov Perfect equilibrium.

<sup>7</sup>It is well-known that if agents condition their decisions on the history of the game, there typically exist many subgame perfect equilibria. Outcomes that give the leader a payoff approximately equal to the first best level can sometimes be sustained using trigger strategies. In this situation the commitment problem vanishes. Chari and Kehoe (1990) and Stokey (1991) explore this method of constructing subgame perfect equilibria in a game between a government and non-atomic agents; Aulsebrook and Deneckere (1989) use this approach in the durable goods monopoly setting. Whether it is reasonable to think of nonstrategic agents as conditioning their beliefs on the history of the game, rather than on the payoff-relevant state variable (as in a Markov equilibrium) is a subjective



## 5 Examples

Using an example with logarithmic utility, we examine the relation between the time-consistent policy and the production function  $f(k)$ . We then illustrate the manner in which the stationarity requirement reduces the set of time-consistent policies, and we discuss the relation between the set of time-consistent stationary policies and the set of Markov Perfect equilibria.

### 5.1 Consistency

Let  $U(c) = \ln c$ . Equation (7) can be written

$$\frac{db}{dk} = \alpha b \frac{-\rho + \frac{df}{dk}}{\alpha f - b} \quad (14)$$

Equation (14) illustrates that (given the utility function) we can treat either the production function  $f(k)$  or the tax function  $b(k)$  as primitive; using (14) to solve for the other function, we obtain a time-consistent tax.

Substituting equation (14) into equation (8) we obtain

$$c = \frac{\rho - \frac{df}{dk}}{\alpha \left(-\rho + \frac{df}{dk}\right)} (\alpha f - b) + f = \frac{b}{\alpha} \quad (15)$$

Thus, under logarithmic utility, the ratio between private and public expenditures (the tax revenue) equals a constant. The government chooses this constant.

(i) If the government uses a linear income tax ( $b(k) \equiv f(k)$ ), the solution to the differential equation (14) is

$$f(k) = \alpha \rho k + \gamma. \quad (16)$$

Thus, a linear income tax is time-consistent under logarithmic utility if and only if the production function is affine.

(ii) For a linear capital tax ( $b(k) \equiv k$ ), the solution to (14) is

$$f(k) = k\mu \ln k + k\gamma \quad \mu \equiv \frac{1 - \alpha\rho}{\alpha}. \quad (17)$$

When  $\mu = 0$ , the production function is linear.

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judgement.

Obviously, for a linear production function, an income and a capital tax differ only by a factor of proportionality, which is absorbed in the coefficient  $\tau$ . Thus, for linear  $f(k)$  the income and capital taxes are equivalent, and both are time-consistent.

Instead of beginning with a tax function and finding the production function for which that tax function is time consistent, we can take the production function as primitive. Suppose, for example, that production is linear:  $f(k) = Ak$ ,  $A > 0$ . In this case, inspection of equation (14) confirms that the affine tax is a particular solution; i.e. the linear tax is time consistent, as we previously showed. This linear wealth tax is  $b = \alpha\rho K$ , and the corresponding income tax is  $b = \frac{\alpha\rho}{A}Ak$ . The general solution to the ODE is given by the implicit function

$$\frac{1}{b\left(\frac{-A}{-\rho+A}\right)}k - \frac{1}{\alpha\rho} \frac{b}{b\left(\frac{-A}{-\rho+A}\right)} = \gamma. \quad (18)$$

Any tax rule that solves this implicit equation is time consistent. Some of these taxes may give the regulator a higher payoff than the linear tax.

## 5.2 Stationarity

We now consider the stationary time-consistent policy for the case where  $U(c) = \ln c$  and  $V(G) = \ln(G) = \ln(b)$ . The last equality uses the budget constraint  $G = \tau b(k)$  and the normalization  $\tau = 1$ . Using the definition of  $\eta$ , (equation (11)) and equation (15) we have

$$\eta = \frac{c \frac{db}{dk}}{b} - \frac{df}{dk} = \frac{\frac{db}{dk} - \alpha \frac{df}{dk}}{\alpha}$$

Using this expression and equation (15), equation (10) simplifies to

$$\frac{db}{dk} = \frac{db}{dk} \frac{c - b}{\alpha c} = \frac{db}{dk} \frac{b - \alpha b}{\alpha b} = \frac{db}{dk} \frac{1 - \alpha}{\alpha}. \quad (19)$$

Since  $\frac{db}{dk} \neq 0$  equation (19) implies that  $\alpha = \frac{1}{2}$ .

