Divergent Time Scales in a Coupled Ecological-Economic Model of Regional Growth *

Yong Chen

Ph.D. Candidate, Department of Agricultural, Environmental and Development Economics,
Ohio State University, 2120 Fyffe Rd., Columbus, OH 43210

Ciriyam Jayaprakash

Professor, Department of Physics, Ohio State University,
West Woodruff Ave., Columbus, Ohio 43210

and

Elena G. Irwin

Associate Professor, Department of Agricultural, Environmental and Development Economics,
Ohio State University, 2120 Fyffe Rd., Columbus, OH 43210

Selected Paper prepared for presentation at the American Agricultural Economics Association Annual Meeting, Orlando, FL, July 27-29, 2008.

April 29, 2008

*This research is supported by the National Science Foundation Biocomplexity Program and NOAA/Ohio Sea Grant Program. We are grateful for helpful discussion and feedback from Mario Miranda, Keith Bedford, David Culver, Jay Martin, Eric Roy, Alan Randall, Keith Warren and other associated with the Ohio State University Biocomplexity Project.

Abstract

This paper establishes a coupled human-ecological model where slow-varying migration is interacting with fast-varying nutrient dynamics in lake ecology. The nonlinearity and fast-slow dynamics built in the model can generate regime shifts (that is, shifts between different equilibrium states) and slowly-reversible ecological changes. Because ecological conditions do affect and are affected by uncoordinated individual decisions on migration and land-use, the policy challenge does not only lie in the optimal use of ecological service but also in the provision of the right incentives that regulates individual behavior. The possibility of regime shifts and slowly-reversible changes in this coupled model makes policy analysis more interesting and technically challenging. Within this framework, this paper shows that specification of relative time scale between the fast and slow dynamic processes is crucial for the analysis of the system dynamics with/without policy intervention. The calculated solutions show that specification of relative time scale can significantly change the cost, magnitude and length of active intervention in optimal policy. This paper shows that optimal policy (even when resilience does not enter into optimization problem) will always increase the resilience of the desirable equilibrium in the coupled system. The extent of this improvement in resilience depends crucially on the relative time scale. It also shows that simplifying assumptions on the relative time scale can lead to incorrect predictions for both the short- and long-run dynamics.

Keywords: time scale/fast-slow dynamics, human-environment interaction, natural amenity, resilience, dynamic optimization, non-convex dynamics

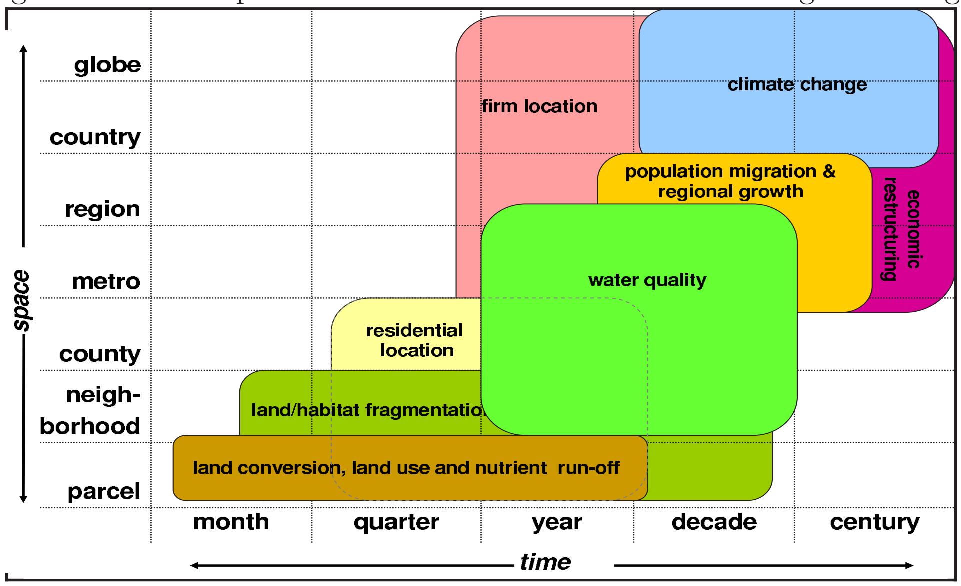
1 Introduction

Recent integration of ecological and economic models (Arrow et al [2], [3], [16], Levin et al [17], [31] ,Carpenter et al [8] and Irwin, Jay and Chen [26] and references therein) has extended traditional resource management literature (like Cropper and Oates [13] and Hahn [22] and references therein) and sustainable economic growth literature (like Brock and Taylor [5] and references therein) by using ecological models based on ecological field experiments. Because the ecological system can have multiple stable states as shown in Carpenter et al [8], it is still unknown how policy should react to the existence of this multi-stability. In other words, the ecological system may resides in different states depending on the level of pollutants people put into the system. Take the lake ecological system as an example, according to Carpenter et al [8], when the nutrient loadings is low, water quality will remain in good state. However, when nutrient loadings are high, lake system will suffer from algae blooms, fish kills and low water quality and remain in this state. The existence of these possible multiple-states imposes both theoretical and technical challenge for policy makers. This is a particularly interesting and important question in the management of coupled human-ecological system, because the trade-off between economic development and environmental protection now has to be considered in the context of possible changes in the final state of the system. However, before we can fully address this issue, this paper argues for a careful specification of the relative time-scales at which economic and ecological processes are taking place. This is not only because divergent time-scale is a common phenomenon in reality but also because it is of crucial importance to our understanding of system dynamics and optimal policy.

Divergence in time-scale (or slow-fast dynamics) is a common phenomenon in reality and has been studied in ecology, physics and sociology (Ludwig et al. [32], Chave and Levin [9] and references therein). In economic theory, its importance is also acknowledged in the literature of habit formation [11] and sticky prices [6] where habit and price is assumed to change slower than other variables in the model. Its presence in coupled human-ecological systems is also evident. On the global level, the interaction between human activities and global warming is a good example. While global warming is a very slow process affected by relatively faster changing human activities, its impact on human economic activities can be imminent and large. Figure 1 vividly illustrates the cross-scale interactions between human activities and water quality. While land conversion on a particular parcel will typically complete within a year, its impact will accumulate at the neighborhood level and result in land/habitat fragmentation. The consequent changes in nutrient run-offs and habitat changes will then affect the water quality on a larger spatial and temporal scale. Because the lake ecological system can dissolve some nutrient loadings and adapt to some changes in habitat through either physical process (sedimentation and outflow) or ecological process (adjustment of the species in the water system), the above-mentioned human impacts may not be evident most of the time. However, as pollution gradually accumulates, dramatic/catastrophic changes in water quality can occur in a short period of 1-2 years, like algae blooms and fish kills. This may then affect population migration and regional growth at even larger spatial and temporal scales. The Cuyahoga River fire (1969) in Cleveland was a result of long-time pollution and has definitely affected people's migration decisions in the following decades. Finally, this population changes on the regional scale is one major driving force behind the land conversion process on the parcel scale.

The impact of divergent time-scale on coupled human-ecological system is fundamental. As noted by Levin [31]: "dynamics on faster time scales, ..., are shaped by slow dynamics on

Figure 1: Scale Dependence: Economic Decisions and Ecological Change



larger times scales, which in turn arise from the aggregate of dynamics on faster and smaller scales." Moreover, divergent time-scale also "holds the key to resilience and potential domain shifts. ... their importance is fundamental to management" (Arrow et al. [2]). This is also acknowledged in the general discussions of Carpenter et al. [7], Dusgupta and Mäler [14], Chave and Levin [9] and Folke [18]. Last year's sudden break-out of massive algae bloom in southeast China is a good exemplification. The Yangtze River Delta is one of the fastest growing regions in China for the past 30 years. Because of the consequent **industrial pollution and residential run-offs [right phrase?]**, a massive algae bloom suddenly broke out in Taihu Lake. While these human impact on the water system builds up only gradually in the last 30 years, the ecological impact on human behavior is immediate and dramatic. In the city of Wuxi, China, because of this algae bloom, city residents had no **usable [right word?]** water and had to rely on bottled water. Many people ran out of the city temporarily. The price of 18-liter purified water increased immediately from RMB$8 to $50. Consequent policy intervention were also dramatic and costly. While the local government is transporting huge amount of bottled water to the city, it issued a policy to freeze the price and specified a fine of RMB$300,000 for any price increase. At the same time, 1.6 million carps were put into the lake in a neighboring city as a counter measure. The recent dramatic decline in catchable wild salmon on the western coast of U.S. is another example, where its amount decreases from 1.5 million in 2005 to only 35,000 this year.[1] This interaction between the slow-and-fast dynamics imposes a even greater challenge for policy makers, because the slow changes is often too slow to grasp the attention of policy makers and the public.

So far, this difference in the time scale between economic and ecological process, as well as the feedback effects, have not been fully explored. In the existing economic literature that deals with human-ecological interactions, the issue of divergent time scale is often neglected or addressed arbitrarily. Different people have made different simplifying assumptions about

[1]See the recent news report on April 14th by MSNBC. The video clip is entitled: "No 'upstream' for salmon fisherman". http://www.msnbc.msn.com/id/21134540/vp/24113279#24113279

the relative time scales of the human and ecological process ([40], [41], [42], [19], [20], [34], [15], [30], [15], [28] and [36], [33]). Based on the ecological work of Ludwig et al [32], Arrow et al [2] pointed out, "Simplification through elimination of slow time scale dynamics can be helpful in many situations; but ultimately, there my be a price to pay if one does not consider the coupling between dynamics on multiple scales". As will be shown in this paper, mis-specification of the relative time scale can lead to either excessive or inadequate policy intervention. With concrete models, Anderies ([1]) and Perrings and Walker ([37]) show that relative time scale between different dynamic processes can be of great significance. However, because of the setups in their models, it is not clear whether the significance is due to purely the impact of time scale or because of the changes in their model structure. Because when a parameter shifts the location of equilibrium, it will definitely affect the associated welfare. Recently, three papers in the literature of resource management (Grimsrud and Huffaker [21], Huffaker and Hotchkiss [25] and Crepin [12]) single out the impact of relative time scale. They propose the application of singular perturbation technique to address the divergent time-scale in general.

This paper establishes a coupled human-ecological model where slow-varying migration is interacting with fast-varying nutrient dynamics in lake ecology. The nonlinearity and fast-slow dynamics built in the model can generate regime shifts (that is, shifts between different equilibrium states) and slowly-reversible ecological changes. Because ecological conditions do affect and are affected by uncoordinated individual decisions on migration and land-use, the policy challenge does not only lie in the optimal use of ecological service but also in the provision of the right incentives that regulates individual behavior. The possibility of regime shifts and slowly-reversible changes in this coupled model makes policy analysis more interesting and technically challenging. Within this framework, this paper shows that specification of relative time scale between the fast and slow dynamic processes is crucial for the analysis of the system dynamics with/without policy intervention. The calculated solutions show that specification of relative time scale can significantly change the cost, magnitude and length of active intervention in optimal policy. This paper shows that optimal policy (even when resilience does not enter into optimization problem) will always increase the resilience of the desirable equilibrium in the coupled system. The extent of this improvement in resilience depends crucially on the relative time scale. It also shows that simplifying assumptions on the relative time scale can lead to incorrect predictions for both the short- and long-run dynamics.

Through the analysis of a coupled model of human migration and lake water quality, this paper shows that relative time-scale is crucial for the understanding of the short-run and long-run dynamics of the coupled system. Simplifying assumptions on time scales have to be done carefully. Moreover, it finds that optimal policy will always increase the resilience of the desirable equilibrium. But how much the improvement will depend crucially on the relative time-scale in the model. In terms of the magnitude and lasting period of active intervention, both will be significantly affected by the relative time-scale. In situations where multiple policy alternatives can lead to the same final state of the system, different specification of relative time-scale can result in different choices of policy. Finally, simulations suggest that the trade-offs among 1) the optimality of policy, 2) robustness of policy and 3) resilience of the system is also affected by the specification of relative time-scale.

For the coupled system without policy intervention, this paper extends the analysis of Anderies [1] by clarifying the conditions under which simplifying assumptions can be made. As for the optimal policy of this coupled system, this paper extends the analysis of Grimsrud and Huffaker ([21]), Huffaker and Hotchkiss ([25]) and Crepin ([12]) by relaxing the extreme assumption that justifies their use of singular perturbation. It thus allows people to study interactions occurring on a broader spectrum of relative time-scales: from extremely divergent time-scale (as required by singular perturbation) to less divergent time-scales. In particular, it sheds light on optimal policy responses under the context of possible regime shifts and slowly-reversible changes. It illustrates that relative time scale will not only affect the decision – "to intervene or not intervene", but also significantly affect the welfare payoffs in case of intervention.

Finally, because bi-stability is built into this paper, it shares similarity with papers that deals with optimal management with multi-stability, like Carpenter et al. [8], Brock and Starrett [4] and Dasgupta and Mäler [14]. These papers essentially fall into the category of resource management. They can be labelled as one-way interaction model. The reason is that while human activities have negative impact on ecological system, the feedback from ecology is only felt by the imaginary manager and does not affect polluter's decision. This paper extends the policy analysis by managing a coupled human-ecological system with two-way interactions. That is, while the economic activities (land use) of uncoordinated individuals are affecting the ecological quality, changes in the ecological conditions will also affect their well-being and indirectly affect their economic decisions. The policy maker has to balance the local economic development and ecological protection. But the optimal policy is not only about the optimal use of resources, but also about provision of the right incentives to individuals, so that their aggregate behavior can result in a balance between economy and ecology.

2 Literature Review

In the economics literature, the relative time scale of environmental changes and human activities is treated quite arbitrarily when divergence in time scale is present.[2] Take the treatment of the environment as an example. On one extreme, environmental changes are regarded as a very slow process and assumed to be constant ([19], [20]). On the other extreme, they are regarded as fast process and assumed to respond to human activity instantaneously ([34], [15] and [33]). In between, they are treated as processes that evolve at the same time scale as human activity ([30]). For instance, in studying regional growth, Fujita et al [19] and Fujita and Thisse [20] treat natural resources and natural amenities as constant even though such a regional growth exerts higher pressure on the local environment. In fact, this is a common practice in regional growth models. On the other extreme, when trying to address economic growth with environmental concerns, Mohtadi [34] assumes that the environment instantaneously responds to human activity by specifying the ecological variable as an explicit function of capital stock. So does Elbasha and Roe [15] when they are analyzing the interaction between environment and economic growth. In their welfare analysis on the optimal management of shallow lakes, Mäler et al. ([33]) also assume instant ecological adjustment. Between these two extremes, Krautkraemer [30], treats the changes in natural environment as a process that evolves at the same time scale as human activity.

