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## **Dynamic Informative Advertising of New Experience Goods**

Alexander E. Saak

Department of Agricultural Economics  
Kansas State University  
331D Waters Hall  
Manhattan, KS 66506-4011  
U.S.A.  
Phone: 515 451-3675  
Fax: 785 532-6925  
alexsaak@agecon.ksu.edu

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## **Dynamic Informative Advertising of New Experience Goods**

### **Abstract**

This paper analyzes the optimal informative advertising and price policies of a monopolist who sells a new experience good over time to a population of heterogeneous buyers. Under certain conditions, the advertising rate first increases and then decreases over the marketing cycle with a peak occurring at the end of the introductory period when prices are low. Advertising lowers introductory prices but also shortens the period during which they are offered. Advertising raises the share of consumers who know their valuation in the long-run but not necessarily in the short-run.

*Keywords:* Monopoly, Private Information, Learning, Informative Advertising

## Dynamic Informative Advertising of New Experience Goods

### I. Introduction

In this paper, we analyze the optimal advertising policy of a monopolist who sells a new experience good over time to a population of heterogeneous buyers. In many markets for experience goods such as food, cosmetics, and pharmaceuticals consumers learn about their valuation for a product by purchasing it, and use the experience to decide whether to buy the product in the future. In addition, the seller may supply information that will help consumers make purchasing decisions. Such information can be provided using informative advertising as well as distributing free samples, coupons, offering smaller package sizes, product trials, and demonstrations. This paper explores at what stages of the marketing cycle the seller reduces the consumers' learning costs by lowering the current price, offering informative advertising, or both.

Recently, Bergemann and Valimaki (2006) (henceforth, BV) developed an infinite-horizon model of monopoly pricing of new experience goods with independent private valuations and consumer learning. In their model, consumption opportunities (or informative signals about the individual match value) arrive at random, and consumers, in aggregate, gradually learn their valuations by purchasing and consuming the product. While BV do not allow for the possibility of advertising (i.e. directly informing consumers about their tastes), the seller indirectly influences the distribution of valuations among the consumers through the dynamic pricing policy. BV show that all markets can be classified as mass and niche markets with qualitatively different learning outcomes and equilibrium patterns for prices and quantities. In a mass market, prices fall over time and eventually all consumers become informed (so called skimming pricing strategy). In a niche market, introductory low prices are followed by higher prices at which point the uninformed consumers stop buying the product (so called penetration pricing strategy).

We consider an extension of BV's model in which the seller may also *directly* influence the size of the segment of consumers who are informed about their true valuation. In our model, in addition to learning by buying and consuming the product, consumers may learn from informative advertising provided by the seller. We assume that the seller cannot target advertising messages to inexperienced consumers (i.e. to a

group of consumers who are uninformed about their match values).<sup>1</sup> Learning from informative advertising is modeled as a perfectly revealing signal that arrives according to a Poisson process to all consumers with the rate of arrival that is controlled at a cost by the seller.

To our knowledge the current paper is the first to accommodate pricing and informative advertising in a fully dynamic model with forward-looking agents. While in a mass market the seller never releases any pre-sale information, this is not the case in a niche market.<sup>2</sup> In a niche market, advertising may be optimal but the equilibrium advertising rate *varies* over the marketing cycle. If the marginal advertising cost is constant or decreasing and the maximum allowable advertising rate is unbounded, the advertising campaign is very brief but intense (so called advertising pulse) and occurs at the end of the period of low introductory prices. If the marginal advertising cost is increasing, the advertising intensity rises during the period of introductory prices, and falls during the period when the price is set at a higher level (bell-shaped advertising schedule).

In a niche market, the optimal path of the advertising intensity is determined by the two considerations: (i) the value of private information to the monopolist and (ii) the diminishing returns to advertising. On the one hand, as the uninformed segment contracts, the marginal value of private information to the seller increases since eventually demand from the informed consumers becomes the main generator of sales and the uninformed consumers drop out of the market. On the other hand, as the size of the uninformed segment shrinks, advertising becomes a less effective tool of delivering product information to consumers. This is due to the inability of the seller to target advertising exposures to a particular segment of consumers. The expenditure needed to achieve a given rate of consumer learning increases when there are fewer uninformed consumers left in the population.

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<sup>1</sup> This assumption is justified as long as the seller cannot screen consumers based on their individual histories of purchasing and exposures to product promotions. For example, this is the case with television advertising and free product samples distributed in the store.

<sup>2</sup> In a mass market, informative advertising is never optimal because the forward-looking uninformed consumers are willing to pay a high premium for being able to make informed purchasing decisions in the future. This makes them more lucrative customers than the informed buyers during all stages of the marketing cycle. Advertising would provide the information (experience) to the uninformed consumers for free, and result in foregone revenues as well as additional expenditures for the seller.

The effects of advertising on the dynamic pricing policy can be summarized as follows. In general, advertising lowers the introductory prices but also shortens the period of time during which they are offered. If the maximum allowable advertising rate is sufficiently low the main qualitative features of the intertemporal pricing policy are not affected by advertising: the price slowly falls in the early stage of the marketing cycle and jumps up once the market matures. However, if the maximum allowable advertising rate is sufficiently high (but finite) the introductory price may slowly rise over time during the advertising campaign. The effect of advertising on the learning outcome is positive in the long-run but not necessarily in the short run. Even though the learning process may temporarily slow down as fewer consumers learn by purchasing, the share of consumers who eventually learn their valuation is greater in equilibrium with informative advertising.

#### *Related Literature*

Our model of informative advertising extends the previous theoretical contributions that analyze the seller's advertising decision in a static setting, and provides micro-foundations for the dynamic models of the aggregate response of demand to advertising that are commonly used in the marketing literature.

In a static setting, the question of whether the monopolist achieves higher profits when consumers are privately more informed about their own valuations for the good has been analyzed by Lewis and Sappington (1994), Johnson and Myatt (2006), Anderson and Renault (2006), and Saak (2008).<sup>3</sup> In Lewis and Sappington (1994) and Johnson and Myatt (2006), the effect of advertising on the aggregate demand is to increase the variance of the distribution of valuations among consumers. If consumers are less informed, they constitute a more uniform group, which makes it easier for the seller to skim consumers. If consumers are more informed, the seller may target consumers with high valuations and raise the price, but information also provides rents to buyers. Their main finding is that the seller's profit is typically maximized under either null or full

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<sup>3</sup> Early theoretical contributions that adopt the view that advertising content directly conveys information on products' existence and attributes are Butters (1977) and Grossman and Shapiro (1984).

private information.<sup>4</sup>

In our model, demand dispersion is determined by the share of the informed consumers. The seller decides whether and how much to spend on advertising that raises the share of the informed consumers in each period while taking into account the long-term effects of advertising on the value of information for the uninformed (inexperienced) consumers and the composition of demand. In Johnson and Myatt's terminology, when the share of consumers who know their valuation is sufficiently low, the monopolist benefits from a "mass-market" posture and offers no informative advertising.<sup>5</sup> However, once their share has reached a certain level the monopolist benefits from a "niche-market" posture and informative advertising becomes profitable. Our analysis of the optimal dynamic advertising policy builds on this insight.<sup>6</sup>

In a dynamic setting, optimal advertising and promotion policies have been extensively studied in the marketing literature.<sup>7</sup> In an early seminal paper, Nerlove and Arrow (1962) model advertising as an investment in the stock of goodwill that increases current revenues and akin to other capital goods depreciates over time. Another class of dynamic advertising models builds on the approach due to Vidale and Wolfe (1957) that postulates a direct relationship between sales and advertising. In the Vidale-Wolfe model, current sales increase due to advertising which affects only the unsold portion of

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<sup>4</sup> Anderson and Renault (2006) and Saak (2008) show the seller typically prefers to convey only partial product information rather than null or full when the set of private information structures is larger. Anderson and Renault (2006) study a static monopoly model with search costs and advertising that may provide information regarding product attributes and price. Johnson (2005) studies the use of product line extensions in a two-period model with young and old consumers who are uncertain of their valuations. He finds that in equilibrium firms may offer entry-level products that allow young consumers to experiment and make better purchasing decisions in the future. Advertising that provides direct information about horizontal attributes in a static duopoly is studied in Meurer and Stahl (1994).

<sup>5</sup> The distinction between "mass" and "niche" markets in Johnson and Myatt (2006) is based on the degree of the dispersion in consumer valuations, while the distinction between "mass" and "niche" markets in BV is based on the comparison between the willingness to pay of uninformed consumers and the profit-maximizing price in the long-run.

<sup>6</sup> In a working paper Johnson and Myatt (2004) discuss the "advertising life-cycle" of a product without repeat purchasing.

<sup>7</sup> For example, optimal dynamic advertising policies in diffusion models of the aggregate sales response to advertising in monopoly are studied in Dockner and Jorgensen (1988), Mesak and Zhang (2001), Sethi et al. (2008). Doganoglu and Klapper (2008) empirically analyze weekly advertising policies of manufacturing firms in consumer goods markets with persuasive advertising that affects the goodwill of a brand. Chintagunta et al. (1993) study dynamic pricing and advertising policies in spatial duopoly for non-durable experience goods. In their model, consumers are uncertain about horizontal attributes, and their beliefs are determined by the consumption experience that evolves according to Nerlove-Arrow and Vidale-Wolfe type advertising effects.

the market. Sales also decrease due to forgetting in the sold portion of the market. An important feature of their model is that the finiteness of the size of the market (the saturation level), rather than the concavity of the seller's revenue function, leads to diminishing returns to advertising. However, these models make ad hoc (sometimes, empirically estimated) specifications of the response of aggregate sales to advertising.<sup>8</sup>

In the present contribution the response of demand to informative advertising shares features of both Nerlove-Arrow and Vidale-Wolfe models. Informative advertising can be seen as an investment in the stock of "goodwill" provided that dealing with more informed consumers benefits the seller (otherwise, advertising would generate a stock of "ill will"). Informative advertising in our model is also subject to diminishing returns because it evokes a response only from the "unsold" (i.e. inexperienced) portion of the market.

In the next section, we present an extension of BV's model in which the seller can directly influence the size of the informed segment through advertising. Section III sets up consumer and the monopolist's optimization problems. In Section IV, we derive optimality conditions that must be satisfied by the dynamic price and advertising policies in niche markets. In Section V, we characterize equilibrium with non-increasing marginal advertising cost and unbounded advertising rate. In Section VI, we consider the case of constant marginal advertising cost with an upper bound on the advertising rate. In Section VII, we characterize equilibrium with increasing marginal advertising cost. Section VIII concludes. The proofs are collected in the Appendix.

## II. Model

We consider an extension of the Bergemann-Valimaki model of dynamic pricing of new experience goods with informative advertising. The model is formulated in continuous time. At each instant  $t \in [0, \infty)$ , a seller (monopolist) with zero marginal cost of production offers a single non-storable, non-returnable product to a unit continuum of risk-neutral consumers with unit demands. Consumption opportunities arrive at random time intervals and follow a Poisson process with parameter  $\lambda$ . This rate of arrival is

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<sup>8</sup> For example, Krishnan and Jain (2006) use an empirically validated demand function to study optimal advertising policies for new products.

constant over time and is common to all consumers. The monopolist and the buyers discount the future rents at rate  $r > 0$ .

Every consumer is characterized by his idiosyncratic willingness to pay for the product  $\theta$ . The good is an experience good, and the true value of  $\theta$  is initially unknown to the buyer and the seller. True valuations,  $\theta$ , do not change over time, and are independently drawn from the probability distribution  $F(\theta)$  with support  $[\theta_l, \theta_h] \subseteq \mathfrak{R}$  in the beginning of the game. This distribution is common knowledge. We assume that  $(1 - F(\theta))\theta$  is strictly quasi-concave in  $\theta$ , and let  $v = E[\theta] = \int_{\theta_l}^{\theta_h} \theta dF(\theta)$ .

