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1 Introduction

The study analyzes a stylized growth scenario, in which the growth rate of produced consumption exceeds that of environmental service streams. I show how limited substitutability in consumption between different classes of goods affects the magnitude and time development of social discount rates. I relate substitutability between environmental and produced goods to the paradigms of weak and strong sustainability. Welfare specifications corresponding to the weak sustainability paradigm imply falling discount rates. Welfare specifications corresponding to the strong sustainability paradigm imply growing discount rates.

The study formalizes and reviews a reasoning put forth by Neumayer (1999) in the climate change debate. He argues that limited substitutability is more critical to long-term evaluation than the effects generally discussed in social discounting, i.e. the rate of pure time preference and decreasing marginal utility under growth. I restate the substitutability effect as a third contribution to the social discount rate. While Neumayer (1999) argues verbally that a stronger limitation of substitutability would increase the attention paid to the long term, the opposite holds true in the modeled growth scenario.

The study also relates to a widely held believe that environmental considerations do not affect the overall discount rate when converted into produced consumption equivalents. I show that even when converting environmental service streams into produced consumption equivalents, limited substitutability can cause a change of individual and numeraire discount rates. I relate the finding to Arrow, Cline, Maler, Munasinghe, Squitieri & Stiglitz's (1995) critique of Weitzman's (1994) reduced and hyperbolic[2] 'environmental discount rate'. Moreover, I establish a connection to Gerlagh & van der Zwaan's (2002) findings concerning the time development of value share of environmental versus produced goods in a comparable growth scenario.

More generally my model relates to a broad field of literature that motivates and works with non-constant discount rates. Groom, Hepburn, Koundouri & Pearce (2005, 7 et seqq.) present an excellent review of reasons that can cause social discount rates to decline. Frederick, Loewenstein & O'Donoghue (2002, 378) survey experiments showing that a falling discount rate describes behavior better than constant discounting. From a different perspective, Chichilnisky (1996) and Li & Löfgren (2000) develop models of hyperbolic discounting based on considerations of intergenerational justice. In 2003 hyperbolic discount rates made their way into applied policy, when the British Green

[2] A hyperbolic discount rate is a rate that falls over time, see page 4.

Book started to prescribe hyperbolic discount rates for the evaluation of long-term projects (HM Treasury 2003, 97 et sqq.).

A phenomenon often analyzed in relation to models of hyperbolic discounting is that of time inconsistency (Phelps & Pollak 1968, Arrow 1999, Karp 2005). These models employ a non-constant rate of pure time preference, which can lead to a continual revision of (formerly) optimal plans. Time inconsistency does not arise if the discount rate falls for other reasons. For example, Weitzman (1998), Azfar (1999), Gollier (2002) and Dasgupta & Maskin (2005) rationalize hyperbolic discounting by introducing uncertainty. My model explains how limited substitutability between different types of consumption can cause optimal discount rates to be non-constant. In these situations optimal programs are time consistent.

The paper is structured as follows. Section 2 derives social discount rates and factors in the multi-commodity setting. Moreover, it relates the concepts of weak and strong sustainability to the degree of substitutability between man-made and environmental goods. Section 3 analyzes the scenario where growth rates for produced goods exceed those of environmental service streams. It derives the magnitude and time behavior of social discount rates and relates them to the concepts of strong versus weak sustainability. Section 4 discusses different perspectives on discounting in a cost benefit evaluation of a small project. It analyzes good-specific discounting, numeraire conversion and social versus market based discounting. Section 5 concludes. Calculations and proofs are gathered in the appendix.

2 Social Discount Rates and the Strength of Sustainability

2.1 Social Discount Rates and Factors

This section derives social discount rates from the trajectory of marginal utility. Consumption quantities of two goods at time t are characterized by positive real numbers, denoted $x_1(t)$ and $x_2(t)$. The time argument will generally be omitted. With $\mathfrak{x}\colon [0,\infty) \to \mathbb{R}^2$ I denote the consumption path of the two goods from the present $t = 0$ to the infinite time horizon. Welfare is

$$\mathcal{U} = \int_0^\infty U(x_1, x_2, t)\, dt \ , \tag{1}$$

with a twice differentiable (instantaneous) utility function $U(x_1, x_2, t)$. I define the *good specific social discount factor* between time t_0 and time t for a given consumption path x by[3]:

$$D_i^{\mathsf{x}}(t, t_0) \equiv \frac{\frac{\partial U(x_1,x_2,t)}{\partial x_i}}{\frac{\partial U(x_1,x_2,t_0)}{\partial x_i}} \quad \Leftrightarrow \quad \frac{\partial U(x_1, x_2, t)}{\partial x_i} = D_i^{\mathsf{x}}(t, t_0) \frac{\partial U(x_1, x_2, t_0)}{\partial x_i}, \tag{2}$$

$i \in \{1, 2\}$. The discount factors $D_i^{\mathsf{x}}(t, t_0)$ capture the value development over time, relating the value of an additional unit of consumption good x_i at time t to the value of an additional unit at time t_0. The $D_i^{\mathsf{x}}(t, t_0)$ are *time propagators of marginal utility*.[4] The discount rates corresponding to the discount factors are

$$\delta_i(t) = -\frac{\frac{d}{dt} D_i^{\mathsf{x}}(t, t_0)}{D_i^{\mathsf{x}}(t, t_0)} = -\frac{\frac{d}{dt}\frac{\partial U(x_1,x_2,t)}{\partial x_i}}{\frac{\partial U(x_1,x_2,t)}{\partial x_i}} = -\frac{\frac{\partial^2 U}{\partial t \partial x_i}(t) + \frac{\partial^2 U}{\partial x_i^2}(t)\dot{x}_i + \frac{\partial^2 U}{\partial x_j \partial x_i}(t)\dot{x}_j}{\frac{\partial U}{\partial x_i}(t)} \tag{3}$$

for $i, j \in \{1, 2\}$ with $i \neq j$. The $\delta_i(t)$ are the *generators* of the propagators $D_i^{\mathsf{x}}(t, t_0)$ and generate the value development of an additional unit of good x_i in the future.[5] The discount factor is recovered from a discount rate by:

$$D_i^{\mathsf{x}}(t, t_0) = \exp\left(- \int_{t_0}^{t} \delta_i(x(t'), \dot{x}(t'), t')\, dt' \right) . \tag{4}$$

2.2 Review of the One Commodity Special Case

In models with a single (aggregate) consumption good, $\delta_i(t)$ is known as the (instantaneous) *social discount rate*. The latter stands out more clearly if instantaneous utility is specified as $U(x_1, x_2, t) = u(x_1, x_2)e^{-\rho t}$. Neglect the second commodity by setting it

[3]For a given consumption path x, $U(t) \equiv U(x_1(t), x_2(t), t)$ and its derivative are evaluated at the implied consumption levels $x_1(t)$ and $x_2(t)$.

[4]The name is based on a general concept in physics and group theory, see footnote 5 for reference. Malinvaud (1974, 234) uses these discount factors in a discrete time setting in a general equilibrium context. The $D_i^{\mathsf{x}}(t, t_0)$ can be calculated even if pure time dependence of instantaneous utility is not multiplicatively separable.

[5]Precisely, the negative of the *dis*count rate $\delta_i(t)$ would be called the generator. See Sakurai (1985, 46 et sqq.,71 et sq.) or Goldstein (1980, chapter 9) for this view on classical and quantum mechanics (e.g. momentum being the generator of translation).

constant. Then, the discount rate $\delta \equiv \delta_1$ becomes

$$\delta(t) = \rho - \frac{\frac{\partial^2 u}{\partial x_1^2}}{\frac{\partial u}{\partial x_1}} \dot{x}_1 = \rho - \frac{\partial \frac{\partial u}{\partial x_1}}{\partial x_1} \frac{x_1}{\frac{\partial u}{\partial x_1}} \frac{\dot{x}_1}{x_1} = \rho + \theta_{(x(t))} \, \hat{x}_{1(x_1(t), \dot{x}_1(t))} \; . \tag{5}$$

This expression for the social discount rate is well known in the literature, see e.g. Arrow et al. (1995, 136) or Groom et al. (2005). The constant ρ is called the *pure rate of time preference*. The term θ is the (absolute value of the) *elasticity of marginal utility of consumption*, which is the inverse of the *intertemporal elasticity of substitution*. Finally, $\hat{x}_1$ denotes the *growth rate* of the consumption commodity. Equation (5) states that the value development of an additional unit of good x_i is generated by the pure rate of time preference as well as a term proportional to the growth rate of consumption and the elasticity of marginal utility. To gain intuition for the second term, assume that consumption is growing over time. Then, an individual with a decreasing marginal valuation of consumption values an additional unit of consumption in the future less than in the present. Therefore, growth increases the rate at which he discounts future consumption. In most macroeconomic models the function u is assumed to exhibit constant elasticity of intertemporal substitution (CIES). The CIES assumption implies that in a steady state, where growth rates are constant, the term $\theta \, \hat{x}_1$ and, thus, the social discount rate $\bar{\delta} = \rho + \theta \hat{x}_1$ are constant. A constant rate of discount implies by equation (4) a social discount factor $D^{\mathsf{x}}(t, t_0) = e^{-\bar{\delta}(t-t_0)}$ and, thus, exponential discounting of future consumption.

In general, expression (5) need not be constant. A non-constant $\theta \, \hat{x}_1$ can lead to hyperbolic discounting. A discount function is said to be *hyperbolic* if it is characterized by a falling instantaneous discount rate (Laibson 1997, 450). Dasgupta (2001, 183 et sqq.) points out that in the face of global climate change, a decline in consumption growth $\hat{x}_1$ would imply a falling social discount rate. This effect is inversely proportional to the intertemporal elasticity of substitution (θ^{-1}). For a given decline in growth, a *lower* intertemporal elasticity of substitution (θ^{-1}) induces a stronger decrease of the social discount rate and, thus, a relatively *higher* weight given to future consumption. Finally, Gollier (2002) derives conditions under which the term $\theta \, \hat{x}_1$ leads to a falling discount rate in a model with uncertainty.