On the other hand, when the focus of research in on the environmental changes, similar simplifying assumptions on human activities are also common in literature. For instance, Keller et al [28] study the impact of climate change on economic growth and assume that population is constant over time. However, on the time scale that significant climate change occurs, population dynamics should be treated as a faster process instead of a constant.

Importance of relative time scale between different dynamic processes are discussed in Anderies [1] and Perrings and Walker [37]. Anderies [1] focuses more on its impact on system dynamics. He develops a general equilibrium model of two sectors where a renewable natural capital is used as an input factor in production, along with labor and human capital. It shows that the relative time scale between savings rate and the regeneration rate of natural capital is important because it may change the stability of the steady state. Perrings and Walker ([37]) are emphasizing the policy impact. They use the steady-state optimal level of harvest to simulate the optimal management of rangeland and the importance of relative time scale is shown in the process. However, because of the way Anderies [1] and Perrings and Walker [37] set up their models, it is not clear whether the significance is due to purely the impact of time scale or due to the consequent structural changes in their dynamic models. The reason is that time-scale changes in their model will at the same time change the location and probably the stability of the steady-state. Because change in equilibrium location will inevitably lead to changes in welfare, it is still not clear whether relative time-scale is the main contributor for the significance.

The impact of time-scale is separated in three recent papers (Grimsrud and Huffaker [21], Huffaker and Hotchkiss [25] and Crepin [12]) in the literature of resource management.

[2] The resource management literature has a long tradition of modelling human-ecological interaction, like the literature on the management of fishery and forestry. In this literature, human interventions (through harvesting) is typically a major factor determining the growth and stock of the resource and is typically coordinated with the changes in it. In this sense, human activity and the ecological process are usually changing on the same time scale. However, the interaction of regional population and local ecological conditions is different. While population migration is affected by the *in situ* ecological quality, its change is governed by uncoordinated individual behavior.

In their models, the ecological process itself has both a fast and a slow component. They all propose the use of singular perturbation with the presence of divergent time-scale. In particular, Grimsrud and Huffaker [21] explore the implication for pesticidal crops management when pest population evolve at fast time scale and the population's genetic composition evolve at slow-time scale. Huffaker and Hotchkiss [25] study the reservoir management by the introduction of a slow sedimentation process into the fast impounding water dynamics. Crepin [12] suggests using singular perturbation theory as a general method to address the issue of divergent time scales and takes the management of coral reefs with fishing pressure as an example. With the assumption that the time scale are vastly different and policy instrument only acts on the fast process, they all use the singular perturbation technique. The advantage of this technique is that it allows for an analytical solution. A disadvantage is that it can only allow simple nonlinear terms.[3] However, all three papers have missed this essence of singular perturbation theory and ended up using numerical methods. Their use of numerical method in the end invalidates the necessity for them to use the singular perturbation method in the first place, because of the readily available numerical algorithms for the original problem and the fact that the dimension of their problem is in no sense too large. Moreover, none of them discuss the matching on the intermediate time scale which is crucial to the singular perturbation theory. In addition, these three papers share a common flaw in their derivation for optimal policy, which leads to doubts about the robustness of their conclusions. The problem is manifested in the contradiction between the assumption and result in Huffaker and Hotchkiss [25]: they assume that the control variable c should be in the range of $[0,\ 6 \times 10^7]$ (Table 1 p.2557), but in Figure 4, their derived optimal c lies in $[-4 \times 10^7,\ 8 \times 10^7]$ (p. 2570).

This paper does not assume the extreme divergence in time scale as required by singular perturbation theory. The technique used here therefore has more general applications and can be used to address a larger variety of inter-temporal interactions. In addition, it extends these three papers by consideration of multiple policy instruments, which act on processes of different time scale. This extension is important because coordination of policies that act on different time scales can now be discussed.

Because bi-stability is built into this paper, it shares similarity with papers that deals with optimal management with multi-stability, like Carpenter et al. [8], Brock and Starrett [4] and Dasgupta and Mäler [14]. These papers essentially fall into the category of resource management. They can be labelled as one-way interaction model. The reason is that while human activities have negative impact on ecological system, the feedback from ecology is only felt by the imaginary manager and does not affect polluter's decision. This paper extends the policy analysis by managing a coupled human-ecological system with two-way interactions. That is, while the economic activities (land use) of uncoordinated individuals are affecting the ecological quality, changes in the ecological conditions will also affect their well-being and indirectly affect their economic decisions. The policy maker has to balance the local economic development and ecological protection. But the optimal policy is not only about the optimal use of resources, but also about provision of the right incentives to individuals, so that their aggregate behavior can result in a balance between economy and ecology.

Finally, in terms of optimal management of system with multi-stability, the closest related work is Brock and Starrett [4]. Brock and Starrett [4] study the optimal management

[3]The idea of this technique is also applied in numerical solutions for large scale dynamical systems, because it can reduce the dimension of the problem.

of an ecological process with nonlinearity, the focus is to prove the existence of multiple equilibria, instead of completely solve the problem. In their paper, while the the positive and negative effects of ecological changes can affect the objective function for policy makers, the human reaction is not explicitly modelled as a dynamic process. In contrast, this paper focus on solving the dynamic optimization problem when human economic activities are responding to ecological changes in a general equilibrium framework. The dynamics of human activities are modelled explicitly. It also shows that changes in relative time scale can transform non-Skiba points into Skiba points.

3 A Migration Model with Coupled Human-Ecological Interaction

This section sets up a coupled model with cross-system and cross-scale interaction between human activities and ecology. In this model, people make migration decisions based on utility differentials between the region and the rest of the wold. High quality of local ecological services will give people more incentive to migrate into the region. As local population increases, due to the increasing return to scale in production, local wage will increase. It will then increase their residential land consumption. Population and the consequent land-use increase will not only increase congestion in the region but also degrade the ecological conditions. When ecological condition degrades, people will have less incentive to migrate into the region. To reflect the change rate of migration and lake water quality in reality, migration is modelled as a slow variable as compared to the water quality change. The model in this paper is a modification of Chen, Irwin and Jayaprakash [10].[4]

3.1 The Economic System

In the regional economy, two goods are traded: a locally produced composite good and land. The composite good (X) is produced with local labor (N), which can be either consumed locally or exported to the rest of the world. Following Fujita and Thisse [20], this composite good X is produced according to the production function:

$$X = E(N)F(N) \tag{3.1}$$

where $F(N)$ is standard neoclassical production function with declining marginal product of labor. Agglomeration is captured by an externality function $E(N)$, which represents scale effects that are external to individual firms.[5] Under the assumption that regional output market is perfectly competitive, local wage rate is determined by

$$w = E(N)F'(N) \tag{3.2}$$

Households that choose to stay outside of the region will receive a constant utility ($\bar{U}$). Those that choose to live in the region will accept the local wage rate and decide on their consumption of the composite good (x) and land (l). Their optimization problem is:

$$U(N,\, P) \quad = \quad \max_{x,\, l} \left\{ x^a l^{1-a} + U_c(N) + U_e(P) \right\} \tag{3.3}$$

subject to

$$x + r \cdot l = w$$

where x and l are household consumption of the composite good and land, respectively. The price of the composite good x has already been normalized to 1. r is the land rent. U_c is the disutility from congestion and U_e is the utility from enjoying the natural amenities in the region. P is an ecological variable which determines the quality of the lake amenity and will

[4]This is also the basic model presented in the conference of the Resources for the Future in 2007.

[5]Because the reason for agglomeration is not the focus of this paper, it is treated in the simplest way.

be described in detail in the next subsection. Assuming that new migrants are allocating one quarter of their annual expenditure on land/house payment, this gives $a = 0.75$.

Substituting the optimal consumption of x and l:

$$x = a\,w, \quad l = (1-a)\frac{w}{r} \tag{3.4}$$

into utility yields the indirect utility as a function of local wage rate w, population N and ecological variable P.

$$U(N,\, P,\, w) = a^a(1-a)^{1-a}r^{a-1}w + U_c(N) + U_e(P) \tag{3.5}$$

Migration incentive is assumed to be determined by the utility differentials between the region and the rest of the world: $\Delta U = U(N,P) - \bar{U}$. Migration will continue until $\Delta U = 0$. For simplicity, migration rate is assumed to be proportional to the utility difference.

$$\dot{N} = \epsilon \Delta U \tag{3.6}$$

where $\dot{N}$ stands for the derivative with respect to time and ϵ controls the rate of change in population N. In order to reflect the reality that regional population is a slow process, ϵ is taken to be a small number. In line with this slowly adjusting local population, I assume that market adjusts in each time period t to a temporary equilibrium that is conditional on the population in the period.

Finally, the land-market clearance condition is needed to close the economic general equilibrium model. Rather than constraining the total amount of land in the region, I assume that, as local population grows, the urban area will expand into the surrounding rural areas where land is used solely for agriculture. Land will be bid away from agriculture for development uses (in this paper, residential use) so long as land rent for development uses r is greater than the agricultural rent r_a. The total amount of developed land will be determined by the equalization of these two rents. The agricultural output is assumed to be produced with constant-return-to-scale technology and sold on a global market, so that output price r_a is exogenously determined. This gives our final closing condition:

$$r = \bar{r}_a \tag{3.7}$$

where r_a is set at value 1.2.[6]

Equations (3.2), (3.3), (3.4) and (3.7), together with full employment condition in the region define a temporary equilibrium for a given N and P. This gives us the reduced form expression for migration rate in equation (3.6)

$$\dot{N} = \epsilon\big(U(N,P) - \bar{U}\big) \tag{3.8}$$

For simplicity, I assume the neoclassical production function of X is constant return to scale, $F(N) = bN$ ($b = 0.67$), production externality is of the simple form: $E(N) = N$, congestion effect is $U_c(N) = -N^2$ and $U_e(P) = -P$. This gives

$$\dot{N} = \epsilon\big(A_0 N - A_1 N^2 - A_2 P - \bar{U}\big) \tag{3.9}$$

[6]The price of the composite good is normalized to 1. $r_a = 1.2$ is to reflect the belief that land is a relative scarcer resource as compared to other goods in general. This is believed to be true particularly in places with high natural amenities.

where $N \geq 0$ and $A_0 = a^a(1-a)^{1-a}br^{a-1}$.

We consider it unlikely that, even with substantial growth, amenity-driven growth will transform small rural areas into large metropolitan areas with large cities. Because moderate-sized metropolitan areas in the U.S. are in the range of 50-100,000 people, the parameters in Equation (3.9) are chosen so that congestion effect is dominating agglomeration effect when local population is at around 90-100,000 people. Moreover, we believe that human impact can become evident before the region develops into a full-fledged metropolitan area. The most interesting case is the one where the region is in its early stage of development yet faces a possible catastrophic ecological change. The coefficient in front of P in Equation (3.9) is set at 0.26, so that population of around 50-60,000 is enough to trigger slowly-reversible or irreversible changes in ecology.

Finally, variation in the value of ϵ reflects the migration rates observed on different spatial scales in the U.S. On the regional level, annual population growth ranges from 0.54% (in northeast) to 1.62% (in west) in 1990-2003 (CEP Report 2006 p.16). On the state level, estimated annual population growth ranges from -1.26% (New York State) to 6.7% (Nevada) between 1995-2000. The states around Lake Erie, except for New York, has an average rate around 0.9%. (Rachel Franklin 2003 pp. 5, 15). On the city level, the fastest growing city between 1990-2000 was Augusta-Richmond County, GA, with an average annual growth of 34%; that in period 2000-2004 was Gilber, AZ, with an rate of 10.6% (CEP Report 2006 p.15).

3.2 The Ecological System

The main interest of this paper lies in the impact of fast-slow dynamics on the dynamics of coupled human-ecological system and its policy implications. We use lake ecological system as an stylized source for natural amenities, because it is tightly coupled with human activities and because it evolves at a faster time-scale than migration. Run-offs from land-development and urbanization are transported to lake through its tributaries. This process occurs on the time scale of months. These nutrient loadings will contribute to the nutrients in the water column of the lake. But part of it will sediment to the bottom. This sedimentation process occurs also on the time scale of months. In spite of the sedimentation and out-flow of nutrients, the nutrient can build-up throughout the year and become sufficiently high in the late summer to spur nonlinear recycling. As a result, the nutrients bounded to sediments can be re-suspended back into the water column and contribute to the transition from oligotrophic (good) to eutrophic (bad) state of the lake in late spring and summer seasons(Carpenter et al. [8], Hein [23] and Scheffer [38]). An oligotrophic lake is characterized by low-nutrient level in the water column, relatively clear water and healthy ecosystem services. On the other hand, an eutrophic lake is characterized by high-nutrient inputs, high concentration of algae, toxicity, turbidity and is much prone to anoxic conditions and undesired events such as fish kills and noxious algal blooms. Hysteresis can occur in nutrient dynamics (Scheffer [38] and Carpenter et al. [8]). That is, once regime shift has occurred, reduced loadings may not bring the lake back to an oligotrophic state or the recovery has to take substantial amount of time. Lake Erie is a good example. After the river fire in 1969, substantial reductions in point sources in the 1970s failed to result in meaningful improvements in water quality until ten years after the controls had been implemented.