Consumers learn their actual valuation for the product either by (i) buying and consuming the product, or (ii) from an informative advertising program sponsored by the seller. We assume that buyers learn their true valuation upon consuming the first unit of the good or upon receiving their first effective advertising message.<sup>9</sup> A perfectly informative (i.e. effective) signal arrives to each consumer at a Poisson rate  $x(t)$ . Consumers who had never purchased the product and had never received a sample during the time interval  $[0, t]$ , remain uninformed in a time interval  $[t, t + dt]$ .<sup>10</sup> The flow expenditures needed to achieve the rate of advertising intensity  $x(t) \in [0, \bar{x}]$  is denoted by  $c(x(t))$ , where  $c$  is a twice differentiable and strictly increasing function with  $c(0) = 0$  and  $\bar{x} > 0$  is the maximum allowable (possibly infinite) advertising rate.

At each instant, the monopolist sets a spot price  $p(t)$  and an advertising rate  $x(t)$ , and the uninformed consumers who received an advertising signal update their beliefs. Upon seeing the price, consumers for whom a consumption opportunity has arrived (both the informed and uninformed ones) decide whether to purchase or not.

Let  $\alpha(t) \in [0, 1]$  denote the share of the informed consumers at time  $t$ . The state variable  $\alpha(t)$  evolves according to

$$\frac{d\alpha(t)}{dt} = \begin{cases} (\lambda + x(t))(1 - \alpha(t)), & \text{if the uninformed consumers buy in period } t \\ x(t)(1 - \alpha(t)), & \text{if the uninformed consumers do not buy in period } t \end{cases} \quad (1)$$

<sup>9</sup> As discussed in BV, the assumption that consumers learn their true valuation upon consuming the first unit of the good can be easily relaxed.

<sup>10</sup> The seller cannot distinguish between informed and uninformed consumers when trade occurs as well as when advertising information is delivered.

since at time  $t$  there are  $1 - \alpha(t)$  currently uninformed consumers and depending on the purchasing decision of the uninformed buyers either a fraction  $(\lambda + x(t))dt$  or  $x(t)dt$  of them become informed in a time interval of length  $dt$ .<sup>11</sup> In the BV's model, the share of the informed consumers does not grow,  $d\alpha(t)/dt = 0$ , if the uninformed consumers are not purchasing. In this extension, advertising provides an additional means of raising  $\alpha(t)$  over time.

Markovian pricing and advertising strategies for the seller are denoted by  $p(\alpha)$  and  $x(\alpha)$ , respectively. The uninformed buyer has a Markovian purchasing strategy  $d^u(\alpha, p)$ . The informed buyer with valuation  $\theta$  also has a Markovian purchasing strategy  $d^\theta(\theta, \alpha, p)$ . The monopolist maximizes her expected discounted profit over the infinite horizon, and the buyers maximize the (expected) discounted value of their utilities from consumption net of price. While there is no aggregate uncertainty, the individual informed buyer faces uncertainty regarding the times at which consumption opportunities will occur. In addition, each uninformed buyer is also uncertain about his actual valuation, and the time at which he will receive an advertising message (if any) from the seller.

### III. Equilibrium

We characterize the Markov perfect equilibrium (MPE) of this advertising monopoly game. In such an equilibrium the decisions of the monopolist and the consumers only depend on the current share of informed consumers,  $\alpha(t)$ , the state variable of the model. As in BV, while consumers and the seller do not directly observe  $\alpha(t)$ , they can infer its value from the equilibrium purchasing and advertising strategies using (1).

For a given price and advertising policies,  $p(\alpha)$  and  $x(\alpha)$ , the value function  $V^\theta(\alpha)$  of the informed buyer satisfies the Bellman equation:

$$(r + \lambda)V^\theta(\alpha) = \lambda \max[\theta - p(\alpha), 0] + \frac{dV^\theta}{d\alpha} \frac{d\alpha}{dt}. \quad (2)$$

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<sup>11</sup> The probability that a consumption opportunity and an informative advertising signal simultaneously arrive to an uninformed consumer,  $\lambda x(t)(dt)^2$ , goes to zero much faster than  $dt$ , and hence, can be ignored.

Because the purchasing decision of a single consumer has no impact on the seller's profits, the myopic decision rule is optimal for the informed buyer: buy if  $p(\alpha) \leq \theta$ . The value function of the uninformed buyers satisfies

$$(r + \lambda)V^u(\alpha) = \lambda \max[v - p(\alpha) + E[V^\theta(\alpha)] - V^u(\alpha), 0] \quad (3)$$

$$+ x(E[V^\theta(\alpha)] - V^u(\alpha)) + \frac{dV^u}{d\alpha} \frac{d\alpha}{dt}.$$

As in BV, a current purchase generates the (expected) flow of consumption,  $v$ , as well as information,  $E[V^\theta(\alpha)] - V^u(\alpha)$ . In the present setting, a buyer who is currently uninformed may become informed even without purchasing the product. This reduces the information value component of the purchase for the uninformed buyers. Let  $w(\alpha) = v + E[V^\theta(\alpha)] - V^u(\alpha)$  denote the maximum willingness to pay of the uninformed buyers.

Let  $V(\alpha)$  denote the value function of the monopolist's program. If the seller targets both the uninformed and some of the informed buyers, the relevant Bellman equation is

$$rV(\alpha) = \max_{p(\alpha), x(\alpha) \in \mathbb{R}_+} \pi(\alpha, p(\alpha)) - c(x(\alpha)) + \frac{dV}{d\alpha} \frac{d\alpha}{dt} \text{ subject to } p(\alpha) \leq w(\alpha), \quad (4)$$

where  $\pi(\alpha, p) = \lambda(1 - \alpha + \alpha(1 - F(p)))p$  is the current revenue when both the uninformed buyers and some of the informed ones purchase the product. If the monopolist only sells to the informed segment, the value function satisfies

$$rV(\alpha) = \max_{p(\alpha), x(\alpha) \in \mathbb{R}_+} \lambda\alpha(1 - F(p(\alpha)))p(\alpha) - c(x(\alpha)) + \frac{dV}{d\alpha} \frac{d\alpha}{dt}. \quad (5)$$

Given the purchasing strategies, the seller seeks optimal dynamic pricing and advertising strategies. A Markov-perfect equilibrium of this game is a quadruple  $(d^u, d^\theta, p, x)$  such that the problems (2) – (5) are simultaneously solved for all  $\alpha$  and  $\theta$ .

BV introduce the following dichotomy. The market is said to be a *mass* market if  $\hat{w} \geq \hat{p}$ , and a *niche* market if  $\hat{w} < \hat{p}$ , where

$$\hat{w} = v + \frac{\lambda}{r + \lambda} E \max[\theta - \hat{p}, 0]$$

is the willingness to pay of the uninformed consumers when the price remains constant at  $\hat{p}$  and there is no advertising in the future, and  $\hat{p} = \arg \max_{p \in \mathbb{R}_+} (1 - F(p))p$  is the price

that maximizes the flow revenues from the informed consumers.

BV show that the pricing policy and the learning outcome are qualitatively different in the case of a mass and a niche market in the absence of informative advertising. In a mass market without advertising, the uninformed consumers buy the product and the equilibrium price path is declining throughout the entire marketing cycle. As a result, the monopolist's flow profit decreases as the share of the informed consumers increases, and it is easy to show that the monopolist does not provide informative advertising in mass markets.<sup>12</sup>

In a niche market without advertising, the uninformed consumers purchase the product and the price is declining in the early stage of the marketing cycle (early niche market). However, the uninformed consumers stop purchasing when the price is permanently raised to  $\hat{p}$  (mature niche market).<sup>13</sup>

#### IV. Niche Markets

For the rest of the paper we assume that  $\hat{w} < \hat{p}$ . In the niche market *with* advertising, the equilibrium pricing policy is similar as the period of introductory prices is also followed by a jump up to  $\hat{p}$ . Using (2) and (3), the differential equation that describes the evolution of the equilibrium price in an early market when the marginal buyer is the uninformed buyer, i.e.  $p(\alpha) = w(\alpha)$ , is

$$\frac{dp}{d\alpha} \frac{d\alpha}{dt} = (r + \lambda + x(\alpha))(p(\alpha) - v) - \lambda E \max[\theta - p(\alpha), 0] \text{ for all } \alpha \leq \hat{\alpha}. \quad (6)$$

<sup>12</sup> We assumed that all consumers are aware of the product existence and that consumption opportunities arrive independently of the exposure to advertising (and both are exogenous to the consumer's problem). We do not consider "persuasive" advertising that raises the frequency with which consumers participate in the market or favorably shifts the distribution of actual tastes.

<sup>13</sup> As pointed out by a referee, the family of the aggregate distributions of valuations parameterized by the size of the informed segment  $G_\alpha(\theta) \equiv \begin{cases} \alpha F(\theta), & \text{if } \theta \leq w(\alpha) \\ 1 - \alpha + \alpha F(\theta), & \text{if } \theta > w(\alpha) \end{cases}$  is ordered by a sequence of rotations around the rotation point  $w(\alpha)$  in the sense of Johnson and Myatt (2006), i.e. for each  $\theta \in [\theta_l, \theta_h]$  we have  $\theta > (<)w(\alpha)$  if and only if  $\partial G_\alpha(\theta)/\partial \alpha < (>)0$  ( $\partial G_\alpha(\theta)/\partial \alpha$  is not defined at  $\theta = w(\alpha)$ ). As the share of the informed increases the aggregate distribution of willingness to pay becomes more dispersed around the willingness to pay of the uninformed consumer. Provided that the rotation point  $w(\alpha)$  is decreasing in  $\alpha$ , Johnson and Myatt (2006) (Proposition 1 on page 762) show that the statically optimal monopoly revenues are quasi-convex in  $\alpha$ , and hence maximized at an extreme  $\alpha \in \{0,1\}$ . In other words, static revenues are "U-shaped" in  $\alpha$ . This preference for extremes explains our finding that the monopolist may "suddenly" start advertising when  $\alpha$  is sufficiently large.

From the uninformed buyers' point of view, the random advertising rate  $x(\alpha)$  (as well as the random purchase rate  $\lambda$ ) acts as an increase in the discount rate. As is explained in BV, (6) represents the trade-off for the uninformed buyer between the benefits of buying today or delaying buying until a consumption opportunity arrives next time. The change in the price equals the difference between the value of information about the buyer's true taste and the expected net utility from making an informed purchasing decision in the following instant.

At the optimal switching point  $\hat{\alpha}$  of the price policy the seller is indifferent between attracting the uninformed consumers for the last time by offering  $p(\hat{\alpha}) = w(\hat{\alpha})$ , and switching to a higher price  $\hat{p}$ , where  $w(\hat{\alpha}) \leq \hat{w} < \hat{p}$ .<sup>14</sup> The payoff from offering  $w(\hat{\alpha})$  and advertising at rate  $x(\hat{\alpha})$  is the flow revenue minus the advertising cost and the gain in the future revenues from a larger segment of the informed consumers who learn by purchasing as well as from advertising:

$$\pi(\hat{\alpha}, w(\hat{\alpha})) - c(x(\hat{\alpha})) + V'(\hat{\alpha})(x(\hat{\alpha}) + \lambda)(1 - \hat{\alpha}). \quad (7)$$

The payoff from offering  $\hat{p}$  and advertising at rate  $x(\hat{\alpha})$  is the flow revenue minus the advertising cost and the gain in the future revenues from a larger segment of the informed consumers due to advertising:

$$\hat{\alpha}\hat{\pi} - c(x(\hat{\alpha})) + V'(\hat{\alpha})x(\hat{\alpha})(1 - \hat{\alpha}), \quad (8)$$

where  $\hat{\pi} = \lambda(1 - F(\hat{p}))\hat{p}$  denotes the flow (static) monopoly revenue when all consumers are informed. The seller stops selling to the uninformed consumers whenever (8) is greater than (7), i.e.

$$\pi(\hat{\alpha}, w(\hat{\alpha})) + V'(\hat{\alpha})\lambda(1 - \hat{\alpha}) \leq \hat{\alpha}\hat{\pi} \text{ (with strict equality if } \hat{\alpha} > 0). \quad (9)$$

Equation (9) determines the switching point of the price policy  $\hat{\alpha}$ . Also, let  $\hat{\alpha}_n$  denote the switching point of the price policy, i.e. the long-run size of the informed segment, in equilibrium without advertising studied in BV.