2.3 Limited Substitutability in Consumption

Returning to equation (3), I analyze how equation (5) changes in the multi-commodity setting. From now on, good x_1 is interpreted as a flow of environmental goods and services, while x_2 represents an aggregate of produced consumption. To assure *time consistency* of the planning functional (1), I assume $U(x_1, x_2, t) = u(x_1, x_2)e^{-\rho t}$ implying a constant rate of pure time preference ρ. Then, the discount rate corresponding to the social discount factor $D_1^{\mathbf{x}}(t, t_0)$ becomes

$$\delta_1(t) = \rho - \frac{\frac{\partial^2 u}{\partial x_1^2}}{\frac{\partial u}{\partial x_1}} \dot{x}_1 - \frac{\frac{\partial^2 u}{\partial x_1 \partial x_2}}{\frac{\partial u}{\partial x_1}} \dot{x}_2 \, . \tag{6}$$

It comprises an additional term that depends on the substitutability $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ between the two classes of goods.[6] To bring out the influence of substitutability in welfare on the social discount rate and its evolvement over time, I take instantaneous utility to be $u(x_1, x_2) = [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{1/s}$ with $s \in \mathbb{R}, a_1, a_2 \in \mathbb{R}_{++}, a_1 + a_2 = 1$ and $u_1, u_2 \geq 0$.[7] This step separates good-specific utility $u_i(x_i)$ from substitutability effects parameterized in a simple form by s. As derived in appendix A, such a welfare specification yields the social discount rate for the environmental service stream

$$\delta_1(t) = \rho - \frac{\frac{\partial^2 u_1}{\partial x_1^2}}{\frac{\partial u_1}{\partial x_1}} \dot{x}_1 - (1-s) \frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s} \left(\frac{\frac{\partial u_2}{\partial x_2}(x_2)}{u_2(x_2)} \dot{x}_2 - \frac{\frac{\partial u_1}{\partial x_1}(x_1)}{u_1(x_1)} \dot{x}_1 \right) . \tag{7}$$

The first and the second term in equation (7) resemble the widely used equation (5). In the following I examine the additional third term that depends on the substitutability parameter s. Focusing on this objective, I simplify the utility function by setting $u_1(x_1) = x_1$ and $u_2(x_2) = x_2$, which leads to the standard CES utility function

$$u(x_1, x_2) = [a_1 x_1^s + a_2 x_2^s]^{1/s} \, . \tag{8}$$

CES functions exhibit a constant elasticity of substitution σ that relates to the substitutability index s by the formula $\sigma = \frac{1}{1-s}$ (Arrow, Chenery, Minhas & Solow 1961). Observe that CES functions are homogeneous of degree one. Thus, proportional overall

[6] Equation (6) has independently been derived by Weikard & Zhu (2005) who also comment on the magnitude effects (see below) but do not analyze time behavior of the discount rates.

[7] $\mathbb{R}_{++}$ denotes the strictly positive real numbers. For $s = 0$ the function is defined by the limit $s \to 0$ yielding $u(x_1, x_2) = u_1(x_1)^{a_1} u_2(x_2)^{a_2}$. For $s \to -\infty, \infty$ the limit functions are $\min\{u_1(x_1), u_2(x_2)\}$ and $\max\{u_1(x_1), u_2(x_2)\}$ respectively. $u_i \geq 0$ abbreviates $u_i(x_i) \geq 0$ for all $x_i \in \mathbb{R}_+$.

growth does not change marginal utility (which is homogeneous of degree zero). Therefore, the chosen functional form is well suited to focus on the new effect, due to limited substitutability and relative difference in growth, filtering out the overall growth effect extensively discussed in the literature in connection with equation (5). This step leads to the discount rate:

$$\delta_1(t) = \rho - (1-s) \underbrace{\frac{a_2 x_2^s}{a_1 x_1^s + a_2 x_2^s}}_{\equiv {V_2}^s(x_1, x_2)} (\hat{x}_2 - \hat{x}_1) \ . \tag{9}$$

The first determinant in the social discount rate for the environmental service stream in equation (9) is the pure rate of time preference ρ. It is reduced by a second term which comprises three different components. The first component $(1-s) = \sigma^{-1}$ *is a measure for the limitedness in substitutability* between the two classes of goods. The second component depicts the value share of the produced consumption stream:

$${V_2}^s(x_1, x_2) = \frac{\frac{\partial u}{\partial x_2} x_2}{\frac{\partial u}{\partial x_1} x_1 + \frac{\partial u}{\partial x_2} x_2} = \frac{a_2 x_2^s}{a_1 x_1^s + a_2 x_2^s} \ . \tag{10}$$

It depends on the ratio $\frac{x_1}{x_2}$ between the environmental services and the produced goods consumed,[8] the utility weights a_1 and a_2 and the substitutability parameter s. The last component in equation (9) is the difference in growth rates between produced and environmental consumption and service streams. Altogether the second term on the right hand side of equation (9) can be summarized as follows. The difference in growth rates is weighted with the value share of produced consumption. This expression is then weighted with the limitedness in substitutability between produced and environmental service streams and subtracted from the pure rate of time preference. Section 3 analyzes the expression for different degrees of substitutability.

Similarly, the social discount rate for produced consumption and service streams is

$$\delta_2(t) = \rho + (1-s) \underbrace{\frac{a_1 x_1^s}{a_2 x_2^s + a_1 x_1^s}}_{\equiv {V_1}^s(x_1, x_2)} (\hat{x}_2 - \hat{x}_1) \ . \tag{11}$$

The interpretation is analogous to that of equation (9). However, depicting the difference in relative growth the same way as in equation (9) implies a sign switch. Therefore, the

[8] That ${V_2}^s$ only depends on the ratio is easily observed by multiplying nominator and denominator on the right hand side of equation (10) with x_2^{-s}.

additional effect, which is weighted with the value share of the environmental services

$$V_1{}^s(x_1, x_2) = \frac{\frac{\partial u_1}{\partial x_1} x_1}{\frac{\partial u_1}{\partial x_1} x_1 + \frac{\partial u_2}{\partial x_2} x_2} = \frac{a_1 x_1^s}{a_1 x_1^s + a_2 x_2^s} \ ,$$

enters the social discount rate for produced consumption positively.

2.4 A Preference for Weak versus Strong Sustainability

This subsection relates the substitutability parameter s to the concepts of weak and strong sustainability. These concepts suggest differing implementations of a sustainable development, i.e. a "development that meets the needs of the present without compromising the ability of future generations to meet their own needs" (WCED 1987). The paradigm of *weak sustainability* translates the latter definition into the demand that overall welfare should not decline over time. To this end, its proponents allow for a substitution between environmental and man-made capital. On the other hand, the advocates of the *strong sustainability* paradigm demand that natural capital (or its service flows) by itself should not decline.[9] They do not believe in substitutability between the different types of capital.

Traditionally, the economic analysis of sustainability is mostly focused on capital and its substitutability *in production*. For a list of environmental assets that are considered non-substitutable by man-made capital see Pearce, Markandya & Barbier (1997, 37) or Neumayer (1999, 39). The claim of non-substitutability of these assets comes down to pointing out that the corresponding service flows cannot be replaced by those of man-made capital. This claim is defensible if we are concerned with a perfect replication of service streams. Take for instance the ozone layer with its UV-protection function. Opponents to the non-substitutabilty assumption would argue that, at least at the margin, the ozone in the stratosphere can be replaced by sunscreen lotion or shelter under glass, both of which protect to some degree from ultraviolet radiation. However, such an argument already involves the *welfare judgment* that taking a sun bath with

[9] Opinions whether natural capital should be non-declining in value or in physical terms differ. Moreover, natural capital is often broken down further into different classes, each of which should be kept non-declining. Often, strong-sustainability is additionally associated with an intrinsic value of nature. The latter can be mapped into 'existence service flows', e.g. proportional to the amount of existing capital. For an overview over the more detailed differences between weak and strong sustainability as well as further differentiations of sustainability demands consult e.g. Neumayer (1999) and van den Bergh & Hofkes (1998).

or without sunscreen are perfect substitutes, or that a glass roof is a substitute for the open air. Assuming a non-perfect replicability of natural capital, I consider the degree of *substitutability in welfare* between man-made and environmental goods and service streams to be the most important difference between the weak and the strong sustainability paradigm.

Neumayer (1999) introduces a similar reasoning into the debate on climate change evaluation. His essay argues that an appropriate characterization of sustainability and limited substitutability would be more critical to long-term evaluation than pure time preference and the growth effect reviewed in equation (5). As I have shown, the substitutability effect can also be translated into the social discount rate. Neumayer (1999) claims that the consideration of strongly limited substitutability would result in a higher weight given to the needs of future generations. The next section formally analyzes the claim.

Neumayer's (1999) verbal discussion relates weak sustainability with perfect substitutability and strong sustainability with 'close to lexicographic preferences'. The model in this paper allows a continuum of different degrees of substitutability. To draw the dividing line between weak and strong sustainability preferences, I adopt the following reasoning. Whenever it is possible to extract positive welfare from *only* consuming man-made goods and service streams, I assign the corresponding preferences to a weak sustainability paradigm.[10] Then, in the model of equation (8), preferences of a weak sustainability proponent are identified with parameters $1 > s > 0$ and an elasticity of substitution $\sigma > 1$. Whenever it is *not* possible to derive positive welfare from only consuming man-made goods and service streams, I identify preferences with the strong sustainability paradigm. In my model, this scenario corresponds to $s < 0$ and $0 < \sigma < 1$.[11] The welfare specification dividing weak and strong sustainability is represented by Cobb-Douglas preferences ($s = 0$, $\sigma = 1$). Here, it is possible to replace any amount of environmental services if (and only if) produced consumption grows to infinity. Note that for my CES model of welfare, this identification of parameters coincides with an assignment also motivated by Gerlagh & van der Zwaan (2002).

[10] In general 'positive welfare' should be thought of as 'welfare above the lowest possible welfare level'.

[11] In the language of Dasgupta & Heal (1974, 4) for production, this assumption corresponds to both goods being *essential*.

3 A Stylized Growth Model

3.1 Assumptions

This section analyzes how the weights for future consumption streams evolve in a scenario where produced consumption grows at a faster rate than consumption of environmental services. The underlying assumption is that technological progress increases the availability of produced consumption at a faster rate than the availability of environmental service and consumption streams can be increased. When thinking about essential life-support services that most advocates of a notion of strong sustainability are concerned about (e.g. climate regulation functions), it is hard to think of a long-term positive growth rate of environmental services at all. When considering environmental goods like those defined in Fisher & Krutilla (1975, 360) as goods that are "generally consumed on site, with little or no transformation by ordinary productive processes", including e.g. scenic views, then by definition these goods are not affected by technological progress in production.[12] The appreciation of biodiversity and its existence value is another example where the growth rate of the corresponding existence service flow is negative and a serious growth within a human planning horizon is hard to imagine. Against this background I introduce

Assumption 1: There exists $\epsilon > 0$ such that $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon$ for all t.