For logarithmic utility, the time-consistent stationary tax is a solution to equation (14) with  $\alpha = \frac{1}{2}$ . The stationarity constraint removes one degree of freedom from the government, by pinning down the value of  $\alpha$ . The government still has one degree of freedom: it chooses the boundary condition to the ODE (14); i.e., the government chooses the parameter  $\gamma$ .

With logarithmic utility, consumption of the private good is twice the level of consumption of the public good under a stationary time-consistent tax. This result does not depend on the production function or on the particular time-consistent stationary tax that the government uses.

If we specialize further by choosing linear production,  $f(k) = Ak$ , the stationary time-consistent tax is a solution to equation (18) with  $\alpha = \frac{1}{2}$ . Suppose, in addition, we assume that government uses an affine tax, i.e. a tax of the form  $b = \beta k + \phi$ . Substituting this expression into equation (14) and equating coefficients of powers of  $k$ , we conclude that  $\gamma = 0$  and  $\beta = \frac{\rho}{2}$ . Since income equals  $f(k) = Ak$ , the unique affine income tax is  $\frac{\rho}{2A}$ . This result reproduces equation (23) in Xie. The result also shows (for logarithmic utility and linear production) that any affine tax is linear: it never involves a lump sum tax/subsidy.

The linear tax does not enable the government to achieve the first best outcome. Under the linear tax  $c = 2b = \rho k$ , and under the first best outcome  $c = b = \frac{\rho}{2}k$ . Consequently we cannot rule out the possibility that the government has a higher payoff if it uses one of the nonlinear taxes that solve equation (18) with  $\alpha = \frac{1}{2}$ .

Remark 2 in the Appendix shows that for the case logarithmic utility, the necessary conditions to the differential game in the previous section imply

$$c(k) = \frac{\sigma f - f + 2g}{2\sigma - 1} \quad (20)$$

where  $\sigma(k) \equiv \frac{d \ln c}{d \ln g}$  and  $g(k)$  is the government's Markov Perfect rule. If  $\sigma = 1$ , equation (20) reproduces our earlier conclusion that the equilibrium level of private consumption is twice the level of public consumption. However, the necessary conditions to the differential game do not determine the value of  $\sigma$ ; in particular, the necessary conditions do not restrict  $\sigma$  to be equal to 1. Thus, for logarithmic utility, the set of stationary time-consistent policies is a *strict* subset of the set of Markov Perfect equilibria.

## 6 Summary and discussion

We studied a situation in which the government maximizes the utility of a representative agent who solves a control problem with an externality. Taking as given any two of the three functions – the utility function, the production function, and the government policy rule – we can identify the third function that makes the policy rule time consistent. In particular, for a given utility function and production function, we can construct the time-consistent policy rule. This procedure shows that the existence of time-consistent open-loop policies is general. The condition for time consistency has an intuitive interpretation: with a time-consistent policy, the effect of the tax on the present discounted value of the agent's utility is independent of the level of wealth.

We also considered a stationary time-consistent policy, and we showed that this policy is subgame perfect. We discussed a differential game in which the government takes the consumption rule as given and agents take the tax rule as given. We explained why the set of stationary time-consistent policies is likely to be a strict subset of the set of Markov Perfect equilibria to this game.

It may be reasonable to assume that the government's leadership role gives it the ability to select the stationary time-consistent equilibrium that gives it the highest payoff. The successful exercise of this leadership ability requires some degree of commitment ability.

## 7 Appendix: proofs and remarks

**Proof. (Lemma 1)** The argument that establishes Xie’s Proposition 1 also demonstrates the “only if” part of the claim in the more general case where  $g(k, t) = b(k)\tau(t)$ , e.g. where  $b(k) \neq f(k)$ ; we do not repeat the argument here. To establish the “if” part, note that in the case where consumption is not controllable (i.e. where  $q_t = q(k_t)$ , so this function is independent of  $\tau(s)$ ,  $s \geq t$ ) the government has a standard control problem with one state variable,  $k$ , the initial value of which is predetermined. The solution to this kind of control problem satisfies the dynamic programming Principle of Optimality, and is thus time consistent. ■

In order to proof Proposition 1 we require an intermediate result. To this end, we consider a perturbation of a reference tax policy:

$$\tau(t) = \tau^*(t) + ah(t)$$

where  $\tau^*(t)$  is the reference tax policy,  $h(t)$  is a continuously differentiable function of time which represents a perturbation, and  $a$  is a scaling parameter for perturbation. When  $a = 0$  we obtain the reference policy. The value function of the consumer is now parameterized by  $a$ , given the perturbation function  $h(t)$  and the policy function  $\tau^*(t)$ . We write this value function as  $\tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty})$  to emphasize that the agent’s payoff depends on the future trajectory of taxes. For a fixed tax trajectory  $\{\tau(s)\}_{s=t}^{\infty}$  the only exogenous time dependent change arises from the change in the minimum value of the time dummy,  $s = t$ . Thus,  $\tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty}) \equiv J(k_t, t)$ . In other words, the second argument in the function  $J(k_t, t)$  “summarizes” the effect, on the agent’s payoff, of the future sequence of taxes,  $\{\tau(s)\}_{s=t}^{\infty}$ .

