Network of Commons

Rahmi İlkiliç

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Rahmi İlkiliç, *Department of Economics, Maastricht University*

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Summary

A tragedy of the commons appears when the users of a common resource have incentives to exploit it more than the socially efficient level. We analyze the situation when the tragedy of the commons is embedded in a network of users and sources. Users play a game of extractions, where they decide how much resource to draw from each source they are connected to. We show that if the value of the resource to the users is linear, then each resource exhibits an isolated problem. There exists a unique equilibrium. But when the users have concave values, the network structure matters. The exploitation at each source depends on the centrality of the links connecting the source to the users. The equilibrium is unique and we provide a formula which expresses the quantities at an equilibrium as a function of a network centrality measure. Next we characterize the efficient levels of extractions by users and outflows from sources. Again, the case of linear values can be broken down source by source. For the case of concave values, we provide a graph decomposition which divides the network into regions according to the availability of sources. Then the efficiency problem can be solved region by region.

Keywords: Tragedy of the Commons, Networks, Nash Equilibrium, Efficiency, Centrality Measures

JEL Classification: C62, C72, D85, Q20

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Address for correspondence:

Rahmi İlkiliç
Department of Economics
Maastricht University
6200 MD Maastricht
The Netherlands
E-mail: r.ilkilic@algec.unimaas.nl

1 Introduction

Many environmental resources which supply the basic inputs of production are owned collectively. Typical examples of such commons are clean air, carbon-dioxide levels in the atmosphere, pastures, forests, fisheries and water sources. One similarity they share is that the availability (or the fertility) of the resource decreases with use, and in some cases over exploitation may even destroy it completely.

When the individual users ignore the cost their activity imposes on the rest, "The Tragedy of The Commons" occurs. It was brought under the spotlight by Hardin (1968), but the analysis of the problem in specific contexts precedes that[1]. Although the term is used for issues relating to the use of natural resources, it is not far from a moral hazard in teams as modeled in Holmstrom (1982). It has been studied widely since Hardin's article.

In a standard model of commons, there exists a single resource exploited by many users. In real life examples, the most representative commons (e.g. pastures, forests, fisheries and water sources) are local, but numerous. Each site from which a natural resource is extracted are utilized by many, but in most of the cases the beneficiaries also have access to many such sites, which they might share with similar or different users. A lake might supply water to many cities, but cities also receive water from many lakes. A country shares fisheries with its coastal neighbors, but many countries have coasts in multiple seas and oceans. The exploitation decision of a user would be affected by the availability of sites, but also by the other users who operate on these sites. When the sources and users are interconnected, the exploitation of each user from each source will depend on the structure of the connections.

We model a bipartite network, where sources and cities are connected through links. We assume that the average cost of extraction at each source increases with the amount of total exploitation from that source. Then each user imposes a cost on all other users. We distinguish between two cases. One where the users value the resource linearly, and the other where they have concave quadratic valuations.

We look at the water extraction game, where agents decide how much to draw from each source they are connected to. They have a value from consuming the resource, but their marginal cost of extraction increases with each extra unit. We assume that at each source, agents share of the total cost is equal to her share from the total extraction. Meaning that

[1] See Gordon (1954) for a model of fisheries. Olson (1965) also alludes to the problem as it relates to collective action.

the users at the same source face the same per unit cost at that source.[2]

We show that for the case of linear values, each source exhibits an isolated tragedy of the commons. Players' actions depend on how many other users there are at each source. In the terminology of networks, only the source centered graph matters. The network effects do not permeate through paths of more than two links.

When the users' values are concave, their actions at a source does not only depend on the number of users they share it with. It also depends on the number of sources their neighbors are linked to. And also on the number of users at the sources which their neighbors are linked to. The externalities diffuse through the paths *ad infinitum*. We write the equilibrium conditions as a linear complementarity problem and show uniqueness. To interpret the equilibrium quantities, we define a centrality index (*link centrality*) that captures the spreading effects of each extraction. We provide an interpretation of this index comparing it with the Katz-Bonacich centrality (Katz (1953), Bonacich (1987)).

We next characterize the efficient amounts of extraction for both cases. The linear case can again be divided source by source. There exists a continuum of flows which give efficiency, but in all of them the outflows from the sources are equal. For concave values, the efficient amounts depend on the whole network. Generically, there exists a continuum of efficient flows, which all give the same amounts of extractions to cities and outflows to sources. To calculate these efficient amounts, we decompose the network into *regions*. Each region is a connected subgraph of the original network. They are cut out from the network, according to the ratio of sources to cities in them.

Given a network, we determine a connected subgraph such that all its cities are among the least privileged with respect to sources. The subgraph will contain all the sources that its cities are connected to in the network. The aim is to favor most the poorest in source. We give them exclusive rights to the sources they are connected to. After cutting out this subgraph from the network, we will find a similar subgraph formed by the least privileged cities in the remaining one. We continue until we reach a network where all cities are equal with respect to source availability.

We bridge two branches of economics literature. On one side we study a tragedy of the commons. In a standard model of a common pool resource (Gordon (1954), Weitzman (1974), Funaki and Yamato (1999)) multiple users exploit a single source. One user's consumption

[2]It is logical that the crowding of the source affects everyone equally. When the fish population decreases, the catch becomes difficult for everyone.

affects others identically. In this paper, we extend this basic model to a network of users and sources. The symmetry between the users is lost (except for exceptional networks like the complete network, the hub, etc.). Given a network, we show how the structure of connections determine users' extraction levels. We also characterize the socially efficient outcomes.

We do not explicitly deal with the question of management of the commons[3]. But we provide a network decomposition such that in each of the subnetworks we obtain, the problem of efficiency is equivalent to the case of one source and many users.

Although we use the metaphor of water, this paper is different from Ambec and Sprumont (2001), because the sources in our model works quite differently from the river in theirs. Moreover, we do not make any cooperative analysis of the problem.

The other related line of literature is the analysis of behavior on networks. We study a bipartite-network as in Corominas-Bosch (2004). She studies the equilibria of a bargaining game in a network of buyers and sellers. The model differs from ours in two basic points. First, both buyers and sellers are active agents, where we only take one side, the users, as strategical. Second, buyers and sellers are bargaining over a single indivisible good. In contrast, we assume that the good transferred between parties is perfectly divisible, allowing a source to supply to many users.

Ballester et al. (2006) analyzes the equilibrium activities at each node of a simple non-directed network. Players create externalities on their neighbors. A player has a single level of activity. Her payoff depends on her activity level and of her neighbors'. They show that the equilibrium levels are given by a network centrality index, which is similar to the Katz-Bonacich centrality. Ballester and Calvó-Armengol (2006) show that the first order equilibrium conditions of games which exhibit cross influences between agents' actions are linear complementarity problems. They study some interesting classes of such games which have a unique equilibrium. In both papers, the agents strategy spaces are subsets of the real line. They choose a real number and a link between two agents shows that they impose externalities on each other. In our model, agents' strategy spaces are multidimensional and a link is not only a qualitative object, but also carries a value.

The basic notation, some of which we borrow from Corominas-Bosch (2004), is introduced

[3]Seabright (1993) gives a survey of the literature on the management of the commons issue. Faysse (2005) provides a survey of game theoretical models of commons management. On the empirical side, many real life examples have been discussed in Ostrom (1991) and Ostrom et al. (1994, 2002). They also provide theoretical and empirical analysis concerning possible solutions for the tragedy of the commons.

in Section 2. Section 3 defines the payoffs and Section 4 defines the water extraction game. We study the equilibrium in section 5 and characterize the efficient outcomes in Section 6. Section 7 discusses the results. The proofs are given in Section 8.

2 Notation

There are n sources $s_1, ..., s_n$, and m cities $c_1, ..., c_m$. They are embedded in a network that links sources with cities, and cities can acquire their water from the sources they are connected to. We will represent the network as a graph.

A non-directed *bipartite graph* $g = \langle S \cup C, L\rangle$ consists of a set of *nodes* formed by sources $S = \{s_1, ..., s_n\}$, and cities $C = \{c_1, ..., c_m\}$ and a set of *links* L, each link joining a source with a city. A link from s_i to c_j will be denoted as (i, j). We say that a node s_i is *linked* to another node c_j if there is a link joining the two. We will use $(i, j) \in g$ and $(i, j) \in L$ interchangeably, meaning that s_i and c_j are connected in g.

A bipartite graph g is *connected* if there exists a path linking any two nodes of the graph. Formally, a path linking nodes s_i and c_j will be a collection of t cities and t sources, $t \geq 0$, $s_1, ...s_t, c_1, ..., c_t$ among $S \cup C$ (possibly some of them repeated) such that

$$\{(i, 1), (1, 1), (1, 2), ..., (t, t), (t, j)\} \in g$$

A *subgraph* $g_0 = \langle S_0 \cup C_0, L_0\rangle$ of g is a graph such that $S_0 \subseteq S, C_0 \subseteq C, L_0 \subseteq L$ and such that each link in L that connects a source in S_0 with a city in C_0 is a member of L_0. Hence a node of g_0 will continue to have the same links it had with the other nodes in g_0. We will write $g_0 \subseteq g$ to mean that g_0 is a subgraph of g.

For a subgraph g_0 of g, we will denote by $g - g_0$, the subgraph of g *that results when we remove the set of nodes* $S_0 \cup C_0$ from g. $g - g_0$ will be defined as the maximal connected parts of the subgraph induced by the set of nodes $(S - S_0) \cup (C - C_0)$.

Given a subgraph $g_0 = \langle S_0 \cup C_0, L_0\rangle$ of g, let $\overleftrightarrow{g_0}$ be the complete bipartite graph with nodes $S_0 \cup C_0$. We call $\overleftrightarrow{g_0}$ *the completed graph of* g_0.

$N_g(s_i)$ will denote the set of cities linked with s_i in $g = \langle S \cup C, L\rangle$, more formally:

$$N_g(s_i) = \{c_j \in C \text{ such that } (i, j) \in g\}$$

and similarly $N_g(c_j)$ stands for the set of sources linked with c_j.

For a set A, let $|A|$ denote the number of elements in A. For s_i in S, we denote $|N_g(s_i)|$ by $m_i(g)$. Similarly for $c_j \in C$, let $|N_g(c_j)| = n_j(g)$, be the number of sources connected to c_j.

