Multi-Period Asset Allocation: An Application of Discrete Stochastic Programming

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Abstract

The issue of modeling farm financial decisions in a dynamic framework is addressed in this paper. Discrete stochastic programming is used to model the farm portfolio over the planning period. One of the main issues of discrete stochastic programming is representing the uncertainty of the data. The development of financial scenario generation routines provides a method to model the stochastic nature of the model. In this paper, two approaches are presented for generating scenarios for a farm portfolio problem. The approaches are based on copulas and optimization. The copula method provides an alternative to the multivariate normal assumption. The optimization method generates a number of discrete outcomes which satisfy specified statistical properties by solving a non-linear optimization model. The application of these different scenario generation methods is then applied to the topic of geographical diversification. The scenarios model the stochastic nature of crop returns and land prices in three separate geographic regions. The results indicate that the optimal diversification strategy is sensitive to both scenario generation method and initial acreage assumptions. The optimal diversification results are presented using both scenario generation methods.

MULTI-PERIOD ASSET ALLOCATION: AN APPLICATION OF DISCRETE STOCHASTIC PROGRAMMING

Introduction

One of the keys to long term success in the food and agribusiness sector is effective asset allocation. Farmers must efficiently allocate their assets including cash, land, equipment and labor over multiple enterprises. The diversification of these assets provides a key risk management strategy to handle production risks (Blank, 1990; Harwood, et al., 1999). The problem faced by many agricultural producers is that a myopic approach such as the standard single period portfolio optimization as proposed by Markowitz and used by many of researchers is often not adequate in the planning process (Lohano and King, 2009). Instead, large capital investments in land and machinery require that producers formulate strategies for multiple periods (Kennedy, 1986; Kim and Omberg, 1996).

The development of discrete stochastic optimization routines provided a method to analyze these dynamic problems (Cocks, 1968). The concept was adopted by agricultural economists who began analyzing multi-period problems in agriculture and particularly began introducing risk into the models (Brink and McCarl, 1978; Rae, 1971). The discrete stochastic modeling approach provided a strong tool for economists but lack of computational power turned out to be a large barrier to continuing advances of research in this area.

As a result of limited computational power, the discrete stochastic programming has had a limited number of applications in agricultural economics (Featherstone, Preckel, and Baker, 1990; Lambert and McCarl, 1989; Leatham and Baker, 1988). Part of the reason for this lack of applications is the difficulty of handling uncertainty. Stochastic programming assumes that random variables follow a given distributional form. This in itself implies that the researcher can accurately estimate this distribution. Price and yield distributions have also been the topic of research (Buccola, 1986; Featherstone and Kastens, 2000; Goodwin and Ker, 2002; Goodwin and Ker, 1998; Just and Weninger, 1999; Ker and Coble, 2003; Preckel and DeVuyst, 1992). Furthermore, the computational complexity of resulting problems requires discrete representation of the random variables to serve as a proxy for the continuous distribution. It is vital that the discrete representation accurately depicts the statistical characteristics of the continuous distribution (Pflug, 2001).

Discrete stochastic programming models assume that all random variables have only a finite number of possible realizations so that any possible state of the world over a finite horizon can be represented as an endpoint of an event tree. In the case of a farm portfolio optimization, the random parameters may include yields, commodity prices, land prices, and interest rates. The quality of results that are produced by the discrete stochastic model depends heavily on the quality of the generated scenarios (Topaloglou, Vladimirou, and Zenios, 2008). The construction of these scenarios are done via scenario tree structure that are commonly seen in stochastic programming. Several different methods have been proposed to generate scenarios. Some of the common techniques used are time series econometric techniques, random sampling, bootstrapping, and moment matching techniques (Kaut and Wallace, 2007). More recently, neural networks and copulas have begun to be used to generate scenarios (Kaut and Wallace, 2009). Each one of these modeling techniques has its strengths and weaknesses.

The topic of scenario generation for multi-period stochastic programming has not been addressed in the agricultural economics literature. Thus, the main objective of this paper is to evaluate current scenario generation techniques and apply a new one using copulas. Copulas model marginal distributions of the random variable separately and still maintain the shape of the multivariate distribution. The concept of using copulas to model scenarios has been applied to a single period setting but not to a multi-period application (Kaut and Wallace, 2009). This research provides an opportunity to extend the current research on scenario generation by using copulas in the multi-period setting.

This scenario generation application is applied to the topic of geographical diversification. A discrete multi-period stochastic program is formulated to optimize a farmer's portfolio, where the portfolio is the acreage allocations between three distinct geographic areas. The farmer has the ability to allocate wheat production acreage over three dry land production regions, Montana, Colorado, and Texas. The portfolio allocation decisions take place at discrete time points (every two years). At these discrete points the farmer evaluates the previous period's market conditions and the composition of the enterprise diversification. At the same time, the farmer evaluates future conditions such as expected future yields and prices. All this information is then used by the farmer to reallocate or adjust the allocation of farmland in the three different regions. This may involve increased short term or long term borrowing because of increased operating expenses, machinery purchases, and land purchases. This same decision process happens annually for a discrete number of years.

The results of the discrete multi-period stochastic optimization are presented. The optimization algorithm consists of maximizing expected utility of wealth by allocating acreage levels in the three different regions. A dynamic analysis of the optimal acreage allocations over time is estimated, as well as, how these allocations change with different levels of risk aversion. The use of copulas to generate the scenarios used in the discrete multi-period program provides a foundation for expanding the standard scenario generation methods to multiple risk factors and multi-periods. The use of copulas to generate the scenarios is compared with moment matching scenario generation using sequential optimization. The effectiveness of the copula scenario generation technique is analyzed by comparing it to the moment matching technique. This modeling technique also provides a framework to analyze other farm financial decisions, farm growth decisions, and even could be applied to loan portfolios from a lender's perspective.

Agriculture is a natural application of sequential decision problems. The current decisions that producers make have implications on future actions. Agricultural economists have employed discrete stochastic programming (dsp) as the tool to work with these problems. One of the first applications and one of the seminal articles dealing with dsp and agriculture was done by Rae (1971). Rae published two articles concerning the application of discrete stochastic programming in agriculture. In his first article, he examined a three-stage fresh vegetable operation. The states were defined both by predefined weather conditions and crop prices. He noted that one of the inherit weaknesses with the application was the lack of more states. The argument against more states was based on the “curse of dimensionality”. There is a tradeoff of complexity and solvability when using dynamic programs.

In that same year, Rae (1971) published his second paper on discrete stochastic programming. This one dwelt with viewing the problem using Bayesian decision theory. He also investigated the use of alternative utility functions within the objective function. Rae (1971) concluded that the ability of discrete stochastic programming to handle alternative utility functions makes itself a useful tool when studying sequential decision problems in agriculture.

Discrete stochastic programming has been used to model multi-period wheat marketing (Lambert and McCarl, 1989), fixed versus adjustable rate loans (Leatham and Baker, 1988), and capital structure (Featherstone, Preckel, and Baker, 1990). In finance research, discrete stochastic programming is now used to model asset allocation and for portfolio optimization routines (Infanger, 2006; Mulvey and Shetty, 2004).