We adapt the simple model of Carpenter et al. [8] to capture the basic dynamics of nutrient loadings and the possible regime shifts between oligotrophic and eutrophic state. Lake water quality is assumed to be affected by nutrient loadings, which is assumed to be proportional to total land-use in the region.

$$\dot{P} = L_0 + L_1\, N\, l - sP + \sigma_1 \frac{P^{\sigma_2}}{P^{\sigma_2} + P_c^{\sigma_2}} \tag{3.10}$$

where P is the mass (or concentration) of phosphorus in the lake water and $\dot{P}$ is the derivative with respect to time. The rate of nutrient input per unit of time is determined by the loading, outflow and recycling. The loading consists of L_0, which is independent of human activities, and $L_1\, N\, l$, which is the total loadings from the urban land-use in the region. L_1 captures the average impact of land-use on lake water quality. The outflow of phosphorous is captured in $s\, P$, where s captures the rate of P-loss from the system. It accounts for processes such as sedimentation and outflow that remove P from the water column. The last term represents the recycling process of P. It captures the fact that after sedimentation, P can be recycled back into lake water from the bottom. Its impact is only evident when P concentration is sufficiently high. The maximum rate of recycling is σ_1; the steepness of the recycling curve is governed by σ_2. P_c is the P value at which recycling reaches half its maximum rate, it is also a rough indicator of the ecological threshold where this nonlinear term comes into play.

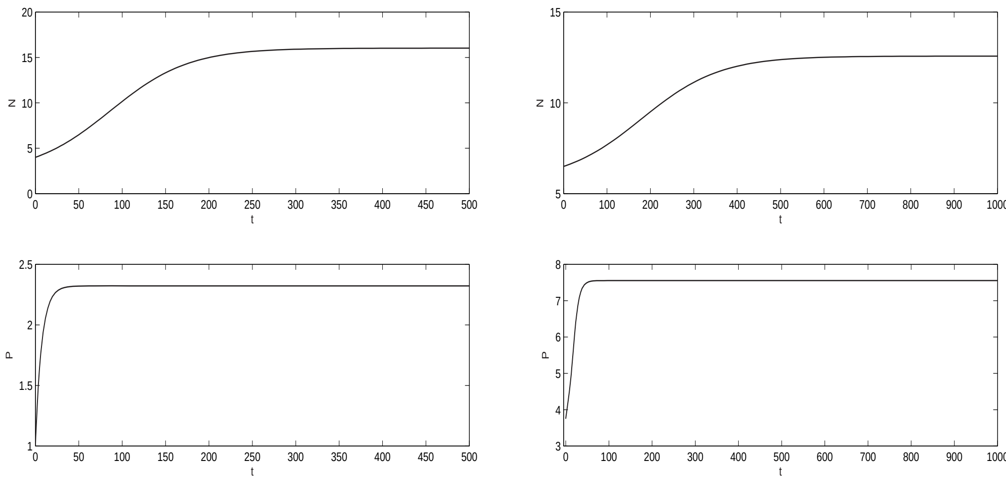
Although the above specification is based on lake phosphorus dynamics, the nonlinear dynamics, embodied in the recycling term that gives rise to threshold responses, is a general feature of many ecosystems (Brock and Starrett [4]). Therefore, with modification of this ecological dynamics, the analysis in this paper is also applicable to the analysis of a more generic human-ecological system with similar nonlinear behavior.

Phosphorus dynamics in lake system evolve on the time scale of approximately one year. The parameters are calibrated to conform with existing ecological literature and generate a change rate in P that roughly corresponds to the time scale of one year. This set of parameters is regarded as the benchmark for comparative analysis on time scales change.

Two exemplary time evolution of the uncoupled system are presented in Figure 2 to illustrate their corresponding time scales. By "uncoupled", I mean that the interacting terms in Equations (3.8) and (3.10) are neglected. In the example, the initial condition is assumed to be $(N_0, P_0) = (4, 1)$ in Figure 2(a) and $(N_0, P_0) = (6, 1)$ in Figure 2(b). The time scale of N and P changes differ roughly by a factor of 10. This is evident from the initial rapid change in P as compared to that in N.

3.3 Coupled System with Divergent Time Scales

This section describes the dynamics of this coupled economic-ecological system. The phosphorous dynamical equation (3.10) in this paper is adapted from ecological studies based on field experiments. While the nonlinear recycling term in Equation (3.10) make it impossible to solve the dynamical system analytically, we consider it more beneficial to retain the term than sacrificing the term for the sake of analytical tractability. Because this nonlinear recycling term is the crucial element that gives rise to the most interesting scenarios like regime shifts and slow-reversible changes. And the fear of these catastrophic events lies in the core of current debate of environmental protection and is the most compelling factor that motivates regulation policies. Maintaining this nonlinear term also makes the following-up policy analysis more meaningful, because the policy analysis now has a concrete real-world



(a) $(N_0, P_0) = (4, 1)$ (b) $(N_0, P_0) = (6, 1)$

Figure 2: Time Evolution of the Uncoupled System

context. The disadvantage is that lack of analytical solution makes comparative static analysis difficult, but it is not at all impossible. With analytical solution, comparative static analysis can be easily done (through tedious analytical calculation of derivatives) and the result is often transparent to the reader. However, with numerical solutions, the comparative static analysis can be done but the result is conditioned on the range of parameters we choose. Theoretically, we can try all the possible parameter values and the result will conform with the analytical solution. But this is impractical in reality. Fortunately, as a researcher, we are not interested in all the possible cases, we are only interested in the cases where interesting phenomenon occur. Finally, there is actually a trade off between the analytical and numerical solution. Analytical solution (if it exists) is more general than numerical solution in terms of parameter values. That is, the numerical solution depends on the particular specification of parameter values. When the specification changes, the solution will change accordingly. The analytical solution does not suffer from this problem. However, in order to ensure the existence of analytical solution, the function space of dynamic equation is limited to extremely simple ones. Yet the numerical solution is free from this disadvantage and can allow more general and realistic function forms to study more complicated phenomenon. Because of the complicated nature of the coupled human-ecological system and our interest in cases where human activities puts ecological system to the verge of catastrophic events, numerical method is preferred in this study to analytical method.

3.3.1 Baseline Results

In this coupled system, the slow-varying population migration responds to ecological quality which is determined by the fast-varying phosphorous loadings. At the same time, the fast-varying phosphorous loading responds to the changes in local population and land-use which evolve slowly. These two-way interactions finally complete a ecological-economic model with fast-slow moving variables. The dynamics of this unregulated system is fully described by

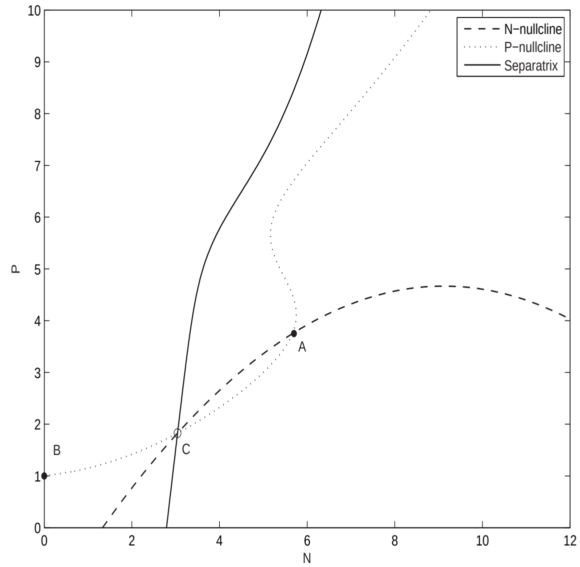


Figure 3: Phase Plot of the Baseline Model

equations (3.8) and (3.10). Substitute in the residential land use l, we have:

$$\begin{cases} \dot{N} = \epsilon\big(A_0 N - A_1 N^2 - A_2 P - \bar{U}\big) \\ \dot{P} = L_0 + L_1\, N\, l - sP + \sigma_1 \dfrac{P^{\sigma_2}}{P^{\sigma_2} + P_c^{\sigma_2}} \end{cases} \tag{3.11}$$

The result for a benchmark case is shown in the phase plot of Figure 3. The broken line is called N-nullcline. Points on this line satisfy the condition of $\dot{N} = 0$. Similarly, the dotted line is called the P-nullcline. The intersection of the two nullclines gives the location of fixed points. The two interceptions give one stable fixed point (A) and an unstable one (C). Given the fact that when N is zero, there can be no further emigration from the region, the third stable fixed point (B) is the intercept of P-nullcline on the vertical axis. Because at point B, local population is zero, corresponding to no economic activity in the region, it is labelled as "undesirable" steady state from the economic perspective. Of course, it is a good state for ecology. Because point A is associated with a reasonable population level and the lake ecology is not in a eutrophic state, it is referred to as "desirable" state. It is evident in the plot that under the current specification, regional development has already pushed the ecology to the edge. An ecological shock that increase phosphorous concentration in the lake (such as a variation in rain fall) or an economic shock (such as government policy to promote further economic development or provision of public services that attract an additional influx of immigrants) can lead to disastrous result. This sets up an interesting case for policy analysis in later sections.

Finally, the solid line is called the separatrix which is the stable manifold of the unstable fixed point. The points on this line will eventually evolve to the unstable fixed point. This separatrix is important because in this model it separates the domains of attraction of the two stable fixed points (points A and B). For example, in Figure 3, all the points on the right of this separatrix will eventually flow to the stable fixed point A. The points that are on the left, will eventually go to the fixed point B. Since this separatrix demarcates the domain of attraction, it can also be used to define the concept of resilience. For a point that is far away from the separatrix, we need a large shock to drive it to the other side of the separatrix and result in a dramatic change in terms of final outcome (referred to as regime shift). In this sense, we say the system is at a point which is quite resilient to shocks in P and N. That is, when the system is further away from separatrix, it is able to absolve larger shocks without changing the final state of the system. On the contrary, when the point is close to separatrix, the system is considered as less resilient, because a small shock can easily drive it across. When the system crosses the separatrix, the final state of the system will be changed, and it is called regime shift. Faced with uncertainty in both economic and ecological processes, a higher resilience associated with a desirable steady state is preferable to a lower one. Similarly, a lower resilience associated with undesirable steady state is preferable. Under the context of current specification, the larger the domain of attraction for fixed point A, the better. Finally, because separatrix is the demarcation of the domain of attraction of the two stable fixed points, any shifts in it will result in changes in the resilience of the corresponding fixed points.

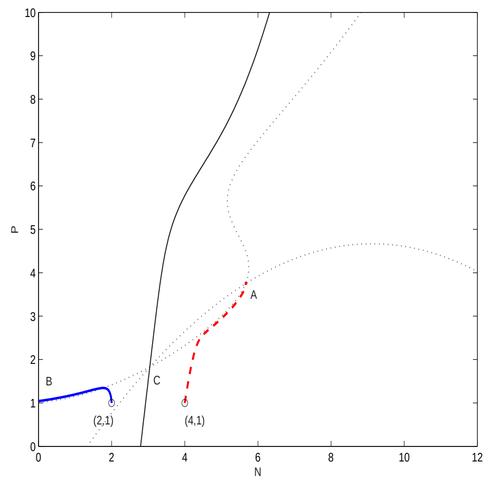
Two exemplary solutions are plotted Figure 4. The first one starts from the initial condition of $(N(0), P(0)) = (2, 1)$. The solution is plotted as the blue line in Figure 4(a). The time path of local population (N) and phosphorous (P) are plotted in Figure 4(b). Because the initial condition lies on the left of the separatrix, the system will eventually approach the 'undesirable' steady state at point B. The second one starts from $(N(0), P(0)) = (4, 1)$. It is the red line in Figure 4(a) and the time path is in Figure 4(c). Because the second initial condition is on the right of the separatrix, the system finally reaches the desirable steady state at point A.

3.3.2 Impact of Divergent Time Scales on System Dynamics

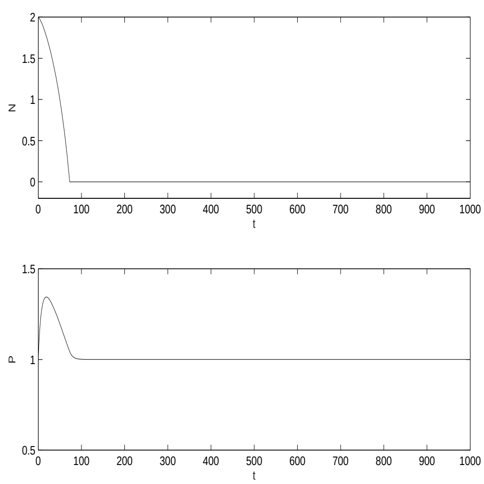
This model with divergent time scales is a special case of models governed by ordinary differential equations. This added feature not only creates transient phenomenon not commonly seen in other ordinary differential equations but also provides more detailed information of the system dynamics. Its mis-specification can result in incorrect predictions on the stability of the fixed point and the location of the sepratrix.

In terms of transient dynamics, a most salient feature of the divergent time scale is that it can give rise to slowly reversible changes. Imagine the state originally resides at the long-run steady state (at point A). A sudden influx of population (due to the improvement of local public services or other reasons), the system is moved to the point where $N = 6.5$ and $P = 3.75$ as shown in Figure 5(a). Without policy intervention, at this point, the local population exerts too much pressure on the lake system. The run-offs from human activities exceeds the carrying capacity of the lake. As a result, phosphorous P) increases to a level around 6.5 and the lake becomes eutrophic. Because the nutrient dynamics in this model occurs much faster than population change, this initial increase in phosphorous is rapid, as shown in Figure 5(a) and Figure 5(b). As lake water quality decreases, the natural amenities becomes less attractive, people will gradually move out of the region. This corresponds to the gradual decline in phosphorous. Because this process depends on the population outflow which changes slowly, the ecological recover will take a long time.[7] As shown in the figure, local population has to decrease by significant amount (more than half of the population)

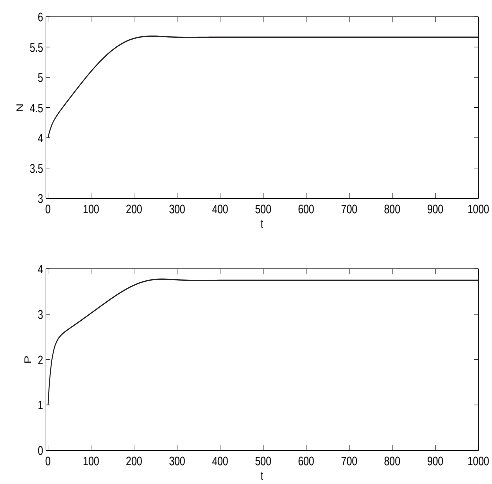
[7] In this model, the slowly reversible nature is driven by the slowly changing population. In reality, the recovery of ecological system itself may take time. Inclusion of this slow ecological recover requires inclusion of a more complicated ecological dynamics and is neglected so far.