The assumption allows for a decline in environmental goods and services. It also allows for a scenario, which is sometimes put forth in relation to climate change, where production and environment decline together and environmental service flows decline at a higher rate. In general, Assumption 1 contains the kind of scenarios that most advocates of a strong sustainability concept are concerned about.[13]

Under this stylized growth assumption, I analyze how different degrees of substitutability between the two classes of goods and services affect the weights given to future consumption. I focus on the effect resulting from the *difference in growth* rates and the limited *substitutability*. As pointed out in Section 2.3, the CES welfare function

[12]One can think of several cases where technological progress helps accessing or enjoying environmental goods. However, such a complementarity between produced and environmental goods and services is captured in the welfare function, i.e. in the parameter s.

[13]Part *i*), *ii*) and *iii*) of Proposition 1 as well as Corollary 1 and Proposition 2 also hold under the slightly weaker assumption that there exist $\epsilon > 0$ and $t^* \in [0, \infty)$ such that $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon$ for all $t \geq t^*$.

facilitates this focus by disregarding the influence of an even overall growth (or decline).

Assumption 2: Welfare is representable in the functional form[14]

$$\mathcal{U} = \int_0^\infty [a_1 x_1^s + a_2 x_2^s]^{1/s} e^{-\rho t}\, dt \quad \text{with } a_1, a_2 \in \mathbb{R}_{++}, a_1 + a_2 = 1 \text{ and } s \in \mathbb{R}, s \leq 1.$$

Assumptions 1 and 2 yield an easily tractable model fleshing out the relation between substitutability and long-term consumption weights. In general however, the somewhat restrictive preference specification in Assumption 2 is not needed to derive a non-constancy of the social discount rates as done below.[15]

3.2 Results

There are four scenarios for the social discount rates. They correspond to the welfare specifications $s = 1$ (perfect substitutability, $\sigma = \infty$), $s = 0$ (Cobb-Douglas preferences, $\sigma = 1$), $s \in (0, 1)$ (moderate substitutability, $\sigma > 1$) and $s < 0$ (strongly limited substitutability, $0 < \sigma < 1$). The interpretation of the social discount rates derived for the different welfare specifications is the following. Take as given an underlying growth scenario that satisfies Assumption 1. A decision-maker or social planner is asked to evaluate a small[16] project that affects environmental service streams and produced consumption streams over some period of time. Then, the social discount rates and factors specify the weight that a planner, subscribing to a particular welfare specification, gives to the corresponding future consumption streams. Section 4 presents a formal setup of such a small project evaluation.

The case of *perfect substitutability* in consumption between environmental service flows and produced consumption is characterized by the substitutability parameter $s = 1$ ($\sigma = \infty$). It implies additivity in welfare between the different classes of goods

[14]For $s = 0$ the integrand is defined by limit, yielding the well known Cobb-Douglas specification: $\lim_{s\to 0}[a_1 x_1^s + a_2 x_2^s]^{1/s} = x_1^{a_1} x_2^{a_2}$ (Arrow et al. 1961, 231). In the range $s \in (1, \infty)$ extreme choices are generally preferred to mixtures. Such an assumption does neither seem reasonable when analyzing environmental and produced consumption and service streams, nor does it correspond to any notion of sustainability.

[15]Note that for less symmetric preference specifications, a non-constancy of the social discount rates can also apply when growth rates for environmental and produced consumption streams coincide. The necessary condition is, however, that substitutability between the two classes of goods is limited.

[16]Smallness of the project assumes that changes brought about by the project do not affect the overall growth scenario.

$u(x_1, x_2) = a_1 x_1 + a_2 x_2$. As there is no limit to substitutability ($1 - s = \sigma^{-1} = 0$), equations (9) and (11) show that the social discount rates for both classes of goods coincide with the pure rate of time preference: $\delta_1 = \delta_2 = \rho$. This result holds by construction (and reduction) of the welfare function carried out in Section 2.3 to focus on the substitutability effect and disregard other growth effects.

In the case of limited substitutability the following result obtains. Recall that I use the term steady state for a scenario where growth rates are constant.

Proposition 1: Let Assumptions 1 and 2 hold with $s < 1$.

Then, the social discount rates are given by equations (9) and (11).

The *social discount rate for the environmental service stream is reduced* proportional to the difference in growth rates, the value share of the produced consumption stream and the limitedness in substitutability expressed by $(1 - s)$.

The *social discount rate for the produced consumption stream is increased* proportional to the difference in growth rates, the value share given to the environmental consumption stream and the limitedness in substitutability expressed by $(1 - s)$.

Moreover, for

i) $s = 0$: In a steady state, *both social discount rates are constant.* In general, the discount rates are $\delta_1(t) = \rho - a_2 \left(\hat{x}_2(t) - \hat{x}_1(t)\right)$ and $\delta_2(t) = \rho + a_1 \left(\hat{x}_2(t) - \hat{x}_1(t)\right)$.

ii) $s \in (0, 1)$: In a steady state, *both social discount rates fall over time.* In general, the long-run discount rates approach the form
$\delta_1(t) = \rho - (1 - s)\left(\hat{x}_2(t) - \hat{x}_1(t)\right)$ and $\delta_2(t) = \rho$.

iii) $s < 0$: In a steady state, *both social discount rates grow over time.* In general, the long-run discount rates approach the form
$\delta_1(t) = \rho$ and $\delta_2(t) = \rho + (1 - s)\left(\hat{x}_2(t) - \hat{x}_1(t)\right)$.

The slower growing environmental consumption good becomes relatively more scarce as time evolves. Expressing its value development over time, the social discount rate is reduced, resulting in a higher weight given to future environmental service streams. On the other hand, the produced good becomes relatively more abundant and, therefore, its social discount rate is increased. The reduction/increase is proportional to the limitedness in substitutability $1 - s$ ($= \sigma^{-1}$) and the difference in growth rates. Moreover, it is proportional to the value share of the other good, characterizing the importance of the relative abundance/scarcity with respect to that good.[17] For Cobb-Douglas preferences in case i) the value share of a commodity x_i corresponds to its utility weight a_i and is independent of the consumption levels. Then, in a steady state, the social discount rate is constant and discounting stays exponential. In general however, the value share V_i^s depends on consumption, implying non-constant social discount rates.

For weak sustainability preferences, where $s \in (0, 1)$ and $\sigma > 1$, statement ii) specifies the time behavior. The change of value shares over time causes both social discount rates to fall. Outside of a steady state, however, a strong fluctuation in the difference in growth rates can counteract this effect and cause the social discount rates to be constant or growing for some period. The discount rate for the environmental service stream x_1 will eventually become negative if there exists t^* such that $(1-s)(\hat{x}_2(t)-\hat{x}_1(t)) > \rho \; \forall \; t > t^*$. That is, if the difference in the growth rates between the two classes of services, weighted with the limitedness in substitutability, dominates the rate of pure time preference ρ.[18] For a strong sustainability preference, where $s < 0$ and $\sigma \in (0, 1)$, the change of value shares over time causes both social discount rates to grow (statement iii). Again, outside of a steady state a strong fluctuation in the difference in growth rates can counteract this effect and cause the social discount rates to be constant or falling for some period.

3.3 Implications

For preferences identified with the paradigm of weak sustainability, the optimal social discount rates fall over time (Proposition 1 ii). The result matches the intuition expressed e.g. in Groom et al.'s (2005, 2) survey on declining discount rates that "It is immediately obvious that using a declining discount rate would make an important contribution towards meeting the goal of sustainable development". Pezzey (2006) even defines sustainable discount rates as falling discount rates.

However, for a strong sustainability preference with strongly limited substitutability between the two classes of goods, part iii) of Proposition 1 no longer supports this intuition. Here, optimal social discount rates are growing. This result seems to be even more surprising in the light of Neumayer's (1999) claim that the strong sustainability

[17]If the other good is important for welfare, relative scarcity is important, too. However, if the other good is of no importance to welfare, the relative scarcity or abundance with respect to that good becomes insignificant as well.

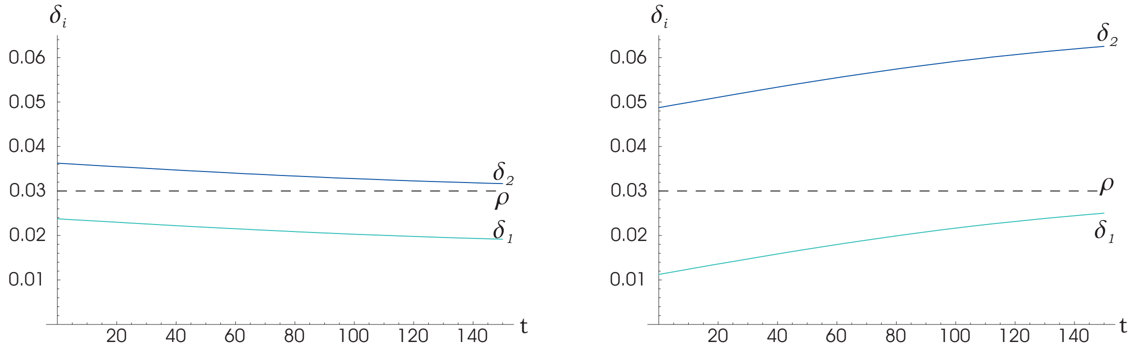
[18]This relation determines only the instantaneous discount rate, in addition it can happen that the social discount factor $D_i^{\mathbf{X}}(t, t_0)$ grows bigger than 1.

paradigm, by implying strongly limited substitutability, would make evaluation models pay more attention to long-run environmental service streams. The following corollary to Proposition 1 fleshes out the relation between the optimal social discount rates in the two scenarios.

Corollary 1: Evaluating the social discount rates for a given growth scenario under Assumptions 1 and 2 the following assertion holds.
There exists $\bar{t} \in [0, \infty)$ such that $\delta_i^{s<0}(t) > \delta_i^{0<s<1}(t)$ for all $t > \bar{t}$ and $i \in \{1, 2\}$.