An *invasive subgraph* $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ *of* g is such that g_0 is connected and,

$$S_0 = \bigcup_{c_j \in C_0} N_g(c_j)$$

An invasive subgraph includes all the sources to which its cities were connected in graph g. We will denote by $W(g) = \{g_0 \subseteq g : g_0 \text{ is invasive}\}$ the set of invasive subgraphs in g. $W(g) \neq \emptyset$ as g is an invasive subgraph of itself. In the network g_1 in Figure 1, the subgraph g_1^0 that we encircle is invasive. It includes c_1 and all the sources that c_1 is connected to.

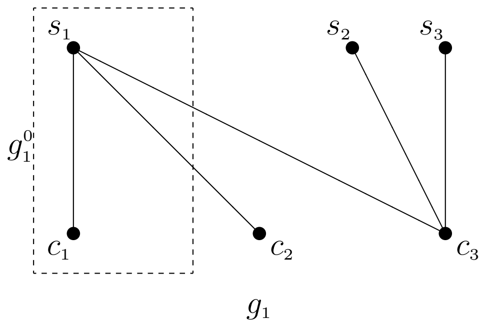


Figure 1

Given a subset of sources $S_0 \subseteq S$ and a subset of cities $C_0 \subseteq C$, $\frac{|S_0|}{|C_0|}$ is the average number of sources per city. A *minimally invasive subgraph* $\widehat{g} = \left\langle \widehat{S} \cup \widehat{C}, \widehat{L} \right\rangle$ *of* g is such that

$$\frac{\left|\widehat{S}\right|}{\left|\widehat{C}\right|} < \frac{|S|}{|C|} \text{ and } \left\langle \widehat{S} \cup \widehat{C}, \widehat{L} \right\rangle \in \underset{\langle S_0 \cup C_0, L_0 \rangle \in W(g)}{\operatorname{argmin}} \frac{|S_0|}{|C_0|}$$

The first requirement for $\widehat{g}$ to be a minimally invasive subgraph of g is for it to have a strictly smaller source/city ratio than g. This means that a graph does not necessarily have a minimally invasive subgraph. For example a complete bipartite graph has no minimally invasive subgraphs. Any invasive subgraph cut out from a complete graph will have a source/city ratio at least as big as the complete graph.

The second requirement is for $\widehat{g}$ to have the smallest source/city ratio among the invasive graphs of g. A minimally invasive subgraph is invasive and formed by a set of least connected cities. There should be no cities in g which are strictly worse than them with respect to source availability.

In Figure 1, the subgraph g_1^0 is not minimally invasive, because the ratio of source to cities in it is 1. But this ratio for the graph g_1 is lower than 1. The subgraph g_1^1 of g_1, as encircled Figure 2 below, is a minimally invasive subgraph. Its source/city ratio is lower than that of g_1, and there is no other subgraph of g_1 with a lower ratio.

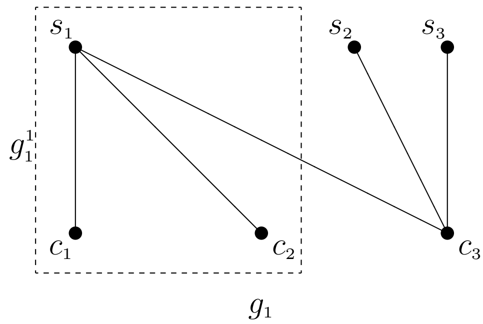


Figure 2

If $\widehat{g}$ is a minimally invasive subgraph of g, then $\widehat{g}$ cannot have a minimally invasive subgraph of its own. Any invasive subgraph of $\widehat{g}$ is also invasive in g. If $\widehat{g}$ had a minimally invasive subgraph with a smaller source/city ratio than $\widehat{g}$, this would have contradicted $\widehat{g}$ having the smallest source/city ratio in g.

We denote by $q_{ij} \geq 0$ the amount of water extracted by city c_j from source s_i.

2.1 Labelling of pairs (i,j)

Let $\tau : \{1, ..., n\} \times \{1, ..., m\} \to \mathbb{N}_+$, be a lexicographic order on $\{1, ..., n\} \times \{1, ..., m\}$ such that:

(*i*) $\tau(1, 1) = 1$,

(*ii*) $(i, j) \neq (k, l) \Rightarrow \tau(i, j) \neq \tau(k, l)$,

(*iii*) $j < l \Rightarrow \tau(i, j) < \tau(k, l)$ for all $i, k \in \{1, .., n\}$,

(*iv*) $i < k \Rightarrow \tau(i, j) < \tau(k, j)$ for all $j \in \{1, .., m\}$,

(*v*) if $\exists(i, j)$ such that $\tau(i, j) = y > 1$ then $\exists(k, l)$ s.t. $\tau(k, l) = y - 1$.

τ orders all possible links such that the links of a city j are assigned a lower number than any city i, for $i > j$, and the links of a city is ordered according to the indices of the sources they come from. For example for 2 cities and 2 sources, the function τ orders the links starting from the first city, and the first source, $\tau(1,1) = 1$. The second ranked link is between the first city and the second source, $\tau(2,1) = 2$. Now, as all links of city c_1 is ranked, τ will next rank the link between c_2 and s_1, $\tau(1,2) = 3$. Next comes the link between city 2 and source 2, $\tau(2,2) = 4$.

For a network g, let $Y(g) = \{y \in \mathbb{N}_+ : y = \tau(i,j) \text{ for some } (i,j) \notin g\}$ be the set of indices that τ assigns to links which are not in g. Assume, without loss of generality that $|Y(g)| = m \times n - r(g)$, for some $1 \leq r(g) \leq m \times n$, where $r(g)$ is the number of links in graph g. For 2 cities and 2 sources, for a graph g, if the only missing link is $(1,2)$, then $Y(g) = \{3\}$ and $r(g) = 3$.

Observe that τ orders all possible links, independent of g, where as $Y(g)$ does depend on g.

We can see how the above definitions work on an example. Suppose that 2 cities and 2 sources, form a completely connected bipartite graph g_2. For graph g_2, $Y(g_2) = \emptyset$.

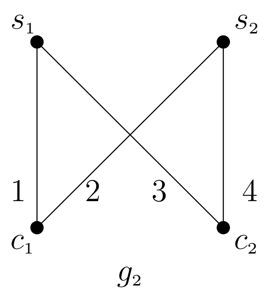


Figure 3

Now we cut the link between c_2 and s_1, to obtain g_3.

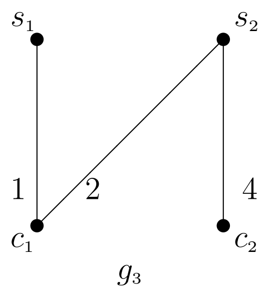


Figure 4

Although link $(1,2)$ does not exist in g_3 it is still labelled equally by τ. $\tau(1,2) = 3$, meaning that $Y(g_3) = \{3\}$.

We will make use of graphs g_2 and g_3 in many examples throughout the paper.

2.2 Some useful matrices

Now we define some matrices which we will use during our analysis.

For $\beta, \gamma \geq 0$, let $A = [a_{ij}]_{n\times n}$ be such that,

$$a_{ij} = \begin{cases} 2\beta + \gamma, & \text{for } i = j \\ \gamma \quad , & \text{for } i \neq j \end{cases}$$

A has $2\beta + \gamma$ on the diagonal and γ off the diagonal.

$$A = \begin{bmatrix} 2\beta + \gamma & & & & \\ & \cdot & & \gamma & \\ & & \cdot & & \\ & \gamma & & \cdot & \\ & & & & 2\beta + \gamma \end{bmatrix}_{n\times n}$$

Let $B = \beta I_{n\times n}$, where $I_{n\times n}$ is the identity matrix of size n. Using matrices A and B, we construct the partitioned matrix $D = [d_{ij}]_{(m\times n)\times(m\times n)}$ such that:

$$D = \begin{bmatrix} A & & & & \\ & \cdot & & B & \\ & & \cdot & & \\ & B & & \cdot & \\ & & & & A \end{bmatrix}_{(m\times n)\times(m\times n)}$$

D has matrix A on its diagonal and matrix B off the diagonal. If we want to write it term by term,

$$d_{ij} = \begin{cases} 2\beta+\gamma, \text{ for } i=j \\ \gamma \quad , \text{ for } i \neq j, \text{ s.t. } (i,j) = (z_1 n + z_2, z_1 n + z_3) \text{ for } z_1, z_2, z_3 \in \mathbb{N} \\ \qquad \text{s.t. } z_2 \neq z_3, \ 1 \leq z_2, z_3 \leq n-1 \text{ and } z_1 \leq m-1 \\ \beta \quad , \text{ for } i \neq j, \text{ s.t. } i+j = (1+z_1)n + 1 + 2z_2, \text{ for } z_1, z_2 \in \mathbb{N} \\ \qquad \text{s.t. } z_1 \leq m-1, z_2 \leq m \\ 0 \quad , \text{ otherwise} \end{cases}$$

For example for 2 cities and 2 sources,

$$D_{4\times4} = \begin{pmatrix} 2\beta+\gamma & \gamma & \beta & 0 \\ \gamma & 2\beta+\gamma & 0 & \beta \\ \beta & 0 & 2\beta+\gamma & \gamma \\ 0 & \beta & \gamma & 2\beta+\gamma \end{pmatrix}$$

The interpretation, when we use it to find the equilibrium quantities flowing from sources to cities, will be that the column z and the row z in D corresponds to the link (i,j) in g such that $\tau(i,j) = z$. Hence, column 1 and row 1 corresponds to the link $(1,1)$, column 2 and row 2 corresponds to the link $(2,1)$, column 3 and row 3 corresponds to the link $(1,2)$, and column 4 and row 4 corresponds to the link $(2,2)$.

Let D_{-j} be the matrix obtained by deleting row j and column j from D. For $J \subset \mathbb{N}_+$, let D_{-J} be the matrix obtained by deleting each row $j \in J$ and column $j \in J$ from D. We will denote $D_{-Y(g)}$ by D_g. We obtain D_g by deleting each row $y \in Y(g)$ and column $y \in Y(g)$ from D. These rows and columns belong to links that are not in g. Then, D_g has size $r(g) \times r(g)$.