One of the key features of a dsp model is that the parameters describing an optimum decision are defined as random variables. Extending a stochastic program to multiple stages requires that the parameter behavior over time be described accurately (Kaut and Wallace, 2007). In continuous time, the randomness of the parameters could be modeled using stochastic processes. The standard way of representing this stochastic process is by first defining a finite time horizon, t = 1,….,T, and a probability space $(\Omega, \Im, \mathrm{P})$; within the defined probability space, where Ω is the sample space, $\Im$ is defined as the σ-field, and P is defined as a finite set of probabilities. The random variable is defined by the function ξ. The sequence of $\xi_1(\omega), \xi_2(\omega), \dots, \xi_T(\omega)$ for a given $\omega \in \Omega$ is the sample path.

Discrete stochastic programming requires that these uncertain parameters are estimated by a finite number of realizations. Given these realizations, a scenario can then be defined as a possible realization of the underlying stochastic process. It is assumed that the probability distribution of ξ is discrete with a finite number of realizations ξ_k . The probability of each realization can then be defined as

$$p_k = P(\xi_k) \; for \; k = 1, \dots, K, where \; k \geq 0 \; and \; \sum_{k=1}^{K} p_k = 1.$$

For discrete multi-stage stochastic programming, it is assumed that the random vector follows a stochastic process over the planning horizon of the model. Given that the process is assumed to be discrete with probability $P(\xi_k)$, the uncertainty of the model parameters can be represented through a multilevel scenario tree. This scenario tree defines the possible sequences of realizations of the data paths.

A scenario tree is shown in figure 1. A scenario tree is defined by its nodes, and branches. The nodes represent the states of nature at a specific point in time. Within the scenario tree there are three different types of nodes. The root node represents the initial period or 'today' and is immediately observable from deterministic data. There is only one root node per scenario tree. Leaf nodes are the final nodes in the scenario tree. These nodes do not have any successors that follow them. In between the root and leaf nodes are the intermediate nodes. In this case, decisions will be made at the root and intermediate nodes. Each branch of the tree represents a possible value of the random variable. An ideal scenario tree would represent the whole universe of possible outcomes of the random variables which would include optimistic and pessimistic projections. Though similar in construction, a scenario tree is different from a decision tree by the fact that a decision tree branches on both decisions and events, the scenario tree only branches for events.

This type of asset allocation model can be viewed a multi-period dynamic decision problem. The decisions take place at discrete time points. At each decision point the farmer has to evaluate the previous period's market conditions and the composition of the enterprise diversification. At the same time, the farmer must evaluate future conditions such as future yields and prices. All this information is then used by the farmer to reallocate or adjust how the land is allocated in three regions. This could involve increased short term or long term borrowing because of increased operating expenses, machinery purchases, and land purchases. This same decision process continues through the time periods of the model.

At the beginning of each decision period, the farm manager is faced with many difficult decisions. Once a farmer makes a decision on land or crop allocation, it is often very costly and difficult to rearrange the land allocation. Some of those decisions are the levels of investment in farmland, capital purchases such as machinery to service new crops or acreage, and debt financing on farmland and capital. These decisions are not limited to one decision period but will be made over a finite horizon time period. Adding more difficulty to this decision process is the fact that these allocation decisions are based on the realization of uncertain events. Because of this uncertainty, the farm manager's objective in making these decisions is to maximize expected utility subject to land and capital constraints. Specifically, for this problem, the farm manager is seeking to maximize the expected utility of terminal net wealth. This model specification follows the specification developed by Lohano and King (2009) a dynamic programming model. Formally, this problem can be specified as:

$$\max_{\{x_{i,t}\}_{t=0}^{T}} E_0[u(W_T)] \tag{1}$$

Subject to

$$R_{i,j,t} = \beta_0 + \beta_1 R_{i,j-1,t-1} + \varepsilon_{1,i,j,t} \tag{2}$$

$$P_{i,j,t} = \alpha_0 + \alpha_1 P_{i,j-1,t-1} + \alpha_2 R_{i,j-1,t-1} + \varepsilon_{2i,j,t} \tag{3}$$

$$L_{i,j,t} = L_{i,j-1,t-1} + x_{i,j-1,t-1} \tag{4}$$

$$Am_{j,t} = Pm_{j,t} + (1 - dm) * Am_{j-1,t-1} \tag{5}$$

$$Pm_{j,t} \geq \sum_{i=0}^{n} \left(L_{i,j,t} + x_{i,j,t}\right) - (1 - dm) * Am_{j-1,t-1} \tag{6}$$

$$M_{j,t} = \tau * Pm_{j,t} + (1 - dm) * M_{j,t-1} \tag{7}$$

$$W_t = (1 + r)[A_t - cL_t] + R_t L_t + (P_t + \tau - tc_s)L_t \tag{8}$$

$$x_{i,t} \in X_{i,t} \tag{9}$$

for t = 0, 1, 2, ….., T, and (R_0, P_0, L_0, W_0) are given

where

$$r = \begin{cases} r_b & if \left[W_t - \sum_{i=o}^{n}\left(\left(P_{jj,t-1} + \tau + tc\right)x_{i,jj,t-1} - cL_{i,jj,t}\right)\right] < 0 \\ r_l & otherwise \end{cases} \tag{10}$$

$$tc = \begin{cases} tc_p & if\ x_{i,t} > 0 \\ -tc_s & otherwise \end{cases} \tag{11}$$

and the variables and parameters of the model are defined in Table 1. Equation (1) specifies the objective function of this model. The objective of this model is to maximize the expected utility of terminal wealth. Maximization of terminal net wealth is used because of the difficulty of implementing an additive utility function (Featherstone, Preckel, and Baker, 1990). The use of maximizing terminal wealth has many advantages when developing a discrete stochastic program. The main advantage is that there is no dependence on an additive utility function. An additive utility function assumes that there is independence between periods. In reality, the assumption of independence is often not the case, so terminal wealth will be used to avoid that problem. In this case, terminal wealth will be defined by owner's equity in the final period. This definition of wealth will then be incorporated into the utility function to maximize utility of wealth. A discussion on the appropriateness of functional form for the utility function has been provided by both Rae (1971) and Featherstone (1989). The power utility function will be used and is defined as

$$U(w) = \frac{w^{1-r}}{1-r} \tag{12}$$

where r is defined as the Pratt-Arrow coefficient of relative risk aversion and w is the wealth of the individual. Implicit in the definition of the power utility function is the assumption of constant relative risk aversion[1]. If r is greater than zero, the preferences are risk averse. If r is equal to zero, then the utility function will exhibit risk neutral preferences. Absolute risk aversion is $\frac{r}{w}$ and relative risk aversion is simply r. When r is equal to zero, the utility function

[1] Constant relative risk aversion is shown in the following manner

$$U(w) = \frac{w^{1-r}}{1-r}$$
$$U'(w) = w^{-r}$$
$$U''(w) = -rw^{-r-1}$$

takes the form of U(w) = w. Thus, when there are risk neutral preferences, the objective function is simply owner's equity. One of the main problems with the power utility function is that it is not defined for negative wealth. Some authors have overcome this weakness using Taylor's series expansion (Featherstone, Preckel, and Baker, 1990). This model will rely on the utility model specification provided by Lohano and King (2009). Lohano and King specified the utility function in two ways to account for negative wealth. The two specifications are

$$When\; r > 0:\;\; u(W) = \begin{cases} U(W) & if\; W \geq b \\ \frac{U(b)}{b}W & if\; W \leq b \end{cases} \tag{13}$$

where U(W) is the utility function specified in equation (12) and b > 0. Under this specification, the utility function is defined for all levels of wealth and is a continuous function.