(a) Trajectories in N-P Space

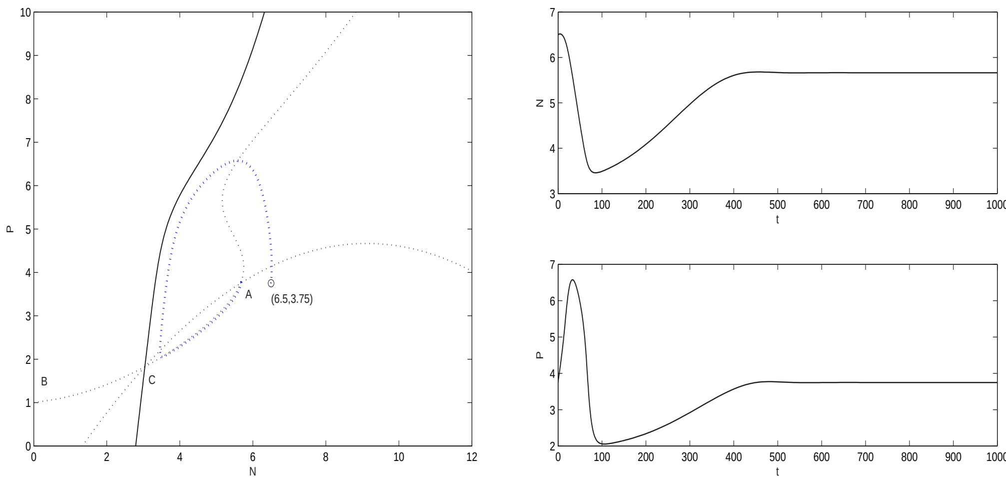


(b) $N(t)$ and $P(t)$ from (2,1)



(c) $N(t)$ and $P(t)$ from (4,1)

Figure 4: Exemplary Evolution of the Coupled System



(a) Trajectories in N-P Space

(b) $N(t)$ and $P(t)$ from (6.5,3.75)

Figure 5: Exemplary Evolution of A Slowly-Reversible Change

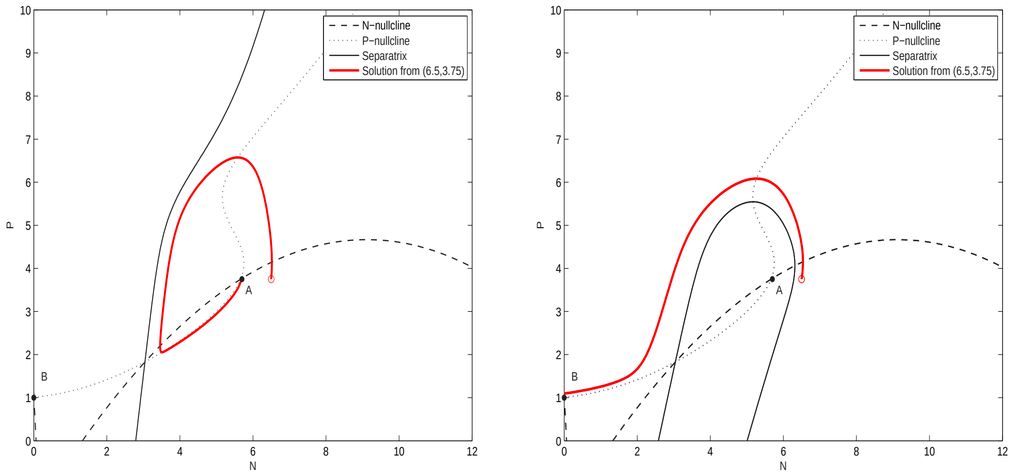
in order to initiate ecological recovery. This improvement will then attract people back into the region. This process will continue until the system come back to the steady state at point A. Because after the initial shock, the system finally goes back to the original steady state and the recovery takes a long period of time, the initial change is called a slowly-reversible change. This is created by the fast-slow dynamics that are changing on different time scales. The slowly-reversible nature and the significant economic decline necessary for the ecological recovery impose an interesting problem for policy makers: when human activity has already put much pressure on the ecology, a small overshoots in human activity or a slightly larger variation in ecological system may initiate slowly reversible changes in ecology in the short run and result in consequent dramatic economic decay that will pertain for a substantial period of time. Confronted with this danger, how should policy makers respond to it? This will be discussed in the policy section.

Divergence in time scale can provide more detailed information about the system dynamics in this model. If the time scales diverge by a lot ($\epsilon \ll 1$), trajectories will rapidly approach the attractive portion of the slow manifold (where $\dot{P} = 0$) at first. It will then move along this slow manifold for some time. When it moves close into the close neighborhood of the fixed point, if the fixed point is stable, the system will converge to it. If it is a saddle point, the trajectory will eventually move away from it. This is essentially a partial reinterpretation of the Fenechil's Theorems of Invariant Manifold ([29], [27] and [35]).

Mis-specification of the relative time scale may change the stability of the fixed point. This hold true even in linear systems. Take a linear dynamic system as an example,

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -0.006 & -0.005 \\ 0.3 & 0.01 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

Here, x is a slow variable and z is a fast variable. This linear system is unstable. If we double the speed of change for x, that is scale the first row up by a factor of 2, the system becomes stable. The coupled human-ecological model in this paper can be used as a nonlinear example. For the case where the reservation utility level is set to be 0.38, when



(a) $\epsilon = 0.08$ (b) $\epsilon = 0.16$

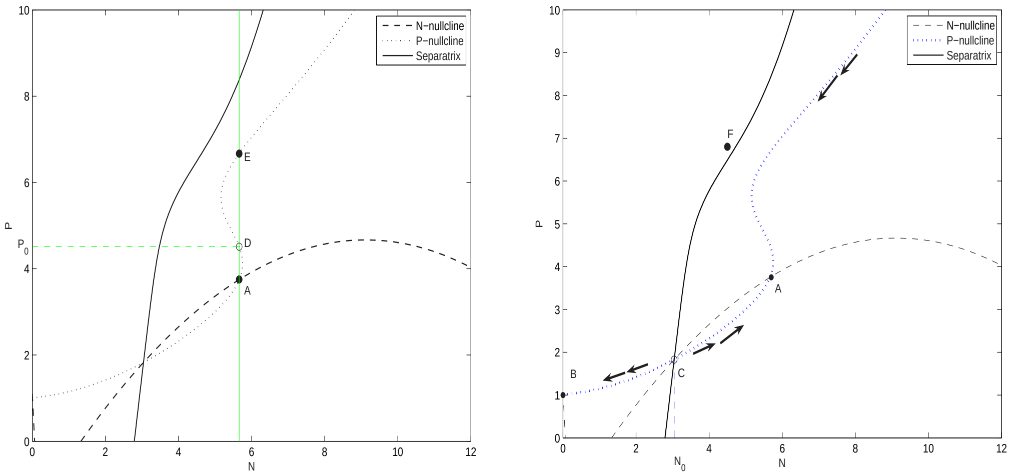
Figure 6: Impact of Time Scale (ϵ) through Shifts in Separatrix. Initial Condition: (6.5,3.75)

the time scale coefficient ϵ decreases from 0.08 to 0.04, the stability of the fixed point (A) is changed from stable to unstable.

In nonlinear system with more than one stable fixed points, changes in relative time scale can shift separatrix. Because the separatrix demarcates the basins of attraction of the two stable fixed points in the current model, such a change has a substantial impact on the resilience of the system, as shown in Figure 6.

The impact on particular trajectories is illustrated in Figure 6. Figure 6(a) gives the trajectory in the benchmark case, where the initial condition is within the domain of the desirable equilibrium A. After some initial oscillations, the system finally reaches the desirable state at point A. The trajectory is plotted as the red line in the figure. As ϵ increases, the separatrix "bend downward" (the black solid line). When $\epsilon = 0.16$, as in Figure 6(b), we find that the initial condition now lies within the domain of attraction for the undesirable equilibrium at point B. Therefore, after some initial oscillation, the system will eventually arrive at the undesirable equilibrium (point B) with the whole region empties out and economy crashed. While the relative time scale is only mis-specified by a magnitude of two, the changes in the resilience (as measured by the size of attraction domain, Holling [24]) of the desirable equilibrium (point A) is much more dramatic.

To sum up, the specification of relative time scale has to be addressed carefully due to the possible change in the stability of the fixed point and the impact on separatrix when there are multiple fixed points. The second effect has important implications for the resilience of the system. When this leads to the expansion/shrinkage of the basin of attraction for the desirable fixed point, such a change means the desirable state of the original system will be able to absolve larger/smaller exogenous shocks.



(a) Slow Process as Constant

(b) Fast Process as Instantaneous

Figure 7: Simplifying Assumptions on Relative Time-Scale

4 Simplification of Divergent Time Scale

This section will discuss simplifications when the two time scales are divergent. It will first discuss the common simplifying assumptions in economic literature: taking the slow process as constant or fast process as instantaneous. Then it will discuss the application of singular perturbation technique. It will also analyze the conditions that such simplifying assumptions can be made. A mathematic interpretation of "divergent time scale" is $\epsilon \ll 1$.

4.1 The Impact of Common Simplifying Assumptions

First, when we are interested in the fast process (like P), it is natural to ask whether we can treat some slow process as constant, especially when we are dealing with already complicated models. In essence, such an assumption can reduce the dimension of the original problem, because the slow dynamic process is degenerated into a single value. Take the baseline model as an example. When we assume the slow variable is a constant, we are imposing the condition that local population N is constant. The impact is illustrated in Figure 7(a): it degenerates the whole N-P plane into a vertical line. Consequently, the simplified model will incorrectly predict the long run equilibria as the intersection of this vertical line with the P-nullcline unless the N level is set exactly at the long run equilibrium level (the solid green line in figure). The multiple equilibria shown in Figure 7(a) further complicates the problem. Therefore, if we are interested in the steady state of the system, we should avoid such a assumption unless we have very good knowledge about the equilibrium level of N, including the information to select the right long-run equilibrium from multiple candidates. But this would require thorough understanding of the interactions between N and P. Thus it would require the same information as determining the exact equilibrium (N, P) of the original system. The point of simplifying the original dynamic system is lost.

Even when the slow-moving variable is fixed right at equilibrium level, it may give false description of the resilience for the coupled system. Again take the baseline model as an example (as shown in Figure 7(a)). The simplified model predicts that there are three fixed points in the system: A, D and E. Point A and E are stable fixed points in the simplified model and Point D is unstable. At fixed point A, the simplified model predicts that the system can only resist an ecological shock of around 20% increase in phosphorous concentration. That is, the phosphorous concentration after shock should be less than P_0. Or the ecological system will be driven to eutrophic state at point E and stay there forever. But the dynamical analysis of the baseline model tells a different story. The location of separatrix shows that the coupled system can resist much larger ecological shocks, so long as the ecological shock does not drive the system across the separatrix. Of course, when large ecological shocks occur (sufficiently larger than 20% increase in P), the ecological system will become eutrophic for some time but this ecological change can be slowly reversed as discussed in the last section. If policy makers cares about the final state of the system, using the simplified model, he is more likely to over-react to ecological shocks then using the un-simplified model.

What if we are interested in the short run behavior of the system? Intuitively, since the slow process is *slow*, the simplified system may provide an approximation for the original system in the short run. But how good is this approximation? Take the dynamic system (4.1) as an example.

Assume that a dynamic system is governed by

$$\begin{cases} \dot{N} &= \epsilon f_1\left(N(t), P(t)\right) \\ \dot{P} &= f_2\left(N(t), P(t)\right) \end{cases} \tag{4.1}$$

As shown in the appendix, an *approximate* under-estimated simplification error (denoted as $\underline{\xi}$) after time period Δt can be written as:

$$\underline{\xi} \approx \frac{1}{2}\left[\epsilon f_1 \frac{\partial f_2}{\partial N}\right]\Bigg|_{(N_0,P_0)} (\Delta t)^2$$

It shows that the slower the slow process (N), the smaller the error after time period Δt. Also, the weaker the impact of the slow process (N) on the fast process (P), through the term $\partial f_2/\partial N$, the smaller the error. The impact of nonlinearity on the error comes through higher order terms as shown in the appendix. It can accumulate rather rapidly. Since the above formula has neglected all the higher order terms, the formula can only be regarded as an approximate under-estimated bound.

This formula can be re-arranged to show how long the simplified model can be used as an approximation. Assume that after initial period of length Δt, the simplification error will exceed η (that is $\xi > \eta$). The length of time Δt satisfies:

$$\underline{\Delta t} \approx \sqrt{\frac{2\,\eta}{\left|\epsilon\, f_1\, \dfrac{\partial f_2}{\partial N}\right|}}$$

This formula only says that when Δt exceed $\underline{\Delta t}$, the error will exceed η. It does not guarantee that within $\underline{\Delta t}$ period, the error is within the bound of η.

In summary, treating the slow process as constant will lead to wrong prediction of the long run equilibria and wrong prediction of the resilience of the system. The quality of the short run approximation will depend on the relative time scale of the slow and fast processes (N and P) and the interaction between the two. Nonlinearity is important and may lead to rapidly growing error through higher order terms of approximation error.

Another simplifying assumption that is also seen in literature is to assume that the fast process instantaneously responds to changes in the slow variable, when research focus is on the evolution of the slow process. Under this assumption, it is self-evident that the simplified model is unable to capture the short run changes. But if done *correctly*, it can correctly predict the location of fixed points. When done properly, this simplification is equivalent to imposing the $\dot{P} = 0$ condition while allowing population N to change. Thus the simplified model can track the population change (N) along the P-nullcline and can give the right prediction on the location of fixed points, as shown by the blue line in Figure 7(b).

However, this simplified model may fail to capture the long run behavior of the system. It is beneficiary to see through a simple example.

Example 4.1. A dynamic system is governed by:

$$\begin{cases} \dot{x} = -z \\ \epsilon \dot{z} = -x + z \end{cases}$$

The assumption of instantaneous adjustment of the fast variable is equivalent to the condition of $\epsilon = 0$. This gives $z = x$. Substitute into the equation of $\dot{x}$, we will have

$$\hat{x}^s = e^{-t} x_0^s$$
$$\hat{z}^s = e^{-t} x_0^s$$

which predicts that the system will converge to $(0,\ 0)$ regardless of the initial condition.