For g_2, since $Y(g_2) = \emptyset$, $D_{g_2} = D_{4\times4}$. For g_3, as $Y(g_3) = \{3\}$, D_{g_3} is formed by taking out the third column and third row of $D_{4\times4}$.

$$D_{g_3} = \begin{bmatrix} 2\beta+\gamma & \gamma & 0 \\ \gamma & 2\beta+\gamma & \beta \\ 0 & \beta & 2\beta+\gamma \end{bmatrix}$$

Let $\overline{B} = 2B$ be the matrix obtained from B by multiplying it with the scalar 2. Similarly we construct the partitioned matrix $F = [f_{ij}]_{(m\times n)\times(m\times n)}$ such that:

$$F = \begin{bmatrix} A & & & \\ & \cdot & \overline{B} & \\ & & \cdot & \\ & \overline{B} & & \cdot \\ & & & & A \end{bmatrix}_{(m\times n)\times(m\times n)}$$

F has matrix A on its diagonal and matrix $\overline{B}$ off the diagonal. If we want to write it term by term,

$$f_{ij} = \begin{cases} 2\beta + \gamma, & \text{for } i = j \\ \gamma, & \text{for } i \neq j, \text{ s.t. } (i,j) = (z_1 n + z_2, z_1 n + z_3) \text{ for } z_1, z_2, z_3 \in \mathbb{N} \\ & \quad \text{s.t. } z_2 \neq z_3,\ 1 \leq z_2, z_3 \leq n-1 \text{ and } z_1 \leq m-1 \\ 2\beta, & \text{for } i \neq j, \text{ s.t. } i + j = (1+z_1)n + 1 + 2z_2, \text{ for } z_1, z_2 \in \mathbb{N} \\ & \quad \text{s.t. } z_1 \leq m-1, z_2 \leq m \\ 0, & \text{otherwise} \end{cases}$$

For example for 2 cities and 2 sources,

$$F_{4\times 4} = \begin{bmatrix} 2\beta+\gamma & \gamma & 2\beta & 0 \\ \gamma & 2\beta+\gamma & 0 & 2\beta \\ 2\beta & 0 & 2\beta+\gamma & \gamma \\ 0 & 2\beta & \gamma & 2\beta+\gamma \end{bmatrix}$$

Similarly, let F_{-j} be the matrix obtained by deleting row j and column j from F. Let $\mathbb{N}_+$ be the set of positive integers. For $J \subset \mathbb{N}_+$, let F_{-J} be the matrix obtained by deleting each row $j \in J$ and column $j \in J$ from F. We will denote $F_{-Y(g)}$ by F_g. We obtain F_g by deleting each row $y \in Y(g)$ and column $y \in Y(g)$ from F. These rows and columns belong to links that are not in g. Then, F_g has size $r(g) \times r(g)$.

For g_1, since $Y(g_2) = \emptyset$, $F_{g_2} = F_{4\times 4}$. For g_3, as $Y(g_3) = \{3\}$, F_{g_3} is formed by taking out the third column and third row of $F_{4\times 4}$.

$$F_{g_2} = \begin{bmatrix} 2\beta+\gamma & \gamma & 0 \\ \gamma & 2\beta+\gamma & 2\beta \\ 0 & 2\beta & 2\beta+\gamma \end{bmatrix}$$

Before going further we show that for $\beta, \gamma > 0$, D_{-J} is positive definite, and F_{-J} is positive semi-definite for any $J \subset \mathbb{N}_+$.

Proposition 1 *For $\beta, \gamma > 0$, D_{-J} is positive define for any $J \subset \mathbb{N}_+$.*

Proposition 2 *For $\beta, \gamma > 0$, F_{-J} is positive semi-definite for any $J \subset \mathbb{N}_+$.*

Now we define the column vector that shows the quantities flowing at each link. Let $Q = [e_z]$ be the column vector of quantities extracted such that for q_{ij}, the quantity extracted from source s_i by c_j, $e_{\tau(i,j)} = q_{ij}$. For 2 cities and 2 sources:

$$Q = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{12} \\ q_{22} \end{bmatrix}$$

Let Q_{-j} be the vector obtained by deleting row j from Q. For $J \subset \mathbb{N}_+$, let Q_{-J} be the vector obtained deleting each row $j \in J$ and column $j \in J$ from Q. For $Y(g) \subset \mathbb{N}$, let Q_g be the matrix obtained by deleting each row $y \in Y(g)$ from Q. Then Q_g has size r. Q_g is the link by link profile of extractions. For the two graphs given above:

$$Q_{g_1} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{12} \\ q_{22} \end{bmatrix} \qquad Q_{g_2} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{22} \end{bmatrix}$$

Let $\mathbb{Q}_{(m \times n)}$ be the set of all non-negative real valued column vectors of size $(m \times n)$. Let $\mathbb{Q}_r$be the set of all non-negative real valued column vectors of size r.

Given a vector of flows Q_g, for a city c_j, we will denote by $E_j(Q_g)$ the total amount extracted by c_j. For a source s_i we will denote by $O_i(Q_g)$ the total outflow from s_i.

3 Payoffs

We will assume that the utility function of players are additively separable into value and cost of extraction. Hence, for a given $Q_g \in \mathbb{Q}_r$,

$$u_j(Q_g) = v_j(E_j(Q_g)) - \sum_{s_i \in N_g(c_j)} T_{ij}(Q_g),$$

where $v_j(Q_g)$ is the value obtained from consuming $E_j(Q_g)$ and $T_{ij}(Q_g)$ is the cost of extraction by c_j from source s_i. We will use quadratic value and cost functions, which will decrease the computational load and help us focus on the effects of the network structure on the equilibrium quantities.

We will assume quadratic costs of extraction, which is uniform for all sources. Hence, for $\beta > 0$ the total cost of extraction from a given source s_i would be

$$T_i(Q_g) = \beta(O_i(Q_g))^2$$

We assume that each player pays her share of the cost proportional to her extraction. The cost of extraction q_{ij} by c_j from s_i would be

$$T_{ij}(Q_g) = \beta q_{ij}(O_i(Q_g))$$

We will assume uniform cost functions among sources, but the results would continue to hold as long as the costs are quadratic at each source.

We will analyze two cases, with two different value functions.

3.1 Linear Values

For $\alpha, \beta > 0$, let

$$u_j(Q_g) = \alpha E_j(Q_g) - \beta \sum_{s_i \in N_g(c_j)} q_{ij}\left[O_i(Q_g)\right]$$

for all cities $c_j \in C$. As the value function is linear, the utility is separable with respect to each source

$$u_j(Q_g) = \sum_{s_i \in N_g(c_j)} q_{ij}[\alpha - \beta O_i(Q_g)]$$

Then, for all $c_j \in C$ and for all $s_i \in N_g(c_j)$, the marginal utility to c_j of extraction from s_i is:

$$\frac{\partial u_j}{\partial q_{ij}} = \alpha - 2\beta q_{ij} - \beta \sum_{c_k \in N_g(s_i) \setminus \{c_j\}} q_{ik}$$

The marginal utility at extraction q_{ij} depends only on the other levels of extraction at source s_i.

3.2 Concave Values

For $\alpha, \beta, \gamma > 0$, let

$$\widetilde{u}_j(Q_g) = \alpha E_j(Q_g) - \frac{\gamma}{2}\left(E_j(Q_g)\right)^2 - \beta \sum_{s_i \in N_g(c_j)} q_{ij}\left(O_i(Q_g)\right)$$

for all cities $c_j \in C$. Now, the utility is not separable with respect to each source. For all $c_j \in C$ and for all $s_i \in N_g(c_j)$, the marginal utility to c_j of extraction from s_i is:

$$\frac{\partial \widetilde{u}_j}{\partial q_{ij}} = \alpha - (2\beta + \gamma) q_{ij} - \gamma \sum_{s_l \in N_g(c_j)\setminus\{s_i\}} q_{lj} - \beta \sum_{c_k \in N_g(s_i)\setminus\{c_j\}} q_{ik}$$

Neither the marginal utilities are separable source by source. The marginal utility at q_{ij} does depend on the amounts extracted by c_j from sources other than s_i.

4 The Water Extraction Game

Given a network g, each city c_j maximizes its utility by extracting a non-negative amount of water through its links from the sources in $N_g(c_j)$. So, the set of players are the set of cities C. The set of strategies of a city c_j is $\mathbb{Q}_j = \mathbb{Q}_{N_g(c_j)}$. We denote a representative strategy of c_j by $Q_j \in \mathbb{Q}_j$. Given that there are $r(g)$ links in g, the strategy space of the game is $\mathbb{Q}_g = \prod\limits_{c_j \in C} \mathbb{Q}_j = \mathbb{Q}_{r(g)}$.

For each city j, in the water extraction game with linear values, we will assume that each player has utility $u_j(Q_g)$. Then a best response Q'_j of city c_j to $Q_g \in \mathbb{Q}_g$ is such that,

$$\text{for all links } (i,j),\ q'_{ij} = \begin{cases} \frac{\alpha - \beta \sum\limits_{c_k \in N_g(s_i)\setminus\{c_j\}} q_{ik}}{2\beta} & \text{, if } \frac{\partial u_j}{\partial q_{ij}}|_{Q_g} \geq 0 \\ 0 & \text{, if } \frac{\partial u_j}{\partial q_{ij}}|_{Q_g} < 0 \end{cases}$$

In the water extraction game with concave values, we assume their utility to be $\widetilde{u}_j(Q_g)$. Then a best response Q'_j of city c_j to $Q_g \in \mathbb{Q}_g$ is such that,

$$\text{for all links } (i,j),\ q'_{ij} = \begin{cases} \frac{\alpha - \gamma \sum\limits_{s_l \in N_g(c_j)\setminus\{s_i\}} q_{lj} - \beta \sum\limits_{c_k \in N_g(s_i)\setminus\{c_j\}} q_{ik}}{2\beta + \gamma} & \text{, if } \frac{\partial u_j}{\partial q_{ij}}|_{Q_g} \geq 0 \\ 0 & \text{, if } \frac{\partial u_j}{\partial q_{ij}}|_{Q_g} < 0 \end{cases}$$

5 The Equilibrium

5.1 Linear Values

For the linear case, the first order condition for q_{ij} does not depend on the amounts extracted from sources other than s_i. Then we can separate the optimization problem source by source. Meaning that equilibrium extractions from a source s_i depend only on how many players are connected to s_i.

Theorem 3 *Water extraction with linear values has a unique Nash equilibrium, such that for any link* (i,j) *the equilibrium flow* $q^*_{ij} = \frac{\alpha}{(m_i(g)+1)\beta}$

Example Suppose we have the graph g_2. Let $\alpha = \beta = 1$. Then the equilibrium flows of the water extraction game are $q^*_{11} = q^*_{21} = q^*_{12} = q^*_{21} = \frac{1}{3}$.

Suppose the graph was g_3. Now, at equilibrium $q^*_{11} = \frac{1}{2}$, and $q^*_{21} = q^*_{22} = \frac{1}{3}$. So, the deletion of the link $(1,2)$ does not change the extraction levels on source s_2.