Equations (2) and (3) model the stochastic processes for the two stochastic variables in the model: gross return for dry land wheat (R_t) and farmland price for the three regions ($P_{i,t}$). It is assumed that both land prices and wheat returns follow an autoregressive process. The land price specification differs from wheat returns in that it also depends on previous year returns. Further details on the estimation of these equations will be presented in the next section of the study. Equation (4) models the land allocation. The amount of land owned $L_{i,t}$ depends on the quantity of owned land from the previous period and the purchase or sale of land from the previous period $x_{i,t-1}$.

Each acre of land that the farmer owns must be serviced by a given level of machinery and equipment. Equations (5) through (7) describe the purchases of machinery and ensure that there are adequate machinery levels to service the production acreage. Equation (5) ensures that the acres with machinery ($Am_{j,t}$) is equal to the acreage needing machinery ($Pm_{j,t}$) plus the acreage with machinery from the previous period ($AM_{j-1,t-1}$). The previous period acreage is assumed to lose productivity because of use. This loss of productivity is represented by the parameter $(1 - dm)$.

Equation (6) constrains the acreage needing machinery $(Pm_{j,t})$ to be greater than or equal to the current acreage level $\left(\sum_{i=0}^{n}(L_{i,j,t})\right)$ less the depreciated machinery from the previous period $(1 - dm) * Am_{j-1,t-1}$. Equation (7) transfers the value of machinery from year to year. It is assumed that there is a fixed value of machinery that is needed to farm each acre of land. This assumption has been relied on for other dynamic modeling applications (Featherstone et al., 1990, Lohano and King, 2009). This level of machinery and equipment required on a per acre basis is given by the parameter τ. It is assumed that τ is fixed and will not change based on acreage levels.

One of the keys in formulating this dynamic problem is the specification of net wealth. Formally, the dynamics are described in equation (8). Net wealth itself is defined by

$$W_{j,t} \equiv A_{j,t} + \sum_{i=0}^{n}(P_{i,j,t} + \tau - tc_s)L_{i,j,t} - Debt_{j,t} \quad (14)$$

where $A_{j,t}$ represents the net cash balance at state j and time t. The net sale of farmland and machinery is represented by $\sum_{i=0}^{n}(P_{i,j,t} + \tau - tc_s)L_{i,j,t}$. The term $P_{i,j,t}$ represents the price of land at node j and time t for region i. The parameter tc_s represents the transaction costs associated with selling land and equipment. The wealth formulation also has to account for the debt level carried at each period and node. The term debt $Debt_{j,t}$ captures the actual debt level carried at each node. The level of debt is calculated as

$$Debt_{j,t} = \sum_{i=0}^{n}(P_{i,j-1,t-1} + \tau + tc)x_{i,j-1,t-1} + Debt_{j-1,t-1} \quad (15)$$

where the terms within the parentheses $\left((P_{i,j-1,t-1} + \tau + tc)x_{i,j-1,t-1}\right)$ captures debt incurred because of land and machinery purchases in the current period and the debt from the previous period is captured by $(Debt_{j-1,t-1})$. To be consistent with the dynamic formulation, the dynamics of the cash balance are specified as

$$A_{j,t} = (1 + r)\left[A_{j-1,t-1} - \sum_{i=o}^{n}\left(cL_{i,j,t}\right) - con - r_b * Debt_{j-1,t-1}\right] + \sum_{i=0}^{n} R_{i,j,t}L_{i,j,t} \qquad (16)$$

Current net cash balance is calculated as previous year's net cash balance $(A_{j-1,t-1})$ less current year's cash allocated for production expenses$\left(cL_{i,j,t}\right)$, consumption (con), and interest paid on debt $\left(r_b * Debt_{j-1,t-1}\right)$, plus the gross revenue from production $\left(\sum_{i=0}^{n} R_{i,j,t}L_{i,j,t}\right)$. The parameter (con) represents the annual cash withdrawals. It is fixed and does not change with wealth level. Current cash balance is represented by $A_{j,t}$. Net cash balance is not constrained to positive values, it can also be negative. If net cash balance is negative (equation (15)), the farmer must use short term debt and thus interest must be paid on the debt (r_b) (see equation (10)). If net cash balance is positive, the farmer can invest the money in a relatively risk free investment. This cash can then be earning interest at the risk free rate (r_l). The cash balance in time period *t* is thus equal to the sum of: net cash balance in time period *t* multiplied by one plus interest rate *r* and revenue from crop production $R_{i,j,t}L_{i,j,t}$.

A constraint on debt levels is also incorporated into the model. This can also be viewed as a credit constraint (Barry, Baker, and Sanint, 1981). This constraint aids in restricting purchases of land and equipment relative to the leverage ratio. The leverage ratio is defined as

$$lev = max\left\{0, \frac{\left|-A_{j,t} - Debt_{j,t}\right|}{\sum_{i=0}^{n}(P_{i,j,t} + \tau - tc_s)L_{i,j,t}}\right\} \qquad (17)$$

where the numerator is the total debt which is a function of both debt from cash shortages $(-A_{j,t})$ and debt from land and machinery purchases $(Debt_{j,t})$. The denominator represents all of the farm assets. The constraint takes the form of

$$\frac{\left|-A_{j,t} - Debt_{j,t}\right|}{\sum_{i=0}^{n}(P_{i,j,t} + \tau - tc_s)L_{i,j,t}} \leq \rho \qquad (18)$$

where ρ represents the maximum level of the leverage ratio which will range from 0 to 1.

One of the key ingredients in an optimization problem is the constraint specification. This problem has the following constraints. The first is a land constraint. The land constraint is formulated so that it satisfies $\underline{L}_{i,j} \leq L_{i,j} \leq \overline{L}_{i,j}$. The variable $\underline{L}_{i,j}$ represents the minimum amount of land the farmer requires in region i and $\overline{L}_{i,j}$ represents the maximum of land the farmer can have in region i. The next constraint is the liquidation constraint. The purpose of this constraint is to handle the case of negative wealth. In the case that wealth is negative, all land will be liquidated

$$x_{i,j,t} = -L_{i,j,t} \qquad if \;\; W_t < 0. \tag{19}$$

The farmer also has the option of selling all land when wealth is positive as well. One of the assumptions of this model is that when all the land is sold or liquidated, the farmer will not re-enter into farming which is represented by

$$x_{i,j,t} = -L_{i,j,t}, \qquad then\; x_{i,j,t+1} = x_{i,j,t+2} = \cdots = x_{i,j,T} = 0. \tag{20}$$

Data

Stochastic equations for gross returns for dry land wheat ($R_{i,t}$) from each of the three production regions (Texas, Montana, and Colorado) and farmland price from each region ($P_{i,t}$) are estimated using time series data covering the years 1973-2008. Gross returns are calculated for each region using county level yields and prices gathered from National Agricultural Statistics Service (USDA, 2008). Farmland prices are gathered from two sources. The farmland values for Montana and Colorado are gathered from Farm Real Estate Values (USDA, 2008) and Texas land values are from Real Estate Center at Texas A&M University (TAMU). The time span covered for Texas and Montana was 1973-2008 while Colorado had limited data and only covered the time span of 1994-2008.