Because the original system is linear, the actual solution can be calculated. It is easy to verify that the system only has one fixed point at $(0,\ 0)$ and it is a saddle point. If we start from $x_0 = 0$ and $z_0 = 1$, the solution explodes as $t \to \infty$, even when $\epsilon \to 0$. The solution with this initial condition is as follows:

$$x(t) = \frac{\epsilon}{\sqrt{4\epsilon + 1}} \left[e^{\frac{(1-\sqrt{4\epsilon+1})}{2\epsilon} t} - e^{\frac{(1+\sqrt{4\epsilon+1})}{2\epsilon} t} \right]$$
$$z(t) = \frac{1}{2\sqrt{4\epsilon + 1}} \left[e^{\frac{(1-\sqrt{4\epsilon+1})}{2\epsilon} t} \big(-1 + \sqrt{4\epsilon + 1} \big) + e^{\frac{(1+\sqrt{4\epsilon+1})}{2\epsilon} t} \big(1 + \sqrt{4\epsilon + 1} \big) \right]$$

Unless we start right on the stable manifold of the fixed point, the simplified version gives incorrect prediction about the long run behavior of the system. Notice that the basin of attraction of the fixed point is changed by the simplification:

$$\frac{d\big(\epsilon \dot{z}\big)}{dz} = 1 > 0$$

This simplification also changes the domain of attraction for the fixed points, which is essentially the reason for the incorrect prediction of long-run behavior in Example 4.1. This can be better explained in the baseline model. The real basin of attraction of the complete model is plotted in Figure 7(b). Under the simplifying assumption, the domains of attraction for stable fixed points A and B are separated by population level n_0, regardless of the P value. This is in effect saying that separatrix is a vertical line in the N-P plane. For instance, starting from point F, the simplified model predicts that P will instantaneously adjust onto the P-nullcline and then evolve to point A. However, in original system, because it is in the domain of attraction for fixed point B, it will eventually go to B.

For a policy maker that is using the simplified model, while the system is at point F, the policy maker will conclude that the system is temporarily out of equilibrium. Based on the observation that population and pollution level in the region is still relatively low, it is unlikely that the policy maker will intervene. But in the un-simplified model, point F is already on the left side of the separatrix. Because the ecological condition is so bad that local population will gradually leave, if no intervention is in place. Since point F is very close to the separatrix, a small policy intervention (either through the improvement of ecological condition or the improvement in local public service) will be sufficient to prevent this eventual economic collapse.

Another complication arises when nonlinearity is present in the model. In the linear world, $\dot{P} = 0$ usually give us a unique solution of $P(N)$ [8]. However, when nonlinearity is present in the model, there can be multiple isolated $P(N)$ as solutions to $\dot{P} = 0$. For instance, in Figure 7(b), when $N = 5.5$, there are three real solutions of P. This will give rise to three possible instantaneous response function for $P(N)$. Then selection among these branches will affect the dynamics of the simplified model. Let's look at another example:

Example 4.2.

$$\begin{cases} \dot{x} = -x + z \\ \epsilon\dot{z} = z(z - 2x) \end{cases}$$

As $\epsilon \to 0$, we have $z_1 = 0$ and $z_2 = 2x$. Substitute z_1 into $\dot{x}$, we have an asymptotically stable solution

$$x(t) = e^{-t}x_0$$

Substitute z_2 into $\dot{x}$, we have an asymptotically unstable solution

$$x(t) = e^{t}x_0$$

Given these two possibilities, which one gives a better approximation of the real model? How shall we determine the selection criteria? Note also that

$$\frac{d(\epsilon\dot{z})}{dz} = 2z - 2x \begin{cases} < 0 & \text{if } z < x \\ > 0 & \text{if } z > x \end{cases}$$

In summary, if we are only interested in the immediate evolution of the system, we may ignore the slow process for a short period, but long run prediction is invalid. If we are interested in the slower process, then treating the fast process as an instantaneous process can give us the right location of equilibrium but it may not give the right long-run dynamics of the system and nor can it be used to approximate the short-run behavior. In both cases, the predicted resilience of the system will be incorrect. The analysis in this section also points out that policy analysis based on these simplified models can be misleading. Complete analysis of the fast-slow dynamics on different time scale is necessary.

[8] The infinite number of solutions or the nonexistence of solution is seldom seen in economic literature.

5 Optimal Policy with Fast-Slow Dynamics

This section analyzes optimal policy when divergent time scale is present. Since the unregulated baseline model described above has bi-stability, optimal policy should provide answers to when and how policy should intervene to carry system across different domains of attraction. In the cases of slowly-reversible changes, the optimal policy should decide whether or not intervention is necessary and beneficial to prevent these slowly-reversible changes. In this world of complex dynamics where 'bad' surprises can arise, one goal of the manager is to identify the policy impact on the resilience of the system (Levin et al. [17] and Anderies et al. [1]). The analysis in this paper shows that optimal policy will always increase the resilience of the desirable equilibrium in the coupled system. The extent of such an improvement, however, depends crucially on the relative time-scale. Numerical simulations also suggest that when the divergence in time scale decreases, in optimal policy, initial intervention tend to be larger in magnitude but shorter in lasting time. Finally, nonlinearity built into the model can give rise to multiple local optimal solutions. The preference of one solution over the other depends crucially on the specification of relative time scale. There are cases where one local optimal solution gives a higher utility than another but is less robust to shocks.

Before discussing the management of the baseline model described in previous sections, a more general policy framework is established and the impact of relative time scale is discussed analytically. As in the baseline model, population migration is an uncoordinated individual behavior, which affects and is affected by ecological changes in the region. Ecological system is governed by nutrient dynamics which is affected by local population through land-use. Policy maker cares about both the local development and environmental quality. He has two policy instruments: one helps to attract immigration to the region ($v_1(t)$) that stimulates local economy; one helps to reduce pollutant in the lake ($v_2(t)$). Adoption of either policy incurs a cost. The optimization problem is formulated as follows:

$$\max_{\{v_1(t), v_2(t)\}} \quad \int_0^\infty e^{-\rho t} \Big\{ U_{pol}(N, P) - C(v_1,\ v_2) \Big\}\, dt \tag{5.1a}$$

subject to

$$\dot{N} = \epsilon\, F(N, P, v_1) \tag{5.1b}$$

$$\dot{P} = G(N, P, v_2) \tag{5.1c}$$

$$N(t) \geq 0,\ P(t) \geq 0 \tag{5.1d}$$

$$N(0) = N_0,\ P(0) = P_0 \tag{5.1e}$$

where $U_{pol}(N, P)$ is the utility that policy maker assigns to a given level of local population and phosphorous concentration. $C(v_1,\ v_2)$ is the cost for policy intervention. Assume $C \geq 0$, $\nabla C > 0$ and $\Delta C > 0$, which are typical assumptions about cost function.[9] Because v_1 is a policy that attracts in-migration, *ceteris paribus*, it increases the utility of local people and leads to higher migration rate. Accordingly, $F(N, P, v_1)$ is assumed to be an increasing function of v_1. Because v_2 is an instrument that improves lake quality, $G(N, P, v_2)$ is assumed to be a decreasing function of v_2. Under the assumption that the marginal impact of policy intervention non-increasing with intervention level, we have $\partial^2 F(N, P, v_1)/\partial v_1^2 \leq 0$ and $\partial^2 G(N, P, v_2)/\partial v_1^2 \leq 0$.

[9] ∇C is the Jacobian of $C(v_1,\ v_2)$ and ΔC is the Hessian.

Assume optimal policy is an interior solution, the current value Hamiltonian is

$$H = [U_{pol}(N,P) - C(v_1,v_2)] + \lambda_1\epsilon\, F(N,P,v_1) + \lambda_2 G(N,P,v_2) \tag{5.2}$$

The necessary conditions are:

$$\frac{\partial H}{\partial v_1} = -\frac{\partial C}{\partial v_1} + \lambda_1\epsilon\,\frac{\partial F}{\partial v_1} = 0 \tag{5.3a}$$

$$\frac{\partial H}{\partial v_2} = -\frac{\partial C}{\partial v_2} + \lambda_2\frac{\partial G}{\partial v_2} = 0 \tag{5.3b}$$

$$\dot{\lambda}_1 = \rho\lambda_1 - \lambda_1\epsilon\,\frac{\partial F}{\partial N} - \lambda_2\frac{\partial G}{\partial N} - \frac{\partial U_{pol}}{\partial N} \tag{5.3c}$$

$$\dot{\lambda}_2 = \rho\lambda_2 - \lambda_1\epsilon\,\frac{\partial F}{\partial P} - \lambda_2\frac{\partial G}{\partial P} - \frac{\partial U_{pol}}{\partial P} \tag{5.3d}$$

Equation (5.3a) determines the level of optimal policy for the slow process. It repeats the basic economic principle of optimality: marginal cost should equal marginal benefit. The first term is the current value marginal cost of the policy on the slow process, the second term is the marginal benefit. The time scale coefficient ϵ affects the marginal benefit. The slower the process, the smaller the marginal benefit. **Under the natural assumptions that marginal cost will increase with the level of intervention (that is, $\Delta C > 0$) and that marginal effect will decrease with it (that is, F is concave in v_1), as the divergence in time scale increases ($\epsilon \downarrow$), the optimal intervention level on the slow process will also decrease ($v_1 \downarrow$).** This confirms the typical practice that, *other things being equal*, policy instrument acting on slower process receives less emphasis.

λ_1 and λ_2 are the shadow price for local population (N) and phosphorous concentration (P), respectively. The inter-temporal changes in these two costate variables are governed by given by Equations (5.3c) and (5.3d), which are simply differences between the marginal benefit and cost for the state variables (N and P). Both equations show that ***ceteris paribus*, as the relative time scales becomes more divergent ($\epsilon \downarrow$), changes in the slow variable will become less important when people are adjusting their valuation over time**. This also justifies the usual practice that when people are adjusting their evaluations over time, they tend to neglect the effect of very slow changes. But this is no justification for the complete neglect of slow changes in all scenarios, as will be explained below.

When the ecological subsystem and economic subsystem are interacting, any policy that neglects it will be suboptimal, see Equations (5.3c) and (5.3d). Take λ_1, the shadow price for population, as an example. Under the current setting, local population pollutes the lake. Valuation of local population should reflect this side effect, so will the optimal policy. This impact is embodied in the third term on the right hand side of Equation (5.3c). When there is nonlinearity in the ecology, where ecologists worry about catastrophic events, the concern should also be incorporated in the shadow price for local population. Essentially, such a nonlinearity means that under certain conditions, a small change in local population (N) can lead to dramatic changes in ecological condition ($\partial G/\partial N$ is large). In an another scenario, the nonlinearity may arise from threshold effect in P. That is, when P exceeds the threshold, ecological system will undergone dramatic changes either due to regime shifts or slowly-reversible changes. This will result in dramatic change in the shadow price for phosphorous (λ_2) and then transmitted to shadow price for local population (λ_1). If such events occur, the optimal policy should adjust accordingly.

Even though the above-mentioned scenarios are among the most interesting/important cases, the optimal policy responses are still unclear based on the first order conditions. Essentially, these first order conditions only spell out the optimality conditions in a small neighborhood of the optimal policy within the policy space. Because the nonlinearity incorporated in the model prohibits analytical solution, these first order conditions are insufficient to answer questions like: when and how shall we adopt/adjust policy when the coupled system is close to a threshold or when we are close to the point where slowly-reversible changes can occur. Nor can they provide adequate policy guidance after the system has received a shock under above-mentioned scenarios. Divergent time scale will exacerbate the problem, because it is very likely that after a short time delay in policy response, the system may have evolved far away from the optimal path, which invalidates the applicability of these first order conditions all together. All these scenarios share one property in common: the system may stay far away from the desirable equilibrium for a fairly long period of time, if not forever.

In order to investigate these interesting and important scenarios, numerical techniques are applied to these representative cases. Based on the belief that all existing coupled systems with divergent time scales have existed for quite a long time, it is unlikely for us to observe initial conditions far off the slow manifold. Since in reality economic and/or ecological shocks may occur, small deviations from the slow manifold is not uncommon. As a result, all of the following cases use initial conditions are close to the slow manifold if not right on it.

Function forms are assumed as follows:

$$U_{pol}(N,P) = \alpha_1\left(N + \frac{3}{P^2+1}\right), \tag{5.4a}$$

$$C(v_1,v_2) = \alpha_2\left(v_1^2 + v_2^2\right), \tag{5.4b}$$

$$F(N,P,v_1) = A_0N - A_1N^2 - A_2P - \bar{U} + \alpha_3 v_1, \tag{5.4c}$$

$$G(N,P,v_2) = L_0 + L_1\,N\,l(N) - sP + \sigma_1\frac{P^{\sigma_2}}{P^{\sigma_2}+P_c^{\sigma_2}} - \alpha_4 v_2. \tag{5.4d}$$

$U_{pol}(N,P)$ is specified so that policy maker prefers larger local economy and better ecological conditions. But his response to ecological changes shares the property of logistic function: when ecological condition is good (P small), policy maker has a relatively high utility payoff; as ecological condition deteriorates (P increases), the utility of policy maker is not significantly affected initially; only when ecological condition has severely deteriorated (P exceed 4 and the lake is transforming into eutrophic state) will the policy maker suffer a significant utility loss. This assumption reflects the reality that policy makers are more responsive to ecological changes after catastrophic events has occurred or when the danger is imminent. Function $F(N,P,v_1)$ is assumed to be linear in v_1. This formulation can be viewed as a first order approximation of more general forms of $F(N,P,v_1)$. $G(N,P,v_2)$ is treated the same way.