5.2 Concave Values

We will write the equilibrium conditions of the water extraction game with concave values as a linear complementarity problem. Given a matrix $M \in \mathbb{R}^{t\times t}$ and a vector $p \in \mathbb{R}^t$, the linear complementarity problem $LCP(p;M)$ consists of finding a vector $z \in \mathbb{R}^t$ satisfying:

$$\begin{aligned} z &\geq 0, \\ p + Mz &\geq 0, \\ z^T(p+Mz) &\geq 0 \end{aligned}$$

Given a graph g, the first order equilibrium conditions of the game define a $LCP(-\alpha\mathbf{1}_r; D_g)$ where $\mathbf{1}$ is a column vector of 1's of size $r(g)$.[4]

[4] The water extraction game with linear values also forms a linear complementarity problem. But it is simpler to find equilibrium flows source by source in that case, rather than work with matrices derived from the network structure.

$$\begin{aligned} Q_g &\geq 0, \\ -\alpha\mathbf{1}_r + D_g Q_g &\geq 0, \\ Q_g^T(q + D_g Q_g) &\geq 0 \end{aligned}$$

Samelson *et al.* (1958) shows that a linear complementarity problem $LCP(p; M)$ has a unique solution for all $p \in \mathbb{R}^t$ if and only if all the principal minors of M are positive. Positive definite matrices satisfy this condition and we showed in Proposition 1 that D_g is positive definite. Then the equilibrium conditions have a unique solution.

We further check for the second order conditions for each agent, which reveals that the solution of the $LCP(-\alpha\mathbf{1}_r; D_g)$ is indeed the equilibrium of the game.

Theorem 4 *Water extraction with concave values has a unique Nash equilibrium.*

Example Suppose we have the graph g_2. Let $\alpha = \beta = \gamma = 1$. Then the link flows at equilibrium are $q^*_{11} = q^*_{21} = q^*_{12} = q^*_{22} = 0.2$.

Suppose the graph was g_3. Now at equilibrium, $q^*_{11} = 0.2857$, $q^*_{21} = 0.1429$, and $q^*_{22} = 0.2857$. Under concave values of extraction, the deletion of the link $(1, 2)$ does change the extraction levels on source s_2, and moreover city c_1 extracts less from the source she shares with city c_2.

5.2.1 The Equilibrium Quantities

Let Q^*_g be an equilibrium of the water extraction game with concave values. There might be some links in g, such that they carry zero flow at equilibrium Q^*_g. Marginal utilities of extractions from those links need not be zero at Q^*_g.

$$\begin{aligned} q^*_{ij} &> 0 \Rightarrow \frac{\partial \widetilde{u}_j}{\partial q_{ij}} = 0 \\ q^*_{ij} &= 0 \Rightarrow \frac{\partial \widetilde{u}_j}{\partial q_{ij}} \leq 0 \end{aligned}$$

To calculate the equilibrium quantities, first we need to weed out the links with zero flow. Let $\rho : L \to \mathbb{N}_+$ be a lexicographic order on L respecting τ such that ρ relabels the (i, j)

pairs from 1 to $r(g)$ by skipping those links which are not in g.[5] Now we delete from Q_g^*, the entries that correspond to links with no flow.

Let $Z\left(Q_g^*\right) = \{z \in \mathbb{N}_+ : z = \rho(i,j) \text{ for some } (i,j) \text{ s.t. } q_{ij}^* = 0\}$. Let $\left|Z\left(Q_g^*\right)\right| = t^*$, then $Q^*_{g-Z\left(Q_g^*\right)}$ is a vector of size $r(g) - t^*$ obtained from Q_g^* by deleting the zero entries. It is the vector of equilibrium quantities for links over which there is a strictly positive flow from a source to a city.

Let's remember the first order conditions. For all $(i,j) \in g$,

$$\frac{\partial \widetilde{u}_j}{\partial q_{ij}} = \alpha - (2\beta + \gamma) q_{ij} - \gamma \sum_{s_l \in N_g(c_j)\setminus\{s_i\}} q_{lj} - \beta \sum_{c_k \in N_g(s_i)\setminus\{c_j\}} q_{ik} = 0$$

Then for any equilibrium Q_g^* of the water extraction game with concave values,

$$D_{g-Z(Q_g^*)}.Q^*_{g-Z(Q_g^*)} = \alpha.\mathbf{1}$$

where $\mathbf{1}$ is a column vector of 1's of size $r(g) - t^*$.

Given a network g let Q_g^* be the equilibrium at g. Then we denote by $g - Z(Q_g^*)$ the network obtained from g by deleting the links which have zero flow at Q_g^*.

Theorem 5 *Given two networks g and g'. Let Q_g^* and $Q_{g'}^*$ be the equilibrium of the water extraction game with concave values in g and g', respectively. If $g - Z(Q_g^*) = g' - Z(Q_{g'}^*)$, then $Q^*_{g-Z(Q_g^*)} = Q^*_{g'-Z(Q_{g'}^*)}$.*

At equilibrium there might be links which carry no flows. For the cities of such links, the marginal utilities of extraction from them are not positive. They are indifferent between having such a link or not. Theorem 5 tells us such links with zero flows play no role while

[5] Explicitly, $\rho : L \to \mathbb{N}_+$ is such that:

(i) $\exists (i,j) \in L$ such that $\rho(i,j) = 1$,

(ii) $(i,j) \neq (k,l) \Rightarrow \rho(i,j) \neq \rho(k,l)$,

(iii) $j < l \Rightarrow \rho(i,j) < \rho(k,l)$ for all $(i,j),(k,l) \in L$,

(iv) $i < k \Rightarrow \rho(i,j) < \rho(k,j)$ for all $(i,j),(k,j) \in L$,

(v) if $\exists (i,j) \text{s.t.} \rho(i,j) = z > 1$ then $\exists (k,l) \in L$ s.t. $\rho(k,l) = y - 1$.

determining equilibrium. They are strategically redundant. Let's see how this result an example. Take graph g_1.

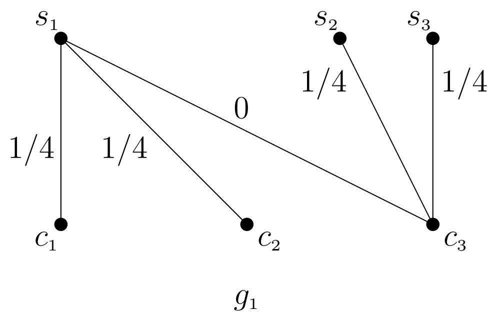


Figure 5

Let $\alpha = \beta = \gamma = 1$. Then for g_1,

$$D_{g_1} = \begin{bmatrix} 3 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

The first order equilibrium conditions form a linear complementarity problem $LCP(-\mathbf{1}_5; D_{g_1})$, where $\mathbf{1}_5$ is the vector of 1's of size 5. Then the link flows at equilibrium are $q^*_{11} = q^*_{12} = \frac{1}{4}$, $q^*_{13} = 0$ and $q^*_{23} = q^*_{33} = \frac{1}{4}$. The cities c_1 and c_2 have no other connections except s_1. At equilibrium the marginal cost of extraction from source s_1 is higher than s_2 and s_3. The difference is too large for city c_3 to make any profitable use of the link $(1, 3)$. The links that carry positive flows give zero marginal utility to their users.

$$D_{g_1 - Z(Q^*_{g_1})} . Q^*_{g_1 - Z(Q^*_{g_1})} - \mathbf{1}_4 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Actually, the equilibrium flows on active links can also be obtained with a matrix operation.

$$
\begin{aligned}
D_{g_1 - Z(Q^*_{g_1})} . Q^*_{g_1 - Z(Q^*_{g_1})} &= \mathbf{1}_4 \\
\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} q^*_{11} \\ q^*_{12} \\ q^*_{23} \\ q^*_{33} \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
\left(3I_{4\times 4} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right) Q^*_{g_1 - Z(Q^*_{g_1})} &= \mathbf{1}_4
\end{aligned}
$$

If we let

$$
G_{g_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

Then,

$$
Q^*_{g_1 - Z(Q^*_{g_1})} = \frac{1}{3} \left[I_{4\times 4} - \left(\frac{1}{3} G_{g_1} \right)^2 \right]^{-1} \left[I_{4\times 4} - \frac{1}{3} G_{g_1} \right]
$$

Now we cut the link $(1,3)$ and denote the new graph by $g_1 - (1,3)$.

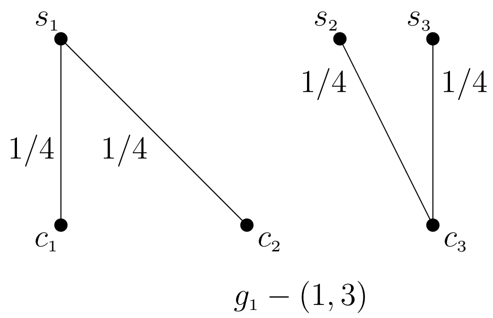


Figure 6

For $\alpha = \beta = \gamma = 1$, Theorem 5 tells us that the flows at equilibrium are $q^*_{11} = q^*_{12} = \frac{1}{4}$ and $q^*_{23} = q^*_{33} = \frac{1}{4}$. At the equilibrium in g_1, the marginal utility to city c_3 from extraction via $(1,3)$ was negative. Deleting it does not change the equilibrium quantities on other links, because the marginal utility on them is the same as in graph g_1.

We can generalize the marginal utility argument used in this example. It will help us give a network interpretation for the flow quantities at equilibrium $Q^*_{g-Z\left(Q^*_g\right)}$ on any given graph g.

5.2.2 Decomposition of $D_{g-Z(Q^*_g)}$

As $D_{g-Z\left(Q^*_g\right)}$ is a symmetric matrix, whose diagonal entries are $2\beta+\gamma$, and non-diagonal entries are either $0, \gamma$, or β we can separate it into $(2\beta+\gamma)I$, and a symmetric matrix G^*, where I is the identity matrix of size $r(g)-t^*$. For example for graph g_3 all links have positive flows at equilibrium. Then,

$$G^*_{g_3} = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & \beta \\ 0 & \beta & 0 \end{bmatrix}$$

For any graph g, G^* has diagonal entries as 0 and non-diagonal entries are either $0, \gamma$ or β. G^* can be interpreted as the weighted adjacency matrix of the network obtained from g, where the active links (i,j) of g with $q^*_{ij} > 0$ at Q^*_g are the vertices, and the cities and sources in g are the edges. An edge in the derived matrix has weight β if it is a source and weight γ if it is a city. From now on we will call G^* *the equilibrium dual of* g. We will use it to denote both the graph derived from g as explained above and the adjacency matrix of that graph.