By (4) and (5), the optimality condition for the advertising rate is

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<sup>14</sup> Clearly, once the monopolist sets  $p(\alpha) = \hat{p}$ , she will not find it optimal to lower the price to  $w(\alpha) < \hat{p}$  in the future because the size of the uninformed segment cannot increase over time.

$$x(\alpha) = \begin{cases} 0, & \text{if } V'(\alpha)(1-\alpha) - c'(0) \leq 0, \\ x \in [0, \bar{x}], & \text{if } V'(\alpha)(1-\alpha) - c'(x) = 0, \text{ for all } \alpha. \\ \bar{x}, & \text{if } V'(\alpha)(1-\alpha) - c'(\bar{x}) \geq 0, \end{cases} \quad (10)$$

Let  $\alpha^-$  and  $\alpha^+$  denote the optimal starting and stopping points of the advertising policy, i.e. the threshold sizes of the informed segment such that the monopolist does not advertise before  $\alpha^-$  and after  $\alpha^+$ . The monopolist stops advertising in the mature market, i.e.  $\alpha^+ \geq \hat{\alpha}$ , so that, by (5),  $rV(\alpha^+) = \alpha^+ \hat{\pi}$  since there is no further growth in sales. Therefore, by (10),  $\alpha^+$  satisfies the optimality condition  $(1/r)\hat{\pi}(1-\alpha^+) - c_0 \leq 0$ , or

$$\alpha^+ = 1 - \frac{rc_0}{\hat{\pi}},$$

where  $c_0 \equiv c'(0)$ . Note that the monopolist stops short of informing the entire population through advertising because of the diminishing effectiveness of advertising expenditures. As the population of the uninformed consumers shrinks it becomes harder to reach them with informative advertising at a given rate.

Calculating the optimal starting point of the advertising policy is more difficult because the monopolist's value function is not monotone in  $\alpha$  during the early stages of the marketing cycle. The analysis to follow will show that the intertemporal pattern of the equilibrium advertising campaign depends on whether the marginal advertising cost is increasing or decreasing.

## V. Constant marginal advertising cost with no upper bound

Suppose that  $c(x) = c_0 x$  for all  $0 \leq x \leq \infty$ , where  $c_0 > 0$ . Because the right-hand sides of (4) and (5) are now linear in the advertising rate, and  $x(\alpha)$  is positive and unbounded, the equilibrium advertising rate takes the form of impulse control or "advertising pulse" (Sethi 1973). To characterize equilibrium when the marginal advertising cost is constant and the advertising rate is unbounded, following Sethi (1973), we let

$$P(\alpha, \alpha_1) = \ln \frac{1-\alpha}{1-\alpha_1}$$

denote the magnitude of an impulse control that instantaneously raises the share of the

informed consumers from  $\alpha$  to  $\alpha_1$ , where  $\alpha \leq \alpha_1$ . An advertising campaign consists of a single advertising pulse  $(\alpha_0, \alpha_1)$  if  $x(\alpha(t)) = P(\alpha_0, \alpha_1)\delta(\alpha_0 - \alpha(t))$ , where  $\delta(\cdot)$  is the Dirac delta function and  $\alpha_0 \leq \alpha_1$ . That is, the monopolist does not advertise before  $\alpha(t)$  reaches  $\alpha_0$  and after the share of the informed consumers has (instantaneously) risen to  $\alpha_1$ . The time  $s < t$  discounted expenditure on an advertising pulse that induces a discontinuous jump in the state variable from  $\alpha(t)$  to  $\alpha_1 > \alpha(t)$  is  $c_0 P(\alpha(t), \alpha_1) e^{-r(t-s)}$ .

Consider an equilibrium with a single advertising pulse of magnitude  $P(\alpha^-, \alpha^+)$  with  $\alpha^- = \hat{\alpha} \leq \alpha^+$  that occurs when  $\alpha(t)$  has reached  $\hat{\alpha}$ . The monopolist's value function at  $\hat{\alpha}$  that marks both the switching point in the price policy as well as the beginning of the (brief but very intense) advertising campaign, satisfies

$$rV(\hat{\alpha}) = \alpha^+ \hat{\pi} - rc_0 P(\hat{\alpha}, \alpha^+). \quad (11)$$

Also, note that at an instant when  $\alpha(t) = \hat{\alpha}$  the uninformed consumers expect to receive an advertising signal with probability  $\alpha^+ - \hat{\alpha}$ , and their willingness to pay is given by

$$v + (1 - (\alpha^+ - \hat{\alpha})) \frac{\lambda}{r + \lambda} E \max[\theta - \hat{p}, 0] = \hat{w} - (\alpha^+ - \hat{\alpha})(\hat{w} - v). \quad (12)$$

It is the sum of the expected value of the flow consumption and the expected value of information about one's actual valuation of the product multiplied by the probability that an informative advertising signal has not been received,  $1 - (\alpha^+ - \hat{\alpha})$ .

Substituting (11), (12), and  $V'(\hat{\alpha}) = -c_0 \lambda (1 - \hat{\alpha}) \partial P(\hat{\alpha}, \alpha^+) / \partial \hat{\alpha}$  in (9), the modified indifference condition that determines  $\hat{\alpha}$  (and  $\alpha^-$ ) becomes

$$\pi(\hat{\alpha}, \hat{w} - (\alpha^+ - \hat{\alpha})(\hat{w} - v)) - c_0 \frac{\partial P(\hat{\alpha}, \alpha^+)}{\partial \hat{\alpha}} \lambda (1 - \hat{\alpha}) \leq \alpha^+ \hat{\pi} - rc_0 P(\hat{\alpha}, \alpha^+). \quad (13)$$

The interpretation of (13) is similar to that of (9). The left-hand side in (13) represents the payoff from attracting the uninformed consumers for one more instant. This payoff consists of the sum of the flow revenue and a reduction in the advertising expenditure that is required to raise the size of the informed segment to  $\alpha^+$  since  $\lambda(1 - \hat{\alpha})$  of the uninformed consumers will become informed. The right-hand side is the flow revenue from price  $\hat{p}$  minus the foregone interest on the expenditure on the advertising pulse.

PROPOSITION 1 (Advertising pulse). If  $c(x) = c_0 x$  for all  $0 \leq x \leq \infty$  and  $c_0 < (1 - \hat{\alpha}_n)\hat{\pi}/r$ , then

1. The advertising campaign consists of a single advertising pulse  $(\alpha^-, \alpha^+)$ , where  $0 \leq \alpha^- = \hat{\alpha} < \alpha^+ < 1$ . Furthermore,  $\alpha^- = \hat{\alpha} > 0$  if  $\lambda v \geq \hat{\pi}$ .
2. The price  $p(\alpha)$  satisfies equation (6) for all  $\alpha \leq \hat{\alpha}$  with  $\lim_{\alpha \uparrow \hat{\alpha}} p(\alpha) = \hat{w} - (\alpha^+ - \hat{\alpha})(\hat{w} - v)$ .
3. The price  $p(\alpha)$  is decreasing for all  $\alpha < \hat{\alpha}$ , and then jumps up and stays at  $\hat{p}$ .

It is also worth observing that, if  $(1 - \hat{\alpha}_n)\hat{\pi}/r \leq c_0 < \hat{\pi}/r$ , the monopolist does not advertise even though the long-run marginal value of advertising at  $t = 0$  (i.e. starting with the completely uninformed population),  $\hat{\pi}/r$ , is greater than the marginal cost of advertising. The reason is that the “true” advertising cost to the monopolist includes the foregone sales to the uninformed consumers that may either make advertising unprofitable or affect its dynamic pattern as we explain next.

Proposition 1 shows that the advertising pulse may occur *after* the product has been offered for sale. A sufficient condition is  $\pi(0, v) = \lambda v > \hat{\pi}$ . It guarantees that the flow revenue is the greatest when the product is first introduced and all consumers are uninformed. When the uninformed consumers are purchasing, the flow revenues are falling as both the number of the uninformed consumers and their willingness to pay decline in an early market. As a result, initially, the monopolist’s value function is decreasing in  $\alpha$ . However, as the segment of the uninformed consumers whittles down in size, the monopolist’s value function begins to increase in  $\alpha$  since the monopolist starts targeting the informed segment at  $\alpha = \hat{\alpha}$ . Because an advertising pulse instantly enlarges the informed segment, the monopolist delays it until the *end* of the period of introductory prices.<sup>15</sup>

A comparative statics analysis that is carried out next uses condition  $\hat{w} \leq v(1 + F(v))$ , which guarantees that the flow revenue from attracting the uninformed consumers for the last time at  $\hat{\alpha}$  is decreasing in  $\hat{\alpha}$ . This condition is satisfied when  $\lambda$  is sufficiently

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<sup>15</sup> Note that advertising can be delayed even if it is costless, i.e.  $\alpha^- = \hat{\alpha} > 0$  if  $c_0 = 0$  and  $\lambda v > \hat{\pi}$ .

small or when it is socially efficient to trade with the average consumer in the left tail, i.e.

$$\int_{\theta_t}^v \theta dF(\theta) \geq 0.$$

PROPOSITION 2 (Comparative statics). If  $c(x) = c_0 x$  for all  $0 \leq x \leq \infty$ ,  $c_0 < (1 - \hat{\alpha}_n)\hat{\pi}/r$ , and  $\hat{w} \leq v(1 + F(v))$ , then  $\hat{\alpha}$  is increasing in  $c_0$ .

Under a regularity condition, as advertising becomes more expensive, the monopolist advertises less: the advertising pulse occurs later and its magnitude is smaller. The magnitude of the advertising pulse  $P(\hat{\alpha}, \alpha^+)$  falls because both  $\hat{\alpha}$  increases and  $\alpha^+$  decreases with  $c_0$ .

In contrast with the case of niche market without advertising, a comparative statics analysis for  $\hat{\alpha}$  as a function of  $\lambda$  and  $r$  is ambiguous (see Proposition 1 in BV). For example, when  $c_0$  is sufficiently small, the advertising pulse occurs later (i.e.,  $\hat{\alpha}$  increases) when the frequency with which consumers have an opportunity to purchase the product,  $\lambda$ , increases. Intuitively, more consumers learn about their valuations by purchasing the product when they shop more often. However, for sufficiently large  $c_0$ ,  $\hat{\alpha}$  may be a decreasing function of  $\lambda$ . Note that an increase in  $\lambda$ , not only speeds up learning-by-purchasing but also raises the returns to advertising. Nonetheless, the effects of  $\lambda$  and  $r$  on the *long-run* size of the informed segment can be easily ascertained:  $\alpha^+$  is increasing in  $\lambda$  and decreasing in  $r$ .

Our last result in this section summarizes the effects of an advertising pulse on the equilibrium prices, purchasing behavior, and the learning outcome. Let  $\alpha_n(t)$  denote the size of the informed segment and  $p_n(\alpha_n(t))$  denote the price in equilibrium without advertising (see BV for the analysis in this case).

COROLLARY 1 (The effects of an advertising pulse).

1. Advertising pulse shortens the period of introductory prices, i.e.  $\hat{t} < \hat{t}_n$  (and  $\hat{\alpha} < \hat{\alpha}_n$ ), where  $\alpha(\hat{t}) = \hat{\alpha}$  and  $\alpha_n(\hat{t}_n) = \hat{\alpha}_n$ .

2. Advertising pulse lowers the introductory prices, i.e.  $p(\alpha(t)) < p_n(\alpha_n(t))$  for all  $t < \hat{t}$ .
3. Advertising pulse increases the share of the informed consumers, i.e.  $\alpha(t) = \alpha_n(t)$  for all  $t \leq \hat{t}$  and  $\alpha(t) = \alpha^+ > \alpha_n(t)$  for all  $t > \hat{t}$ .

Because the advertising pulse speeds up the consumer learning process and substitutes for learning by purchasing, the monopolist stops offering introductory prices that attract the uninformed consumers *sooner*. On the other hand, the introductory prices are *lower*. The value of information about the desirability of future purchases is lower because the period of low introductory prices is shorter and advertising provides a “free” alternative to learning by purchasing. And so, the advertising pulse has an ambiguous effect on prices: introductory prices are lower, but the jump in the price occurs sooner. Nonetheless, the effect of the advertising pulse on the share of the informed consumers is positive. Even though the uninformed consumers stop purchasing sooner, more of them learn about their preferences from advertising. Example 1 illustrates.

EXAMPLE 1 (Constant marginal advertising cost and unbounded advertising rate).