Because the final state of the system (when $t=\infty$) has no salvage value for policy makers, except for the initial intervention that may be necessary to drive the system across separatrix or to prevent slowly-reversible changes, the policy intervention is expected to be small in magnitude. As intervention level decreases, the instantaneous cost $C(v_1,\, v_2)$ also decreases. On the other hand, the desirable equilibrium at point A is associated with a medium level of population and a fairly good ecological condition. This will provide positive utility to policy maker. Assume that the system is approaching point A, because migration is a slow process, as time proceeds, the instantaneous utility for policy maker $U_{pol}(N, P)$ will not decrease as rapidly as the instantaneous cost function $C(v_1,\ v_2)$. This difference between the instantaneous cost and benefit will accumulate in the aggregate cost $(\int_0^\infty e^{-\rho t} C(v_1,\ v_2)\, dt)$ and aggregate benefit $(\int_0^\infty e^{-\rho t} U_{pol}(N, P)\, dt)$ over time. Because the optimization horizon is $T = \infty$, the difference in total cost and benefit can be large. This issue is handled by the assumption that $\alpha_1 = 0.001$. This parameter value also helps to illustrate all three representative scenarios in one parameter settings of the model. The results in this paper does not depend crucially on this parameter value. For simplicity, I assume that $\alpha_2 = 0.5$, $\alpha_3 = 1$ and $\alpha_4 = 1$.

Substitute the function forms in Equations (5.4) into optimization problem (5.1). In order to simplify the discussion, notations for constant coefficients in front of state variables are changed. All the coefficients in $\dot{N}$ equation are labelled as θ's and those in $\dot{P}$ equation are labelled as δ's. The Hamiltonian now becomes:

$$H = e^{-\rho t}[N - \mu(P) - 0.5v_1^2 - 0.5v_2^2] \tag{5.5}$$

$$+ \lambda_1 \epsilon \left[\theta_1 + \theta_2 N + \theta_3 N^2 + \theta_4 P + v_1\right] \tag{5.6}$$

$$+ \lambda_2 \left[\delta_1 + \delta_2 N + \delta_3 N^2 + \delta_4 P + \delta_5 \frac{P^8}{P^8 + P_c^8} - v_2\right] \tag{5.7}$$

The necessary conditions are:

$$\frac{\partial H}{\partial v_1} = -e^{-\rho t} v_1 + \lambda_1 \epsilon = 0 \tag{5.8a}$$

$$\frac{\partial H}{\partial v_2} = -e^{-\rho t} v_2 - \lambda_2 = 0 \tag{5.8b}$$

$$\dot{\lambda}_1 = -\lambda_1 \epsilon \left[\theta_2 + 2\theta_3 N\right] - \lambda_2 \left[\delta_2 + 2\delta_3 N\right] - e^{-\rho t} \tag{5.8c}$$

$$\dot{\lambda}_2 = -\lambda_1 \epsilon\, \theta_4 - \lambda_2 \left[\delta_4 + \delta_5 \big[\frac{P^8}{P^8 + P_c^8}\big]'\right] + e^{-\rho t} \mu'(P) \tag{5.8d}$$

With constraints

$$\dot{N} = \epsilon \left[\theta_1 + \theta_2 N + \theta_3 N^2 + \theta_4 P + v_1\right] \tag{5.8e}$$

$$\dot{P} = \delta_1 + \delta_2 N + \delta_3 N^2 + \delta_4 P + \delta_5 \frac{P^8}{P^8 + P_c^8} - v_2 \tag{5.8f}$$

Initial conditions

$$N(t) \geq 0,\ P(t) \geq 0 \tag{5.8g}$$

$$N(t_0) = N_0,\ P(t_0) = P_0 \tag{5.8h}$$

Transversality conditions

$$0 = \lim_{T \to +\infty} \lambda_1(T) N(T) \tag{5.8i}$$

$$0 = \lim_{T \to +\infty} \lambda_2(T) P(T) \tag{5.8j}$$

The first order conditions along with the initial and transversality conditions formulates a boundary value problem. This boundary value problem has mixed boundary conditions, with initial conditions specified for state variables N and P and ending conditions for costate variables λ_1 and λ_2. The problem is solved with boundary value problem solver bvp4c.m in Matlab.

The first conclusion from policy analysis is that optimal policy will increase the resilience of the desirable equilibrium even when resilience is not considered in policy analysis. Following Holling [24], resilience is defined as the size of the basin of attraction in which the system resides. In current model with no policy intervention, we have three fixed points (A, B and C), as shown in Figure 3. As explained earlier, equilibrium A is more preferable than both B and C. Therefore, a policy that can successfully shift the system from the domain of attraction of B or C [10] to that of A will generate a positive gain in utility so long as $\epsilon \neq 0$. Imagine that the dynamic system lies right on the separatrix, an instantaneous and infinitely small push in the right direction can push the system into the domain of attraction of A. While there is a positive gain in utility, the cost is infinitely small. Therefore, optimal policy will at least extend the domain of attraction of A to include the separatrix. Therefore, a purely utility driven policy can increase the resilience of the desirable equilibrium. Following the same argument, when the initial state of the system reside on the left side of the separatrix but is infinitely close to it, policy intervention will also be welfare improving. As we move further away from the separatrix into the domain of attraction of fixed point B, the cost of intervention will increase. There will be a maximum distance from separatrix that policy intervention to cause the regime shift is beneficial. Numerical simulations show that this maximum distance depends crucially on the relative time scale. When the divergence in time scale increases (time-scale coefficient ϵ decreases from 0.08 to 0.06), the maximum beneficial extension (as measured by horizontal distance to separatrix) can change by 15-40%.[11] In conclusion, purely utility driven optimal policy will increase the resilience for the desirable equilibrium in the N-P space.

Optimal policy in three typical scenarios are studied. **In all these three scenarios, as the relative time-scale becomes more divergent ($\epsilon \downarrow$), the initial policy intervention (both v_1 and v_2) becomes smaller in magnitude and longer in lasting period.** The first scenario is where the initial condition lies deep in the domain of attraction of the desirable fixed point A. In this scenario, initial condition is set to (4, 2.32). The optimal policy (v_1 and v_2) is plotted on the first row of Figure 8. The second scenario is where the initial condition is on the left side of the separatrix, so that without policy intervention, the system will converge to the undesirable equilibrium at point B. In this scenario, initial condition is set to (2.8, 1.72). The optimal policy is plotted on the second row of Figure 8. The third one is where slowly-reversible changes will occur if there is no policy intervention. In this scenario, initial condition is set to (6.5, 3.75). The optimal policy is plotted on the third row. In each of these three scenarios, the coefficient controlling the relative time-scale (ϵ) is changed from 0.04 to 0.08 and then to 0.16. The corresponding optimal policy is denoted as the blue, red and pink lines respectively. As shown in the figure, as the divergence in time-scale reduces ($\epsilon \uparrow$), the initial policy intervention becomes larger in magnitude and

[10] The domain of attraction for unstable fixed point C is simply the separatrix.

[11] Because separatrix is a curve in N-P space, the curve that demarcate the maximum beneficial intervention (call it maximum extension curve) will also be a curve in N-P space. As divergence in time scale varies, this maximum extension curve will not change uniformly. Thus the distance to separatrix may vary along this extension curve.

shorter in lasting period. This result reflects the fact that implementation cost depends on the relative time scale. *Ceteris paribus*, as the relative time scale becomes less divergent ($\epsilon \uparrow$), with the same initial condition and policy, the system can achieve any state in N-P space in less time. Therefore, it is unnecessary for the policy to last for the same period of time. At the same time, because the cost of policy intervention has been reduced, a more active intervention in the initial period is affordable. This active intervention in the initial period will make it unnecessary to maintain the same level of intervention later. As a result, the level of intervention in later period will decrease. The period of active intervention is reduced further. This change in the magnitude and lasting-period of intervention highlights the substitution between these two aspects of policy intervention.

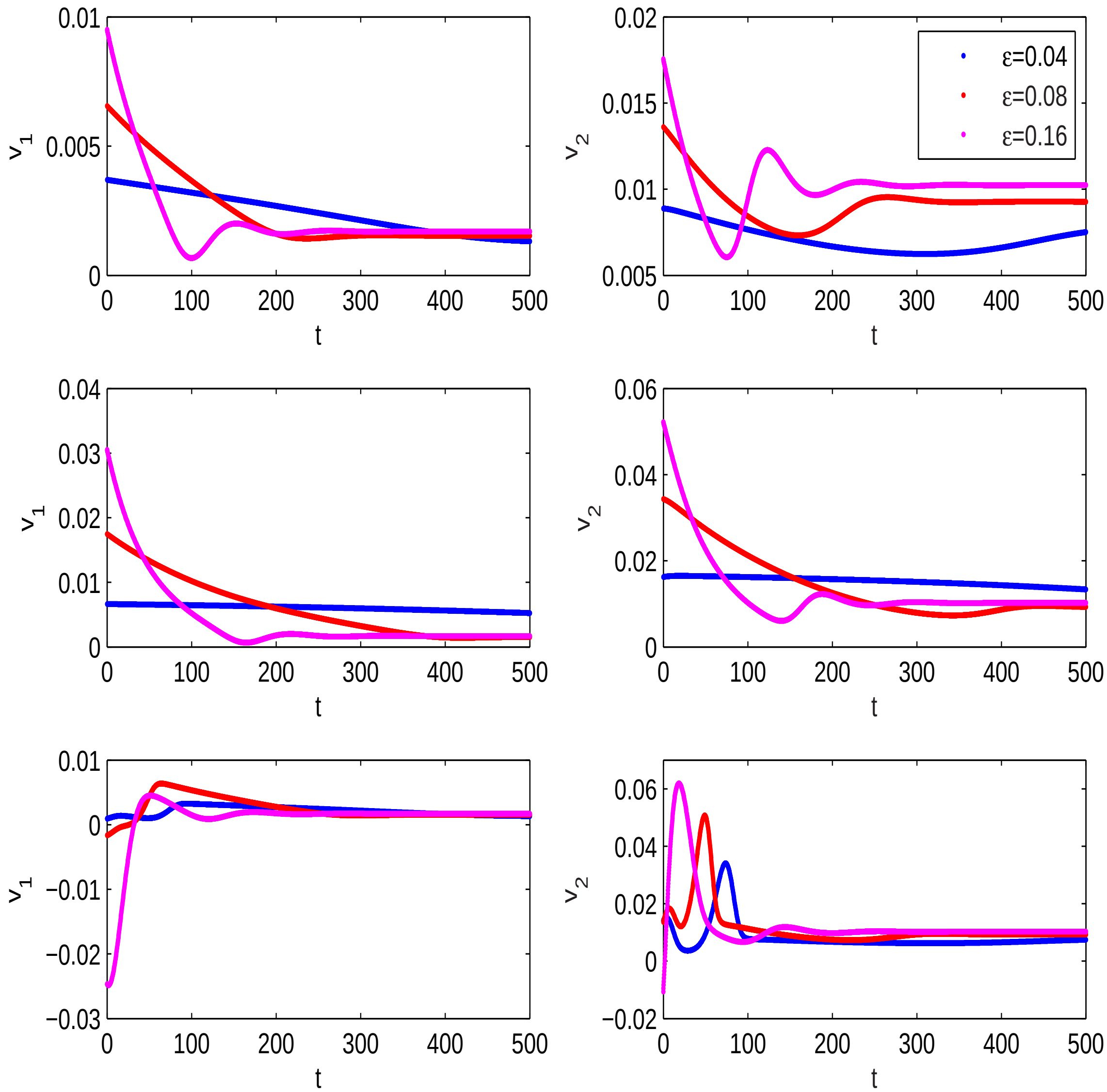


Figure 8: Impact of Relative Time-Scale on Optimal Policy.

The net gain from policy intervention is highly sensitive to the changes in relative time scale (ϵ). Table 1 lists the difference in utility with and without policy

Table 1: Net Utility Gain and Divergence in Time Scale

	$(N_0\ ,\ \ P_0)$	$\epsilon = 0.04$	$\epsilon = 0.08$	$\epsilon = 0.16$
Scenario I	(4.0 , 2.32)	0.8%	1.4%	1.5%
Scenario II	(2.8 , 1.72)	4.8%	26.2%	80.0%
Scenario III	(6.5 , 3.75)	1.8%	6.1%	67.4%

intervention. In the first scenario, when divergence in time scale is set at $\epsilon = 0.04$, optimal policy intervention generates 0.8% utility gain as compared to the case with no policy intervention. When the relative time scale changes from 0.04 to 0.08, policy intervention can generate 1.4% increase in utility. The relative gain has almost doubled as compared with the case when $\epsilon = 0.04$. The result is more sensitive in the second and third scenarios. In the second scenario, when $\epsilon = 0.04$, the net gain is 4.8%, but when ϵ doubles, the net gain increases to 26.2% This large gain utility is due to the fact that optimal policy has change final state of the system from the undesirable state (point B) to the desirable state (point A). When ϵ increase further to 0.16, the net gain increase to 80.0%. This is mainly due to the fact that the system reaches the good state (A) in much shorter period. The third scenario is the case where ecology suffers from a slowly-reversible change when there is no policy intervention. When ϵ increase from 0.04 to 0.08, because the system recovers faster, the net gain increase from 1.8% to 6.1%. When ϵ increase further 0.16, the net gain increase more dramatically, this is because at this specification of relative time scale, it is optimal to eliminate the slowly-reversible change altogether.

The third scenario gives rise to multiple local solutions. As shown in Figure 9, both sol_a and sol_b are local optimal solutions. Results show that different specification of relative time scale can lead to different preference among these policy candidates. This result is closely related to the concept of Skiba point. According to Skiba [39] and Brock and Starrett [4], Skiba point is the dividing point where on one side sol_a is the global optimal solution and on the other side sol_b is global optimal. Simulations show that changes in relative time scale can transform a non-Skiba point into a Skiba point. When $\epsilon = 0.04$, sol_a gives a highest utility than sol_b and is global optimal. When $\epsilon = 0.08$, the utility difference between sol_a and sol_b narrows, but sol_a still dominates. When $\epsilon = 0.16$, sol_b gives higher utility than sol_a and becomes global optimal. Detailed simulations show that for an ϵ in the range of [0.08, 0.09], the initial condition $(N_0, P_0) = (6.5, 3.75)$ becomes Skiba point.