Hence,

$$\begin{aligned} D_{g-Z(Q^*_g)}.Q^*_{g-Z(Q^*_g)} &= \left[(2\beta+\gamma)I + G^*\right].Q^*_{g-Z(Q^*_g)} \\ &= (2\beta+\gamma)\left[I + aG^*\right].Q^*_{g-Z(Q^*_g)} \end{aligned}$$

where $a = \frac{1}{2\beta+\gamma}$. Remember that Q^*_g is the solution to $LCP(-\alpha\mathbf{1}_r; D_g)$. Then, when we invert $D_{g-Z(Q^*_g)}$, the matrix multiplication $\alpha.\left[D_{g-Z(Q^*_g)}\right]^{-1}\mathbf{1}$ will give us a strictly positive vector. Now, for $a \geq 0$,

$$\begin{aligned}
[I + aG^*] &= [I - aG^*]^{-1}\left[I - (aG^*)^2\right] \\
[I + aG^*]^{-1} &= \left[I - (aG^*)^2\right]^{-1}[I - aG^*] \\
&\text{and} \\
\left[I - (aG^*)^2\right]^{-1} &= \sum_{k=0}^{\infty}(aG^*)^{2k}
\end{aligned}$$

Substituting this into $D_{g-Z(Q_g^*)}.Q^*_{g-Z(Q_g^*)} = \alpha.\mathbf{1}$,

$$\begin{aligned}
Q^*_{g-Z(Q_g^*)} &= a\alpha\left[I - (aG^*)^2\right]^{-1}[I - aG^*].\mathbf{1} \\
&= a\alpha\sum_{k=0}^{\infty}(aG^*)^{2k}[I - aG^*].\mathbf{1} \\
&= a\alpha\left[\sum_{k=0}^{\infty}(aG^*)^{2k}.\,\mathbf{1} - \sum_{k=0}^{\infty}(aG^*)^{2k+1}.\mathbf{1}\right]
\end{aligned}$$

The last expression is a centrality measure for the network with adjacency matrix G^*. Although it is not a standard centrality index, we can understand it by comparing it with a known one. For $a \geq 0$, and a network adjacency matrix G^*, let

$$M(G^*, a) = [I - aG^*]^{-1} = \sum_{k=0}^{\infty}(aG^*)^k$$

If $M(a, G^*)$ is non-negative, its entries $m_{ij}(G^*, a)$ counts the number of paths in the network, starting at i and ending at j, where paths of length k are weighted by a^k.

Definition 1 *For a network adjacency matrix G, and for scalar $a > 0$ such that $M(G, a) = [I - aG]^{-1}$ is well-defined and non-negative, the vector Katz-Bonacich centralities of parameter a in G is:*

$$\boldsymbol{b}(G, a) = [I - aG^*]^{-1}.\mathbf{1}$$

In a graph with z nodes, the Katz-Bonacich centrality of node i,

$$b_i(G, a) = \sum_{j=1}^{z} m_{ij}(G, a)$$

counts the total number of paths in G starting from i.

Using the Katz-Bonacich centrality as a benchmark, let's define the link centrality of a network of commons g.

Definition 2 *For scalars $\alpha, a \geq 0$, a network of commons g and its equilibrium dual G^*, such that $[I + aG^*]^{-1}$ is well-defined and non-negative, the vector link centralities of in g is:*

$$\boldsymbol{l}(G, a) = a\alpha \left[I + aG^*\right]^{-1} .\mathbf{1}$$

Hence, in the expression

$$a\alpha \left[\sum_{k=0}^{\infty}(aG^*)^{2k}.\,\mathbf{1} - \sum_{k=0}^{\infty}(aG^*)^{2k+1}.\mathbf{1}\right]$$

the first summation counts the total number of even paths that start from the corresponding node in G^*, and the second summation counts the total number of odd paths that start from it.

The first sum tells that the equilibrium extraction from a link is positively related with the number of even length paths that start from it. The links which have an even distance between them are strategical complements. In contrast, the negative sign on the second summation means the equilibrium extraction from a link is negatively related with the number of odd length paths that start from it. The links which have an odd distance between them are strategical substitutes.

For example, in graph g_2,

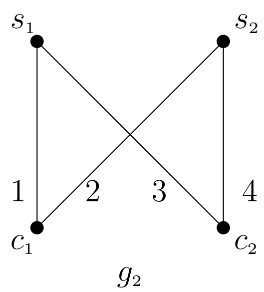


Figure 7

links $(1, 1)$ and $(2, 2)$ are strategical complements. The extraction from source s_2 by city c_2 increases incentives for city c_1 to extract more from source s_1, because the former increases the marginal cost on s_2. This makes s_1 a better option. Links $(1, 1)$ and $(2, 1)$ are strategical substitutes, because extraction from one decreases the marginal value of water to city c_1. This decreases city's incentives to extract more.

In general, the links of a city are substitutes for each other (e.g. $(1, 1)$ and $(2, 2)$ at graph g_1). Similarly, the links of a source are substitutes for each other, too (e.g. $(1, 1)$ and $(1, 2)$

at graph g_1). If two cities are sharing a source, then their links to sources they don't share are complements (e.g. $(1,1)$ and $(2,2)$ at graph g_1). Moreover, if a link (i_1, j_1) is a strategic substitute of a link (i_2, j_2) and (i_2, j_2) is a strategic substitute of (i_3, j_3), then (i_1, j_1) and (i_3, j_3) are strategic complements. Therefore, the strategic effect depends on the parity of the distance between two links.

In the water extraction game the adjacency matrix G^* does not necessarily have binary entries, neither its non-zero entries are all equal. Each link in G^* has a weight. While counting the number of paths, these weights are taken into account as well. The extraction by a city c_j is calculated by summing up the link centralities of the elements in $N_g(c_j)$.

6 The Efficient Extraction

We will assume that cities have comparable and identical utilities. Such an assumption is not far fetched from reality, in particular for the many commons which are not end products.[6] Indeed in most setups, commons receive their value from being a productive input for firms that supply to a market (Weitzman (1974), Funaki and Yamato(1999)).

6.1 Linear Values

When cities value water linearly, the sum of their utilities is

$$U = \sum_{c_j \in C} u_j(Q_g) = \alpha \sum_{(i,j)\in g} q_{ij} - \beta \sum_{s_i \in S} (O_i(Q_g))^2$$

Then the first order condition implies that at an efficient vector of flows Q_g^e ,for any source s_i, $O_i(Q_g^e) = \frac{\alpha}{2\beta}$.

As the values are linear, it does not matter to whom the water goes, as long as no source's total outflow exceeds the efficient amount.

Example Suppose we have the graph g_1. Let $\alpha = \beta = 1$. Then the efficient flows are

$$\{q_{11}^e, q_{21}^e, q_{12}^e, q_{21}^e \geq 0 : q_{11}^e + q_{12}^e = \frac{1}{2} \text{ and } q_{21}^e + q_{22}^e = \frac{1}{2}\}$$

[6]Though such a comparison lacks sense for commons which are imperative for their users. To compare a catastrophically dehydrated city with a well provided one would be inacceptable, both in economic and ethical terms.

Suppose the graph was g_2. Now, the efficient flows are

$$\{q^e_{11}, q^e_{21}, q^e_{21} \geq 0 : q^e_{11} = \frac{1}{2} \text{ and } q^e_{21} + q^e_{22} = \frac{1}{2}\}$$

There exists a continuum of flows which give an efficient outcome in both graphs. All the efficient flows lead to the same amounts of outflows from sources.

6.2 Concave Values

When cities have concave values, the network structure determines the efficient levels of extraction in a non trivial fashion. The sum of utilities is

$$\widetilde{U} = \sum_{c_j \in C} \widetilde{u}_j(Q_g) = \alpha \sum_{(i,j) \in g} q_{ij} - \frac{\gamma}{2} \sum_{c_j \in C} (E_j(Q_g))^2 - \beta \sum_{s_i \in S} (O_i(Q_g))^2$$

Then the first order condition that an efficient vector of flows Q^e_g has to satisfy is,

$$\text{for all } (i,j) \in g \quad \begin{cases} \text{if } q^e_{ij} \neq 0, \text{ then } \alpha = \gamma E_j(Q^e_g) + 2\beta O_i(Q^e_g) \\ \text{if } q^e_{ij} = 0, \text{ then } \alpha < \gamma E_j(Q^e_g) + 2\beta O_i(Q^e_g) \end{cases}$$

Hence given a city c_j, and 2 different sources $s_i, s_k \in N_g(c_j)$

$$\begin{aligned} q^e_{ij}, q^e_{kj} &\neq 0 \implies O_i(Q^e_g) = O_k(Q^e_g) \\ q^e_{ij} &= 0 \text{ and } q^e_{kj} \neq 0 \implies O_i(Q^e_g) > O_k(Q^e_g) \end{aligned}$$

Similarly, given a source s_i, and 2 different cities $c_j, c_k \in N_g(s_i)$

$$\begin{aligned} q^e_{ij}, q^e_{ik} &\neq 0 \implies E_j(Q^e_g) = E_k(Q^e_g) \\ q^e_{ij} &= 0 \text{ and } q^e_{ik} \neq 0 \implies E_j(Q^e_g) > E_k(Q^e_g) \end{aligned}$$

Observe that this is also a linear complementarity problem with $LCP(-\alpha\mathbf{1}_r; F_g)$. But F_g is positive semi-definite, and not necessarily positive definite. We are not guaranteed a unique solution. Indeed, we will see that, in general, there exists a continuum of solutions to the problem of efficient flows. To solve it, we first characterized the first order conditions above and now we look at the Hessian of the sum $\widetilde{U}$. The Hessian matrix of $\widetilde{U}$ is so that $H_{\widetilde{U}} = -F_g$. As F_g is positive semi-definite, $H_{\widetilde{U}}$ is negative semi-definite. Meaning that any Q^e_g that satisfies the first order conditions maximizes $\widetilde{U}$.