Consider  $\theta \in \{0,1,2\}$  with  $\Pr(\theta = 1) = z$ ,  $\Pr(\theta = 2) = \varepsilon$ , and  $\Pr(\theta = 0) = 1 - z - \varepsilon$ . We assume that  $\varepsilon = 0 < z < 1$  and  $c_0 < \hat{c} = z(1 - z) / (r(2 - z) + \lambda)$ . With this distribution of preferences, we have  $\hat{w} = z < 1 = \hat{p}$ ,  $\lambda v = \hat{\pi}$ , and  $\hat{w} = z < v(1 + F(v)) = z(2 - z)$ . The time when the advertising pulse occurs and the intertemporal equilibrium prices with and without advertising are graphically depicted in Figure 1, where  $z = 0.8$ ,  $\lambda = 1$ ,  $r = 0.05$ . The figure demonstrates that the overall dynamic pattern of the price policy is preserved under advertising, but the learning by purchasing stops sooner and the introductory prices are lower.

When  $\alpha(t)$  reaches  $\hat{\alpha}$ , the advertising pulse raises the share of the informed consumers to

$$\alpha^+ = 1 - rc_0 / z > 1 - r(1 - z) / (r(2 - z) + \lambda) = \hat{\alpha}_n \geq \alpha_n(t) \text{ for all } t > \hat{t},$$

where the first inequality follows because  $c_0 < \hat{c}$ . ■

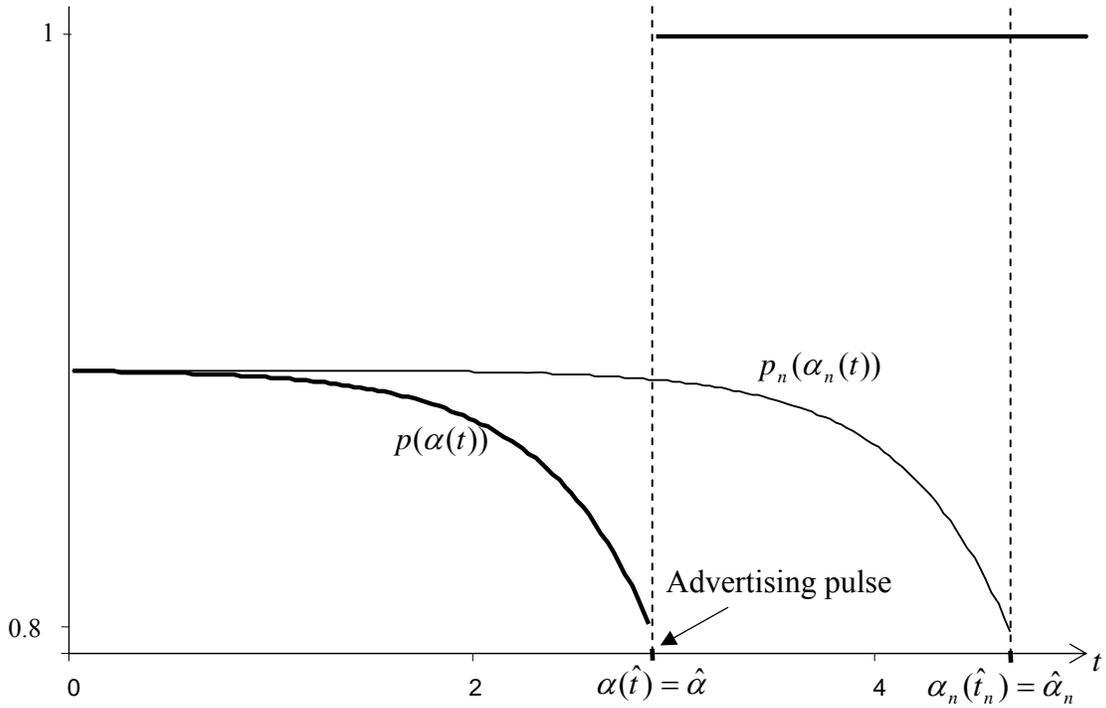


FIGURE 1. Advertising pulse and prices with and without advertising

It is easy to see that the dynamic pattern of equilibrium advertising and prices is qualitatively similar when the marginal advertising cost is decreasing, i.e.  $c''(x) \leq 0$  for all  $x$  with strict inequality for some  $x > 0$ . However, multiple equilibria will exist when optimality equation for the long-run size of the informed segment,  $\alpha^+$ ,

$$(\hat{\pi}/r)(1 - \alpha^+) - c'(P(\hat{\alpha}, \alpha^+)) = 0$$

has more than one solution.

## VI. Constant marginal advertising cost with an upper bound

The assumption that there is no upper limit on the advertising rate is not a realistic representation of the real-world advertising (e.g., Feichtinger et al 1994). In this section we consider the equilibrium patterns of advertising and prices when it is not possible to advertise at a rate higher than a certain rate  $\bar{x} < \infty$ . We continue to assume that the marginal advertising cost is constant for  $x \leq \bar{x}$ , i.e.  $c(x) = c_0 x$  for  $0 \leq x \leq \bar{x}$ .

If  $\bar{x} < \infty$ , the advertising campaign is optimally spread out over the marketing cycle. When the monopolist cannot instantaneously raise the share of the informed consumers, advertising starts during the period of introductory prices, and continues after the price is raised and the uninformed consumers stop purchasing. It is optimal to start advertising when the flow revenue is decreasing in  $\alpha$  because the monopolist's value function is increasing when  $\alpha$  is sufficiently close to  $\hat{\alpha}$ . It is optimal to continue to advertise for some time after  $\hat{\alpha}$  because the long-run returns to advertising are diminishing slowly as  $\alpha$  increases.

By (10), the equilibrium advertising takes the form of ‘bang-bang’ control that results in the “Min-Max-Min” policy (see Krishnan and Jain 2006). The equilibrium advertising rate is zero during the initial state  $\alpha < \alpha^-$ , maximum during the intermediate stage  $\alpha^- < \alpha < \alpha^+$ , and zero for  $\alpha > \alpha^+$ :<sup>16</sup>

$$x(\alpha) = \begin{cases} 0, & \text{if } \alpha < \alpha^- \text{ or } \alpha > \alpha^+ \\ [0, \bar{x}], & \text{if } \alpha = \alpha^- \text{ or } \alpha = \alpha^+ \\ \bar{x}, & \text{if } \alpha^- < \alpha < \alpha^+ \end{cases} \quad (14)$$

The following examples illustrate the patterns of intertemporal pricing policies that may arise in the constant marginal cost case with a finite maximum allowable advertising rate.

**EXAMPLE 2** (Constant marginal advertising cost with an upper bound). Consider the environment analyzed in Example 1 with  $x(\alpha) \leq \bar{x}$  for all  $\alpha$ . The period of time during which advertising occurs, and the intertemporal equilibrium prices are graphically depicted in Figure 2 for  $\bar{x} = 2$  (the values of the other parameters are unchanged), where  $\alpha(t^-) = 1 - e^{-\lambda t^-} = \alpha^-$ ,  $\alpha(\hat{t}) = 1 - e^{-\lambda \hat{t} - \bar{x}(\hat{t} - t^-)} = \hat{\alpha}$ , and  $\alpha(t^+) = 1 - e^{-\lambda t^+ - \bar{x}(t^+ - t^-)} = \alpha^+$ .

When  $t \leq t^-$  the monopolist does not provide informative advertising, and sets the price low enough to attract the uninformed consumers. When  $t^- < t < \hat{t}$  the monopolist advertises at the maximum allowable rate, but continues to attract the uninformed

<sup>16</sup> It can be shown that the constraint  $x(\alpha) \leq \bar{x}$  binds and  $V'(\alpha)(1 - \alpha) - c_0 > 0$  for all  $\alpha \in (\alpha^-, \alpha^+)$ .

Proposition 3 in Section VII provides a characterization of the intertemporal advertising policy in a more general case with a strictly increasing marginal cost. Using a limiting argument, the optimality condition (14) can be obtained as a limit of smooth advertising policies that are studied in Section VII.

consumers. The kink at  $t = t^-$  occurs because the price declines at a slower rate at the start of the advertising campaign. From the point of view of an uninformed consumer, the probability that an informative advertising signal will arrive in the future diminishes as the advertising campaign progresses. This tends to raise the value of information about individual tastes, and the willingness to pay of the uninformed consumers falls more slowly in the beginning of the advertising campaign.

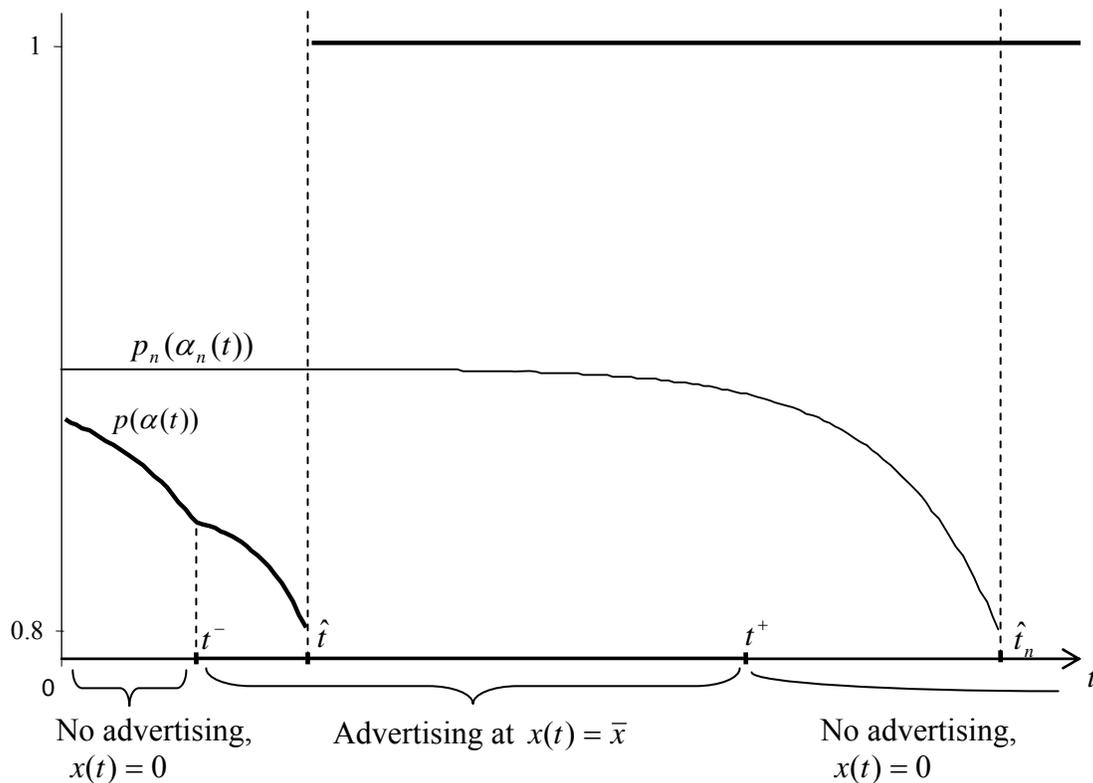


FIGURE 2. “Min-Max-Min” advertising policy and prices

When  $\hat{t} \leq t < t^+$  the monopolist continues to advertise at the maximum feasible rate but the price jumps up and stays at  $\hat{p} = 1$ . Now the uninformed consumers learn only from advertising. When the share of the uninformed consumers is in the intermediate range,  $\alpha \in (\hat{\alpha}, \alpha^+)$ , it is too “costly” for the monopolist to lower the spot prices to attract the uninformed consumers, but the long-run returns to advertising are still sufficiently high. Finally, when  $t \geq t^+$  there is no more advertising, the price remains at  $\hat{p}$ , and the monopolist’s sales stops growing. ■

### VI(i). The Share of the Informed Consumers in the Short-Run

Also, it is worth observing that, if  $\bar{x} < \lambda$ , then after  $\hat{\alpha}$  there may be a period of time during which the current size of the informed segment is *smaller* as a result of advertising. In other words, it is possible that a policy that prohibits advertising would *raise* the share of the informed consumers in the short-run. This may happen because the uninformed consumers stop learning by purchasing and only learn from advertising during  $t \in (\hat{t}, t^+)$  but continue to learn by purchasing during  $t \in (\hat{t}, \hat{t}_n)$  in equilibrium without advertising. If  $t_a \equiv (\lambda \hat{t} - \bar{x} t^-) / (\lambda - \bar{x}) \in (\hat{t}, \hat{t}_n)$  solves  $\alpha_n(t_a) = \alpha(t_a)$  and  $t_b \equiv t^- + (\lambda / \bar{x})(\hat{t}_n - \hat{t}) \in (\hat{t}_n, t^+)$  solves  $\alpha_n(t_b) = \alpha(t_b)$ , then  $\alpha(t) < \alpha_n(t)$  for all  $t \in (t_a, t_b)$  and  $\alpha(t) \geq \alpha_n(t)$  for all  $t \leq t_a$  and  $t \geq t_b$ . For example, if we set  $\bar{x} = 0.2$  and keep the other parameters in Example 2 unchanged, then  $t_a = 3.419$  and  $t_b = 9.64$ . However, Corollary 1 shows that  $\alpha^+ > \hat{\alpha}_n$ , so that it is still true that advertising enlarges the size of the informed segment in the *long run*. Figure 3 illustrates.