The *long-term social discount rates* corresponding to a *strong sustainability preference* ($s < 0$) are *higher* than those implied by a *weak sustainability preference* ($0 < s < 1$). A numerical example of the time evolvement of the social discount rates for the two different scenarios is drawn in Figure 1. In the left diagram the substitutability parameter is chosen to be $s = .5$ corresponding to moderate substitutability and a weak sustainability preference. In the right diagram the substitutability parameter is chosen to be $s = -.5$, corresponding to strongly limited substitutability and a strong sustainability preference. The other parameters are chosen equally for both scenarios as $\rho = 3\%, \hat{x}_2 - \hat{x}_1 = 2.5\%$ and $a_1 = a_2 = .5$.[19] As the model is constructed to only depend on the relative growth difference, this scenario depicts equally well a situation where both growth rates of consumption are positive (e.g. $\hat{x}_2 = 3\%$ and $\hat{x}_1 = .5\%$), a scenario where produced consumption grows and environmental services decline (e.g. $\hat{x}_2 = 1.5\%$ and $\hat{x}_2 = -1\%$), or one where both forms of cosumptions are subject to a decrease over time. The reduction/increase with respect to pure time preference (complete substitutability) as well as the time behavior pointed out in Proposition 1 are clearly observed. Moreover, after $t = 88$ years, the (instantaneous) discount rate for the environmental service stream grows bigger in the strong sustainability scenario than in the weak sustainability scenario. Note that the latter does not immediately imply that the weight given to the environmental service stream is lower with a strong sustainability preference. As derived in Section 2.1, the evaluation of an extra unit of environmental services is captured by the corresponding discount factor. Figure 2 depicts the discount factors for the same scenario specifications as in Figure 1. By equation (4), the discount factor relates to the rate as $D_i^{\mathbf{x}}(t, t_0) = \exp\left(-\int_{t_0}^{t} \delta_i(x(t'), \dot{x}(t'), t') dt'\right)$. Hence, a small discount rate at the beginning is 'memorized' in the discount factor for all times and, therefore, raises the weight given to the future not only at early times, but also in the long run. Therefore, the second figure matches the intuition better than the first that environmental goods,

[19]The initial values in the example are $x_1(0) = x_2(0) = 1$.



Weak sustainability preference   Strong sustainability preference

Figure 1 : Numerical example for the time development of social discount rates over time in years. The upper line represents the social discount rate δ_2 for the produced consumption stream, the lower line represents the discount rate δ_1 for the environmental service stream. The dashed line reflects the pure rate of time preference ρ, corresponding to the common discount rate if perfect substitutability in consumption is assumed. In the left diagram the substitutability parameter is chosen to be $s = .5$, on the right it is $s = -.5$. The other parameters coincide for both scenarios and are $\rho = 3\%$, $\hat{x}_2 - \hat{x}_1 = 2.5\%$ and $a_1 = a_2 = .5$.

which in relative terms become increasingly scarce over time, should be valued higher in the long term in a setting with strong sustainability preferences than in a setting with weak sustainability preference. However, the following proposition shows that, in the long run, the development of the discount factors does not agree with this intuition, either.

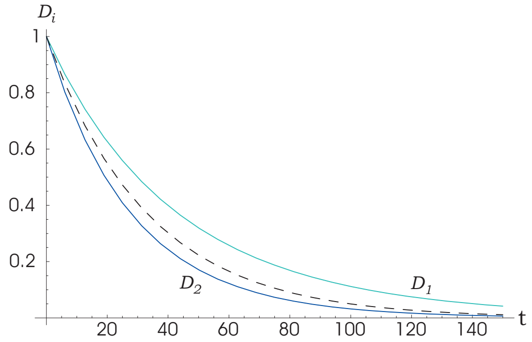
Proposition 2: Evaluating the social discount rates for a given growth scenario under Assumptions 1 and 2 the following assertion holds.
For any $t_0 \in [0, \infty)$ there exists $\bar{\bar{t}} \in [0, \infty)$ such that $D_i^{\boldsymbol{\chi}^{s<0}}(t, t_0) < D_i^{\boldsymbol{\chi}^{0<s<1}}(t, t_0)$ for all $t > \bar{\bar{t}}$ and $i \in \{1, 2\}$.

The proposition implies that a strong sustainability decision-maker gives less weight to long-run environmental service streams than does a weak-sustainability decision-maker.[20]

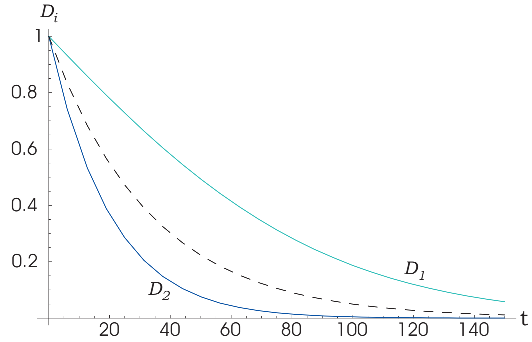
3.4 Explanation and Value Share

The key to the latter puzzle lies in the time development of value share and relates closely to an observation by Gerlagh & van der Zwaan (2002). The authors find in

[20] The proof even shows that in the long run $\frac{D_i^{\boldsymbol{\chi}^{s<0}}(t,t_0)}{D_i^{\boldsymbol{\chi}^{0<s<1}}(t,t_0)} \to 0$.



Weak sustainability preference



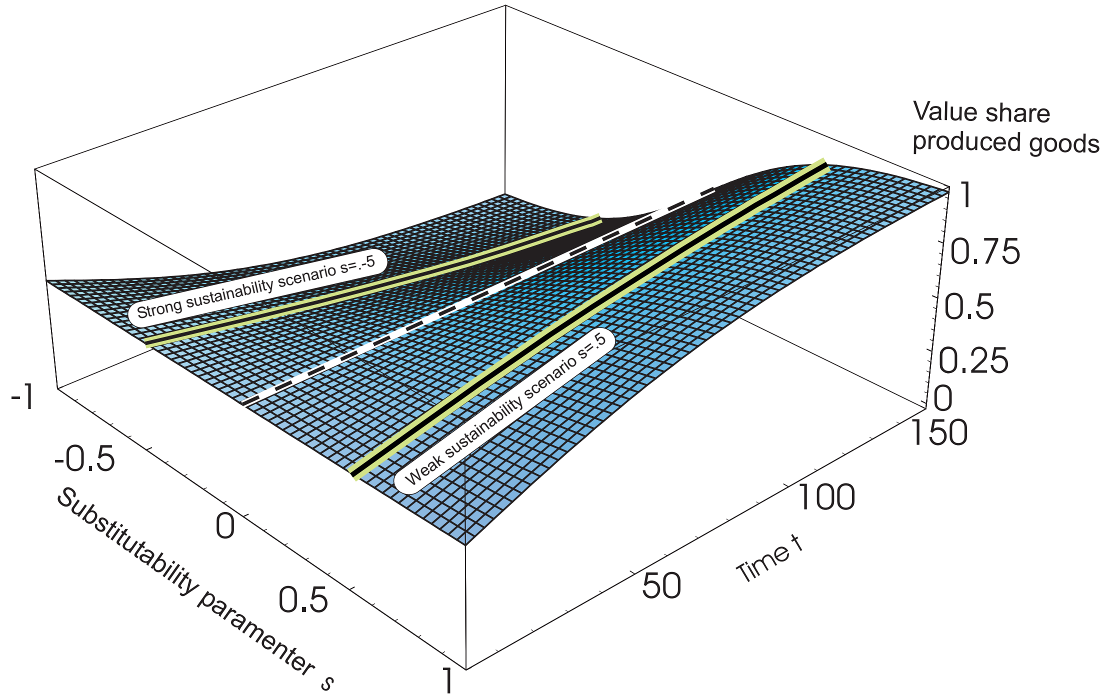
Strong sustainability preference

Figure 2 : Numerical example continued (same specifications as for Figure 1). Drawn are the social discount factors for the environmental (upper line, D_1) and the produced (lower line, D_2) good. The dashed line reflects exponential discounting corresponding to the pure rate of time preference. In the depicted scenario, D_1 for the strong sustainability scenario falls below D_1 for the weak sustainability scenario after $t = 195$ years.

a similar stylized growth scenario that for *strongly limited substitutability* between the two classes of commodities, the *value share of man-made consumption goes to zero in the long run*.[21] Figure 3 depicts how the value share of the produced consumption stream evolves in the scenario underlying Figures 1 and 2. The value share of produced consumption grows for a weak sustainability scenario and falls for a strong sustainability scenario. Only for the specification at the dividing line between the two different regions ($s = 0$), does the value share stay constant over time (Proposition 1*i*).

The value share is a combination of the amount consumed and its evaluation. In the analyzed growth scenario, the environmental service stream grows relatively scarce over time while produced consumption becomes relatively more abundant. At the same time, the limited substitutability causes a unit of environmental services to be increasingly more valuable than a unit of produced consumption. For weak sustainability preferences (moderate substitutability), the relative physical scarcity of the environmental service stream dominates its value share development. Thus, the value share of environmental services declines, while that of produced consumption grows. For strong sustainability preferences (strongly limited substitutability), however, the increase in unit value dominates the relative physical scarcity in determining the value share of the environmental service stream. Therefore, the total amount of environmental services consumed be-

[21]Precisely, Gerlagh & van der Zwaan (2002) assume that produced consumption grows to infinity while environmental service streams are bounded. My analysis implies the same result building only on the difference in growth rates (see proof of Proposition 1).



Development of the value share of produced consumption over time

Figure 3: Numerical example continued (same specifications as for figure 1). Drawn is the value share of the produced consumption stream. The thick lines correspond to the substitutability parameters used for the weak and strong sustainability preference scenario drawn in figures 1 and 2.

comes more valuable than the total amount of produced goods consumed. The value share of produced consumption declines to zero.

In the analyzed CES model, the substitution effect in the social discount rate for the environmental good - i.e. the influence of x_2 on the value development of x_1 - is proportional to the value share of x_2. The lower the value share of produced consumption, the less influence has an increase in relative scarcity of environmental services with respect to produced consumption. Consider e.g. the extreme of a strong sustainability preference with $s \to -\infty$, where the evaluation functional converges to $\mathcal{U} = \int_0^\infty \min\{x_1, x_2\} e^{-\rho t}\, dt$. Once the economy is scarcer in environmental service flows than in produced consumption, the decision-maker will only pay attention to the environmental service streams. Therefore, the time development of his valuation for an extra unit of environmental services is *solely generated by the pure rate of time preference* ($\delta_1 = \rho$). With a growing relative scarcity of the environmental services and a declining value share of the produced services, the other preference specifications in the strong sustainability domain converge towards a similar evaluation. In consequence, less attention is paid to the increase in relative scarcity and $\lim_{t\to\infty} \delta_1 = \rho$. On the other hand, in the weak sustainability scenario, produced consumption stays important for (marginal) welfare. Then, an increase in relative scarcity lowers the social discount rate proportional to the limitedness in substitutability.