In this third scenario, the fundamental problem lies in the excessive population in the region. The ecological pressure exerted by human activities is beyond the carrying capacity of the ecological system and an ecological catastrophe is in place. In order to prevent the catastrophe, policy maker has to implement a policy that will eventually reduce human population and local pollution. Intuitively, there are two ways to achieve this goal. One choice of policy involves active intervention to prevent the catastrophic ecological degradation altogether. sol_b in Figure 9 is such a policy. Expenditure of this policy intervention mainly concentrate in the initial period. Another choice of policy is to intervene less actively. The aim of intervention is not to prevent the initial catastrophic event, but to accelerate the recovery process. sol_a is such a policy. Even though the level of initial intervention in sol_a is not as high as in sol_b, the intervention in sol_a will last for longer time. As a result, the total expenditure of policy sol_a is more evenly spread over time than sol_b. Which of these two policies gives higher utility depends on the relative time scale in the model. Since excessive local population is the essence of problem in this third scenario, when relative time scale is less divergernt (ϵ is sufficiently large), sol_b becomes less costly. Because the essence of sol_b is that it tries to reduce population in order to prevent the immediate ecological degradation, when ϵ is large, the cost of this policy become small. When ϵ is large enough, it would be more desirable to implement this policy and prevent the immediate catastrophic event and the slow recovery that follows. When the relative time scale becomes more divergent ($\epsilon \downarrow$), The cost of this intervention will increase because population now changes more slowly. When the the relative time scale is divergent enough, the cost of sol_b can become too high that completely eliminating the ecological catastrophe is too costly. As a result, a moderate intervention aiming at the acceleration of the recovery process (sol_a) can become more beneficial. This economic motivation is the underlying reason that changes in relative time scale can transform a non-Skiba point into Skiba point.

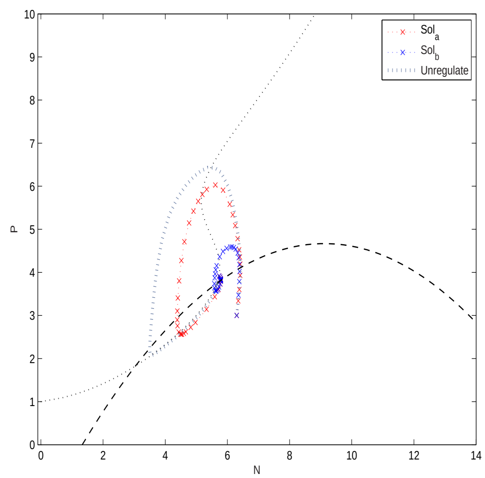


Figure 9: Multiple Local Solutions

6 Conclusion

This paper establishes a coupled human-ecological model where the slow-varying migration process is interacting with a fast-varying nutrient dynamics in lake ecology. The multiple equilibria built into the model can generate regime shifts that can lead to the collapse of local economy. Optimal management of such a coupled system with multiple equilibria is studied and the dynamic solution is derived. The fast-slow dynamics build in the model creates multiple time scales that give rise to slowly-reversible changes. The impact of relative time scale on the dynamical system with and without policy intervention is analyzed. The result shows that:

1) Relative time scale is important for the description of system dynamics. The fast-slow dynamics in the model can give rise to catastrophic ecological degradation that is slowly reversible. Mis-specification of relative time scale can change the stability of the fixed point and shift the location of separatrix. These changes can fundamentally change our understanding of the resilience of the system.

2) Simplification of relative time scale has to be conducted careful. Simplifying the slow process as a constant can give an approximation for the original system only in the short run. The length of this initial period depends on the relative time scale in the original system. On the other hand, treating the fast process as an instantaneous process can give the right location of equilibrium. But the simplified model can lead to incorrect prediction about both the short- and long-run dynamics. Policy analysis based on these simplifying assumptions can lead to either excessive or inadequate policy intervention when the system receives an exogenous shock and faces the danger of regime shifts.

3) Relative time scale is an important factor in policy analysis. It has significant impact on the net gains from policy intervention. Moreover, as the relative time scale becomes more divergent, intervention in optimal policy becomes smaller in magnitude and longer in lasting period. When policy maker is confronted with multiple policy alternatives, different specification of the relative time scale can lead to different choice of policy. It can also transform non-Skiba point into Skiba point.

4) The analysis also shows that even when resilience does not show up in optimization problem for policy makers, optimal policy will increase the resilience of the most desirable equilibria.

The analysis in this paper shows that relative time scale is a crucial element in models of coupled human-ecological systems. Future work can proceed along two lines. One is finding empirical evidence to support the analysis here. The major concern is that the slow process varies so slowly and its data typically are collected so infrequently, that the current available data may not be sufficient for empirical studies. The second is further exploration of multiple time scales. In this model, there are essentially two time scales. In case of three distinct time scales, more interesting result may come out of the analysis.[12] One case would be that the policy intervention may have its own time scale, because all policy maker have finite service period and because choice and implementation of policy all takes time.

[12]Irwin, Jayaprakash and Chen [26] have briefly touched upon the issue.

A Parameter Values

The parameter specifications in baseline model is:

a	b	r_a	$\overline{u}$	A_1	A_2
0.75	0.67	1.2	0.45	0.02	0.26
L_0	L_1	s	σ_1	σ_2	P_c
0.14	0.06	0.14	0.5	8	5
ϵ	ρ	α_1	α_2	α_3	α_4
0.1	0.01	0.001	0.5	1	1

B Derivation of the Simplification Error

B.1 Second Order Errors

We have a dynamic system governed by

$$(A) \quad \begin{cases} \dot{N} &= f_1\left(N(t), P(t)\right) \\ \dot{P} &= f_2\left(N(t), P(t)\right) \end{cases} \tag{B.1}$$

Note that I have omitted ϵ in $\dot{N}$ equation for simplicity. Because ϵ is a constant, all the results derived in this appendix holds when we substitute f_1 with ϵf_1. This gives the formula in the paper.

With simplification assumption that N can be treated as constant because it is changing slowly, we have a new system governed by

$$(B) \quad \begin{cases} \dot{\underline{N}} &= 0 \\ \dot{\underline{P}} &= f_2\left(\underline{N}(t_0), \underline{P}(t)\right) \end{cases} \tag{B.2}$$

Let the initial condition be N_0 and P_0 at time t_0 for both systems. Denote $P(t)$ and $N(t)$ as the correct solution; $\underline{P}(t)$ and $\underline{N}(t)$ as the solution when N is treated as constant. We want to determine the scale of Δt such that $\left|P(t_0+\Delta t) - \underline{P}(t_0+\Delta t)\right| \leq \eta$.

It can be rewritten as:

$$\begin{aligned} P(t_0+\Delta t) - \underline{P}(t_0+\Delta t) = &\left[P(t_0+\Delta t) - \hat{P}(t_0+\Delta t)\right] - \left[\underline{P}(t_0+\Delta t) - \hat{\underline{P}}(t_0+\Delta t)\right] \\ &+ \left[\hat{P}(t_0+\Delta t) - \hat{\underline{P}}(t_0+\Delta t)\right] \end{aligned} \tag{B.3}$$

where "$\wedge$" stands for an approximation.

In general, a ordinary differential equation system,

$$\dot{X} = F(X)$$

can be integrated as

$$X(t_0+\Delta t) = X(t_0) + \int_{t_0}^{t_0+\Delta t} F(X(\tau))\, d\tau$$

The integral can be approximate by

$$\hat{X}(t_0+\Delta t) \approx \hat{X}(t_0) + \sum_{j=0}^{k-1} F(\hat{X}(j))\Delta s$$

where $X(0) = \hat{X}(t_0)$, $\hat{X}(j) = \hat{X}(t_0 + j\,\Delta s)$, and $\hat{X}(k) = X(t_0+\Delta t)$.

According to Atkinson (1989) (page 346 theorem 6.3), the error of this approximation is approximately $o(\Delta^2 s)$. This proves that the error introduced by the first two terms in eq (B.3) are of the magnitude $o(\Delta^2 s)$. Now, let's focus on the third term. Intuitively, as time evolves, we expect that the error introduced by the simplification should accumulate and this accumulation should be present in this third term. Therefore, we should expect that this third term in general will not be mitigated by reducing the size of Δs.

Apply the same approximation to $P(t)$ and $\underline{P}(t)$, we have the iterative formula:

$$(A) \quad \begin{cases} \hat{N}_{k-j} = \hat{N}_{k-j-1} + f_1(\hat{N}_{k-j-1}, \hat{P}_{k-j-1})\,\Delta s \\ \hat{P}_{k-j} = \hat{P}_{k-j-1} + f_2(\hat{N}_{k-j-1}, \hat{P}_{k-j-1})\,\Delta s \end{cases} \tag{B.4a}$$

$$(B) \qquad \underline{\hat{P}}_{k-j} = \underline{\hat{P}}_{k-j-1} + f_2(N_0, \underline{\hat{P}}_{k-j-1})\,\Delta s \tag{B.4b}$$

Assume $f_1(N(\tau),\, P(\tau)),\ f_2(N(\tau),\, P(\tau)) \in \mathbb{C}^3$ for $\tau \in [t_0,\, t_0 + \Delta t]$ and Δs is small enough that Taylor series expansion is applicable. We can then expand eq (B.4a) around $\left(\hat{N}_{k-j-2},\, \hat{P}_{k-j-2}\right)$. Apply the iterative formula again and then expand around $\left(\hat{N}_{k-j-3},\, \hat{P}_{k-j-3}\right)$ and so on, until we get to $\left(\hat{N}_0,\, \hat{P}_0\right)$. If we keep track only of the second order terms, we have

$$\begin{aligned} \hat{P}_k - \underline{\hat{P}}_k &\approx \frac{1}{2}(k-1)k\left[f_1\frac{\partial f_2}{\partial \hat{N}}\right]\bigg|_{(N_0,P_0)} (\Delta s)^2 \\ &\approx \frac{1}{2}\left[f_1\frac{\partial f_2}{\partial \hat{N}}\right]\bigg|_{(N_0,P_0)} (\Delta t)^2 \end{aligned}$$

But because the higher order terms does not vanish except for very special function forms, the above formula can be regarded as an approximate estimate of the lower bound for the error and thus an approximate upper bound for Δt in question.

B.2 Third Order Simplification Errors

In this process, we only keep track of the third order terms. Let's assume the following formula is correct to the third order.

$$
\begin{aligned}
\hat{P}_k \approx\ & \hat{P}_{k-j} + j\, f_2(\hat{N}_{k-j}, \hat{P}_{k-j})\Delta s \\
& + n_1 \left[\frac{\partial f_2}{\partial \hat{N}}(\hat{N}_{k-j}, \hat{P}_{k-j}) f_1(\hat{N}_{k-j}, \hat{P}_{k-j}) + \frac{\partial f_2}{\partial \hat{P}}(\hat{N}_{k-j}, \hat{P}_{k-j}) f_2(\hat{N}_{k-j}, \hat{P}_{k-j}) \right] (\Delta s)^2 \\
& + \frac{1}{2}\, n_2 \left(\Delta s \right)^3 \cdot \\
& \left\{ \frac{\partial^2 f_2}{\partial \hat{N}^2}(\hat{N}_{k-j}, \hat{P}_{k-j}) \left[f_1(\hat{N}_{k-j}, \hat{P}_{k-j}) \right]^2 \right. \\
& \quad + 2\, \frac{\partial^2 f_2}{\partial \hat{N} \partial \hat{P}}(\hat{N}_{k-j}, \hat{P}_{k-j}) f_1(\hat{N}_{k-j}, \hat{P}_{k-j}) f_2(\hat{N}_{k-j}, \hat{P}_{k-j}) \\
& \quad \left. + \frac{\partial^2 f_2}{\partial \hat{P}^2}(\hat{N}_{k-j}, \hat{P}_{k-j}) \left[f_2(\hat{N}_{k-j}, \hat{P}_{k-j}) \right]^2 \right\} \\
& + n_3 \left(\Delta s \right)^3 \cdot \\
& \left\{ \left[\frac{\partial^2 f_2}{\partial \hat{N}^2}(\hat{N}_{k-j}, \hat{P}_{k-j}) f_1(\hat{N}_{k-j}, \hat{P}_{k-j}) + \frac{\partial f_2}{\partial \hat{N}}(\hat{N}_{k-j}, \hat{P}_{k-j}) \frac{\partial f_1}{\partial \hat{N}}(\hat{N}_{k-j}, \hat{P}_{k-j}) \right] f_1(\hat{N}_{k-j}, \hat{P}_{k-j}) \right. \\
& \quad + \left[\frac{\partial^2 f_2}{\partial \hat{N} \partial \hat{P}}(\hat{N}_{k-j}, \hat{P}_{k-j}) f_1(\hat{N}_{k-j}, \hat{P}_{k-j}) + \frac{\partial f_2}{\partial \hat{N}}(\hat{N}_{k-j}, \hat{P}_{k-j}) \frac{\partial f_1}{\partial \hat{P}}(\hat{N}_{k-j}, \hat{P}_{k-j}) \right] f_2(\hat{N}_{k-j}, \hat{P}_{k-j}) \\
& \quad + \left[\frac{\partial^2 f_2}{\partial \hat{N} \partial \hat{P}}(\hat{N}_{k-j}, \hat{P}_{k-j}) f_2(\hat{N}_{k-j}, \hat{P}_{k-j}) + \frac{\partial f_2}{\partial \hat{P}}(\hat{N}_{k-j}, \hat{P}_{k-j}) \frac{\partial f_2}{\partial \hat{N}}(\hat{N}_{k-j}, \hat{P}_{k-j}) \right] f_1(\hat{N}_{k-j}, \hat{P}_{k-j}) \\
& \quad \left. + \left[\frac{\partial^2 f_2}{\partial \hat{P}^2}(\hat{N}_{k-j}, \hat{P}_{k-j}) f_2(\hat{N}_{k-j}, \hat{P}_{k-j}) + \left(\frac{\partial f_2}{\partial \hat{P}} \right)^2 (\hat{N}_{k-j}, \hat{P}_{k-j}) \right] f_2(\hat{N}_{k-j}, \hat{P}_{k-j}) \right\}
\end{aligned}
$$