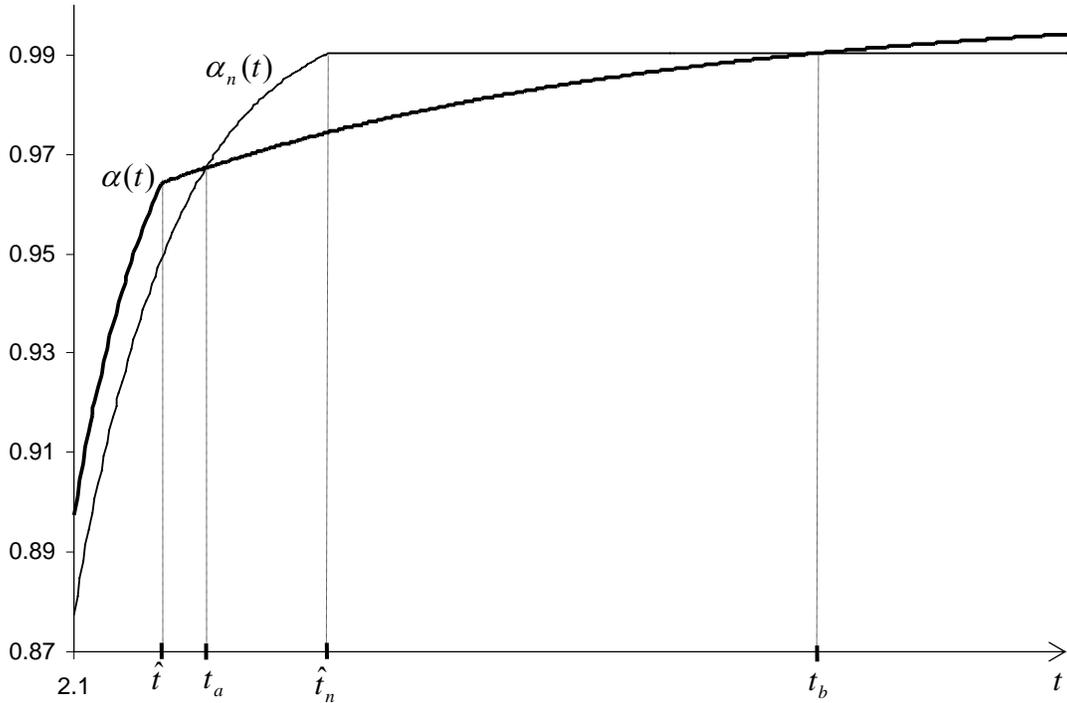


FIGURE 3. The size of the informed segment with and without advertising

## VI(ii). V-shaped Introductory Prices

Our analysis so far suggests that the general intertemporal pattern of the pricing policy is not altered by advertising. This is not necessarily the case. When the monopolist advertises the price may *rise* over time in an early market. This is a dynamic pattern of spot prices that *cannot* emerge in the BV's model. Note that in Example 2 we assumed that  $\theta$  takes only two values (with positive probability),  $\theta = 0$  and  $\theta = 1$ . As a result, *none* of the consumers retain a positive information rent and  $w(\hat{\alpha}) = \hat{w} = v$  for any advertising policy.

Next we will consider a more general case where some of the informed consumers retain a positive information rent in a mature market. To proceed it will be useful to let  $\bar{w}(x)$  denote the rest point of (6) for a given advertising rate  $x$ , which uniquely solves

$$(r + \lambda + x)(\bar{w}(x) - v) - \lambda E \max[\theta - \bar{w}(x), 0] = 0. \quad (15)$$

$\bar{w}(x)$  is the maximum willingness to pay of the uninformed consumers if the future advertising rate is set at  $x$  in each instant and all future prices are given by the willingness to pay of the uninformed consumers. If  $p(\alpha) = \bar{w}(x)$  and  $x(\alpha) = x$  for all  $\alpha$ , then  $V^u(\alpha) = 0$  for all  $\alpha$  because the uninformed consumers are left indifferent between purchasing and not purchasing the good in each instant.

As discussed in BV, in a niche market  $\hat{w} < \bar{w}(0)$  because the value of information obtained from consumption experience,  $E[V^\theta(\alpha)] - V^u(\alpha) = E[V^\theta(\alpha)] = (\lambda/(r + \lambda)) E \max[\theta - p, 0]$ , increases when the future spot prices decrease from  $p(\alpha) = \hat{p}$  to  $p(\alpha) = \bar{w}(0)$  for all  $\alpha$  (since  $\bar{w}(0) < \hat{p}$  in a niche market). Note that  $\bar{w}(x)$  is decreasing in  $x$ . And so, as long as the maximum advertising intensity  $\bar{x}$  is not too great, it is still true that  $\hat{w} \leq \bar{w}(x)$  for all  $x \leq \bar{x}$ . However, when  $\bar{x}$  is sufficiently large, the willingness to pay of the uninformed at  $\alpha = \hat{\alpha}$  (i.e. immediately before the price jumps up and the uninformed consumers drop out of the market) may exceed the rest point of (6), i.e.  $p(\hat{\alpha}) = w(\hat{\alpha}) > \bar{w}(x(\hat{\alpha}))$ . The intertemporal pricing and advertising policies in this case are illustrated in Example 3.

EXAMPLE 3 (Non-monotone introductory prices). Consider the environment analyzed in Example 2, but now we suppose that  $\varepsilon > 0$ , where  $\varepsilon < z < 1 - (2 + \lambda / (r + \lambda))\varepsilon$ .<sup>17</sup> The intertemporal advertising and price policies are shown in Figure 4 for  $z = 0.71$ ,  $\varepsilon = 0.09$  (the values of the other parameters are unchanged). While the equilibrium advertising policy takes the same form as in (14), the price policy exhibits a new dynamic pattern: As depicted in Figure 4, the price *slowly rises* in the early market when the monopolist advertises during  $t^- < t < \hat{t}$ .<sup>18</sup>

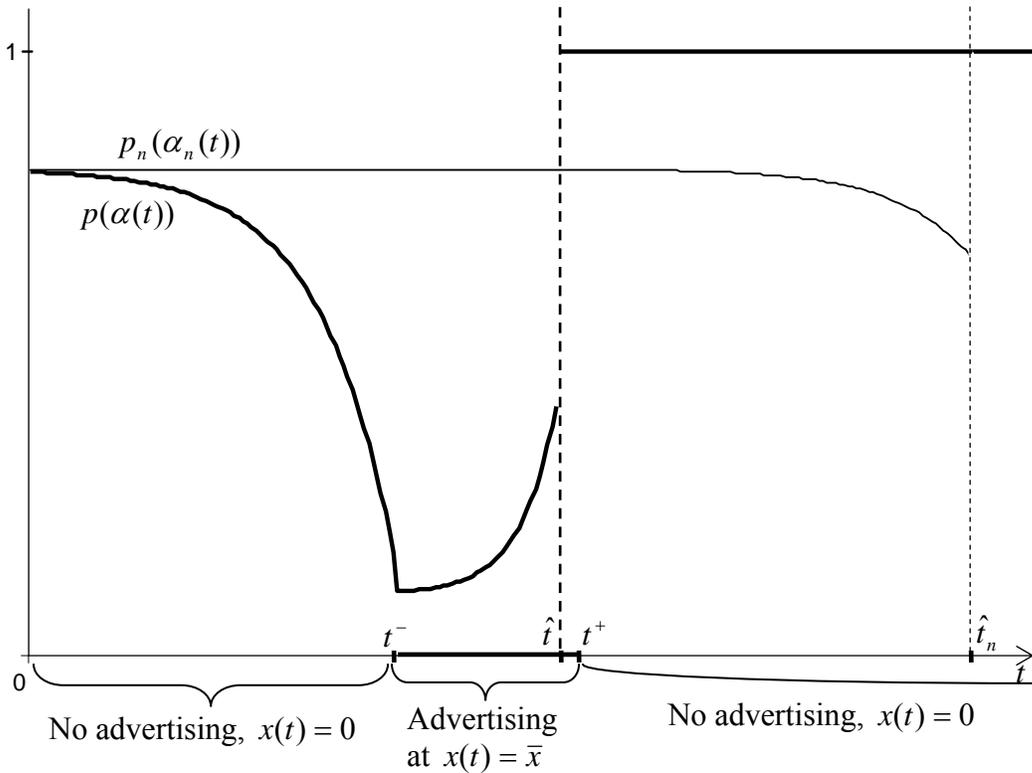


FIGURE 4. Advertising and V-shaped introductory prices

Intuitively, the evolution of the willingness to pay of the uninformed consumers is driven by two considerations. On the one hand, as the period of low introductory prices draws to an end, their willingness to pay falls because the point at which the price jumps up is closer, which *lowers* the long-run expected information rent (see BV). On the other

<sup>17</sup> The first condition assures that  $\hat{p} = 1$  and the second condition assures that  $\hat{w} < \hat{p}$ .

<sup>18</sup> Still, as shown in Figure 4, compared with the equilibrium price without advertising, the willingness to pay of the uninformed consumer is reduced.

hand, as the advertising campaign progresses, the probability of receiving an informative advertising message during the remainder of the advertising campaign diminishes, which *raises* the value of information obtained from purchasing and trying the product. If  $\varepsilon$  falls into some intermediate range and  $\bar{x}$  is sufficiently large, the second effect dominates for all  $t \in (t^-, \hat{t})$ , and the spot price rises while the advertising campaign is in progress in the early market. Specifically, the price rises as  $\alpha$  approaches  $\hat{\alpha}$  if

$$\begin{aligned} p(\hat{\alpha}) &= v + \int_i^{t^+} \lambda e^{-(r+\lambda+\bar{x})(t-\hat{t})} E \max[\theta - \hat{p}, 0] dt + \int_{t^+}^{\infty} \lambda e^{-(r+\lambda+\bar{x})(t^+-\hat{t})-(r+\lambda)(t-t^+)} E \max[\theta - \hat{p}, 0] dt \\ &= z + 2\varepsilon + \frac{\lambda\varepsilon}{r + \lambda + \bar{x}} \left(1 + \frac{\bar{x}e^{-(r+\lambda+\bar{x})(t^+-\hat{t})}}{r + \lambda}\right) > \bar{w}(\bar{x}) = \frac{(r + 2\lambda + \bar{x})(z + 2\varepsilon)}{r + \bar{x} + \lambda(1 + z + \varepsilon)}. \end{aligned}$$

If  $\varepsilon$  is too small, this inequality will not hold because the long-run instantaneous expected information rent  $E \max[\theta - \hat{p}, 0] = \varepsilon$  also becomes small. If  $\varepsilon$  is too large, i.e.  $\varepsilon \geq z$ , then  $\hat{p} = 2$ , and the information rent in the mature market drops to zero. Then we have  $p(\hat{t}) = \hat{w} = v < \bar{w}(\bar{x})$ , which, by (6), implies that the spot price is decreasing as  $\alpha$  approaches  $\hat{\alpha}$  in the early market. ■

## VII. Increasing Marginal Advertising Cost

The simplifying assumption that the marginal cost is constant within the range of the allowable advertising rates obscures how the trade-off between the effects of advertising on the short-run and long-run profits evolves over time. We now consider equilibrium with a *strictly* convex advertising cost function  $c(x)$  with  $c'(\bar{x}) = \infty$ . This advertising cost function has the properties of the “constant-reach, independent readership” technology used in Grossman and Shapiro (1984). Also, as pointed out in Heiman et al. (2001), convex costs of informative advertising via a free sample promotion arise if a firm pursues the cheapest locations or time slots first.