3.5 The Two Sustainability Paradigms Revisited

The focus of my analysis has been the time development of the weight given to future consumption streams. To interpret the results of this section, first observe that *at a given point of time* the social discount rates for the different scenarios reflect the different concepts of sustainability. At any given point of time, the *difference in evaluation* between an extra unit of environmental services and an extra unit of produced consumption *increases* in the *relative scarcity* of the environmental service as well as in the *limitedness in substitutability.* This fact is observed by taking the difference between equations (11) and (9) yielding

$$\delta_2(t) - \delta_1(t) = (1-s)\left(\hat{x}_2(t) - \hat{x}_1(t)\right) . \tag{12}$$

The difference in social discount rates in equation (12) generates a relative difference in weights given to the consumption streams corresponding to

$$\begin{aligned}
\frac{D_1^{\chi}(t,t_0)}{D_2^{\chi}(t,t_0)} &= \exp\left(-\int_{t_0}^{t} \delta_1(t') - \delta_2(t')\,dt'\right) \\
&= \exp\left(\int_{t_0}^{t} (1-s)\left(\hat{x}_2(t') - \hat{x}_1(t')\right)\,dt'\right) .
\end{aligned} \tag{13}$$

A *stronger notion of sustainability* corresponds to a reduced substitutability in the welfare function. As equations (12) and (13) show, such an increase in $(1-s)$ implies also an *increase in the weight given to environmental services as opposed to produced consumption.* Moreover, the difference in weight is monotonically growing over time as relative scarcity of the environmental service stream increases.

However, this section has derived a second implication of a differentiation between a weak and a strong notion of sustainability through parametrization of the substitutability between environmental services and produced consumption streams. A *stronger notion of sustainability* results in a *reduced weight given to the future* as opposed to the present. Whenever environmental services or both consumption streams are declining over time ($\hat{x}_1 < \min\{\hat{x}_2, 0\}$), such a reduced attention paid to future service and consumption streams seems to oppose the fundamental objective of a sustainable development as expressed in the Brundtland report (see Section 2.4). I offer two alternative perspectives on the relationship discussed in this section.

From the *first perspective*, the notions of *strong* versus *weak* sustainability only relate to the substitutability between the different classes of goods and services. Thus, the concepts are concerned *only* with distributing weight between produced and environmental goods at a given point of time. When concerned with intertemporal comparisons in a growth scenario as analyzed in this section, applying a weak sustainability preference for project evaluation corresponds to a stronger sustainability demand in the sense that a higher weight is given to long-run future consumption and service streams. In any case, limited substitutability gives more weight to future environmental services than a welfare function assuming perfect substitutability (which does not pay attention at all to an increase in relative scarcity). Attaching a relatively higher weight to the scarcer environmental goods by specifying substitutability accordingly, comes at the cost of shifting weight from the future to the present.

From the *second perspective*, strong sustainability should be mapped into a strongly limited substitutability between the two classes of goods *and* a lower intertemporal substitutability. As discussed on page 4 (equation 5), a decrease in intertemporal substitutability implies an increase in weight given to future consumption streams whenever growth is declining. That way, the latter increase counteracts the substitutability effect analyzed in this section.[22]

4 Social Discount Rates in a Multi-Commodity Cost Benefit Analysis of a Small Project

4.1 Good-Specific Discounting

This section relates different perspectives on discounting. It makes precise how the social discount rates and factors derived in the previous sections have to be applied in project evaluation. First, I take the perspective of good-specific discount rates. Then, I discuss the effects of a choice of numeraire and consumption conversion. Finally, I relate social discount rates to market evaluation. The project analyzed here is characterized

[22]A similar effect can be obtained by making the elasticity of substitution between the two classes of goods dependent on the consumption level.

as a small change $\Delta\mathsf{x}$ of a consumption plan x^0. Exercising the project yields the new consumption and service stream $\mathsf{x}=\mathsf{x}^0+\Delta\mathsf{x}$ with $x_i = x_i^0(t) + \Delta x_i(t)$, $i \in \{1,2\}$, $t \in [0,T]$.[23] At each point of time $\Delta x_i(t)$ should be small with respect to $x_i(t)$ so that $U(x_1(t)+\Delta x_1(t), x_2(t)+\Delta x_2(t), t)$ can be expanded first order in the $\Delta x_i(t)$ (small project assumption). Denoting the welfare corresponding to consumption path x^0 by $\mathcal{U}^0$, the welfare of the new consumption path can be written as

$$\begin{aligned}\mathcal{U} &= \int_0^T U(x_1^0(t)+\Delta x_1(t), x_2^0(t)+\Delta x_2(t), t)\, dt \\ &= \mathcal{U}^0 + \int_0^T \frac{\partial U}{\partial x_1}(t)\Delta x_1(t) + \frac{\partial U}{\partial x_2}(t)\Delta x_2(t) + O(\Delta x(t)^2)\, dt\ , \end{aligned} \tag{14}$$

where the marginal utilities are evaluated along x^0. Equation (14) states that neglecting terms of second order in Δx, the project raises welfare, if and only if,

$$\int_0^T \frac{\partial U}{\partial x_1}(t)\Delta x_1(t) + \frac{\partial U}{\partial x_2}(t)\Delta x_2(t)\, dt > 0\ . \tag{15}$$

The integral represents a cost benefit functional in continuous time with valuation derived from the social welfare objective given in equation (1). If the path x^0 is optimal, all feasible projects $\Delta\mathsf{x}$ should yield an evaluation smaller or equal to zero. To evaluate whether a particular project raises welfare, assume that for some reference time t_0 there exist prices $p_1(t_0)$ and $p_2(t_0)$ satisfying $\frac{p_1(t_0)}{p_2(t_0)} = \frac{\frac{\partial U}{\partial x_1}(t_0)}{\frac{\partial U}{\partial x_2}(t_0)}$. Generally t_0 will be the present ($t_0 = 0$) and prices $p_1(t_0) = p_1(0)$ and $p_2(t_0) = p_2(0)$ are either market prices or, more likely for the environmental service streams, prices derived from direct and indirect methods of evaluation like contingent valuation or hedonic price studies (see e.g. Hanley, Shogren & White 1997, 383 et sqq., or Mäler & Vincent 2005). Equation (2) relates the marginal utilities in equation (15) for different points of time, yielding the project evaluation criteria

$$\int_0^T D_1^{\mathsf{x}^0}(t,t_0)p_1(t_0)\Delta x_1(t) + D_2^{\mathsf{x}^0}(t,t_0)p_2(t_0)\Delta x_2(t)\, dt > 0\ . \tag{16}$$

[23] T denotes the end of the consumption change induced by the project or the end of the planning horizon. It is allowed to be infinite.

Equation (16) takes the prices at t_0 to determine the relative value of x_1 and x_2 at t_0 and propagates both prices over time by means of the marginal utility propagators $D_1^{\mathbf{x}}(t,t_0)$ and $D_2^{\mathbf{x}}(t,t_0)$ respectively. The prices $D_i^{\mathbf{x}}(t,t_0)\,p_i(t_0)$ could be referred to as social accounting prices.[24] Another interpretation is to take the factors $D_i^{\mathbf{x}}(t,t_0)$ as *good-specific social discount factors*. This view underlies the analysis of Sections 2 and 3. Applying equation (16) to the growth scenario in Section 3 with a weak sustainability preference implies an increased and falling discount rate for the produced consumption stream and a reduced and falling discount rate for the environmental service stream. It is important to be aware that *either* one can argue that prices of the environmental service stream rise due to its increasing relative scarcity, *or* one can apply the good-specific discount rates discussed earlier. Doing both at the same time yields a wrong evaluation.

An interesting special case is the evaluation of a project that affects only consumption of the environmental service streams at different points of time. Then equation (16) is equivalent to

$$\int_0^T D_1^{\mathbf{x}^0}(t,t_0)\Delta x_1(t)\,dt > 0\,.$$

An important consequence of the discussion in Section 3 is the following. Considering a *partial model of the environmental sector*, optimal discounting can be *hyperbolic* with a reduced discount rate, even in a steady state.[25] Moreover, for the evaluation of such a project, the *relative* weight given to environmental services as opposed to produced consumption is of no importance. In such a situation, it appears particularly surprising that an evaluation based on a strong sustainability preference gives less weight to long-term environmental service flows than an evaluation based on a weak sustainability preference.

[24] Note that these prices $D_i^{\mathbf{x}}(t,t_0)\,p_i(t_0)$, in general, do not coincide with the capital measured market prices that will be studied in Section 4.3.

[25] Recall that the model in Section 3 uses a constant rate of pure time preference and, thus, the evaluation is time consist.

4.2 Choice of Numeraire and Consumption Conversion

By factoring out $D_1^{\mathsf{X}^0}(t,t_0)$ or $D_2^{\mathsf{X}^0}(t,t_0)$ in equation (16) the evaluation functional becomes

$$\int_0^T \left[p_1(t_0)\Delta x_1(t) + \frac{D_2^{\mathsf{X}^0}(t,t_0)}{D_1^{\mathsf{X}^0}(t,t_0)} p_2(t_0)\Delta x_2(t) \right] D_1^{\mathsf{X}^0}(t,t_0)\, dt > 0 \quad \text{or} \tag{17}$$

$$\int_0^T \left[\frac{D_1^{\mathsf{X}^0}(t,t_0)}{D_2^{\mathsf{X}^0}(t,t_0)} p_1(t_0)\Delta x_1(t) + p_2(t_0)\Delta x_2(t) \right] D_2^{\mathsf{X}^0}(t,t_0)\, dt > 0\,. \tag{18}$$

Equations (17) and (18) take the common view that there is one common discount rate applicable to all goods. Setting $t_0 = 0$, the reference period becomes the present and equation (17) can be interpreted as follows. The first good is taken to be the *numeraire*, and its price is kept constant over time. Hence $D_1^{\mathsf{X}^0}(t,t_0)$, expressing the change of marginal utility of the first good, becomes *the* discount factor. The value of the second good must be propagated by the relative change of marginal utility of good two relative to good one, i.e. by $\frac{D_2^{\mathsf{X}^0}(t,t_0)}{D_1^{\mathsf{X}^0}(t,t_0)}$. Applied again to the setup of Section 3 and the example of moderately limited substitutability between environmental and produced consumption streams, the social discount rate for the environmental service stream would be *the* discount rate and discounting would take place with the lower hyperbolic discount rate δ_1.