Substitute in the iterative formula eq(B.4a). Expand around $(\hat{N}_{k-j-1}, \hat{P}_{k-j-1})$ and keep the third order terms. We have

$$
\begin{aligned}
\hat{P}_k = &\hat{P}_{k-j-1} + (j+1)\, f_2(\hat{N}_{k-j-1}, \hat{P}_{k-j-1})\Delta s \\
&+ (n_1 + j)\, (\Delta s)^2 \cdot \\
&\left\{ \frac{\partial f_2}{\partial \hat{N}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) f_1(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) + \frac{\partial f_2}{\partial \hat{P}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) f_2(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \right\} \\
&+ \frac{1}{2}\, (n_2 + j) \Big(\Delta s\Big)^3 \cdot \\
&\Bigg\{ \frac{\partial^2 f_2}{\partial \hat{N}^2}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \Big[f_1(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \Big]^2 \\
&\quad + 2\, \frac{\partial^2 f_2}{\partial \hat{N} \partial \hat{P}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) f_1(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) f_2(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \\
&\quad + \frac{\partial^2 f_2}{\partial \hat{P}^2}(\hat{N}_{k-(j+1)}, \hat{P}_{k-j-1}) \Big[f_2(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \Big]^2 \Bigg\} \\
&+ (n_3 + n_1) \Big(\Delta s\Big)^3 \cdot \\
&\Bigg\{ \Bigg[\frac{\partial^2 f_2}{\partial \hat{N}^2}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) f_1(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) + \frac{\partial f_2}{\partial \hat{N}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \frac{\partial f_1}{\partial \hat{N}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \Bigg] \cdot \\
&\quad \cdot f_1(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \\
&\quad + \Bigg[\frac{\partial^2 f_2}{\partial \hat{N} \partial \hat{P}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) f_1(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) + \frac{\partial f_2}{\partial \hat{N}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \frac{\partial f_1}{\partial \hat{P}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \Bigg] \cdot \\
&\quad \cdot f_2(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \\
&\quad + \Bigg[\frac{\partial^2 f_2}{\partial \hat{N} \partial \hat{P}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) f_2(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) + \frac{\partial f_2}{\partial \hat{P}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \frac{\partial f_2}{\partial \hat{N}}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \Bigg] \cdot \\
&\quad \cdot f_1(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \\
&\quad + \Bigg[\frac{\partial^2 f_2}{\partial \hat{P}^2}(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) f_2(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) + \Big(\frac{\partial f_2}{\partial \hat{P}}\Big)^2 (\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \Bigg] \cdot \\
&\quad \cdot f_2(\hat{N}_{k-j-1}, \hat{P}_{k-j-1}) \Bigg\}
\end{aligned}
$$

The coefficients in the above formula is determined through induction. The process is summarized in the following table.

Table 2: Derivation of coefficients in $\hat{P}_j$

Expansion of $\hat{P}_k$	$j = j+1$	$n_1 = n_1 + j$	$n_2 = n_2 + j$	$n_3 = n_3 + n_1$
$\hat{P}_{k-1}$	1	0	0	0
$\hat{P}_{k-2}$	2	1	1	0
$\hat{P}_{k-3}$	3	3	$1+2$	1
$\hat{P}_{k-4}$	4	6	$1+2+3$	1+3
$\hat{P}_{k-5}$	5	10	$1+2+3+4$	1+3+6
$\hat{P}_{k-j}$	j	$\frac{1}{2}(j-1)j$	$\frac{1}{2}(j-1)j$	$\frac{1}{6}(j-2)(j-1)j$
$\hat{P}_0$	k	$\frac{1}{2}(k-1)k$	$\frac{1}{2}(k-1)k$	$\frac{1}{6}(k-2)(k-1)k$

Therefore,

$$\begin{aligned}
\hat{P}_k = {} & P_0 + k\, f_2(N_0, P_0)\Delta s \\
& + \frac{1}{2}(k-1)k\left[f_1 \frac{\partial f_2}{\partial \hat{N}} + f_2 \frac{\partial f_2}{\partial \hat{P}}\right]\Big|_{(N_0,P_0)} (\Delta s)^2 \\
& + \frac{1}{4}(k-1)k\left(f_1 \frac{\partial}{\partial \hat{N}} + f_2 \frac{\partial}{\partial \hat{P}}\right)^2 [f_2]\Big|_{(N_0,P_0)} \Big(\Delta s\Big)^3 \\
& + \frac{1}{6}(k-2)(k-1)k\left(\frac{\partial}{\partial \hat{N}}\left[f_1 \frac{\partial f_2}{\partial \hat{N}} + f_2 \frac{\partial f_2}{\partial \hat{P}}\right] f_1\right)\Big|_{(N_0,P_0)} \Big(\Delta s\Big)^3 \\
& + \frac{1}{6}(k-2)(k-1)k\left(\frac{\partial}{\partial \hat{P}}\left[f_1 \frac{\partial f_2}{\partial \hat{N}} + f_2 \frac{\partial f_2}{\partial \hat{P}}\right] f_2\right)\Big|_{(N_0,P_0)} \Big(\Delta s\Big)^3
\end{aligned}$$

For the simplified system, the above formula is also applicable. We only need to assume: $f_1(\cdot,\cdot) \equiv 0$ and $N_t = N_0$. This also implies that the error will include those terms involving f_1:

$$\begin{aligned}
\hat{P}_k - \underline{\hat{P}}_k \approx {} & \frac{1}{2}(k-1)k\left[f_1 \frac{\partial f_2}{\partial \hat{N}}\right]\Big|_{(N_0,P_0)} (\Delta s)^2 \\
& + \frac{1}{6}(k-2)(k-1)k\Big(\Delta s\Big)^3 \cdot \\
& \cdot \left\{\Big[\frac{\partial f_2}{\partial \hat{N}}\frac{\partial f_1}{\partial \hat{N}} + \frac{\partial^2 f_2}{\partial \hat{N}^2} f_1\Big] f_1 + \Big[\frac{\partial f_2}{\partial \hat{P}}\frac{\partial f_2}{\partial \hat{N}} + \frac{\partial^2 f_2}{\partial \hat{N}\partial \hat{P}} f_2\Big] f_1 + \Big[\frac{\partial f_2}{\partial \hat{N}}\frac{\partial f_1}{\partial \hat{P}} + \frac{\partial^2 f_2}{\partial \hat{N}\partial \hat{P}} f_1\Big] f_2\right\}\Big|_{(N_0,P_0)}
\end{aligned}$$

Note that the third order error does not vanish even without nonlinearity. This is due to the accumulation of error in the the approximation procedure. However, error of nonlinearity only comes to play from the third order terms.

As the third order term does not vanish, neither does the other higher order terms, the second order error can be regarded as a lower bound of the error in general. [13] In other words, it imposes an upper bound for the length of the initial period that the simplification

[13] Theoretically, there is the possibility that the errors can offset each other. In reality, however, I believe it is very unlikely and therefore regard the second order term as an approximate lower bound.

can be used as an approximate. Denote this upper bound as $\overline{\Delta t}$ and let $\overline{\Delta t} = k\,\Delta s$, we have

$$\begin{aligned}\left|\hat{P}(t_0 + \overline{\Delta t}) - \underline{\hat{P}}(t_0 + \overline{\Delta t})\right| &\approx \frac{1}{2}(k-1)k\left|\frac{\partial f_2}{\partial N}\, f_1\right|(\Delta s)^2 \\ &\approx \frac{1}{2}k^2\left|\frac{\partial f_2}{\partial N}\, f_1\right|(\Delta s)^2 \le \eta\end{aligned}$$

which implies

$$\overline{\Delta t} \approx \sqrt{\frac{2\,\eta}{\left|\dfrac{\partial f_2}{\partial N}\, f_1\right|}}$$

References

[1] John M. Anderies, *Economic development, demographics, and renewable resources: A dynamical ssytems approach*, Environment and Development Economics **8** (2003), 219–246.

[2] Kenneth Arrow, Gretchen Daily, Partha Dasgupta, Simon Levin, Kal-Göran Mäler, Eric Maskin, David Starrett, Thomas Sterner, and Thomas Tietenberg, *Managing ecosystem resources*.

[3] Kenneth Arrow, Partha Dasgupta, and Kal-Göran Mäler, *Evaluating projects and assessing sustainable development in imperfect economies*, Environmental and Resource Econometics **26** (2003), no. 4, 647–685.

[4] W. A. Brock and D. Starrett, *Managing systems with non-convex positive feedback*, Environmental and Resouce Economics **26** (2003), 575–602.

[5] William A. Brock and M. Scott Taylor, *Economic growth and the environment: A review of theory and empirics*, NBER Working Paper 10854, NBER, October 2004.

[6] Guillermo A. Calvo, *Staggered prices in a utility maximizing framework*, **XII** (1983), 383–398.

[7] S. Carpenter, B. Walker, JM Anderies, and N Abel, *From metaphor to measurement: Resilience of what to what?*, Ecosystems **4** (2001), no. 8, 765–781.

[8] S. R. Carpenter, D. Ludwig, and W. A. Brock, *Mangement of eutrophication for lakes subject to potentially irreversible change*, Ecological Applications **9** (1999), no. 3, 751–771.

[9] Jérôme Chave and Simon Levin, *Scale and scaling in ecological and economic systems*, Environmental and Resource Econometics **26** (2003), no. 4, 527–557.

[10] Yong Chen, Elena G. Irwin, and Ciriyam Jayaprakash, *Dynamic modelling of environmental amenity-driven migration with ecological feedbacks*, Submitted to Ecological Economics, August 2008.

[11] George M. Constantinides, *Habit formation: A resolution of the equity premium puzzle*, Journal of Political Economy **98** (1990), no. 3, 519–543.

[12] Anne-Sophie Crepin, *Using fast and slow processes to manage resources with thresholds*, Environmental and Resource Economics **36** (2007), 191–213.

[13] M. Cropper and W. Oates, *Environmental economics: A survey*, Journal of Economic Literature **30** (1992), 675–740.

[14] Partha Dasgupta and Karl-Göran Mäler, *The economics of non-convex ecosystems: Introduction*, Environmental and Resource Econometics **26** (2003), no. 4, 499–525.

[15] Elamin H. Elbasha and Terry L. Roe, *On endogenous growth: The implications of environmental externalities*, Journal of Environmental Economics and Management **31** (1996), 240–268, Treat environment change as instantaneous.

[16] Kenneth Arrow et al, *Are we consuming too much?*, Journal of Economic Perspectives **18** (2004), no. 3, 147–172.

[17] Simon Levin et al, *Resilience in natural and socioeconomic systems*, Environment and Development Economics **3** (1998), no. 2, 222–35.

[18] Carl Folke, *Resilience: The emergence of a perspective for social-ecological systems analyses*, Global Environmental Change **16** (2006), 253–267.

[19] M. Fujita, P. Krugman, and A. J. Venables, *The spatial economy: Cities, regions and international trade*, MIT Press, Cambridge, MA, 1999.

[20] M. Fujita and J. F. Thisse, *Economics of agglomeration: Cities, industrial location and regional growth*, Cambridge University Press, Cambridge, U.K., 2002.

[21] Kristine M. Grimsrud and Ray Huffaker, *Solving multidimensional bioeconomic problems with singular-perturbation reduction methods: Application to managing pest resistance to pesticidal crops*, Journal of Environmental Economics and Management **51** (2006), 336–353.

[22] R. Hahn, *The impact of economics on environmental policy*, Journal of Environmental Economics and Management **39** (2002), no. 3, 328–350.

[23] Lars Hein, *Cost-efficient eutrophication control in a shallow lake ecosystem subject to two steady states*, Ecological Econometics **59** (2006), no. 4, 429–439.

[24] C.S. Holling, *Resilience and stability of ecological systems*, Annual Review of Ecology and Systematics **4** (1973), 1–23.

[25] Ray Huffaker and Rollin Hotchkiss, *Economic dynamics of reservoir sedimentation management: Optimal control with singularly perturbed equations of motion*, Journal of Economic Dynamics and Control **30** (2006), 2553–2575.

[26] Elena G. Irwin, Ciriyam Jayaprakash, and Yong Chen, *A dynamic model of household location, regional growth, and endogenous natural amenities with cross-scale interactions*, The Frontiers of Environmental Economics (Washington, DC), Resources for the Future, Feburary 2007.

[27] Russell Johnson (ed.), *Geometric singular perturbation theory*, first ed., pp. 44–118, Springer, Berlin, Germany, 1994.

[28] Klaus Keller, Benjamin M. Bolker, and David F. Bradford, *Uncertain climate thresholds and optimal economic growth*, Journal of Environmental Economics and Management **48** (2004), no. 1, 723–741.

[29] Petar V. Kokotovic, *Applications of singular perturbation techniques to control problems*, SIAM Review **26** (1984), no. 4, 501–550.

[30] Jeffrey A. Krautkraemer, *Optimal growth, resource amenities and the preservation of natural environments*, The Review of Economic Studies **52** (1985), no. 1, 153–170.

[31] Simon Levin, *Learning to live in a global commons: socioeconomic challenges for a sustainable environment*, Ecological Research **21** (2006), 328–333.

[32] D. Ludwig, D. D. Jones, and C. S. Holling, *Qualitative analysis of insect outbreak systems: The spruce budworm and forest*, The Journal of Animal Ecology **47** (1978), no. 1, 315–332.

[33] Karl-Göran Mäler, Anastasios Xepapadeas, and Aart De Zeeuw, *The economics of shallow lakes*, Environmental and Resource Econometics **26** (2003), no. 4, 603–624.

[34] Hamid Mohtadi, *Environment, growth, and optimal policy design*, Journal of Public Economics **63** (1996), 119–140.

[35] D. Subbaram Naidu, *Singular perturbations and time scales in control theory and applications: An overview*, Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications and Algorithms **9** (2002), 233–278.

[36] Nuria Osés-Eraso and Montserrat Viladrich-Grau, *On the sustainability of common property resources*, Journal of Environmental Economics and Management **53** (2007), 393–410.

[37] Charles Perrings and Brian Walker, *Conservation in the optimal use of rangelands*, Ecological Economics **49** (2004), 119–128.

[38] Marten Scheffer, Chapman and Hall, London, 1998.

[39] A. Skiba, *Optimal growth with a convex-concave production function*, Econometrica **46** (1978), no. 3, 527–539.

[40] JunJie Wu, *Environmental amenities and the spatial pattern of urban sprawl*, American Journal of Agricultural Economics **83** (2001), no. 3, 691–697.

[41] ———, *Environmental amenities, urban sprawl, and community characteristics*, Journal of Environmental Economics and Management **52** (2006), 527–547.

[42] JunJie Wu and Andrew J. Plantinga, *The influence of public open space on urban spatial structure*, Journal of Environmental Economics and Management **46** (2003), 288–309.