We suppose that the advertising cost function is “sufficiently” convex in the sense that  $\hat{w} \leq \bar{w}(\bar{x})$ . Now the dynamic advertising policy combines the elements found in the intertemporal patterns in Examples 1 and 2. Namely, as in Example 1 the intensity of advertising peaks at the end of the period of the falling introductory prices. In addition, as in Example 2, advertising starts in the early market and continues in the mature

market. However, in contrast with Examples 1 and 2, the advertising intensity smoothly varies throughout the marketing cycle: it slowly rises (possibly, with a delay) in the early market and slowly falls in the mature market.

PROPOSITION 3 (Long-lasting advertising campaign). If  $\hat{w} \leq \bar{w}(\bar{x})$  and  $c(x)$  is a strictly convex function with  $c'(0) = c_0 < (1 - \hat{\alpha}_n)\hat{\pi}/r$  and  $c'(\bar{x}) = \infty$ , then

1. There exist  $0 \leq \alpha^- \leq \hat{\alpha} < \alpha^+ < 1$  such that either (i)  $x(\alpha) > 0$  if and only if  $\alpha \in (\alpha^-, \alpha^+)$ , or (ii)  $x(\alpha) > 0$  for all  $\alpha \in [0, \alpha^+)$  and  $\alpha^- = 0$ . Furthermore,  $0 < \alpha^- < \hat{\alpha} < \alpha^+ < 1$  if  $\lambda v \geq \hat{\pi}$ .

2. The advertising rate  $x(\alpha)$  satisfies

$$c''(x(\alpha)) \frac{dx}{d\alpha} \frac{d\alpha}{dt} = rc'(x(\alpha)) - (1 - \alpha) \frac{d\pi(\alpha, p(\alpha))}{d\alpha} \text{ for all } \alpha^- \leq \alpha < \hat{\alpha}, \quad (16)$$

$$c''(x(\alpha)) \frac{dx}{d\alpha} \frac{d\alpha}{dt} = rc'(x(\alpha)) - (1 - \alpha)\hat{\pi} \text{ for all } \hat{\alpha} \leq \alpha \leq \alpha^+. \quad (17)$$

3. The advertising rate  $x(\alpha)$  is strictly increasing for all  $\alpha^- \leq \alpha < \hat{\alpha}$ , and is strictly decreasing for all  $\hat{\alpha} < \alpha < \alpha^+$ .
4. The price  $p(\alpha)$  is decreasing and satisfies equation (6) for all  $\alpha \leq \hat{\alpha}$  with  $p(\hat{\alpha}) = w(\hat{\alpha}) < \hat{w}$ , and  $p(\alpha) = \hat{p}$  for all  $\alpha > \hat{\alpha}$ .

The monopolist begins to advertise when the end of the period of introductory prices is sufficiently close. To see the intuition, let  $c_0 = 0$ , so that the monopolist advertises whenever her value function is increasing in  $\alpha$ . Then condition  $\lambda v \geq \hat{\pi}$  assures that the monopolist's value function is decreasing in  $\alpha$  for all  $\alpha < \alpha^-$ , and increasing for all  $\alpha > \alpha^-$ . At  $\alpha = \alpha^-$ , the marginal value of private information to the monopolist changes sign from negative to positive. When the size of the informed segment is small,  $\alpha < \alpha^-$ , the monopolist wants to slow down the learning process as much possible and does not advertise. However, the lack of advertising does not stop the learning process while the uninformed consumers are purchasing. And so, when  $\alpha^- < \alpha \leq \hat{\alpha}$ , the monopolist benefits from a larger informed segment even though the current revenues

decrease with  $\alpha$  because she anticipates that only the informed consumers will be purchasing after  $\hat{\alpha}$ .

The optimal advertising path that satisfies the differential equations (16) and (17) balances the intertemporal effects of advertising on the flow revenue and the marginal advertising costs. Shifting a marginal dollar of advertising expenditures from today into tomorrow saves the marginal cost and yields the additional flow revenue from the uninformed consumers who do not become informed when the current advertising rate is lower in the early market. But postponing advertising by an instant also raises the future marginal cost.

And so, the equilibrium advertising rate *rises* during the early market, because the monopolist foregoes less and less current revenue by reducing the share of the uninformed consumers whose number and willingness to pay decline over time. Note that during the early market the monopolist increases the expenditures on advertising even though it becomes harder to reach the remaining uninformed consumers when their number is smaller.

In the mature market with  $\alpha > \hat{\alpha}$ , the advertising rate *falls* over time. The reason is that while the rate at which the flow revenues increase with  $\alpha$  remains constant, the effectiveness of advertising diminishes as the share of the uninformed consumers shrinks. Thus, the U-shape of the flow revenue as a function of the share of the informed consumers and the diminishing effectiveness of advertising *together* give rise to the *bell-shaped* advertising path. If  $\lambda v \leq \hat{\pi}$ , the advertising campaign may start when the product is first introduced, but the other qualitative features of the intertemporal advertising and pricing policies remain unchanged.<sup>19</sup>

Like an advertising pulse, a long-lasting advertising campaign lowers the introductory prices but shortens the time during which they are offered. As shown in Figure 3, while advertising may slow down the learning process for some time after  $\alpha(t)$  has reached  $\hat{\alpha}$ , it raises the share of the informed consumers in the long-run.

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<sup>19</sup> If the expected value of the good is sufficiently low, subsidizing consumer learning by selling the good to the uninformed consumers may become too costly and the seller may skip the introductory period during which she offers low prices and the uninformed consumers buy the product. Then only the informed consumers with high valuations purchase the product since the beginning and the advertising rate falls over the entire life-time of the product.

COROLLARY 2 (The effects of a long-lasting advertising campaign).

1. The effects of advertising on prices described in Corollary 1 continue to hold when the advertising campaign is long-lasting.
2. Advertising raises the share of the informed consumers in the long-run but not necessarily in the short run.

When the maximum feasible advertising intensity is sufficiently high, i.e.  $\hat{w} > \bar{w}(\bar{x})$ , it is possible that the equilibrium price  $p(\alpha)$  rises before it jumps up to  $\hat{p}$  (this is illustrated in Example 3 in the constant marginal cost case). Then the advertising rate may begin to decrease during the early market and the peak of the advertising rate may occur before the seller begins to exclude the uninformed consumers from the market.

### VIII. Conclusion

This paper shows that while it is optimal not to offer advertising in mass markets, the value of private information to the monopolist and the advertising effort vary over the marketing cycle in niche markets. We find that the equilibrium advertising rate peaks at or before the end of the introductory period during which the uninformed consumers purchase the good. We demonstrate that advertising reduces introductory prices but also shortens the time during which they are offered. While the learning process may temporarily slow down because the uninformed consumers stop purchasing sooner, advertising raises the share of the informed consumers in the long-run. We find that the overall intertemporal pattern of the price policy is preserved under advertising: the introductory prices are followed by a higher price. However, in contrast with the model without advertising, the introductory prices may follow a V-shaped path during the early stage of the marketing cycle.

This model of informative advertising with forward-looking consumers provides a micro-foundation for the aggregate advertising response models used in the marketing literature. Informative advertising has the Nerlove-Arrow “goodwill” (or the advertising “carryover”) effect because upon exposure to advertising some of the uninformed consumers learn that their true valuations are above average, which has a long-term effect on demand. On the other hand, because advertising only affects the uninformed

consumers, it also exhibits diminishing returns as does advertising that elicits a direct sales response only from the unsold segment of the market in the Vidale-Wolfe model.

Our modeling strategy can be used to investigate not only the dynamics of the advertising efforts but also the evolution of the optimal marketing mix between informative and persuasive advertising over time (Bagwell 2007).<sup>20</sup> In the paper, we assumed that (i) the size of the consumer population is fixed, and (ii) consumption opportunities arrive at an exogenous constant rate, and the distribution of actual tastes is stable and exogenously given. Relaxing assumption (i) makes it possible to distinguish between consumer information about product existence (awareness) and product attributes. Then an increase in the number of potential consumers will correspond to awareness advertising while an informative signal about individual match values will correspond to informative advertising that was studied in this paper. Relaxing assumption (ii) makes it possible to distinguish between “quantity” and “quality” effects of persuasive advertising. The former can be modeled as an increase in the Poisson arrival rate of consumption opportunities, while the latter can be modeled as a parameterized change in the distribution of the “actual” tastes that is influenced by the seller.<sup>21</sup>

We also assumed that consumers learn their true tastes upon consuming the first unit of the good, and the effectiveness of learning from advertising does not depend on the cumulative number of advertising exposures. As is explained in BV, allowing for a perfectly informative signal to arrive at random intervals to the active buyers, will slightly complicate the exposition but will not change the qualitative nature of the results. Allowing the accumulated stock of advertising to influence the information acquisition rate would require the introduction of imperfections into the consumer learning process, and may obscure the informative role of advertising.<sup>22</sup>

In contrast, the simplifying assumption that both consumption experience and informative advertising provide consumers with perfectly revealing signals about their

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<sup>20</sup> As suggested in Johnson and Myatt (2004), one can expect that the optimal marketing strategy mix will shift from awareness advertising towards promotions that provide information about product attributes and persuasive advertising as the market matures and consumers gain more direct product experience.

<sup>21</sup> We can still stay within the Bayesian framework as long as consumers know the aggregate parameterized probability distribution of tastes.

<sup>22</sup> Also, this is probably less important for television advertising which is consumed in real time.

true valuation for the product is difficult to relax in a fully dynamic model. Clearly, a more realistic and natural assumption is that not only the speed of acquiring but also the precision of information about individual valuations varies depending on whether a buyer learns by consuming the product or exposure to advertisements. The latter, typically, convey information about some but not all of the good's experience characteristics via product specifications, descriptions of content or likely users, illustrative analogies, etc.<sup>23</sup> Allowing consumers to acquire *partial* information about their true tastes, i.e. learn slowly not only at the aggregate but also at an *individual* level, is an interesting topic for future research.

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<sup>23</sup> Osborne (2007) provides empirical estimates of the effects of hypothetical informative advertising and free samples on a new product's market share in a packaged goods market. Barroso (2008) estimates the dynamic effects of advertising by a profit-maximizing firm that raises product awareness in a durable goods market.

## Appendix

It will be convenient to let

$$p^m(\alpha) \equiv \arg \max_p \pi(\alpha, p),$$

and

$$w(\alpha; \alpha^+) \equiv v + (1 - (\alpha^+ - \alpha)) \frac{\lambda}{r + \lambda} E \max[\theta - \hat{p}, 0].$$

Also note that, as shown in BV, in equilibrium without advertising the optimal switching point  $\hat{\alpha}_n$  satisfies

$$g(\hat{\alpha}_n) \equiv \pi(\hat{\alpha}_n, \hat{w}) + \frac{\hat{\pi}}{r} \lambda (1 - \hat{\alpha}_n) - \hat{\alpha}_n \hat{\pi} = 0. \quad (\text{A1})$$

It will also be convenient to let

$$\hat{c} \equiv (1 - \hat{\alpha}_n) \hat{\pi} / r = \frac{\hat{\pi} - \pi(1, \hat{w})}{r + \lambda + r \lambda F(\hat{w}) \hat{w} / \hat{\pi}},$$

where we used (A1) to solve for  $\hat{\alpha}_n$ .

### *Proof of Proposition 1*

Part 1. We prove Part 1 in two steps. First, we show that the switching point  $\hat{\alpha} \in [0, \alpha^+)$  exists. Second, we show that the intertemporal advertising policy must take the form of a single advertising pulse.