A more common perspective on cost benefit analysis corresponds to equation (18), which is the analogue taking x_2 to be the numeraire. Such a cost benefit evaluation takes the social discount rate of produced consumption as *the* discount rate. Time development of the accounting price for the environmental service stream is characterized by the expression $\frac{D_1^{\mathsf{X}^0}(t,t_0)}{D_2^{\mathsf{X}^0}(t,t_0)}$. Normalize $p_2(0)$ to unity and define $p_1^*(0) = \frac{\frac{\partial U}{\partial x_1}(0)}{\frac{\partial U}{\partial x_2}(0)}$ as the value of a unit of environmental services in units of produced goods in the present. Then, choosing $t_0 = 0$, equation (18) together with equation (13) imply that for the scenario analyzed in Section 3, the pricing of the environmental service stream in units of produced consumption develops as:

$$p_1^*(t) = p_1^*(0) \exp\left(\int_0^t (1-s)\left(\hat{x}_2(t') - \hat{x}_1(t')\right) dt' \right)\,. \tag{19}$$

In words, the accounting price of environmental services in units of produced goods increases over time due to increasing scarcity and limited substitutability.

Such a conversion to (produced[26]) consumption equivalents is promoted by Arrow et al. (1995). From this point of view, the authors criticize Weitzman's (1994) derivation of a reduced, hyperbolic 'environmental discount rate' for its lack of such a conversion.[27] In fact, Weitzman (1994) neither models the environmental good explicitly, nor does he state a functional form of preferences. Moreover, he does not account separately for environmental changes and growth of produced consumption. Instead, Weitzman derives an overall discount rate under the assumption that produced consumption is growing at the cost of degrading the environment.[28] The assumed functional form for this relationship renders an overall discount rate that is smaller than in the absence of environmental externalities and falling over time. This so called 'environmental discount rate' does not explicitly distinguish between the value development of environmental service streams and that of produced consumption. Therefore, it could at most be applied to a project where the assumed fixed relationship between production growth and environmental decline holds. More generally, Groom et al. (2005, 458) criticize that "in many ways Weitzman's environmental discount rate is difficult to interpret in light of the reduced form set up and, in particular, the absence of an explicit modeling of preferences, environmental goods and externalities."

Preferences and both goods are explicitly modeled in my setup. Equation (18) gives a cost benefit analysis in the perspective of Arrow et al. (1995) and equation (19) shows how, for the scenario in Section 3, a 'proper relative pricing of environmental goods over time' converts the environmental service stream into (produced) consumption equivalents. Criticizing Weitzman's (1994) reduced and hyperbolic discount rate, Arrow et al. (1995, 140) express the widely[29] held belief that when properly converting into

[26]Arrow et al. (1995) use the one commodity equation (5) as point of departure for their discussion on discounting. In the section on 'the environment and discounting' the authors state that environmental benefits have to be converted into consumption equivalents. In my explicit two commodity setup, I identify their (non-environmental) consumption with my produced consumption stream x_2.

[27]Arrow et al. (1995, 139) state that the "essence of social discounting is to convert all effects into their consumption equivalents and then to discount the resulting stream of consumption equivalents at the social rate of time preference. Incorporating environmental effects does not change the discount rate itself but does require special attention to the proper relative pricing of environmental goods over time". Note that Arrow et al. (1995) use the term 'social rate of time preference' for the social discount rate δ in the sense of the one commodity equation (5).

[28]An alternative interpretation offered by Weitzman is that the environment is a luxury good whose demand grows over time.

[29]See Groom et al. (2005, 459, 486 endnote 25).

consumption equivalents, the environmental considerations do "not change the discount rate to apply to the consumption stream". However, the model of Section 3 shows that under limited substitutability in consumption *the increasing relative scarcity of the environmental service stream also changes the discount rate* δ_2 *that has to be applied to the (produced) consumption stream* x_2. In particular under the assumption of moderate substitutability, such an environmental consideration still can give rise to hyperbolic discount rates for the produced consumption stream and, thus, the overall consumption stream under numeraire conversion. In contrast to Weitzman's (1994) results, though, the produced consumption stream discount rate is increased and not reduced. Moreover, under the assumption of strongly limited substitutability, a growing relative scarcity in the environmental service stream can result in a growing value share of the environmental goods and go along with a growing discount rate.

4.3 Relation to a Complete Market Evaluation

After having worked out the evaluative structure for the setting of incomplete future markets, I want to point out how it relates to a scenario where markets are complete and evaluation is reflected in the corresponding market prices. The prices are derived by setting up the budget constraint of a representative consumer. Welfare is assumed to be of the general form of equation (1), though restricted by the assumptions that follow below. I assume that the social optimum can be decentralized by an appropriate price system. Prices are measured in units of capital which can be regarded as money, assets or real capital. These current value prices are denoted by $p_1(t)$ and $p_2(t)$.[30] The interest rate on capital is $r(t)$. Remuneration for a fixed offer of one unit of labor $w(t)$ is only introduced for 'completeness' of the budget constraint. All these variables are exogenous to the representative consumer. His choice is between saving $\dot{k}(t)$ units of the capital good k and consuming the amounts $x_1(t)$ and $x_2(t)$. For x_1 being essential life support services provided by the environment, such an immediate choice of a representative agent is, of course, fictitious. For environmental goods like a hiking trip or a scenic view one might get closer to existing future markets. However, the setting is only meant to relate discounting in markets, or with respect to market based prices, to a social cost benefit setup. With the above assumptions the budget constraint of the representative

[30]Equation (23) for t_0 will justify using the same notation for prices as in the preceding subsections.

agent is given by the equation

$$\dot{k}(t) = r(t)k(t) + w(t) - p_1(t)x_1(t) - p_2(t)x_2(t) \ .$$

Together with equation (1) it follows that the Hamiltonian describing the optimization problem of the representative agent is

$$H = U(x_1, x_2, t) + \lambda(t) \ [r(t)k(t) + w(t) - p_1(t)x_1(t) - p_2(t)x_2(t)] \ .$$

In the following, I assume that a sufficiency condition for the optimization problem is met.[31] Moreover, I assume a continuous control (consumption) path and an interior solution. The solution for the consumption path is denoted by $\mathbf{x}$. Along this path the following necessary conditions for an optimum must be satisfied:

$$\frac{\partial H}{\partial x_1} = \frac{\partial U}{\partial x_1} - \lambda(t)\, p_1(t) \stackrel{!}{=} 0\,, \tag{20}$$

$$\frac{\partial H}{\partial x_2} = \frac{\partial U}{\partial x_2} - \lambda(t)\, p_2(t) \stackrel{!}{=} 0\,, \tag{21}$$

$$\frac{\partial H}{\partial k} = \lambda(t)\, r(t) \stackrel{!}{=} -\dot{\lambda}(t) \ . \tag{22}$$

From equations (20) and (21) I obtain the relations:

$$\frac{\frac{\partial U}{\partial x_1}(t)}{\frac{\partial U}{\partial x_2}(t)} = \frac{p_1(t)}{p_2(t)} \qquad \text{and} \tag{23}$$

$$\frac{\frac{\partial U}{\partial x_i}(t)}{\frac{\partial U}{\partial x_i}(t_0)} = \frac{\lambda(t)}{\lambda(t_0)} \frac{p_i(t)}{p_i(t_0)} \quad i \in \{1, 2\} \ . \tag{24}$$

Integration of equation (22) yields the present value shadow price of capital

$$\lambda(t) = c \exp\left(- \int_0^t r(t')dt' \right) \tag{25}$$

with the integration constant $\lambda(0) = c \in \mathbb{R}_+$. Analogous to the social discount factors describing marginal utility propagation on the preference side, let me define the *time*

[31] See Takayama (1994, 660 sqq.), Chiang (1992, 214 et sqq.) and Seierstad & Sydsaeter (1977) for different sufficiency conditions.

propagator of capital as

$$R(t_0, t) = \exp\left(\int_{t_0}^{t} r(t')dt'\right) .$$

It describes how much capital in t can be derived from an extra unit of capital in t_0. Just as the discount rate in Section 2.1, the productivity of capital $r(t)$ can be interpreted as the *generator of capital propagation.* I have defined $R(t_0, t)$ in a way that $R(t, t_0) = \frac{1}{R(t_0,t)} = exp\left(-\int_{t_0}^{t} r(t')dt'\right)$ is again the factor which is *dis*counting with capital productivity. I refer to the latter as the inverse capital propagator. Equation (25) shows that the shadow value of capital at time t is inversely proportional to the productivity of capital between the present and time t, i.e. $\lambda(t) \propto R(t, t_0)$.[32] This relation is straight forward as a unit of capital today can be turned into $R(t_0, t)$ units of capital in period t. Therefore a unit of capital in time t is worth $\frac{1}{R(t_0,t)} = R(t, t_0)$ units of capital today.

Inserting $R(t_0, t)$ into equation (24) the following relation between the time propagator of marginal utility $D_i^{\mathsf{X}}(t, t_0)$ of good i, the capital propagator, and the price of good i is obtained:

$$p_i(t) = D_i^{\mathsf{X}}(t, t_0)\, p_i(t_0)\, R(t_0, t) . \tag{26}$$

Equation (26) shows that time development of (capital measured) prices depends on two influencing factors. One is the effect discussed in the previous sections depending on the change of marginal utility expressed by $D_i^{\mathsf{X}}(t, t_0)$. In addition, the units of measurement of the *current value prices* $p_i(t_0)$ and $p_i(t)$, corresponding to different periods, have to be related. As prices of the goods are measured in units of the capital good, this is achieved by the capital propagator $R(t_0, t)$.

For the one commodity setting it is often assumed that capital is measured in units of consumption (e.g. Barro & Sala-i-Martin 1995, 62). Similarly, I could assume in the two commodity setting that capital is measured in units of produced consumption. This assumption makes the current value price of produced consumption constant over time.[33] Then, equation (26) implies for $i = 2$ that the inverse capital propagator $R(t, t_0)$ and the propagator of marginal utility $D_2^{\mathsf{X}}(t, t_0)$ coincide. Capital is now measured in units

[32]The shadow value reflects the value of an extra unit of capital in units of welfare along the optimal path. For a closer discussion and the derivation of this interpretation of a shadow price (costate variable) see e.g. Kamien & Schwartz (2000, 136 et sqq.). Note that λ is the present value shadow price.