We have the following

**Lemma 2** *Suppose Assumptions 1-3 hold. The open-loop optimal tax policy is time consistent if and only if*

$$\frac{\partial}{\partial a} \frac{\partial \tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty})}{\partial k} = 0 \quad (21)$$

for all admissible  $\tau^*(t)$  and  $h(t)$ .

**Proof.** By the first order condition to the H-J-B equation (4), consumption equals  $U'^{-1}(J_k)$ , i.e. consumption depends only on the shadow value of capital. By the identity  $\tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty}) \equiv J(k_t, t)$ , the shadow value of the state,  $J_k$  (and thus consumption) is uncontrollable if and only

if equation (21) holds for all admissible  $\tau^*(t)$  and  $h(t)$ . By lemma 1, equation (21) is therefore necessary and sufficient for time consistency. ■

**Proof. (Proposition 1)** We first establish the “only if” part of the proposition. From lemma 2, time consistency implies equation (21), which implies that  $\frac{\partial \tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty})}{\partial K} = \psi(k)$  for some function  $\psi(\cdot)$ . Taking the integral of both sides,

$$\tilde{J} = \int^k \psi(k) dk + Z(t) \quad (22)$$

where  $Z(t)$  is the constant of integration. The identity  $\tilde{J}(k_t, \{\tau(s)\}_{s=t}^{\infty}) \equiv J(k_t, t)$  completes the demonstration.

To establish the “if” part of the proposition we can simply note that equation (22) implies equation (21), and then invoke lemma 2. However, the following detailed argument is more intuitive, and it will also be used for a subsequent result.

Suppose that the agent’s value function is additively separable in the state variable and time. In this case, the H-J-B equation (4) is rewritten as

$$\rho(W(k) + Z(t)) = \max_c \{U(c) + W_k(k) [f(k) - b(k)\tau(t) - c]\} + Z_t(t). \quad (23)$$

The first order condition is given by:

$$W_k(k) = U'(c) \implies c = c^*(k) \quad (24)$$

while the transversality condition is given by:

$$\lim_{t \rightarrow \infty} e^{-\rho t} W_k(k) k = 0. \quad (25)$$

By the first order condition (24), consumption depends only on the current state and is therefore uncontrollable. By Lemma 1, the tax policy is time consistent. ■

**Proof. (Proposition 2)** The proof of the sufficient part is an adaptation of Xie (Proposition 3 on page 419). Suppose that equation (7) is satisfied. It suffices to show that under this condition, the consumption path is independent of the time dependent part of tax policy.

Consider the consumption plan given by equation (8). Using the necessary conditions to the agent’s control problem, We can verify that this consumption plan satisfies the first order conditions for the agent’s control problem and the transversality condition. (This verification

uses the same steps as Xie's proof.) Moreover this consumption plan is independent of the time dependent part of tax policy,  $\tau(t)$  and thus uncontrollable.

The necessary part follows from Corollary 1. Using (5) in (6) we obtain an equation for  $W(k)$

$$\rho W(k) = U(c(k)) + W_k(k) [f(k) - c(k)]. \quad (26)$$

Since equations ((5)) and ((26)) hold identically, we can differentiate them with respect to  $k$  (using the envelope theorem and (24)), to obtain

$$W_{kk}b + W_k b_k = 0 \quad (27)$$

$$\rho W_k = W_{kk} (f - c) + W_k f_k. \quad (28)$$

Using (27) and (28) to eliminate  $W_k$  and  $W_{kk}$  we can solve for the optimal consumption rule  $c^*(k)$  to obtain equation (8). Finally, using equations (24), (5), and (8) we obtain equation (7). ■

The proof of Proposition 3 uses the following lemma:

**Lemma 3** *Denote  $\tau^o(t; k_0)$  as the open-loop representation of the solution to the problem (9), and denote  $\tau^{fb}(k)$  as the feedback representation of the solution to this problem. Given Assumptions 4 and 5, and given a time-consistent policy rule  $b(k)$ , the optimal  $\tau^o(t)$ , is a constant for all initial conditions  $k_0$  if and only if the feedback form of the policy,  $\tau^{fb}(k)$  is independent of  $k$ .*