Step 1. Suppose that, at  $\hat{\alpha}$ , an advertising pulse of magnitude  $P(\hat{\alpha}, \alpha^+)$  occurs and for all  $\alpha > \alpha^+$  the marginal buyer is an informed buyer. Let

$$h(\alpha, c_0) \equiv c_0 (r \ln(\hat{\pi}(1 - \alpha)/(c_0 r)) + r + \lambda) - (\hat{\pi} - \pi(\alpha, w(\alpha; \alpha^+))).$$

Substituting  $\partial P(\alpha, \alpha^+) / \partial \alpha = -1/(1 - \alpha)$  in (13) and rearranging it, we can rewrite (13) as  $h(\hat{\alpha}, c_0) \leq 0$ . It is easy to verify that  $\alpha^+ > 0$  and  $h(\alpha^+, c_0) < 0$  when  $c_0 < \hat{c}$ . Hence, if  $h(0, c_0) \leq 0$ , then there is an equilibrium with  $0 = \hat{\alpha} < \alpha^+ \leq 1$ . If  $h(0, c_0) > 0$  then, by continuity of  $h(\alpha, c_0)$ , there exists an equilibrium with  $0 < \hat{\alpha} < \alpha^+ \leq 1$ . Furthermore, if  $\lambda v \geq \hat{\pi}$ , then it must be that  $\hat{\alpha} > 0$  since

$$h(0, c_0) = c_0 (r \ln(\hat{\pi}/(c_0 r)) + r + \lambda) - (\hat{\pi} - \lambda w(0; \alpha^+)) > \lambda v - \hat{\pi} \geq 0.$$

The first inequality follows because, by  $c_0 < \hat{c}$ , the first term is positive, and

$w(0; \alpha^+) \geq v$ . The second inequality follows by assumption.

Step 2. Because  $V'(\alpha)(1-\alpha) = (\hat{\pi}/r)(1-\alpha)$  is decreasing in  $\alpha$  for all  $\alpha > \alpha^+$ , there can be only one advertising pulse in the mature market. So it only remains to show that the monopolist does not advertise in the early market. Suppose, to the contrary, that there is another advertising pulse  $(\alpha_0^-, \alpha_0^+)$  with  $\alpha_0^- < \alpha_0^+ < \hat{\alpha}$  and  $p(\alpha_0^+) \leq w(\alpha_0^+)$ . Then it must be that there exists  $\tilde{\alpha} \in (\alpha_0^+, \hat{\alpha})$  such that the monopolist's value function  $V(\alpha)$  satisfies equation (4) and  $V'(\alpha)(1-\alpha)$  is strictly decreasing in  $\alpha$  for all  $\alpha \in (\alpha_0^+, \tilde{\alpha})$  since, by assumption,  $V'(\alpha_0^+)(1-\alpha_0^+) - c_0 = 0$  and  $V'(\alpha)(1-\alpha) - c_0 \leq 0$  for all  $\alpha \in (\alpha_0^+, \hat{\alpha})$ . But differentiating (4), we have

$$\begin{aligned} \lambda \frac{d^+ [V'(\alpha_0^+)(1-\alpha_0^+)]}{d\alpha} &= r \frac{d^+ V(\alpha_0^+)}{d\alpha} - \frac{d^+ \pi(\alpha_0^+, p(\alpha_0^+))}{d\alpha} \\ &= \frac{rc_0}{1-\alpha_0^+} - \pi_\alpha(\alpha_0^+, p(\alpha_0^+)) - \pi_p(\alpha_0^+, p(\alpha_0^+)) \frac{dp^+(\alpha_0^+)}{d\alpha} > 0. \end{aligned}$$

The inequality follows because  $\pi_\alpha(\alpha_0^+, p(\alpha_0^+)) = -\lambda F(p(\alpha_0^+))p(\alpha_0^+) < 0$ ,

$\pi_p(\alpha_0^+, p(\alpha_0^+)) = \lambda(1-\alpha_0^+ F(p(\alpha_0^+)) - \alpha_0^+ f(p(\alpha_0^+))p(\alpha_0^+)) > 0$  since  $(1-F(p))p$  is assumed to be quasi-concave and  $p(\alpha_0^+) = \min[p^m(\alpha_0^+), w(\alpha_0^+)]$ , and as shown in BV,  $d^+ p(\alpha_0^+)/d\alpha < 0$  when the seller does not advertise.<sup>24</sup> This yields the desired contradiction.

Parts 2 and 3. The proof is the same as that of Parts 1 and 2 in Proposition 3 on p. 727 in BV because in our equilibrium there is no advertising during the period of introductory prices, i.e.  $x(\alpha) = 0$  for all  $\alpha < \hat{\alpha}$  (we just need to replace  $\hat{w}$  with  $w(\hat{\alpha}; \alpha^+)$  in the text

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<sup>24</sup>  $p(\alpha) = w(\alpha)$  must be decreasing for all  $\alpha \in (\alpha_0^+, \hat{\alpha})$  because, by assumption,  $w(\hat{\alpha}; \alpha^+) \leq \hat{w} < \bar{w}(0)$ , where  $\bar{w}(0)$  is defined in (15). To see why, note that if there exists  $\tilde{\alpha} \in (\alpha_0^+, \hat{\alpha})$  such that  $p(\tilde{\alpha}) > \bar{w}(0)$ , then, by (6),  $p(\alpha) = w(\alpha) > \bar{w}(0)$  for all  $\alpha \in [\tilde{\alpha}, \hat{\alpha}]$ . But this is contradiction because, at  $\hat{\alpha}$ ,  $p(\hat{\alpha}) = w(\hat{\alpha}) = w(\hat{\alpha}; \alpha^+) \leq \hat{w} < \bar{w}(0)$ . Therefore, we must have  $p(\alpha) = w(\alpha) < \bar{w}(0)$  and  $dp(\alpha(t))/d\alpha(t) = (r+\lambda)(p(\alpha(t)) - v) - \lambda E \max[\theta - p(\alpha(t)), 0] < (r+\lambda)(\bar{w}(0) - v) - \lambda E \max[\theta - \bar{w}(0), 0] = 0$  for all  $\alpha(t) \in (\alpha_0^+, \hat{\alpha})$ .

of the proof on pp.740-741). We state the proof here for completeness. For  $x(\alpha) = 0$  the differential equation (6) has a unique rest point,  $dp(t)/dt = 0$ , at  $p(t) = \bar{w}(0)$ , where  $\bar{w}(0)$  is defined in (15). To argue by contradiction, we suppose that  $p(0) > \bar{w}(0)$ . Then, by (6),  $p(\alpha) > \bar{w}(0)$  for all  $\alpha \leq \hat{\alpha}$ . It follows that at  $\alpha = \hat{\alpha}$ , we have  $p(\hat{\alpha}) > \bar{w}(0)$ ; but at  $\alpha = \hat{\alpha}$ , it must be that  $p(\hat{\alpha}) = w(\hat{\alpha}; \alpha^+)$  for the uninformed buyer to be willing to buy. Since we have  $w(\hat{\alpha}; \alpha^+) \leq \hat{w} < \bar{w}(0)$ , where the last inequality follows because  $\hat{w} < \hat{p}$ , this leads to the desired contradiction. ■

### *Proof of Proposition 2*

Note that  $h(\hat{\alpha}, c_0)$  is strictly decreasing in  $\hat{\alpha}$  if  $\hat{w} < v(1 + F(v))$ . Differentiation yields

$$\begin{aligned} \frac{\partial h(\hat{\alpha}, c_0)}{\partial \hat{\alpha}} &= -\frac{c_0 r}{1 - \alpha} + \frac{\partial \pi(\hat{\alpha}, w(\hat{\alpha}; \alpha^+))}{\partial \hat{\alpha}} + \frac{\partial \pi(\hat{\alpha}, w(\hat{\alpha}; \alpha^+))}{\partial p} \frac{\partial w(\hat{\alpha}; \alpha^+)}{\partial \hat{\alpha}} \quad (\text{A2}) \\ &\leq -\lambda(F(w(\hat{\alpha}; \alpha^+))w(\hat{\alpha}; \alpha^+) - (1 - \hat{\alpha}F(w(\hat{\alpha}; \alpha^+)) - \hat{\alpha}f(w(\hat{\alpha}; \alpha^+))w(\hat{\alpha}; \alpha^+))(\hat{w} - v)) \\ &\leq -\lambda(F(v)v - \hat{w} + v) < 0, \end{aligned}$$

where we used  $\partial w(\hat{\alpha}; \alpha^+)/\partial \hat{\alpha} = (\lambda/(r + \lambda))E \max[\theta - \hat{p}, 0] = \hat{w} - v$ . The first inequality follows because  $v \leq w(\hat{\alpha}; \alpha^+)$  and  $v \leq \hat{w} < \hat{p}$ . The second inequality follows by assumption. Now, by implicitly differentiating  $h(\hat{\alpha}, c_0) = 0$ , it is easy to verify that

$$\frac{\partial \hat{\alpha}}{\partial c_0} = -\frac{r \ln \frac{\hat{\pi}(1 - \hat{\alpha})}{c_0 r} + \lambda + \pi_p(\hat{\alpha}, w(\hat{\alpha}; \alpha^+)) \frac{r}{\hat{\pi}} \frac{\lambda}{r + \lambda} E \max[\theta - \hat{p}, 0]}{-c_0 r / (1 - \hat{\alpha}) + \pi_\alpha(\hat{\alpha}, w(\hat{\alpha}; \alpha^+)) + \pi_p(\hat{\alpha}, w(\hat{\alpha}; \alpha^+)) \frac{\lambda}{r + \lambda} E \max[\theta - \hat{p}, 0]} > 0.$$

The numerator is positive because, by Proposition 1,  $1 - \hat{\alpha} < \hat{\pi}/(c_0 r)$  and

$\pi_p(\hat{\alpha}, w(\hat{\alpha}; \alpha^+)) > 0$ , by quasi-concavity of  $(1 - F(p))p$  and  $w(\hat{\alpha}; \alpha^+) \leq \hat{w} < \hat{p}$ . The denominator is negative by (A2). ■

### *Proof of Corollary 1*

Part 1. The statement follows immediately from Propositions 1 and 2. If  $c_0 \geq \hat{c}$  then there is no advertising. For  $c_0 < \hat{c}$ , Proposition 2 implies that the period of time during

which the uninformed consumers purchase is shorter compared with the equilibrium without advertising because  $\partial \hat{\alpha} / \partial c_0 > 0$ .

Part 2. To show that  $p(\alpha(t)) < p_n(\alpha_n(t))$  for all  $\alpha(t) < \hat{\alpha}$ , suppose, to the contrary, that there exists  $\alpha(\tilde{t}) < \hat{\alpha}$  such that  $p(\tilde{t}) > p_n(\tilde{t})$ . Note that  $p_n(\alpha_n(t))$  satisfies the differential equation (6) with  $x(\alpha(t)) = 0$ , and by (6),  $dp_n / dt \leq dp / dt$  if  $p(t) \geq p_n(t)$ .

Hence, it must be that  $p(\alpha(t)) = w(\hat{\alpha}; \alpha^+) \geq p_n(\alpha_n(t))$  at  $\alpha(t) = \hat{\alpha}$ . But this is impossible since  $w(\hat{\alpha}; \alpha^+) \leq \hat{w} < p_n(\alpha_n(t))$  for all  $\alpha(t) < \hat{\alpha} < \hat{\alpha}_n$ , where the last inequality follows because  $p_n(\hat{\alpha}_n) = \hat{w}$  and  $dp_n / dt < 0$  for all  $t < \hat{t}_n$ .

Part 3. It is easy to verify that  $g(\alpha^+) < 0$  if  $c_0 < \hat{c}$ . Hence, because  $g(\hat{\alpha})$  is decreasing in  $\hat{\alpha}$ , it must be that  $\hat{\alpha}_n < \alpha^+$ . The proof is completed by observing that  $x(\alpha) = 0$  for all  $\alpha < \hat{\alpha}$ . ■

### *Proof of Proposition 3*

Parts 1, 2, and 3. Suppose that the monopolist attracts the uninformed buyers if and only if  $\alpha \leq \hat{\alpha}$ . Then for all  $\alpha \leq \hat{\alpha}$  the seller's value function satisfies equation (4), and for all  $\alpha > \hat{\alpha}$  the seller's value function satisfies equation (5), or more explicitly

$$V(\alpha(t)) = \int_t^{\infty} e^{-r(s-t)} [(1 - (1 - \alpha(t))e^{-\int_t^s x(\alpha(z))dz})\hat{\pi} - c(x(\alpha(s)))] ds \text{ for all } \alpha(t) \in (\hat{\alpha}, 1)$$

with

$$\frac{dV(\alpha(t))}{d\alpha(t)} = \hat{\pi} \int_t^{\infty} e^{-\int_t^s (r+x(z))dz} ds \in (0, \hat{\pi} / r], \quad (\text{A3})$$

and

$$x(\alpha) \frac{d[V'(\alpha)(1 - \alpha)]}{d\alpha} = rV'(\alpha) - \hat{\pi} \leq 0 \text{ for all } \alpha > \hat{\alpha}, \quad (\text{A4})$$

where the inequality follows by (A3).