Example Suppose we have the graph g_2. Let $\alpha = \beta = \gamma = 1$. Observe that g_2 has no minimally invasive subgraphs. Indeed, it is complete. Then the efficient flows are

$$\{q^e_{11}, q^e_{21}, q^e_{12}, q^e_{21} \geq 0 : q^e_{11} + q^e_{12} = \frac{1}{3}, q^e_{21} + q^e_{22} = \frac{1}{3}, q^e_{11} + q^e_{21} = \frac{1}{3} \text{ and } q^e_{12} + q^e_{22} = \frac{1}{3}\}$$

There exists a continuum of flows which give an efficient outcome. The total extractions at each city and the total outflows at each source are the same for all the efficient flow levels.

Now we will find a vector of extractions that satisfies the first order conditions of efficiency. Given a subgraph $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ of g, consider the efficient amount of extractions and outflows in its completed graph $\overleftrightarrow{g_0}$. Clearly the levels are identical across cities and across sources. Let $\overleftrightarrow{E_0}$ be the efficient amount of total extraction by a city in $\overleftrightarrow{g_0}$ and $\overleftrightarrow{O_0}$ the efficient amount of total outflow from a source in $\overleftrightarrow{g_0}$. If $|S_0| = n_0$ and $|C_0| = m_0$, then direct calculation shows that

$$\overleftrightarrow{E_0} = \frac{\alpha n_0}{\gamma n_0 + 2\beta m_0} \text{ and } \overleftrightarrow{O_0} = \frac{\alpha m_0}{\gamma n_0 + 2\beta m_0}$$

These values depend only on the source/city ratio. For two graphs $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ and $g_1 = \langle S_1 \cup C_1, L_1 \rangle$,

$$\frac{|S_0|}{|C_0|} = \frac{|S_1|}{|C_1|} \Rightarrow \overleftrightarrow{E_0} = \overleftrightarrow{E_1} \text{ and } \overleftrightarrow{O_0} = \overleftrightarrow{O_1}$$

We will use the efficient levels of the complete graph as benchmarks while calculating the efficient amounts in non-complete bipartite graphs.

The feasible set of flows in a graph g_0 is a subset of the feasible set of flows in its completed graph $\overleftrightarrow{g_0}$. Then given efficient levels of extraction $\overleftrightarrow{E_0}$ and outflow $\overleftrightarrow{O_0}$ at $\overleftrightarrow{g_0}$, if these amounts are possible in g_0, then they must be efficient for g_0 also.

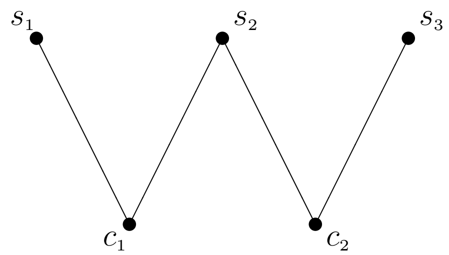
Proposition 6 *Let $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ be a connected subgraph of g. If the extraction of $\overleftrightarrow{E_0}$ by each city in C_0 is possible without exceeding the outflow $\overleftrightarrow{O_0}$ in any source in S_0, then these levels are efficient in g_0.*

Now we show that if a subgraph $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ of g has no minimally invasive subgraph, then the extraction of $\overleftrightarrow{E_0}$ by each city in C_0 is possible without exceeding the outflow $\overleftrightarrow{O_0}$ in any source in S_0

Proposition 7 *Let $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ of g be an invasive subgraph. If g_0 has no minimally invasive subgraphs, then the extraction of $\overleftrightarrow{E_0}$ by each city in C_0 is possible without exceeding the outflow $\overleftrightarrow{O_0}$ in any source in S_0.*

We prove Proposition 7 by induction. We start with a city c_j of a graph g_0 with no invasive subgraphs. This city must be able to extract $\overleftrightarrow{E_0}$, without exceeding the outflow $\overleftrightarrow{O_0}$ in any of its sources. If not, that city with its sources would have formed a minimally invasive subgraph in g_0. Next, we add a new city to this subgraph and iteratively show that such extractions must be possible for all invasive subgraphs of g_0 that contain c_j. As g_0 is an invasive subgraph of itself, this proves that such extractions are possible in g_0.

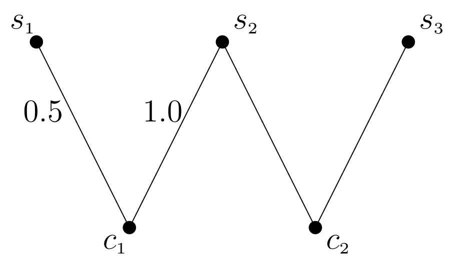
To prove the iteration we manipulate the flows in the following way. Let $\alpha = \frac{7}{2}, \beta = 1$ and $\gamma = 1$. Suppose that our graph is g_4 in Figure 5.



g_4

Figure 8

Then according to Proposition 7, extraction 1.5 by each city is possible without exceeding outflow 1.0 in any source. Let's take c_1. Let's take the vector of flows $(q_{11}, q_{21}) = (0.5, 1)$. Then c_1 extracts 1.5 without exceeding 1 at any of its sources. Let's depict those flows on the graph, by writing the quantities that correspond to each link.



g_4

Figure 9

In Figure 6, s_2 supplies 1.0 to c_1. To extend the argument to the subgraph that contains c_2, we manipulate the flows, so that the slack in source s_1 can be transferred to city c_2 through the path that connects s_1 with c_2. Such a change of flows should be possible, because if not, we could find a minimally invasive subgraph, which leads to a contradiction.

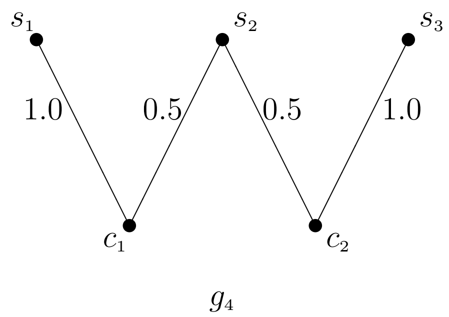


Figure 10

6.3 Decomposing the network

Now we will break down the given network g, so that the commons problem in each subnetwork is independent from the other ones. We will sequentially cut out minimally invasive subgraphs. Hence, they will not have any minimally invasive subgraphs of their own. We will continue until we reach a subgraph which has no minimally invasive subgraphs. Then, given Proposition 7, in each subgraph, the efficient amounts of total extractions at each city and total outflows at each source will be equal to the efficient amounts in their completed graphs.

Step 1: Take g. Suppose $g = \langle S \cup C, L \rangle$ has no minimally invasive subgraph. Then the efficient total extraction by a city c_j, $E_j(Q_g^e)$, and the efficient total extraction from a source s_i,$O_i(Q_g^e)$, is equal to the extraction of a city in a complete bipartite graph with nodes $S \cup C$, and we are done.

Suppose $g = \langle S \cup C, L \rangle$ has a minimally invasive subgraph $g_0 = \langle S_0 \cup C_0, L_0 \rangle$. Then, the efficient total extraction by a city $c_j \in C_0$ is $\overleftrightarrow{E_0}$, and the efficient total extraction from a source $s_i \in S_0$ is $\overleftrightarrow{O_0}$.

Step 2: Now, for the rest of the cities and sources apply *Step 1* to $g - g_0$.

In this way we will obtain a series of *regions* $g_0, g_1...$ of g, with a non-decreasing source per city ratio. In each of them, the efficient levels of extractions would equal to the levels in their respective completed graphs.

So, given a subgraph $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ obtained from the above decomposition, the efficient extraction by a city in g_0 is

$$\overleftrightarrow{E_0} = \frac{\alpha n_0}{\gamma n_0 + 2\beta m_0}$$

and the efficient outflow from each source in g_0 is

$$\overleftrightarrow{O_0} = \frac{\alpha m_0}{\gamma n_0 + 2\beta m_0}$$

These levels satisfy the first order conditions within each region. Moreover, less resourceful regions have lower amounts of extractions per city and higher amounts of outflows per source. Since there are no flows between different regions the first order conditions hold for graph g as well.

The idea of redundant links reappears while calculating efficiency. Take two graphs g and g' such that their decomposition gives the same subgraphs. The efficient amounts of total extractions at each city and total outflows at each source are the same for both of them.

Example Suppose we have the graph g_1. Let $\alpha = \beta = \gamma = 1$. The decomposition would give us two regions, g_1^1 and $g_1 - g_1^1$. Then the efficient flows are

$$\{q_{11}^e, q_{12}^e, q_{13}^e, q_{23}^e, q_{33}^e \geq 0 : q_{11}^e = \frac{1}{5}, q_{12}^e = \frac{1}{5}, q_{13}^e = 0, q_{23}^e = \frac{1}{4} \text{ and } q_{33}^e = \frac{1}{4}\}$$

Suppose the graph was $g_1 - (1,3)$. The decomposition leads to the same subgraphs. The efficient flows are

$$\{q_{11}^e, q_{12}^e, q_{23}^e, q_{33}^e \geq 0 : q_{11}^e = \frac{1}{5}, q_{12}^e = \frac{1}{5}, q_{23}^e = \frac{1}{4} \text{ and } q_{33}^e = \frac{1}{4}\}$$

The link $(1,3)$ is redundant from an efficiency point of view, just as it was for equilibrium.

7 Discussion

We have analyzed a situation where the tragedy of the commons is embedded in a network. We have shown that when players have concave valuations, their equilibrium actions will depend on the whole structure. The quantity extracted by a user from a source depends on the centrality of the links she has. The centrality index which determines the quantities is calculated using the equilibrium dual of the original network. Then the quantity flowing from a resource to a city is positively proportional to the total number of even paths and negatively proportional to the total number of odd paths starting from it.

We next characterize the efficient amounts of extractions. Again when players have concave valuations, these amounts depend on the whole network. We find a network decomposition which separates the efficiency problem into subgraphs. These subgraphs, which we call regions, are taken out from the network starting with the one with the lowest source/city ratio. Each region consumes only from its sources. The sources are distributed between regions, so that the less resourceful ones are assigned to the most possible number of sources.

The model we studied can also be used to analyze Cournot competition among firms which are linked through markets. If we think of cities as firms with quadratic costs, and sources as markets with linear demands, the results in this paper shows what the equilibrium quantities would be in such a setup. The efficiency in our story would be equivalent to the profit maximization of a cartel that the suppliers might form.

A further research agenda would be the case where the sources, which can be thought as exporters of the resources, behave strategically as well. Their strategies can be prices they charge and/or quantities they sell through each of their links. The users would be the consumers of the market, buying according to the prices charged. Such a model would be a close approximation of the international petrol and natural gas markets.