The method to estimate the gross return equations for the three individual regions follows the Box-Jenkins methodology (Box and Jenkins, 1976). The Box-Jenkins methodology consists of identifying the order of the autoregressive process, estimation of the model, and finally testing to ensure that the error terms are white noise. Three separate equations are estimated for each region. All three regions were identified with a first order autoregressive model, AR(1). OLS was used to estimate each equation. All three autoregressive terms were found to be significant (Table 2) but the explanatory power of the models is low. For this reason, an AR(1) model will not be used to model the gross return equations. It is assumed that the scenario generated gross returns are independent of the previous period returns.

The purpose of this research is not to expand the literature surrounding the topic of land valuation. This topic has been heavily researched (Chavas and Thomas, 1999; Fontnouvelle and Lence, 2002). For the purpose of this research, the farmland price equations will be based on the following specification

$$P_t = \alpha_0 + \alpha_1 P_{t-1} + \alpha_2 R_{t-1} + \varepsilon_{2t.} \tag{21}$$

After the first pass of estimations, it was found that the lagged values of gross returns (R_{t-1}) were not statistically significant, so the equation was re-estimated using the simple first order autoregressive AR(1) form with only lagged land prices and not lagged returns(Table 3). All three AR(1) terms were found to be significant and the models illustrate that current land prices are explained by previous period land prices.

Scenario Generation using Copulas

The estimated models above give a starting point in generating the scenarios that are used to model the scenario tree. For modeling purposes, the estimated coefficients are used as constants and use alternative methods in modeling the error terms from each model. The simplest method is to assume independence and model each error term as normally distributed. An estimated correlation matrix shows that the error terms are not independent and this relationship must be taken into account (Table 4). The standard method for accounting for the dependency of the variables is to use the estimated correlation. The use of copulas allows the estimation of alternative dependency measures beyond linear correlation. These are discussed below and are used to generate the alternative scenarios.

Copulas provide a flexible method of separating the marginal distributions from the joint distribution. This maintains the “shape” of the joint distribution. The use of copulas has been used in finance, statistics, and recently in the agricultural economics literature (Bai and Sun, 2007; Clemen and Reilly, 1999; Joe, 1997; Patton, 2002; Rank, 2000; Trivedi and Zimmer, 2005; Vedenov, 2008; Xu, 2005; Zhu, Ghosh, and Goodwin, 2008). Copulas are used to model multivariate distributions. An extensive treatment of copulas can be found in numerous books and research articles (Patton, 2002).

The origin of copulas can be traced back to the Sklar theorem (Sklar, 1959). The Sklar theorem allows one to construct joint distribution of several random variables based on their marginal distributions and a copula. By definition there are an infinite number of copula functions, therefore an infinite number of joint distributions that may be generated for given marginal distributions. Various copula families have been used in risk research (e.g. Gaussian, Archimedean, etc. (Hennessy and Lapan, 2002)). However, it is not the purpose of this research to investigate various copula functions. Instead, the Gaussian and t copula will be used to estimate alternative scenarios to be used in the optimization model.

The Gaussian copula is an extension of the multivariate normal distribution but it can be used to model multivariate data that may exhibit non-normal dependencies and fat tails. The Gaussian copula is formally defined as

$$C(u_1, \dots, u_n; \Sigma) = \Phi^K(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n); \Sigma) \,, \qquad (22)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution and Σ is the variance-covariance matrix.

In the two-dimensional case, the Gaussian copula density can be written as

$$c(u, v) = \frac{1}{\sqrt{1-\rho^2}} exp\left(\frac{\left(\Phi^{-1}(u)\right)^2+\left(\Phi^{-1}(v)\right)^2}{2} + \frac{2\rho\Phi^{-1}(u)\Phi^{-1}(v)-\left(\Phi^{-1}(u)\right)^2-\left(\Phi^{-1}(v)\right)^2}{2(1-\rho^2)}\right), \qquad (23)$$

where ρ is the linear correlation between the two variables and $\Phi(\cdot)$ is the cumulative density function of the standard normal distribution. One of the useful features of the Gaussian copula is that it is parameterized by a single parameter (correlation coefficient) which can be estimated from historical data.

The copula based joint cdf is obtained by transforming the margins to standard uniform distribution. One can view this joint cdf as the joint distribution stripped of all information about the margins. The only thing remaining is the information about the multivariate structure. Therefore, copulas enable the decoupling of the marginal distributions from the multivariate structure. This gives the modeler much more flexibility in modeling multivariate relationships. In this study, the marginal distributions are modeled using an empirical distribution. This does not enforce any assumed distributional form on the margins. The copula will then be modeled using the Gaussian copula.

The copula based scenario generation consists of two parts. The first part consists of creating the scenario copula. This copula will be described in terms of the ranks of the margins. This will be accomplished by using the Gaussian copula. The parameters for the copula will be estimated from the historical data. The next step consists of generating the values of each margin. This will be accomplished by using empirical cdf estimated in Matlab. One of the benefits of using copulas is that the multivariate data can be simulated based on the estimated copula. Two samples will be generated, one based on the multivariate normal distribution and one based on the Gaussian copula. Using these random samples, values will be generated for each period based on the sample and the stochastic processes of the variables.

The next step was to then discretize the outcomes for each period. This was done using Gauss-Hermite multivariate quadrature[2]. Essentially this method uses a specified number of nodes and weights to evaluate the specified function. This method has been shown to be particularly useful for portfolio allocation problems (Judd, 1998) and for discretizing continuous data (Miranda and Fackler, 2002). In addition, this method is also consistent with moment matching techniques, which ensures that the simulated scenarios are consistent with the original data. The CompEcon Matlab toolbox (Miranda and Fackler, 2002) is used for this method. At each time period, a specified number of nodes are used to estimate the branches on the scenario tree and its associated probability. Five nodes are used for each decision period (every two years), with five total decision periods. This combination will lead to a total of 3,125 final scenarios and 3,905 nodes in the decision tree.

Scenario Generation through Optimization

The scenario generation approach using optimization requires that the statistical properties of the random variables be specified. The scenario tree is then constructed so that these pre-specified statistical properties are satisfied. These properties are maintained by letting

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[2] Gauss-Hermite Quadrature is defined by

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx = \sum_{i=1}^{n} \omega_i f(x_i) + \frac{n!\sqrt{\pi}}{2^n} \cdot \frac{f^{(2n)}(\varphi)}{(2n)!}$$

the stochastic variables and probabilities be decision variables in a non-linear optimization problem. The objective function in the non-linear problem is to minimize the square distance between the specified statistical properties and the statistical properties of the scenario tree. The non-linear problem is often not convex, which implies that the solution may be a local optimum. In many cases this is not satisfactory, but in the case of scenario generation, the local optimum is sufficient.