Also, note that, using (4) and (5) to differentiate the optimality condition (10) with  $x(\alpha) \in (0, \bar{x})$  yields equations (16) and (17), respectfully. Using (A4), from (17) and

convexity of the advertising cost function it follows that  $x(\alpha)$  is non-increasing for all  $\alpha > \hat{\alpha}$ .

Because the equilibrium advertising rate and price are determined jointly, to show that  $x(\alpha)$  is non-decreasing for all  $\alpha < \hat{\alpha}$  and complete the proof of Parts 1-3, we will first prove Part 4.

Part 4. The proof of Part 4 proceeds in three steps. In Step 1, we show that for all  $\alpha < \hat{\alpha}$  the flow revenue is decreasing in  $\alpha$  if the price  $p(\alpha) \leq p^m(\alpha)$  is decreasing. In Step 2, we show that  $p(\alpha)$  is decreasing as  $\alpha \uparrow \hat{\alpha}$ . In Step 3, we show that  $p(\alpha)$  is decreasing for all  $\alpha < \hat{\alpha}$ .

Step 1. Note that  $\pi(\alpha, p(\alpha))$  is decreasing in  $\alpha$  if the price  $p(\alpha)$  is non-increasing and  $p(\alpha) \leq p^m(\alpha)$  because differentiating  $\pi(\alpha, p(\alpha))$  yields

$$\frac{d\pi(\alpha, p(\alpha))}{d\alpha} = \pi_{\alpha}(\alpha, p(\alpha)) + \pi_p(\alpha, p(\alpha)) \frac{dp(\alpha)}{d\alpha} < 0. \quad (\text{A5})$$

The inequality follows because  $\pi_{\alpha}(\alpha, p(\alpha)) < 0$ , and quasi-concavity of  $(1 - F(p))p$  implies that  $\pi_p(\alpha, p(\alpha)) \geq 0$  when  $p(\alpha) \leq p^m(\alpha)$ .

Step 2. Next we show that  $p(\alpha) = w(\alpha)$  is decreasing as  $\alpha$  approaches  $\hat{\alpha}$  from the left, i.e. we demonstrate that condition  $\hat{w} \leq \bar{w}(\bar{x})$  assures that  $d^- p(\hat{\alpha})/d\alpha < 0$ . By the assumption that the marginal buyer is an informed buyer for all  $\alpha > \hat{\alpha}$ , at  $\alpha = \hat{\alpha}$ , the uninformed agents are willing to pay

$$w(\hat{\alpha}) = v + \lambda \int_i^{\infty} e^{-\int_i^t (r + \lambda + x(\alpha(s))) ds} E \max[\theta - \hat{p}, 0] dt \leq \hat{w} \leq \bar{w}(\bar{x}) < \bar{w}(x(\hat{\alpha})).$$

The second inequality follows by the maintained assumption that  $\hat{w} \leq \bar{w}(\bar{x})$ . The third inequality follows because  $d\bar{w}(x)/dx < 0$  and  $x(\hat{\alpha}) < \bar{x}$ .

Step 3. Next we show that in equilibrium

$$p(\alpha) = w(\alpha) \leq \bar{w}(x(\alpha)) \text{ for all } \alpha \leq \hat{\alpha}. \quad (\text{A6})$$

We argue by contradiction. Suppose to the contrary that  $w(\alpha) > \bar{w}(x(\alpha))$  for some  $\alpha \in [0, \hat{\alpha})$ . Then, because  $w(\hat{\alpha}) < \bar{w}(x(\hat{\alpha}))$ , continuity of  $w(\alpha)$  and  $\bar{w}(x(\alpha))$  implies

that there exists  $\alpha_0 \in (0, \hat{\alpha})$  and  $\varepsilon > 0$  such that  $p(\alpha) = w(\alpha) > \bar{w}(x(\alpha))$  for all  $\alpha \in (\alpha_0 - \varepsilon, \alpha_0)$  and  $p(\alpha_0) = w(\alpha_0) = \bar{w}(x(\alpha_0))$ . Then from (16) and (A5) it follows that  $\frac{dx(\alpha_0)}{d\alpha} \geq 0$  since, by assumption,  $\frac{dp(\alpha_0)}{d\alpha} = \frac{dw(\alpha_0)}{d\alpha} = 0$ , and hence,  $\frac{d\bar{w}(x(\alpha_0))}{dx(\alpha_0)} \frac{dx(\alpha_0)}{d\alpha} \leq 0$ . But this implies that  $p(\alpha) \leq \bar{w}(x(\alpha))$  for some  $\alpha \in (\alpha_0 - \varepsilon, \alpha_0)$ , which gives us the desired contradiction. Hence, from (6) and (A6) it follows that the equilibrium price  $p(\alpha)$  is decreasing for all  $\alpha < \hat{\alpha}$ . This completes the proof of Part 4.

Now we can return to the proof of the statements in Parts 2 and 3. Because, as we just showed,  $p(\alpha)$  is decreasing for all  $\alpha < \hat{\alpha}$ , by (A5),  $\pi(\alpha, p(\alpha))$  is decreasing in  $\alpha$  for all  $\alpha < \hat{\alpha}$  since  $p(\alpha) \leq p^m(\alpha)$ . Hence, from the optimality condition (16) it follows that  $x(\alpha)$  is non-decreasing for all  $\alpha < \hat{\alpha}$ . Also, note that from  $c'(0) < (1 - \hat{\alpha}_n)\hat{\pi}/r$  it follows that  $x(\hat{\alpha}) > 0$ . To see why, suppose that, to the contrary, if  $x(\hat{\alpha}) = 0$ . Then, by Part 4, it must be that  $x(\alpha) = 0$  for all  $\alpha$ ,  $\hat{\alpha} = \hat{\alpha}_n$ , and  $V'(\hat{\alpha}_n)(1 - \hat{\alpha}_n) - c'(0) > 0$ , where the inequality follows because  $V'(\hat{\alpha}_n) = \hat{\pi}/r$  in equilibrium without advertising. This leads to the desired contradiction. Hence, it must be that either (i) there exists an  $\alpha^- \in [0, \hat{\alpha}]$  such that  $x(\alpha) > 0$  if and only if  $\alpha \in (\alpha^-, \alpha^+)$ , or (ii)  $x(\alpha) > 0$  for all  $\alpha \in [0, \alpha^+)$  and  $\alpha^- = 0$ . This completes the proof of Parts 2 and 3.

Next we show that in equilibrium  $0 < \alpha^- < \hat{\alpha}$  if  $\lambda v \geq \hat{\pi}$ . Suppose that, to the contrary,  $\alpha^- = 0$ . Then, by (10) and continuity of  $V'(\alpha)$ , it must be that  $V'(0) - c'(x(0)) = 0$ , and we have

$$\begin{aligned} c'(x(0))x(0) - c(x(0)) &< c'(x(0))(\lambda + x(0)) - c(x(0)) = V'(0)(\lambda + x(0)) - c(x(0)) \\ &= rV(0) - \pi(0, p(0)) = r\left(\int_0^{\hat{\alpha}} e^{-rs} (\pi(\alpha(s), p(\alpha(s))) - c(x(\alpha(s)))) ds\right. \\ &\left. + \int_{\hat{\alpha}}^{\infty} e^{-rs} (\alpha(s)\hat{\pi} - c(x(\alpha(s)))) ds\right) - \pi(0, p(0)) \leq r\int_0^{\infty} e^{-rs} \pi(0, p(0)) ds - \pi(0, p(0)) = 0. \end{aligned}$$

The first equality follows from the optimality condition (10) evaluated at  $\alpha = 0$ . The second equality follows by rearranging (4) evaluated at  $\alpha = 0$ . The second inequality follows because, by (A5),  $\pi(0, p(0)) > \pi(\alpha, p(\alpha))$  for all  $\alpha \leq \hat{\alpha}$  and, by assumption,  $\pi(0, p(0)) = \lambda w(0) \geq \lambda v \geq \hat{\pi}$ . But this yields the desired contradiction because convexity of  $c(x)$  implies that  $xc'(x) > c(x)$  for all  $x > 0$ . Therefore, there exists  $\alpha^- \in (0, \hat{\alpha})$  such that  $V'(\alpha) < 0$  for all  $\alpha < \alpha^-$ . Also,  $V'(\alpha)$  is single-crossing, i.e.  $V'(\alpha) > 0$  for all  $\alpha > \alpha^-$  because differentiating (4) and rearranging yields

$$V''(\alpha)(\lambda + x(\alpha))(1 - \alpha) = (r + \lambda + x(\alpha))V'(\alpha) - \frac{d\pi(\alpha, p(\alpha))}{d\alpha} > 0 \text{ for all } \alpha < \hat{\alpha},$$

where the inequality follows by (A5).

Finally, we verify our conjecture that the monopolist attracts the uninformed buyers if and only if  $\alpha \leq \hat{\alpha}$ . Because  $\pi(\alpha, p(\alpha))$  is decreasing in  $\alpha$ , and by (A4),  $V'(\alpha)(1 - \alpha)$  is decreasing in  $\alpha$  when  $V(\alpha)$  satisfies (5), the left-hand side of the indifference condition in (9) is decreasing in  $\hat{\alpha}$ , while the right-hand side is increasing. This verifies the conjecture and completes the proof of Part 1. ■

### *Proof of Corollary 2*

Part 1. To show that  $\hat{\alpha} < \hat{\alpha}_n$ , suppose that, to the contrary  $\hat{\alpha} \geq \hat{\alpha}_n$ . Then we have

$$0 \geq g(\hat{\alpha}) = \pi(\hat{\alpha}, \hat{w}) + \frac{\hat{\pi}}{r} \lambda(1 - \hat{\alpha}) - \hat{\alpha}\hat{\pi} > \pi(\hat{\alpha}, w(\hat{\alpha})) + V'(\hat{\alpha})\lambda(1 - \hat{\alpha}) - \hat{\alpha}\hat{\pi}.$$

The first inequality follows because  $g(\alpha)$  is decreasing in  $\alpha$ . The second inequality follows because  $w(\hat{\alpha}) \leq \hat{w}$  and by (A3). This yields the desired contradiction.

The proof that  $p(\alpha(t)) < p_n(\alpha_n(t))$  for all  $\alpha(t) < \hat{\alpha}$  is almost identical to the proof of Part 2 of Corollary 1. Suppose, to the contrary, that there exists  $\alpha(\tilde{t}) < \hat{\alpha}$  such that  $p(\tilde{t}) > p_n(\tilde{t})$ . Note that, by (6),  $dp_n/dt \leq dp/dt$  if  $p(t) \geq p_n(t)$  since  $x(\alpha(t)) \geq 0$  for all  $\alpha(t) < \hat{\alpha}$ . Hence, it must be that  $p(t) = w(\alpha(t)) \geq p_n(\alpha_n(t))$  at  $\alpha(t) = \hat{\alpha}$ . But this is impossible because  $w(\hat{\alpha}) \leq \hat{w} < p_n(\alpha_n(t))$  for all  $\alpha(t) < \hat{\alpha} < \hat{\alpha}_n$ .

Part 2. If  $c_0 \geq \hat{c}$ , then the monopolist does not advertise. If  $c_0 < \hat{c}$ , then from Part 3 of Corollary 1 it follows that  $\lim_{t \rightarrow \infty} \alpha(t) = \alpha^+ > \hat{\alpha}_n = \lim_{t \rightarrow \infty} \alpha_n(t)$ . An example where  $\alpha(t) < \alpha_n(t)$  on a set of positive measure is provided in Section VI(i). ■

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