**Proof.** Taking as given the parameters  $\alpha$  and  $\gamma$ , equations (8) and (7) determine the functions  $c(k; \alpha, \gamma)$  and  $b(k; \alpha, \gamma)$ . Given these functions, the government solves an autonomous control problem. The value of its program is a function  $S(k; \alpha, \gamma)$  that satisfies H-B-J equation

$$\rho S(k) = \max_{\tau} \{U(c(k)) + V(\tau b(k)) + S_k(k) (f(k) - c(k) - \tau b(k))\}. \quad (29)$$

where we suppress the arguments  $\alpha$  and  $\gamma$ .

The solution to the government's stationary control problem is a policy rule  $\tau^{fb}(k)$ . Using this function and the agent's control rule, we can solve the equation for  $\frac{dk}{dt}$  to obtain  $k^e(t; k_0)$ , the equilibrium value of the state. ("e" denotes equilibrium) at time  $t$ . Substituting  $k^e(t; k_0)$  into the government's policy rule, we obtain the open-loop representation of the policy,  $\tau^o(t, k_0) \equiv$

$\tau^{fb}(k^e(t; k_0))$ . By Assumption 4  $\frac{dk^e}{dt} \neq 0$  along the optimal trajectory. Therefore  $\tau^o(t, k_0)$  is a constant if and only if  $\tau^{fb}(k)$  is independent of  $k$ . ■

**Proof. (Proposition 3)** In view of Lemma 3 we need to show that the feedback form of the government's control rule,  $\tau^{fb}(k)$  is independent of  $k$ . We begin with some notation, for the purpose of simplifying the derivations. Use equation (8) to write

$$f - c = -\frac{\rho - f'}{b'}b. \quad (30)$$

Define

$$\eta \equiv \frac{d}{dk} \left( \frac{\rho - f'}{b'}b \right). \quad (31)$$

With this notation we have

$$f' - c' = -\eta. \quad (32)$$

Differentiating both sides of equation (7) with respect to  $k$ , using the definition (31), implies

$$U'b' + U''[\eta + f']b = 0.$$

Rearranging this expression gives equation (11).

We now proceed with the main argument. Using equation (30), we rewrite the government problem (29) as

$$\rho S = \max_{\tau} \left\{ U + V - S' \left[ \frac{\rho - f'}{b'} + \tau \right] b \right\}. \quad (33)$$

The first order condition to (33) is

$$V'(\tau b(k)) = S'(k). \quad (34)$$

Equation (34) implicitly defines the optimal tax rule  $\tau(k)$ . Substituting this tax rule into equation (34) and taking the derivative of both sides, we obtain

$$S'' = V''(\tau b' + b\tau'). \quad (35)$$

Substituting the optimal tax rule  $\tau(k)$  into (33) we obtain the maximized H-J-B equation. We take the derivative with respect to  $k$  of both sides of the maximized H-J-B equation, using the envelope theorem. The resulting equation is

$$\rho S' = U'c' + V'\tau b' - S'(\eta + \tau b') - S'' \left( \frac{\rho - f'}{b'}b + \tau b \right). \quad (36)$$



We then use equations (32) to eliminate  $c'$ , and equations (34) and (35) to eliminate  $S'$  and  $S''$  from equation (36). The resulting equation and the identity  $\tau'(k) \equiv 0$  are both satisfied if and only if equation (10) holds identically in  $k$ . ■

**Proof. (Corollary 2)** Suppose that the function  $b(k; \alpha, \gamma)$  and the constant  $\tau$  satisfy the consistency constraint and the stationarity constraint. Define the function  $\tilde{b}(k) = \tau b(k; \alpha, \gamma)$ . Clearly  $\tilde{b}(k)$  is a solution to (7), i.e. it satisfies the consistency constraint. By construction,  $\tilde{b}(k)$  and  $\tau = 1$  satisfy the stationarity constraint. Consumption and tax revenues are the same under the two formulations. ■

**Proof. (Proposition 4)** To prove the proposition we need to show that when equations (7), (8), and (10) are satisfied, then: (i)  $c^*(k)$  satisfies the necessary conditions for a best response to  $b(k)$  and (ii)  $b(k)$  satisfies the necessary conditions for a best response to  $c^*(k)$ .