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Appendix

Proof of Proposition 1 We first show that for the matrix D we can find a matrix R with independent columns such that $D = R^T R$.[7] We will write columns of R so that the entries in D appear in squareroots in R. For example, let's take $D_{4\times 4}$:

$$D_{4\times 4} = \begin{bmatrix} 2\beta+\gamma & \gamma & \beta & 0 \\ \gamma & 2\beta+\gamma & 0 & \beta \\ \beta & 0 & 2\beta+\gamma & \gamma \\ 0 & \beta & \gamma & 2\beta+\gamma \end{bmatrix}$$

We write R as

$$R = \begin{bmatrix} \sqrt{\beta} & 0 & 0 & 0 \\ 0 & \sqrt{\beta} & 0 & 0 \\ 0 & 0 & \sqrt{\beta} & 0 \\ 0 & 0 & 0 & \sqrt{\beta} \\ \sqrt{\gamma} & \sqrt{\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{\gamma} & \sqrt{\gamma} \\ \sqrt{\beta} & 0 & \sqrt{\beta} & 0 \\ 0 & \sqrt{\beta} & 0 & \sqrt{\beta} \end{bmatrix}$$

Then clearly $D_{4\times 4} = R^T R$. Now, we generalize this to all possible D.

Let $R = [r_{ij}]_{[3(m\times n)]\times(m\times n)}$ be such that,

$$r_{ij} = \begin{cases} \sqrt{\beta} & , \text{ for } i = j \\ \sqrt{\frac{\gamma}{n}}, & \text{for } i \neq j, \text{ s.t. } (i,j) = (z_1 n + (m\times n) + z_2, z_1 n + z_3) \text{ for } z_1, z_2, z_3 \in \mathbb{N} \\ & \quad \text{s.t. } 1 \leq z_2, z_3 \leq n \text{ and } z_1 \leq m-1 \\ \sqrt{\frac{\beta}{m}}, & \text{for } i \neq j, \text{ s.t. } (i,j) = (z_2 m + 2(m\times n) + z_1 + 1, z_3 n + z_2 + 1), \text{ for } z_1, z_2, z_3 \in \mathbb{N} \\ & \quad \text{s.t. } z_1, z_3 \leq m-1, \text{and } z_2 \leq n-1 \\ 0 & , \text{ otherwise} \end{cases}$$

If we let $K = \sqrt{\beta} I_{(m\times n)\times(m\times n)}$, $L = \sqrt{\frac{\gamma}{n}} \mathbf{1}_{n\times n}$, where $\mathbf{1}_{n\times n}$ is the square matrix of 1's of size n, and we define $M = [m_{ij}]_{(m\times n)\times n}$ such that,

[7] This is equivalent to checking that D is positive definite. For other characterizations of positive definete-ness see Strang (1988).

$$m_{ij} = \begin{cases} \sqrt{\frac{\beta}{m}} & \text{, for } (i,j) = ((z_1 m + z_2, z_1 + 1), \text{ for } z_1, z_2 \in \mathbb{N} \\ & \quad \text{s.t. } z_1 \leq n-1, \text{and } 1 \leq z_2 \leq m \\ 0 & \text{, otherwise} \end{cases}$$

Then R can be written as a partitioned matrix,

$$R = \begin{bmatrix} & K & & \\ L & & & \\ & . & 0 & \\ & & . & \\ & 0 & & . \\ & & & L \\ M & & \dots & M \end{bmatrix}_{[3(m \times n)] \times (m \times n)} ,$$

As K is a diagonal matrix of size $m \times n$, the row space of R has dimension $m \times n$, meaning that the column space also has dimension $m \times n$. Then the columns of R are independent. It is straight forward to check that $D = R^T R$.

Now, we show that D_{-J} can be proven to be positively definite in a similar way. For example, let's take D_{g3}:

$$D_{g3} = \begin{bmatrix} 2\beta + \gamma & \gamma & 0 \\ \gamma & 2\beta + \gamma & \beta \\ 0 & \beta & 2\beta + \gamma \end{bmatrix}$$

We write R_{g3} as

$$R_{g3} = \begin{bmatrix} \sqrt{\beta} & 0 & 0 \\ 0 & \sqrt{\beta} & 0 \\ 0 & 0 & \sqrt{\beta} \\ \sqrt{\gamma} & \sqrt{\gamma} & 0 \\ 0 & 0 & \sqrt{\gamma} \\ 0 & \sqrt{\beta} & \sqrt{\beta} \\ \sqrt{\beta} & 0 & 0 \end{bmatrix}$$

Then clearly $D_{g3} = (R_{g3})^T R_{g3}$. Now, we generalize this to all possible D_{-J}.

For any $J \subset \mathbb{N}_+$, we can find a matrix R_J with independent columns such that $D_{-J} = (R_J)^T R_J$. D_{-J} has dimension $m \times n - |J|$. For any $J \subset \mathbb{N}_+$, there exists a bipartite graph

$g(J)$ such that D_{-J} is derived by deleting from D the columns and rows that correspond to the links which are missing in $g(J)$. Now, let $R_J = [t_{ij}]_{[3(m\times n-|J|)]\times(m\times n-|J|)}$ be such that,

$$r_{ij} = \begin{cases} \sqrt{\beta} & , \text{ for } i = j \\ \sqrt{\frac{\gamma}{n_{z_1}(g)}} , & \text{for } i \neq j, \text{ s.t. } (i,j) = (\sum\limits_{0\leq k<z_1} n_k(g) + (m \times n - |J|) + z_2, \sum\limits_{0\leq k<z_1} n_k(g) + z_3) \\ & \text{for } z_1, z_2, z_3 \in \mathbb{N} \text{ s.t. } 1 \leq z_2, z_3 \leq n_{z_1}(g) \text{ and } 1 \leq z_1 \leq m \\ \sqrt{\frac{\beta}{m_{z_1}(g)}} , & \text{for } i \neq j, \text{ s.t. } (i,j) = (\sum\limits_{0\leq k<z_1} m_k(g) + z_2 + 2(m \times n - |J|), \sum\limits_{k=0}^{k=z_3} m_{z_3}(g) + 1), \\ & \text{for } z_1, z_2, z_3 \in \mathbb{N} \text{ s.t. } 1 \leq z_1, z_3 \leq m, \text{and } 1 \leq z_2 \leq m_{z_1}(g) \\ 0 & , \text{ otherwise} \end{cases}$$

If we let $K_J = \beta I_{(m\times n-|J|)(m\times n-|J|)}$, for $i \in \{1, ..., m\}$. $L_i = \sqrt{\frac{\gamma}{n_i(g)}}\mathbf{1}_{n_i(g)}$, where $\mathbf{1}$ is the square matrix of 1 And for $k \in \{1, ..., m\}$, we define $\left[m_{ij}^k\right]_{(m\times n-|J|)\times n_k(g)}$ such that,

$$m_{ij}^k = \begin{cases} \sqrt{\frac{\beta}{m_{z_1}(g)}} & , \text{ for } i \neq j, \text{ s.t. } (i,j) = ((\sum\limits_{0\leq k<z_1} m_k(g) + z_2, z_1 + 1), \text{ for } z_1, z_2 \in \mathbb{N} \\ & \text{s.t. } 1 \leq z_1 \leq m, \text{and } 1 \leq z_2 \leq m_{z_1}(g) \\ 0 & , \text{ otherwise} \end{cases}$$

Then R_J can be written as a partitioned matrix,

$$R_J = \begin{bmatrix} & K_J & & \\ L_1 & & & \\ & . & 0 & \\ & & . & \\ & 0 & & . \\ & & & L_m \\ M^1 & & ... & M^k \end{bmatrix}_{[3(m\times n-|J|)]\times(m\times n-|J|)}$$

As K_J is a diagonal matrix of size $m \times n - |J|$, the row space of R_J has dimension $m \times n - |J|$, meaning that the column space also has dimension $m \times n - |J|$. Then the columns of R_J are independent. It is straight forward to check that $D_{-J} = (R_J)^T R_J$. ∎

Proof of Proposition 2 For any $J \subset \mathbb{N}_+$, we can find a matrix R_J such that $F_{-J} = (R_J)^T R_J$. F_{-J} has dimension $m \times n - |J|$. For any $J \subset \mathbb{N}_+$, there exists a bipartite graph $g(J)$ such that F_{-J} is derived by deleting from F the columns and rows that correspond to the links which are missing in $g(J)$. Now, let $R_J = [t_{ij}]_{[3(m\times n-|J|)]\times(m\times n-|J|)}$ be such that,

$$r_{ij} = \begin{cases} \sqrt{\frac{\gamma}{n_{z_1}(g)}} & \text{, for } i \neq j, \text{ s.t. } (i,j) = (\sum\limits_{0\leq k<z_1} n_k(g) + (m \times n - |J|) + z_2, \sum\limits_{0\leq k<z_1} n_k(g) + z_3) \\ & \quad \text{for } z_1, z_2, z_3 \in \mathbb{N} \text{ s.t. } 1 \leq z_2, z_3 \leq n_{z_1}(g) \text{ and } 1 \leq z_1 \leq m \\ \sqrt{\frac{2\beta}{m_{z_1}(g)}} & \text{, for } i \neq j, \text{ s.t. } (i,j) = (\sum\limits_{0\leq k<z_1} m_k(g) + z_2 + 2(m \times n - |J|), \sum\limits_{k=0}^{k=z_3} m_{z_3}(g) + 1), \\ & \quad \text{for } z_1, z_2, z_3 \in \mathbb{N} \text{ s.t. } 1 \leq z_1, z_3 \leq m, \text{and } 1 \leq z_2 \leq m_{z_1}(g) \\ 0 & \text{, otherwise} \end{cases}$$

If we let $L_i = \sqrt{\frac{\gamma}{n_i(g)}} \mathbf{1}_{n_i(g)}$, where $\mathbf{1}$ is the square matrix of 1 And for $k \in \{1, ..., m\}$, we define $\left[m_{ij}^k\right]_{(m\times n-|J|)\times n_k(g)}$ such that,

$$m_{ij}^k = \begin{cases} \sqrt{\frac{2\beta}{m_{z_1}(g)}} & \text{, for } i \neq j, \text{ s.t. } (i,j) = ((\sum\limits_{0\leq k<z_1} m_k(g) + z_2, z_1 + 1), \text{ for } z_1, z_2 \in \mathbb{N} \\ & \quad \text{s.t. } 1 \leq z_1 \leq m, \text{and } 1 \leq z_2 \leq m_{z_1}(g) \\ 0 & \text{, otherwise} \end{cases}$$

Then R_J can be written as a partitioned matrix,

$$R_J = \begin{bmatrix} & 0 & \\ L_1 & & \\ & . \quad 0 & \\ & . & \\ & 0 \quad . & \\ & & L_m \\ M^1 & ... & M^k \end{bmatrix}_{[3(m\times n-|J|)]\times(m\times n-|J|)}$$

It is straight forward to check that $F_{-J} = (R_J)^T R_J$. ■

Proof of Theorem 1 As the first order condition for q_{ij} does not depend on the amounts extracted from sources other than s_i, the optimization problem is separable source by source. Meaning that equilibrium extractions from a source s_i depend only on how many players are connected to s_i.