[33]Then the current value price of produced consumption is measured in units proportional to itself.

of produced consumption and reflects the value development of produced consumption over time. With regard to the non-constancy of the social discount rates in the scenario of Section 3, such a measurement of capital implies that $r(t)$ exhibits the same non-constant form as derived for $\delta_2(t)$.

With this background, let me finally analyze how the evaluation of the small project from the preceding section would be evaluated if complete markets existed for all times.[34] Applying equation (26) to equation (16), the following evaluation functional for the project is obtained:

$$\int_0^T \Big[p_1(t)\Delta x_1(t) + p_2(t)\Delta x_2(t)\Big] R(t,t_0)\, dt > 0 \; . \tag{27}$$

This time, the social discount factors $D_i^{\mathbf{x}}(t,t_0)$ are not needed for evaluation. The price development accounts already for the change in welfare. But prices are measured in capital and capital is generally productive. Thus, the present value of a unit of capital in the future is less than the value of a unit of capital in the present and, therefore, the future prices have to be *dis*counted with capital productivity. In consequence, capital productivity can be regarded as *the* common discount rate for both goods. Finally, for the case where capital is measured in terms of produced consumption it was seen that $R(t,t_0) = D_2^{\mathbf{x}}(t,t_0)$. In these units, equation (27) coincides with equation (18) and discounting shows the same time behavior.

5 Conclusions

To evaluate long-term projects, an expression for the development of valuation over time is needed. Social discount rates represent such an expression. They allow the economist to think in rates and elasticities, and lay out different contributions in a convenient additive form. This study elaborates one such contribution to value development over time, which emerges in a multi-commodity world with limited substitutability between different forms of consumption. The latter was shown to affect the magnitude as well as time evolvement of social discount rates.

I have related the result to the concepts of weak and strong sustainability. When the

[34] Having assumed that the social optimum can be decentralized in a complete market system, such an evaluation is of theoretical interest only to compare the resulting cost benefit functional to that of sections 4.1 and 4.2.

difference between these two paradigms is translated into the degree of substitutability between environmental and produced goods and services, the strong sustainability paradigm will generally give less weight to future service and consumption streams than the weak sustainability paradigm. If this implication of strong sustainability is unwanted, I suggested that the mathematical formulation of the concept should not only lower the substitutability between the different classes of goods, but also decrease the intertemporal substitutability.

I have shown that numeraire conversion of environmental services into produced good equivalents does not eliminate the dependence of the social discount rate on the substitutability effect. Moreover, when environmental services are not converted into consumption equivalents, e.g. in a partial model or cost benefit analysis of the environmental sector, the developed model specifies conditions under which a reduced discount rate should be used. Finally, the social discounting perspective was related to a market based setting, where markets determine the price paths of goods and services and discounting takes place with capital productivity. I pointed out that in any of these approaches non-constant optimal discount rates can arise, even in a steady state and with a constant rate of pure time preference.

Appendix

A Calculations for Section 2

Calculation of the social discount rate for
$U(x_1, x_2, t) = [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}} e^{-\rho t}$:

The derivatives needed for the computation of δ_1 are for $s \notin \{0, 1\}$:

$$\frac{\partial U}{\partial x_1} = a_1 u_1(x_1)^{s-1} u_1'(x_1) \Big[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s\Big]^{\frac{1}{s}-1} e^{-\rho t} \ ,$$

$$\frac{\partial^2 U}{\partial x_1^2} = \Big(a_1 u_1(x_1)^{s-1} u_1''(x_1) - (1-s) a_1 u_1(x_1)^{s-2} u_1'(x_1)^2\Big)$$

$$\cdot \Big[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s\Big]^{\frac{1}{s}-1} \cdot e^{-\rho t}$$

$$+(1-s)\Big(a_1 u_1(x_1)^{s-1}\Big)^2 u_1'(x_1)^2 \Big[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s\Big]^{\frac{1}{s}-2} e^{-\rho t} \quad \text{and}$$

$$\frac{\partial^2 U}{\partial x_1 \partial x_2} = (1-s)\Big(a_1 u_1(x_1) a_2 u_2(x_2)\Big)^{s-1} u_1'(x_1) u_2'(x_2)$$

$$\cdot \Big[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s\Big]^{\frac{1}{s}-2} e^{-\rho t} \,.$$

Inserting these into equation (6) yields:

$$\delta_1(t) = \rho - \frac{(a_1 u_1(x_1)^{s-1} u_1''(x_1) - (1-s) a_1 u_1(x_1)^{s-2} u_1'(x_1)^2)\,[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1-s}{s}}}{a_1 u_1(x_1)^{s-1} u_1'(x_1) [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1}} \cdot \dot{x}_1$$

$$- \frac{(1-s)(a_1 u_1(x_1)^{s-1})^2 u_1'(x_1)^2 [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-2}}{a_1 u_1(x_1)^{s-1} u_1'(x_1) [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1}}\, \dot{x}_1$$

$$- \frac{(1-s)\,(a_1 u_1(x_1) a_2 u_2(x_2))^{s-1}\, u_1'(x_1) u_2'(x_2) [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-2}}{a_1 u_1(x_1)^{s-1} u_1'(x_1) [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1}}\, \dot{x}_2$$

$$= \rho - \frac{u_1''(x_1)}{u_1'(x_1)}\, \dot{x}_1 + (1-s) u_1(x_1)^{-1} u_1'(x_1)\, \dot{x}_1$$

$$-(1-s)\frac{a_1 u_1(x_1)^{s-1} u_1'(x_1)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\, \dot{x}_1 - (1-s)\frac{a_2 u_2(x_2)^{s-1} u_2'(x_2)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\, \dot{x}_2$$

$$= \rho - \frac{u_1''(x_1)}{u_1'(x_1)}\, \dot{x}_1$$

$$+(1-s)\frac{u_1(x_1)^{-1} u_1'(x_1)(a_1 u_1(x_1)^s + a_2 u_2(x_2)^s) - a_1 u_1(x_1)^{s-1} u_1'(x_1)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\, \dot{x}_1$$

$$-(1-s)\frac{a_2 u_2(x_2)^{s-1} u_2'(x_2)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\, \dot{x}_2$$

$$= \rho - \frac{u_1''(x_1)}{u_1'(x_1)}\, \dot{x}_1 + (1-s)\frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\, \frac{u_1'(x_1)}{u_1(x_1)}\, \dot{x}_1$$

$$-(1-s)\frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\, \frac{u_2'(x_2)}{u_2(x_2)}\, \dot{x}_2 \,.$$

Which brings about equation (7):

$$\delta_1(t) = \rho - \frac{u_1''(x_1)}{u_1'(x_1)}\dot{x}_1 - (1-s)\frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\left(\frac{u_2'(x_2)}{u_2(x_2)}\dot{x}_2 - \frac{u_1'(x_1)}{u_1(x_1)}\dot{x}_1\right).$$

For $s = 1$ with $\frac{\partial^2 U}{\partial x_1 \partial x_2} = 0$ and $1 - s = 0$ it is easily observed that the same equation has to hold. For the case $s = 1$ it is $u(x_1, x_2) = [a_1 x_1^s + a_2 x_2^s]^{1/s}$. The derivatives needed for the computation of δ_1 are

$$\begin{aligned}
\frac{\partial U}{\partial x_1} &= a_1 u_1(x_1)^{a_1 - 1} u_1'(x_1) u_2(x_2)^{a_2} e^{-\rho t} \,, \\
\frac{\partial^2 U}{\partial x_1^2} &= a_1(a_1 - 1) u_1(x_1)^{a_1 - 2} u_1'(x_1)^2 u_2(x_2)^{a_2} e^{-\rho t} \\
&\quad + a_1 u_1(x_1)^{a_1 - 1} u_1''(x_1) u_2(x_2)^{a_2} e^{-\rho t} \qquad \text{and} \\
\frac{\partial^2 U}{\partial x_1 \partial x_2} &= a_1 u_1(x_1)^{a_1 - 2} u_1'(x_1) a_2 u_2(x_2)^{a_2 - 1} u_2'(x_2) e^{-\rho t} \,.
\end{aligned}$$

These derivatives deliver the social discount rate

$$\delta_1(t) = \rho - \frac{u_1''(x_1)}{u_1'(x_1)} \dot{x}_1 - a_2 \left(\frac{u_2'(x_2)}{a_2 u_2(x_2)} \dot{x}_2 - \frac{u_1'(x_1)}{a_1 u_1(x_1)} \dot{x}_1 \right)$$

which coincides with equation (7) for $s = 0$ as $a_1 + a_2 = 1$.

B Proofs for Section 3

Proof of Proposition 1: By Assumtpion 1, all of the terms in equations (9) and (11) are positive. Therefore, the verbal statements in the proposition concerning the reduction/increase and its proportionality merely summarize the equations.[35]

For cases *ii*) and *iii*), the proof makes use of a transformation of the value share V_2^s. This transformation employs the relation

$$\begin{aligned}
\frac{d \ln x_i(t)}{dt} &= \frac{\dot{x}_i(t)}{x_i(t)} \\
\Rightarrow \quad \ln x_i(t) &= \int_0^t \hat{x}_i(t')\, dt' + c \\
\Rightarrow \quad x_i(t) &= x_i(0) e^{\int_0^t \hat{x}_i(t')\, dt'} \\
\Rightarrow \quad x_i(t)^s &= x_i(0)^s e^{s \int_0^t \hat{x}_i(t')\, dt'} \,.
\end{aligned} \tag{28}$$

[35]This is the only statement in Section 3 which does not necessarily hold true if Assumption 1 is relaxed to the form stated in footnote 13. Then, for $t < t^*$ the term $(\hat{x}_2(t) - \hat{x}_1(t))$ can flip sign.

Using equation (28) the value share V_2^s can be transformed to

$$V_2^s(x_1, x_2) = \frac{a_2 x_2(0)^s e^{s\int_0^t \hat{x}_2(t')\,dt'}}{a_1 x_1(0)^s e^{s\int_0^t \hat{x}_1(t')\,dt'} + a_2 x_2(0)^s e^{s\int_0^t \hat{x}_2(t')\,dt'}}$$
$$= \frac{1}{\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s}\frac{e^{s\int_0^t \hat{x}_1(t')\,dt'}}{e^{s\int_0^t \hat{x}_2(t')\,dt'}} + 1} = \frac{1}{\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s\int_0^t \hat{x}_2(t')-\hat{x}_1(t')\,dt'} + 1}\,. \tag{29}$$

Case i: Given that the utility weights a_1 and a_2 sum to unity, it is $V_2^0 = a_2$ and $V_1^0 = a_1$, which yields the equations stated in the proposition. In a steady state, also $\hat{x}_1$ and $\hat{x}_2$ are constant over time and, thus, the social discount rates are constant.