(i) The proof of the first part (sufficiency) of Proposition 2 showed that  $c^*(k)$ , defined by equation (8), is a best response to  $b(k)$  when  $b(k)$  satisfies equation (7).

(ii) Suppose that  $c^*(k)$  satisfies equation (8). The government takes  $c^*(k)$  as given; it does not recognize that the consumption rule depends on the tax rule. We show that the necessary conditions to the government's control problem, (13) is equivalent to (10).

The government's H-J-B equation is

$$\rho S = \max_b \{U(c^*(k)) + V(b) + S' [f(k) - c^*(k) - b]\} \quad (37)$$

The first order condition is

$$V'(b) = S'(k). \quad (38)$$

Recognizing that the optimal  $b$  is a function of  $k$ , we take the derivative of the first order condition to obtain

$$V''(b)b'(k) = S''(k). \quad (39)$$

We take the derivative of both sides of (37) using the envelope theorem and using equation (32) to eliminate  $f' - c^{*'}$ . The result is

$$\rho S' = U' c^{*'} - S' \eta + (f - c^* - b) S''. \quad (40)$$

Use equation (32) to eliminate  $c^*$  and use equations (38) and (39) to eliminate  $S'$  and  $S''$  from (40), and then rearrange to obtain

$$(\rho + \eta) V' = (f' + \eta)U' + (f - c - b) V''b'. \quad (41)$$

Finally, use equation (30) to eliminate  $f - c$  and rewrite equation (41) as

$$(\rho + \eta) V' = (f' + \eta)U' - (\rho - f' + b') V''b \quad (42)$$

Using the normalization  $\tau = 1$ , equations (10) and (42) are identical. ■

**Remark 1** We denote the Markov Perfect equilibrium tax rule as  $g(k)$  in order to distinguish it from the stationary time consistent policy. The H-J-B equations for the agent and the government are

$$\rho W(k) = \max_c [U(c) + W_k(k) (f(k) - g(k) - c)] \quad (43)$$

$$\rho S(k) = \max_g [U(c(k)) + V(g) + S_k(k) (f(k) - c(k) - g)]. \quad (44)$$

(For ease of comparison we denote the value functions as  $W(k)$  and  $S(k)$ , although these functions are obviously different than in the open-loop equilibrium.) Manipulation of the necessary conditions lead to the system of ODEs  $c'(k)$  and  $g'(k)$ :

$$c' = \frac{U' \rho - f' + g'}{U'' f - c - g} \quad (45)$$

$$g' = \frac{(\rho - f' + c') V' - U' c'}{V''(f - c - g)}. \quad (46)$$

For example, to obtain equation (45) we take the derivative with respect to  $k$  of the agent's first order condition  $U'(c) = W_k$  to obtain  $U''c' = W_{kk}$  and of the H-J-B equation to obtain  $\rho W_k = W_{kk} (f(k) - g(k) - c) + W_k (f' - g')$ . Using these two equations and (43) implies equation (45).

**Remark 2** For  $U = \ln c$   $V = \ln g$ , the above ODE system can be written as

$$c' = -c \frac{\rho - f' + g'}{f - c - g} \quad (47)$$

$$g' = -(c\rho - cf' + cc' - c'g) \frac{g}{c(f - c - g)}. \quad (48)$$

Some tedious manipulation of this system implies

$$\sigma(k) \equiv \frac{d \ln c}{d \ln g} = \frac{\frac{d \ln c}{dk}}{\frac{d \ln g}{dk}} = \frac{-f + c + 2g}{-f + 2c} \quad (49)$$

Solving equation (49) for  $c$  gives equation (20) in the text.

In the special case where  $f(k)$  is linear, the unique affine solution to equations (47) and (48) is the linear equilibrium:  $c = \rho k$ ,  $g = \frac{\rho}{2}k$ .<sup>8</sup> If we allow  $f(k)$  to be nonlinear, but restrict attention to linear control rules ( $c(k) = ck$ ,  $g(k) = gk$  for positive constants  $c, g$ ), equations (47), (48) imply<sup>9</sup> that  $c = 2g$  and

$$f(k) = k\mu \ln k + k\gamma, \quad (50)$$

which reproduces equation (17) in the text.

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<sup>8</sup>In order to verify this claim, substitute linear rules  $c = \chi k + \varphi$  and  $g = \phi k + \gamma$  into equations (47), (48) and equate coefficients of powers of  $k$  to obtain  $\gamma = \varphi = 0$ ,  $\chi = \rho$ ,  $\phi = \frac{\rho}{2}$ .

<sup>9</sup>To verify this claim we merely substitute the linear control rules into equations (47), (48).

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