The equilibrium at each source is unique, because for all players at a source s_i,

$$q_{ij} = \frac{\alpha - \beta \sum_{c_k \in N_g(s_i) \setminus \{c_j\}} q_{ik}}{2\beta}$$

For all links $(i,j) \in L$,

$$\frac{\partial^2 u_j}{\partial q_{ij}^2} = -2\beta < 0$$

meaning that the second order conditions are satisfied for all cities.

Around each source s_i there is a symmetric amount of flow at equilibrium, such that

$$q_{ij}^* = \frac{\alpha}{(m_i(g)+1)\,\beta}$$

■

Proof of Theorem 2 Given a graph g, the equilibrium conditions of the game is a $LCP(-\alpha \mathbf{1}_r; D_g)$ where $\mathbf{1}$ is a column vector of 1's of size r.

$$\begin{aligned} Q_g &\geq 0, \\ -\alpha \mathbf{1}_r + D_g Q_g &\geq 0, \\ Q_g^T(q + D_g Q_g) &\geq 0 \end{aligned}$$

Samelson *et al.* (1958) shows that a linear complementarity problem $LCP(p; M)$ has a unique solution for all $p \in \mathbb{R}^t$ if and only if all the principal minors of M are positive. Positive definite matrices satisfy this condition and we showed in Proposition 1 that D_g is positive definite. Then the first order equilibrium conditions have a unique solution.

Let's check the second order condition. For city c_k, denote the Hessian matrix of u_k by $H_{u_k} = [h_{ij}]_{n_k(g) \times n_k(g)}$ such that

$$h_{ij=} \begin{cases} -2\beta - \gamma, & \text{for } i = j \\ -\gamma \quad , & \text{for } i \neq j \end{cases}$$

We will show that for any $z \in \mathbb{N}_+$, the matrix $H_z = [h_{ij}]_{z\times z}$ such that

$$h_{ij} = \begin{cases} -2\beta - \gamma, & \text{for } i = j \\ -\gamma \quad , & \text{for } i \neq j \end{cases}$$

is negative definite.

$H_z = -(2\beta+\gamma)H_z^{'}$, where $H_z' = [h_{ij}']_{z\times z}$ such that,

$$h_{ij}' = \begin{cases} 1, & \text{for } i = j \\ \phi, & \text{for } i \neq j \end{cases}, \text{ where } \phi = \frac{\gamma}{2\beta+\gamma}$$

If we denote the determinant of H_z by $Det(H_z)$.

$$Det(H_z) = (2\beta+\gamma)(-1)^n Det(H_z')$$

Now, we show by induction that for all $z \in \mathbb{N}_+$, $Det(H_z') > 0$.

$Det(H_1') = 2\beta + \alpha > 0$. Assume $Det(H_{z-1}') > 0$.

$$Det\begin{pmatrix} 1 & \phi & . & . & . & \phi \\ \phi & 1 & . & . & . & . \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \phi & \phi & . & . & . & 1 \end{pmatrix} = (1 - \frac{\phi^2(n-1)}{1+(n-2)\phi})Det(A_{z-1}')$$

Then, H_z is negative definite and the water extraction game with concave values has a unique Nash equilibrium. ■

Proof of Theorem 3 Assume $Q^*_{g-Z(Q_g)}, Q^*_{g-Z(Q_g'')}$ are equilibria of the game at g and g', respectively. Let

$$g - Z(Q_g^*) = g' - Z(Q_{g'}^*)$$

Then,

$$D_{g-Z(Q_g^*)}.Q^*_{g-Z(Q_g^*)} = \alpha.\mathbf{1} = D_{g'-Z(Q_{g'}^*)}.Q^*_{g'-Z(Q_{g'}^*)} = D_{g-Z(Q_g^*)}.Q^*_{g'-Z(Q_{g'}^*)}$$

As we showed in Proposition 1 $D_{g-Z(Q_g^*)}$ is positive definite, hence invertible.

$$Q^*_{g-Z(Q_g)} = Q^*_{g-Z(Q_g'')}$$

■

Proof of Proposition 3 We know that the extraction of $\overleftrightarrow{E_0}$ and the outflow $\overleftrightarrow{O_0}$ satisfies the first order conditions in $\overleftrightarrow{g_0}$. Since g_0 and $\overleftrightarrow{g_0}$ have the same set of nodes, they also satisfy the conditions in $g_{0.}$ ■

Proof of Proposition 4 By assumption, g_0 has no minimally invasive subgraphs.

Take a city c_j in g_0. Let c_j extract a total of $\overleftrightarrow{E_0}$, such that none of the sources supply more than $\overleftrightarrow{O_0}$. $\overleftrightarrow{E_0}$ and $\overleftrightarrow{O_0}$ are functions of the source/city ratio. If c_j is not linked to enough sources to achieve such an extraction, then city c_j and the sources $N_g(c_j)$ form a minimally invasive subgraph in g_0, which is a contradiction with g_0 having no minimally invasive subgraphs.

Now, we are going to show by induction that $\overleftrightarrow{E_0}$ extraction by a city in g_0 such that no source supplies more than $\overleftrightarrow{O_0}$ is possible in any invasive subgraph of g_0 that contains c_j. As g_0 is an invasive subgraph of itself, this will imply that such levels of extraction is possible in g_0.

We know that it is possible for the invasive subgraph with city c_j and the sources $N_g(c_j)$. Take an invasive subgraph g_{k-1} of g_0 that contains $k-1$ cities including c_j. Suppose that such levels of extractions are possible in g_{k-1}. Denote by $Q_{g_{k-1}}$ such a possible amount of flows in g_{k-1}.

Now take an invasive subgraph g_k of g_0 that contains k cities, $k-1$ which were in g_{k-1} and a fixed city c_k which was not in g_{k-1}.

Assume that in g_k, $\frac{|\hat{S}_k|}{|\hat{C}_k|} < \frac{|\hat{S}|}{|\hat{C}|}$. Then g_k is a minimally invasive subgraph of g_0, which is a contradiction.

Then, $\frac{|\hat{S}_k|}{|\hat{C}_k|} \geq \frac{|\hat{S}|}{|\hat{C}|}$. Take $Q_{g_{k-1}}$ which delivers $\overleftrightarrow{E_0}$ to all cities in g_{k-1}. As g_k contains g_{k-1} we can supply the cities in g_{k-1} with $\overleftrightarrow{E_0}$ without exceeding outflow $\overleftrightarrow{O_0}$ in any source. Now let c_k extract through its links such that the outflow from each source in $N_g(c_k)$ is $\overleftrightarrow{O_0}$. If the total extraction of c_k exceeds $\overleftrightarrow{E_0}$, then we are done.

If not, denote by Q^1 the flow vector for g_k such that flows for the links which were already in g_{k-1} equals to $Q_{g_{k-1}}$, and the flows for the links which were not in g_{k-1} equals to 0. Now, given that $c_k \notin g_{k-1}$, let Q^2 be the flow vector for g_k such that

$$\begin{aligned} q^2_{jk} &= \overleftrightarrow{O_0} - O_i(Q^1), \text{ for } j \in N_g(c_k) \\ q^2_{jl} &= q^1_{jl}, \text{ for } l \neq k \end{aligned}$$

Since $\frac{|\hat{S}_k|}{|\hat{C}_k|} \geq \frac{|\hat{S}|}{|\hat{C}|}$, there must be a source s_i in g_k not connected to c_k, such that its outflow in Q^2 is strictly less than $\overleftrightarrow{O_0}$. Let S_k^- be the set of sources in g_k which not connected to c_k and which have outflows in Q^2 strictly less than $\overleftrightarrow{O_0}$.

$$S_k^- = \{s_i \in g_k : s_i \notin N_g(c_k) \text{ and } O_i(Q^2) < \overleftrightarrow{O_0}\}$$

Suppose that for any source $s_i \in S_k^-$ and for all paths

$$P = \{(s_i, c_1), (c_1, s_1), ..., (c_t, s_t), (s_t, c_k)\}$$

that connects s_i with c_k, $\exists (c_j, s_j) \in P$ such that $q_{jj}^2 = 0$. Given such a path P, let s_P denote the source s_l such that $(c_l, s_l) \in P$, $q_{ll}^2 = 0$ and there exists no other source s_j in P, closer to c_k than s_l such that $(c_j, s_j) \in P$ and $q_{jj}^2 = 0$. Let $\overline{C}_k = \{c_j \in g_k : \exists$ a path P from s_i to $c_k$for some $s_i \in S_k^-$ and in P, c_j is between s_P and $c_k\}$. Then the invasive subgraph with cities $\overline{C}_k \cup c_k$is minimally invasive in g_k, which is a contradiction.

Then there exists a source $s_i \in S_k^-$ such that there exists a path

$$P = \{(s_i, c_1), (c_1, s_1), ..., (c_t, s_t), (s_t, c_k)\}$$

that connects $s_i$with c_k and $\min_{(c_j, s_j) \in P} q_{jj}^2 \neq 0$. Let

$$d = \min_{(c_j, s_j) \in P} \{q_{jj}^2, O_i(Q^2)\}$$

Now, given such a path P, let Q^3 be the flow vector for g_k such that

$$\begin{aligned}
q_{i1}^3 &= q_{i1}^2 + d, \\
q_{jj}^3 &= q_{jj}^2 - d, \\
q_{j(j+1)}^3 &= q_{j(j+1)}^2 + d \\
q_{tk}^3 &= q_{tk}^2 + d \\
q_{ll'}^3 &= q_{ll'}^2, \text{ for all other links } (l, l')
\end{aligned}$$

It is possible to make c_k extract at least $\overleftrightarrow{E_0}$ by finding such paths from sources in $\hat{S}_k^-$ to c_k and changing the flows as explained above for each path from a source in $\hat{S}_k^-$ to c_k. If after using all such paths, c_k could still not extract $\overleftrightarrow{E_0}$, then we could use the reasoning above to get a contradiction.

Then the desired levels of extractions are possible in g_0. ■

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