Case ii: *First*, I show that V_2^s is strictly increasing. By Assumption 1, it is $\hat{x}_2(t) - \hat{x}_1(t) > 0 \,\forall\, t$. As derived above, equation (29) holds. For $s > 0$ the expression $\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s\int_0^t \hat{x}_2(t')-\hat{x}_1(t')\,dt'}$ is strictly falling in time. Therefore, the value share of the produced consumption stream V_2^s is strictly increasing over time.

Second, in a steady state, such an increasing V_2^s causes the second term in the social discount rate for the environmental service stream $(1-s)V_2^s\left(x_1(t), x_2(t)\right)(\hat{x}_2 - \hat{x}_1)$ to increase over time. As this term is subtracted from the constant rate of pure time preference, the social discount rate for the first commodity class $\delta_1(t)$ *declines* in a steady state.

Third, a strictly increasing term V_2^s implies a strictly decreasing value share of the environmental service stream $V_1^s = 1 - V_2^s$. In a steady state, such a strictly decreasing term V_1^s implies that the expression $(1-s)V_1^s\left(x_1(t), x_2(t)\right)(\hat{x}_2 - \hat{x}_1)$ strictly decreases. This expression is added to the constant rate of pure time preference to yield the social discount rate for the produced consumption stream. Thus, the social discount rate $\delta_2(t)$ *declines* as well in a steady state.

Finally, by Assumption 1 there exists $\epsilon > 0$ such that $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon \,\forall\, t$. In consequence, the expression $\lim_{t\to\infty} \frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s\int_0^t \hat{x}_2(t')-\hat{x}_1(t')\,dt'}$ falls to zero and the value share V_2^s grows to unity.[36] Therefore, in a steady state the discount rate δ_1 monotonously falls to $\delta_1 = \rho - (1-s)(\hat{x}_2 - \hat{x}_1)$ for $t \to \infty$. In general, it approaches the form $\delta_1(t) = \rho - (1-s)(\hat{x}_2(t) - \hat{x}_1(t))$. At the same time the value share of the environmental service stream V_1^s falls to zero, implying that the social discount rate for the produced consumption stream falls to $\delta_2 = \rho$.

Case iii: *First*, I show that V_2^s is strictly decreasing. From $\hat{x}_2(t) - \hat{x}_1(t) > 0 \,\forall\, t$. As de-

[36]Note that already the slightly weaker assumption in footnote 13 assures this limit.

rived above, equation (29) holds. For $s < 0$ the expression $\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s \int_0^t \hat{x}_2(t') - \hat{x}_1(t')\, dt'}$ is strictly increasing in time. Therefore, the value share of the produced consumption stream ${V_2}^s$ strictly decreases over time.

Second, such a decreasing ${V_2}^s$ implies that the second term in the social discount rate for the environmental service stream $(1-s){V_2}^s\,(x_1(t), x_2(t))\,(\hat{x}_2 - \hat{x}_1)$ is decreasing in a steady state. As this term is subtracted from the constant rate of pure time preference, the social discount rate for the first commodity class $\delta_1(t)$ *grows* in a steady state.

Third, a strictly decreasing term ${V_2}^s$ implies a strictly increasing value share of the environmental service stream ${V_1}^s = 1 - {V_2}^s$. Such a strictly increasing term ${V_1}^s$ implies that the the expression $(1-s){V_2}^s\,(x_1(t), x_2(t))\,(\hat{x}_2 - \hat{x}_1)$ strictly increases in a steady state. This expression is added to the constant rate of pure time preference to yield the social discount rate for the produced consumption stream. Thus, the social discount rate $\delta_2(t)$ *grows* as well in a steady state.

Finally, by Assumption 1 there exists $\epsilon > 0$ such that $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon\, \forall\, t$. In consequence, the expression $\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s \int_0^t \hat{x}_2(t') - \hat{x}_1(t')\, dt'}$ grows without bounds and the value share ${V_2}^s$ falls to zero. Therefore, in a steady state, the discount rate δ_1 monotonously *grows* to $\delta_1 = \rho$ for $t \to \infty$. At the same time the value share of the environmental service stream ${V_1}^s$ grows to one, implying that the discount rate for the produced consumption stream *grows* to $\delta_2 = \rho + (1-s)(\hat{x}_2 - \hat{x}_1)$. Outside of a steady state, the same reasoning implies for $t \to \infty$ that $\delta_1 = \rho$ and δ_2 approaches the form $\delta_2(t) = \rho + (1-s)(\hat{x}_2(t) - \hat{x}_1(t))$. □

Proof of Corollary 1: Consider the long run social discount rate for the environmental service stream. In the proof of Proposition 1 case *ii*) I have shown that the term ${V_2}^{0<s<1}$ monotonously grows to unity as $t \to \infty$. In particular, there has to exist $t_1 \in [0, \infty)$ such that ${V_2}^{0<s<1} > \frac{2}{3}\, \forall\, t > t_1$, implying

$$
\begin{aligned}
&(1-s)\, {V_2}^{0<s<1}\, (\hat{x}_2(t) - \hat{x}_1(t)) > (1-s)\, \frac{2}{3}\, (\hat{x}_2(t) - \hat{x}_1(t)) \\
\Rightarrow\ &\delta_1^{0<s<1}(t) = \rho - (1-s)\, {V_2}^{0<s<1}(t)\, (\hat{x}_2(t) - \hat{x}_1(t)) \\
&\qquad < \rho - (1-s)\, \frac{2}{3}\, (\hat{x}_2(t) - \hat{x}_1(t))
\end{aligned}
$$

for all $t > t_1$.[37] Similarly, the fact that for $s < 0$ the proof of Proposition 1 case *iii*) has

[37] For the relaxation of Assumption 1 to the form pointed out in footnote 13 replace $t > t_1$ by $t > \max\{t_1, t^*\}$.

shown that ${V_2}^{s<0}$ monotonously falls to zero as $t \to \infty$, implies the existence of t_2 such that ${V_2}^{s<0} < \frac{1}{3}$. Then, for the social discount rate of the environmental service stream in the strong sustainability scenario it follows

$$\begin{aligned} &(1-s)\, {V_2}^{s<0}\; (\hat{x}_2(t) - \hat{x}_1(t)) < (1-s)\, \frac{1}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ \Rightarrow\; &\delta_1^{s<0}(t) = \rho - (1-s)\, {V_2}^{s<0}(t)\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &\qquad\quad > \rho - (1-s)\, \frac{1}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \end{aligned}$$

for all $t > t_2$. Setting $t_3 = \max\{t_1, t_2\}$ I find

$$\begin{aligned} \delta_1^{s<0}(t) &> \rho - (1-s)\, \frac{1}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &> \rho - (1-s)\, \frac{2}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &> \delta_1^{0<s<1}(t) \end{aligned}$$

for all $t > t_3$. Analogously, one derives for the social discount rate of the produced consumption stream the existence of $t_1' \in [0, \infty)$ such that for $0 < s < 1$ it holds

$$\begin{aligned} \delta_2^{0<s<1}(t) &= \rho + (1-s)\, {V_1}^{0<s<1}(t)\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &< \rho + (1-s)\, \frac{1}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \end{aligned}$$

for all $t > t_1'$ (as ${V_1}^s$ goes to zero), and the existence of $t_2' \in [0, \infty)$ such that for $s < 0$ it holds

$$\begin{aligned} \delta_2^{s<0}(t) &= \rho + (1-s)\, {V_1}^{s<0}(t)\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &> \rho + (1-s)\, \frac{2}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \end{aligned}$$

for all $t > t_2'$ (as ${V_1}^s$ grows to unity). Then setting $t_3' = \max\{t_1', t_2'\}$ delivers the relation

$$\begin{aligned} \delta_1^{s<0}(t) &> \rho + (1-s)\, \frac{2}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &> \rho + (1-s)\, \frac{1}{3}\; (\hat{x}_2(t) - \hat{x}_1(t)) \\ &> \delta_1^{0<s<1}(t) \end{aligned}$$

for all $t > t_3'$. Setting $\bar{t} = \max\{t_3, t_3'\}$ yields the statement of the proposition. □

Proof of Proposition 2: The proof of Corollary 1 brings about the existence of $\bar{t}$ and $\epsilon > 0$ such that

$$\begin{aligned} & \delta_i^{s<0}(t) - \delta_i^{0<s<1}(t) && > \epsilon && \text{for all } t > \bar{t} \\ \Leftrightarrow \ & \exp\left(\int_{\bar{t}}^{t} \delta_i^{s<0}(t) - \delta_i^{0<s<1}(t)\, dt'\right) && > \exp\left(\int_{\bar{t}}^{t} \epsilon\, dt'\right) && \text{for all } t > \bar{t} \\ \Leftrightarrow \ & \frac{D_i^{\chi^{0<s<1}}(t,\bar{t})}{D_i^{\chi^{s<0}}(t,\bar{t})} && > \exp\left(\int_{\bar{t}}^{t} \epsilon\, dt'\right) && \text{for all } t > \bar{t}\,. \end{aligned}$$

Define the strictly positive constant $C = \frac{D_i^{\chi^{0<s<1}}(\bar{t},t_0)}{D_i^{\chi^{s<0}}(\bar{t},t_0)} \in \mathbb{R}_{++}$.[38] Then, for any $t_0 \in [0, \infty)$ the following relation has to hold:

$$\begin{aligned} \frac{D_i^{\chi^{0<s<1}}(t,t_0)}{D_i^{\chi^{s<0}}(t,t_0)} &= \frac{D_i^{\chi^{0<s<1}}(\bar{t},t_0)}{D_i^{\chi^{s<0}}(\bar{t},t_0)}\, \frac{D_i^{\chi^{0<s<1}}(t,\bar{t})}{D_i^{\chi^{s<0}}(t,\bar{t})} \\ &> \quad C \quad \exp\left(\int_{\bar{t}}^{t} \epsilon\, dt'\right)\,. \end{aligned} \tag{30}$$

As the right hand side of equation (30) grows to infinity for $t \to \infty$ the left hand side in particular grows bigger than one. Hence it exists $\bar{\bar{t}}$ such that

$$D_i^{\chi^{0<s<1}}(t,t_0) > D_i^{\chi^{s<0}}(t,t_0) \quad \text{for all } t > \bar{\bar{t}}\,.$$

□

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