The advantage of using an optimization approach to generate the scenario tree is that any central moments and co-moments can be part of the statistical specifications of the distribution. The first four moments will be considered in this study. The dependency between variables will be modeled using the covariance. Let I = {1,2,...n} denote the set of random variables. Let M_{ik}, for k = 1, 2, 3, 4, be the first four central moments of the continuous distribution of random variable i. The covariance between random variable i and l (such that $i, l \in I$ and $i < l$) is denoted by C_{il}.Let N_t be the number of branches from a node at stage $t = 1, \dots, T-1$. The scenarios x_{ij} for random variable $i \in I$ and probabilities p_j for $j = 1, \dots, N_t$ of the continuous distribution are decision variables in the following non-linear optimization problem:

$$\min_{x,p} \sum_{i=1}^{n} \sum_{k=1}^{4} w_{ik} (m_{ik} - M_{ik})^2 + \sum_{i,l \in I, i<l} w_{il} (c_{il} - C_{il})^2 \qquad (24)$$

$$s.t. \quad \sum_{j=1}^{N_t} p_j = 1, \qquad (25)$$

$$m_{i1} = \sum_{j=1}^{N_t} x_{ij} p_j, \quad i \in I, \qquad (26)$$

$$m_{ik} = \sum_{j=1}^{N_t} \left(x_{ij} - m_{i1}\right)^k p_j, \quad i \in I, \; k = 2,3,4, \qquad (27)$$

$$c_{il} = \sum_{j=1}^{N_t} \left(x_{ij} - m_{i1}\right)\left(x_{lj} - m_{l1}\right) p_j, \quad i, l \in I \; and \; i < l, \qquad (28)$$

$$p_j \geq 0, \; j = 1, \dots, N_t, \qquad (29)$$

where w_{ik} are weights in which $w_k^{'}$ for $(k = 1, \ldots, 4)$ are the relative importance of the central moments and $w_{il}^{'}$ for covariances of the random variables $i, l \in I$. The first constraint shows that the probabilities must sum to one at each branch. The rest of the constraints are used to formulate the first four central moments and the covariance. The last constraint is to ensure that probabilities are non-negative.

It is important to note that the estimated moments of the distributions are conditional on past history and are conditional on the associated path of the scenario tree. This implies that the historical data is recalculated after each scenario estimation. Thus the updated historical mean of each distribution contains both the historical observations and the new observations generated through the scenarios.

Results

The model was solved using various starting points and scenarios. The model was first solved assuming that geographical diversification was not an option. The production was constrained to the base region. Next, the model was solved varying the base region. The base region began with the most acres and could not drop below a specified level. The results of each will be discussed below.

The certainty equivalents of wealth for each scenario generation method and each model scenario is shown in Table 7. When production is limited to one region, as expected Montana has the highest wealth levels and Texas has the lowest wealth levels. The results are consistent with the multivariate normal scenarios but not with the moment matching scenario generation method. Colorado has the highest wealth level followed closely by Texas and then Montana. The higher levels of wealth for both Gaussian and multivariate normal methods could represent the inability of both methods to capture the higher moments of the historical distributions. The wealth levels when allowing for geographical diversification are also shown in Table 7. The results indicate an interest point. The incentive to diversify depends on both base region and scenario generation method. Diversification decreased wealth levels when Texas is the base region under both Gaussian and multivariate normal methods but gains wealth under the moment matching scenarios. Diversification increased wealth levels for Montana and Colorado for all scenario generation methods.

The results for the optimal acreage allocations when production is limited to Texas is shown in Table 8. The assumption is that the farmer begins with 3,000 acres and has the opportunity to expand acreage to 10,000 acres. In all three scenario methods, the farmer expands slowly for the first three periods. The farmer then reduces acreage in the final two periods. It is important to remember that these are mean allocations. The maximum and minimum allocations are shown in Table 8. The standard deviation of the acreage allocations are also shown in Table 8. The Gaussian method has the smallest standard deviations. The multivariate normal method has the highest standard deviations.

The results for the optimal acreage allocations when production is limited to Colorado is shown in Table 9. Unlike Texas, there is a difference between the scenario methods. The farmer increases acreage slowly and maintain acreage levels over the planning period for both Gaussian and moment matching methods. The same behavior is not seen with the multivariate normal method. The farmer increases acreage slightly over the first two periods but then reduce acreage levels over the subsequent periods. As was seen with Texas, multivariate normal scenarios have the highest standard deviations. The moment matching method had lower standard deviations than the Gaussian method in periods two and three but higher standard deviations for periods four and five.

The results for acreage allocations in Montana produce different results than both Colorado and Texas. The results are found in Table 10. The farmer increases acreage slowly over the planning period for the Gaussian and multivariate normal scenario methods. For the moment matching method, the acreage increases slightly for the first three periods but then the acreage decreases over the last two periods

The results of the optimal acreage allocation based on the assumption that the farmer is based in Texas and has the opportunity to acquire land for production in Montana and Colorado is shown in Table 10. This means that the majority of the initial acreage allocation is in Texas and that the farmer will maintain a given level of acreage in the base region. The farmer purchases land in Montana and sell land in Texas at the beginning of the planning period under the Gaussian and multivariate normal scenarios. The farmer maintains the same acreage level in Texas over the rest of the planning period. The Montana acreage will remain relatively constant until period four when the acreage will be sold and acreage will be purchased in Colorado. Under the moment matching scenarios, the farmer does not purchase land in Montana but increases acreage in Texas until period four when additional acreage is purchased in Colorado and acreage is sold in Texas.

The optimal acreage allocations when Colorado is the base region are shown in Table 12. This implies that the base acreage is now shifted to Colorado. In this case, land will be purchased in Texas only under the moment matching scenarios. Additional acreage is purchased in Montana under the Gaussian and multivariate normal scenarios but not the moment matching method.   The amount of acreage in Colorado is decreased at the beginning but then is increased in periods four and five for all three scenario methods. These results illustrate the sensitivity of the results to the scenario generation method. Montana and Texas allocations are the most influenced in all three base acreage location scenarios.

The results for the acreage allocations given that Montana is the base location is shown in Table 13. Under both the Gaussian and multivariate normal scenarios, the amount of land in Texas is basically constant. In periods four and five, acreage will shifted from Montana to Colorado for both scenarios. Under the moment matching method, acreage is transferred to Texas from Montana in period one and then is transferred from Texas to Colorado in period 4. These results are consisted with the previous acreage allocations results.

Conclusions

A multi-period discrete stochastic programming model was formulated to analyze geographical diversification. Specifically, it analyzed whether a farmer would expand by buying more land locally or expand to other regions. The production of dry land wheat consisted of three different regions: Texas, Colorado, and Montana. The objective function consisted of maximizing terminal net wealth. The model analyzed the decision of how a farmer would allocate land to different production regions. Land is one of the most important resources a farmer has. Land traditionally composes a large share of the farmer's balance sheet. It is the base for loan collateral and future wealth. Not only is it important to consider the revenue stream from production on the land but also returns from land appreciation. The inclusion of both aspects is critical to effectively model geographical diversification decisions.

Discrete stochastic programming models both land prices and production revenue in a dynamic setting. As a farmer looks to make large investments in land and machinery, it is important to consider the results of the investment over multiple periods and not just look at the single period consequences. Discrete stochastic programming breaks away from the single period methodology of the traditional portfolio optimization and analyzes the optimal investments in a dynamic setting.

This research looked at three different methods to handle the joint distributions of the random variables. The random variables were modeled using a multi-variate normal distribution, Gaussian copula, and moment matching methods. The acreage allocation results illustrate the importance of properly specifying the distribution of the random variables used in the discrete stochastic program. When the third and fourth moments of the historical distributions are taken into consideration, the acreage allocations to Texas and Montana are drastically different. Under the moment matching method of scenario generation, the majority of land is allocated to Texas and Colorado, whereas under the other two methods, the majority of land is allocated to Colorado and Montana.

In addition, the use of copulas provides an alternative method to estimate the dependence between the random variables. The results from these joint distributions were then used as the stochastic inputs into the model. Future work could look at alternative scenario generation methods beyond the two copulas used in this model and also additional methods to match the first four moments closer. The inclusion of non-parametric copulas could overcome the limitation of the two parametric copulas used for this research. In addition, future work could focus on new techniques that are being used to reduce the number of scenarios in the model.

The results of this research also indicate that there are possible gains from geographical diversification. Wealth levels are increased for all three regions when production is diversified over the different regions. One important factor of geographic diversification is the additional costs incurred. Future research could take into consideration not only the wealth benefits but also the additional management, transportation, and labor costs that may occur.

References

Bai, M., and L. Sun. 2007. *Application of Copula and Copula-CVaR in the Multivariate Portfolio Optimization*, vol. Volume 4614/2007. Berlin / Heidelberg, Springer, pp. 231-242.

Barry, P. J., C. B. Baker, and L. R. Sanint. 1981. "Farmer's Credit Risks and Liquidity Management." *American Journal of Agricultural Economics* 63 (2):216-227.

Blank, S. C. 1990. "Returns to Limited Crop Diversification." *Western Journal of Agricultural Economics* 15 (2):204-212.

Box, G. E. P., and G. M. Jenkins. 1976. *Time Series Analysis: Forecasting and Control*. Revised ed. San Francisco, CA: Holden-Day.

Brink, L., and B. McCarl. 1978. "The Tradeoff Between Expected Return and Risk Among Cornbelt Farmers." *American Journal of Agricultural Economics* 60 (2):259-263.

Buccola, S. T. 1986. "Testing for Nonnormality in Farm Net Returns." *American Journal of Agricultural Economics* 68 (2):334-343.

Chavas, J. P., and A. Thomas. 1999. "A Dynamic Analysis of Land Prices." *American Journal of Agricultural Economics* 81 772-784.

Clemen, R. T., and T. Reilly. 1999. "Correlations and Copulas for Decision and Risk Analsysis." *Management Science* 45 (2):208-224.

Cocks, K. D. 1968. "Discrete Stochastic Programming." *Management Science* 15 (1968):72-79.

Featherstone, A. M., and T. L. Kastens. 2000. "Non-Parametric and Semi-Parametric Techniques for Modeling and Simulating Correlated, Non-Normal Price and Yield Distributions: Applications to Risk Analysis in Kansas Agriculture." *Journal of Agricultural and Applied Economics* 32 (2):267-281.

Featherstone, A. M., P. V. Preckel, and T. G. Baker. 1990. "Modelling Farm Financial Decisions in a Dynamic and Stochastic Environment." *Agricultural Finance Review* 50 80-99.

Fontnouvelle, P. d., and S. H. Lence. 2002. "Transaction Costs and the Present Value "Puzzle" of Farmland Prices." *Southern Economic Journal* 68 (3):549-565.

Goodwin, B. K., and A. P. Ker. 2002. *Modeling Price and Yield Risk*, ed. R. E. Just, and R. D. Pope, vol. 1. Boston, Kluwer Academic Publishers, pp. 289-323.

Goodwin, B. K., and A. P. Ker. 1998. "Nonparametric Estimation of Crop Yield Distributions: Implications for Rating Group-Risk Crop Insurance Contracts." *American Journal of Agricultural Economics* 80 (1):139-153.

Harwood, J., R. G. Heifner, J. Perry, and A. Somwaru. 1999. Managing Risk in Farming: Concepts, Research and Analysis, United States Department of Agriculture Economic Research Service.

Hennessy, D. A., and H. E. Lapan. 2002. "The Use of Archimedean Copulas to Model Portfolio Allocations." *Mathematical Finance* 12 (2):143-154.

Hibiki, N. 2006. "Multi-Period Stochastic Optimization Models for Dynamic Asset Allocation." *Journal of Banking & Finance* 30 (2):365-390.

Infanger, G. 2006. *Dynamic Asset Allocation Strategies Using a Stochastic Dynamic Programming Approach*. Handbook of Asset and Liability Management. Edited by S. A. Zenios, and W. T. Ziemba.

Joe, H. 1997. *Multivariate Models and Dependence Concepts*. Monographs on Statistics and Probability. Edited by D. R. Cox. London: Chapman & Hall.

Judd, K. L. 1998. *Numerical Methods in Economics*. Cambridge: MIT Press.

Just, R. E., and Q. Weninger. 1999. "Are Crop Yields Normally Distributed?" *American Journal of Agricultural Economics* 81 (2):287-304.

Kaut, M., and S. W. Wallace. 2007. "Evaluation of Scenario-Generation Methods for Stochastic Programming." *Pacific Journal of Optimization* 3 (2):257-271.

Kaut, M., and S. W. Wallace. 2009. "Shape-Based Scenario Generation Using Copulas." *Computational Management Science*.

Kennedy, J. O. S. 1986. *Dynamic Programming: Applications to Agriculture and Natural Resources*. New York: Elsevier Applied Science.

Ker, A. P., and K. Coble. 2003. "Modeling Conditional Yield Densities." *American Journal of Agricultural Economics* 85 (2):291-304.

Kim, T. S., and E. Omberg. 1996. "Dynamic Nonmyopic Portfolio Behavior." *The Review of Financial Studies* 9 (1):141-161.

Lambert, D. K., and B. A. McCarl. 1989. "Sequential Modeling of White Wheat Marketing Strategies." *North Central Journal of Agricultural Economics* 11 (1):105-115.

Leatham, D. J., and T. G. Baker. 1988. "Farmer's Choice of Fixed and Adjustable Interest Rate Loans." *American Journal of Agricultural Economics* 70 (4):803-812.

Lohano, H. D., and R. P. King. 2009. "A Stochastic Dynamic Programming Analysis of Farmland Investment and Financial Management." *Canadian Journal of Agricultural Economics* 57 575-600.

Miranda, M., and P. L. Fackler. 2002. *Applied Computational Economics and Finance*. Cambridge, MA: The MIT Press.

Mulvey, J. M., and B. Shetty. 2004. "Financial Planning Via Multi-Stage Stochastic Optimization." *Computers & Operations Research* 31 1-20.

Patton, A. J. 2002. "Applications of Copula Theory in Financial Econometrics." PhD dissertation, University of California, San Diego."

Pflug, G. C. 2001. "Scenario Tree Generation for Multiperiod Financial Optimization by Optimal Discretization." *Mathematical Programming* 89 251-271.

Preckel, P. V., and E. DeVuyst. 1992. "Efficient Handling of Probability Information for Decision Analysis under Risk." *American Journal of Agricultural Economics* 74 (3):655-662.

Rae, A. N. 1971. "Decision Problems in Farm Management." *American Journal of Agricultural Economics* 53 (1971b):448-460.

Rae, A. N. 1971. "An Empirical Application and Evaluation of Discrete Stochastic Programming in Farm Management." *American Journal of Agricultural Economics* 53 (4):625-638.

Rank, J. 2000. "Copulas in Financial Risk Management." Unviversity of Oxford.

Sklar, A. 1959. "Fonction de Repartiton a n Dimensions et Leurs Marges." *Publications de I'Institut de Statistique de L'Universite de Paris* 8 229-231.

TAMU. *Real Estate Center*, http://recenter.tamu.edu. (Accessed November 2009).

Topaloglou, N., H. Vladimirou, and S. A. Zenios. 2008. "Pricing Options on Scenario Trees." *Journal of Banking & Finance* 32 (2):283-298.

Trivedi, P. K., and D. M. Zimmer. 2005. "Copula Modeling: An Introduction for Practitioners." *Foundations and Trends in Econometrics* 1 (1):1-111.

USDA. 2008. National Agriculture Statistics Service/State Facts/County Facts. Internet Site: www.nass.usda.gov/QuickStats/ (Accessed March 2008).

Vedenov, D. 2008. "Application of Copulas to Estimation of Joint Crop Yield Distributions." Paper presented at AAEA annual meeting, Orlando FL, 27-29 July. Orlando, Florida.

Xu, Y. 2005. "Applications of Copula-Based Models in Portfolio Optimization." Dissertation, University of Miami.

Zhu, Y., S. K. Ghosh, and B. K. Goodwin. 2008. "Modeling Dependence in the Design of Whole Farm---A Copula-Based Model Approach." Paper presented at AAEA annual meeting, Orlando FL, 27-29 July.

Table 1. Variable and Parameter Descriptions

Variable/Parameter	Definition
t	Time index for the beginning of each year where t = 0,1,….,T
$R_{i,j,t}$	Gross return per acre from dry land wheat production in region i, state j, and year t
$P_{i,j,,t}$	Farmland price per acre for region i, state j, and in year t
$L_{i,j,t}$	Farmland acreage owned by the farmer in region i, state j, and year t
$Pm_{j,,t}$	Acreage needing machinery in state j and year t
$Am_{j,,t}$	Acres with machinery in state j and year t
$M_{j,,t}$	Value of machinery in dollars in state j and year t
τ	Value of machinery and equipment required per acre of farmland
dm	Depreciation rate for machinery
W_t	Net wealth in year t
$x_{i,j,t}$	Farmland acreage purchased or sold by the farmer in region i and year t
tc_p	Total transaction costs per acre on purchasing farmland, machinery and equipment
tc_s	Transaction costs per acre on selling farmland and equipment
r_b	Interest rate on short term borrowing
r_l	Interest rate on lending
$E_{i,j,t}$	Error term for region I, node j, and time t.

Table 2. Estimation of gross return equation for three states: Dependent variable R_t

State	Variable	Coefficient Estimate	Standard Error	t-statistic
	Constant	0.0653	0.0861	0.76
Texas	R_{t-1}	-0.4421**	0.1504	-2.94
		$\widehat{Var}[\varepsilon_{1t}] = 0.2442, R^2 = 0.213, Adjusted\ R^2 = 0.188$		
	Constant	2.4899	0.7361	3.38
Colorado	R_{t-1}	0.4614**	0.1598	2.89
		$\widehat{Var}[\varepsilon_{1t}] = 0.0806, R^2 = 0.202, Adjusted\ R^2 = 0.178$		
	Constant	1.8796	0.7312	2.57
Montana	R_{t-1}	0.6084**	0.1540	3.95
		$\widehat{Var}[\varepsilon_{1t}] = 0.1337, R^2 = 0.321, Adjusted\ R^2 = 0.301$		

** Significant at 5% level.

Note: Refer to Table 1 for variable definitions.

Table 3. Estimation of farmland price equation for each state: Dependent variable P_t

State	Variable	Coefficient Estimate	Standard Error	t-statistic
Texas	Constant	-0.380	0.672	-0.57
	P_{t-1}	1.070**	0.113	9.42
		$\widehat{Var}[\varepsilon_{1t}] = 0.155, R^2 = 0.73, Adjusted\ R^2 = 0.72$		
Colorado	Constant	-0.089	0.484	0.86
	P_{t-1}	1.022**	0.070	14.67
		$\widehat{Var}[\varepsilon_{1t}] = 0.001, R^2 = 0.96, Adjusted\ R^2 = 0.96$		
Montana	Constant	0.197	0.213	0.93
	P_{t-1}	0.979**	0.038	25.48
		$\widehat{Var}[\varepsilon_{1t}] = 0.011, R^2 = 0.96, Adjusted\ R^2 = 0.96$		

** Significant at 5% level.

Table 4. Correlation Matrix of Land Price Error Terms and Gross Returns from Each Region

	Tx Land Price	Co Land Price	Mt Land Price	Tx Returns	Co Returns	Mo Returns
Mt Land Price	1.000	-0.123	0.796	-0.404	0.030	0.335
Tx Land Price	-0.123	1.000	-0.052	0.700	0.721	0.390
Co Land Price	0.796	-0.052	1.000	-0.386	-0.111	0.076
Tx Returns	-0.404	0.700	-0.386	1.000	0.460	0.170
Co Returns	0.030	0.721	-0.111	0.460	1.000	0.716
Mt Returns	0.335	0.390	0.076	0.170	0.716	1.000

Table 7. Certainty Equivalents of Wealth (1,000s)

	Production in One Region		
Region	Gaussian Scenarios	MVN Scenarios	Moment Matching
Texas	968.64	972.79	880.24
Colorado	1030.41	1065.09	887.59
Montana	1138.10	1144.46	851.22
	Production in Multiple Regions		
Base Region	Gaussian Scenarios	MVN Scenarios	Moment Matching
Texas	921.12	927.54	967.76
Montana	1225.10	1228.13	951.46
Colorado	1287.60	1309.35	1102.10

Table 8. Optimal Acreage with Land Only in Texas

	Mean	Min	Max	SD
	Gaussian Copula Scenario Generation			
Period 0	3000	3000	3000	0
Period 1	3316	3316	3316	0
Period 2	3522	3518	3526	3
Period 3	3702	3477	3740	65
Period 4	3595	2612	3907	307
Period 5	3498	2000	4091	583
	Multivariate Normal Scenario Generation			
Period 0	3000	3000	3000	0
Period 1	3316	3316	3316	0
Period 2	3520	3427	3611	66
Period 3	3721	3529	3916	132
Period 4	3442	2843	4060	456
Period 5	3169	2000	4296	943
	Moment Matching Scenario Generation			
Period 0	3000	3000	3000	0
Period 1	3316	3316	3316	0
Period 2	3531	3527	3545	7
Period 3	3697	3562	3839	114
Period 4	3202	2649	4084	432
Period 5	2799	2000	4355	934

Table 9. Optimal Acreage with Land Only in Colorado

	Mean	Min	Max	SD
		Gaussian Copula Scenario Generation		
Period 0	3000	3000	3000	0
Period 1	3133	3133	3133	0
Period 2	3265	3183	3336	56
Period 3	3323	3116	3540	120
Period 4	3285	2724	3557	150
Period 5	3314	2000	3599	205
		Multivariate Normal Scenario Generation		
Period 0	3000	3000	3000	0
Period 1	3156	3156	3156	0
Period 2	3302	3262	3345	30
Period 3	2884	2667	3543	327
Period 4	2316	2000	3520	463
Period 5	2256	2000	3499	397
		Moment Matching Scenario Generation		
Period 0	3000	2867	250	0
Period 1	2867	2867	2867	0
Period 2	3026	3021	3037	6
Period 3	3221	3211	3244	9
Period 4	3319	2619	3462	266
Period 5	3417	2000	3685	555

Table 10. Optimal Acreage with Land Only in Montana

	Mean	Min	Max	SD
	Gaussian Copula Scenario Generation			
Period 0	3000	3000	3000	0
Period 1	3216	3216	3216	0
Period 2	3462	3398	3532	47
Period 3	3691	3556	3839	91
Period 4	3896	3697	4138	126
Period 5	4092	3842	4448	167
	Multivariate Normal Scenario Generation			
Period 0	3000	3000	3000	0
Period 1	3312	3312	3312	0
Period 2	3580	3500	3657	56
Period 3	3842	3663	4024	118
Period 4	3840	2958	4392	468
Period 5	3834	2000	4758	928
	Moment Matching Scenario Generation			
Period 0	3000	3000	3000	0
Period 1	2938	2938	2938	0
Period 2	3313	3311	3320	3
Period 3	3351	2837	3771	455
Period 4	2729	2000	4306	784
Period 5	2419	2000	4925	967

Table 11. Optimal Acreage Allocations Given Texas is Base Region

	Colorado				Texas				Montana			
	Mean	Min	Max	SD	Mean	Min	Max	SD	Mean	Min	Max	SD
	Gaussian Copula Scenario Generation											
Period 0	250	250	250	0	3000	3000	3000	0	250	250	250	0
Period 1	250	250	250	0	2000	2000	2000	0	1505	1505	1505	0
Period 2	250	250	250	0	2000	2000	2000	0	1764	1728	1799	26
Period 3	250	250	250	0	2000	2000	2000	0	1964	1708	2083	103
Period 4	1623	1401	1747	94	2000	2000	2000	0	250	250	250	0
Period 5	1633	1517	1760	77	2000	2000	2000	0	250	250	250	0
	Multivariate Normal Scenario Generation											
Period 0	250	250	250	0	3000	3000	3000	0	250	250	250	0
Period 1	250	250	250	0	2000	2000	2000	0	1552	1552	1552	0
Period 2	250	250	250	0	2000	2000	2000	0	1813	1728	1894	59
Period 3	250	250	250	0	2000	2000	2000	0	1944	1681	2256	171
Period 4	1441	250	1922	496	2199	2000	3602	503	250	250	250	0
Period 5	1635	1346	1904	171	2000	2000	2000	0	250	250	250	0
	Moment Matching Scenario Generation											
Period 0	250	250	250	0	3000	3000	3000	0	250	250	250	0
Period 1	250	250	250	0	3316	3316	250	0	250	250	250	0
Period 2	250	250	250	0	3582	3577	250	9	250	250	250	0
Period 3	250	250	250	0	3901	3683	250	45	250	250	250	0
Period 4	1564	250	1923	629	2085	2000	1923	264	428	250	2763	622
Period 5	1687	250	2046	605	2000	2000	2046	0	250	250	250	0

Table 12. Optimal Acreage Allocations Given Colorado is Base Region

	Colorado				Texas				Montana			
	Mean	Min	Max	SD	Mean	Min	Max	SD	Mean	Min	Max	SD
					Gaussian Copula Scenario Generation							
Period 0	3000	2000	250	0	250	250	250	0	250	250	250	0
Period 1	2000	2000	2000	0	250	250	250	0	1907	1907	1907	0
Period 2	2000	2000	2000	0	250	250	250	0	2162	2084	2245	57
Period 3	2000	2000	2000	0	415	250	2380	563	2187	250	2597	600
Period 4	3518	2000	3956	506	331	250	2290	393	343	250	2692	410
Period 5	3674	2000	4204	226	250	250	250	0	253	250	1922	67
					Multivariate Normal Scenario Generation							
Period 0	3000	2000	250	0	250	250	250	0	250	250	250	0
Period 1	2000	2000	2000	0	250	250	250	0	2037	2037	2037	0
Period 2	2000	2000	2000	0	250	250	250	0	2345	2321	2364	15
Period 3	2000	2000	2000	0	269	250	517	59	2525	1878	2695	185
Period 4	3209	2000	4102	686	350	250	2650	462	592	250	2528	717
Period 5	3325	2000	4286	551	250	250	250	0	252	250	1560	52
					Moment Matching Scenario Generation							
Period 0	3000	2000	250	0	250	250	250	0	250	250	250	0
Period 1	2000	2000	2000	0	2056	2056	2056	0	250	250	250	0
Period 2	2000	2000	2000	0	2382	2378	2400	9	250	250	250	0
Period 3	2000	2000	2000	0	2730	2719	2751	10	250	250	251	0
Period 4	3335	2000	4060	793	667	250	2647	769	312	250	3318	361
Period 5	3388	2000	4138	737	254	250	2446	88	250	250	250	0

Table 13. Optimal Acreage Allocation Given Montana is Base Region

	Colorado				Texas				Montana			
	Mean	Min	Max	SD	Mean	Min	Max	SD	Mean	Min	Max	SD
	Gaussian Copula Scenario Generation											
Period 0	250	250	250	0	250	250	250	0	3000	3000	3000	0
Period 1	250	250	250	0	250	250	250	0	3208	3208	3208	0
Period 2	250	250	250	0	250	250	250	0	3468	3398	3540	51
Period 3	250	250	250	0	299	250	1336	213	3656	2147	3877	323
Period 4	1356	250	1970	667	338	250	2124	361	2266	2000	4015	544
Period 5	1681	250	2216	530	250	250	250	0	2000	2000	2000	0
	Multivariate Normal Scenario Generation											
Period 0	250	250	250	0	250	250	250	0	3000	3000	3000	0
Period 1	250	250	250	0	250	250	250	0	3312	3312	3312	0
Period 2	250	250	250	0	250	250	250	0	3612	3529	3690	58
Period 3	250	250	250	0	258	250	452	40	3851	3532	4091	149
Period 4	1490	250	2065	616	383	250	2362	460	2279	2000	4288	696
Period 5	1902	250	2495	229	250	250	250	0	2007	2000	4263	122
	Moment Matching Scenario Generation											
Period 0	250	250	250	0	250	250	250	0	3000	3000	3000	0
Period 1	250	250	250	0	1177	1177	1177	0	2000	2000	2000	0
Period 2	250	250	250	0	1510	1508	1517	3	2000	2000	2000	0
Period 3	250	250	250	0	1455	250	1865	681	2472	2000	3922	816
Period 4	1547	250	1885	443	250	250	250	0	2361	2000	4494	647
Period 5	2004	250	2208	460	250	250	250	0	2000	2000	2000	0

Figure 1. Scenario Tree