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METHODS FOR SELECTING THE OPTIMAL DYNAMIC HEDGE  
WHEN PRODUCTION IS STOCHASTIC

by

Larry S. Karp

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Abstract

A dynamic hedging problem with stochastic production is solved. The optimal feedback rules recognize that future hedges will be chosen optimally based on the most current information. The resulting distribution of revenue is analyzed numerically. This analysis enables the hedger to select his appropriate level of risk aversion.

Larry S. Karp is an assistant professor of agricultural and resource economics, University of California, Berkeley.

## METHODS FOR SELECTING THE OPTIMAL DYNAMIC HEDGE WHEN PRODUCTION IS STOCHASTIC

### Introduction

Futures and forward markets insulate producers from the risk inherent in output and price uncertainty. The possibility of hedging influences the production decision. There is considerable literature on the joint problems of optimal hedging and production. McKinnon derived the hedge which minimizes the variance of income. The papers by Danthine; Feder, Just, and Schmitz; and Holthausen consider the more general expected utility maximization problem with hedging. In a series of papers, Anderson and Danthine (1981, 1983a, 1983b) derive theoretical results using a mean-variance criterion. Hildreth considers more general utility functions and stochastic production. Batlin concentrates on the case where the date of maturity of the futures contracts and the time of harvest do not coincide (the "imperfect time hedge"). Ho; Karp (1986); and Marcus and Modest consider dynamic hedging in continuous time. The papers by Berck (1981); Nelson; Peck; and Rolfo present empirical results.

With the exception of Anderson and Danthine (1983b), Ho; Marcus and Modest; and Karp (1986), these papers view the hedging decision as static. The static approach includes the situation where hedges can be made at different points in time, but the current hedge is determined as if a commitment were being made concerning future hedges. In this case there is no recognition that future hedges will be conditioned on information which will become available in the future. The dynamic strategy, on the other hand, does anticipate that future hedges will be optimally chosen. The solution to the dynamic problem consists of rules which determine the hedge as a function of the most current information.

There have been two approaches to characterizing the dynamic hedge. Marcus and Modest consider the case of a public firm which is able to make its total return free of all systematic risk. Using arbitrage relations, the optimal hedge is shown to be independent of firm-specific characteristics such as risk aversion. These results are not applicable to the unincorporated private firm. The second approach is to choose a specific utility function and specific stochastic processes and solve the resulting optimal control problem. Anderson and Danthine (1983b) use a mean-variance criterion in a two-period problem. Ho and Karp (1986) both use a constant absolute risk aversion (CARA) utility function with continuous time. The former paper assumes that prices and harvest are lognormal and the latter that they are normal; the former paper obtains an approximate solution and the latter an exact solution. Ho considers the joint hedging and consumption decision and assumes a 0 expected change in price. Karp allows a nonzero expected change in price so that there is a speculative motive in the hedge; consumption decisions are ignored.

The objective of this paper, which recasts Karp's paper in discrete time, is to provide a practical decision-making tool. The continuous time problem permits a closed-form solution in the case of one crop and a simple form of price expectations. The discrete time version, which relies on numerical methods, has two advantages. First, it accommodates a more general problem: It is possible to treat  $n$  crops, transactions costs, and more complicated forecasting equations. Second, it leads to ways of identifying key parameters in the decision problem. For example, the hedger is unlikely to know his (constant absolute) risk aversion parameter. To each value of this parameter, there corresponds a set of optimal control rules and, hence, a distribution of

profits. In selecting his preferred distribution, the decision-maker identifies his aversion to risk and chooses the optimal hedging strategy. The next two sections discuss these two aspects of the problem. The subsequent section contains an example that illustrates the methods. A conclusion follows.

### The Optimal Dynamic Hedge

Given the assumption of additive normal errors and a CARA utility function, the dynamic hedging problem can be written as a variation of the Linear Exponential Gaussian (LEG) control problem first solved by Jacobson. This section formulates the dynamic hedging problem as a LEG problem and mentions the modifications needed in Jacobson's solution. The production aspect of the problem is then considered. The solution to the single-period version of the hedging problem is given by Bray.

To simplify notation, suppose there is a single crop; generalization to  $n$  crops requires replacing scalars by vectors. The purchase or sale of a futures contract involves no exchange of assets; any price change is debited or credited from the agent's account. This is referred to as "marking to market" (Cox, Ingersoll, and Ross). Let the discount rate for one week be  $\beta$ . Let futures positions be marked to market at intervals of arbitrary length; for concreteness, take this interval to be one week. Initially, assume that the farmer plans on revising his hedge every week. This assumption is later relaxed. Let the production season be  $T + 1$  weeks. At the beginning of weeks 1, 2, ...,  $T$  the farmer can buy or sell futures contracts. At the beginning of week  $T + 1$ , he closes his futures position and sells his crop on the cash market.

The time of harvest,  $T + 1$ , need not coincide with the date of maturity of the futures contract. In this case the futures price at  $T + 1$  will not equal

the cash price. Define  $b_{T+1}$  as the basis at harvest (basis = futures - cash price) and  $h_{T+1}$  as the harvest. Let  $p_t$  and  $f_t$  be the futures price and the hedge at  $t$ , respectively. The present value of the farmer's profits, discounted to period 1, are

$$(1) \quad \pi = - \sum_{t=1}^T \beta^t (p_{t+1} - p_t) f_t + \beta^T (p_{T+1} - b_{T+1}) h_{T+1}.$$

Futures prices are assumed to be governed by

$$p_{t+1} = a p_t + \eta_{1,t}, \quad p_1 \text{ given}$$

where  $\eta_{1,t}$  is normal. Setting  $a > 1$  implies normal backwardation;  $a < 1$  implies contango;  $a = 1$  implies that the current futures price is an unbiased estimator of the futures price of the next period.

Designate the farmer's forecast at  $t$  of his harvest and the basis at  $T + 1$  as, respectively,  $h_t$  and  $b_t$ . If his estimates are unbiased, then

$$h_{t+1} = h_t + \eta_{2,t}, \quad h_1 \text{ given}$$

$$b_{t+1} = b_t + \eta_{3,t}, \quad b_1 \text{ given}$$

where  $\eta_{i,t}$  is normal. It is convenient to write the state vector as  $y_t = (f_t, h_{t+1}, p_{t+1}, p_t, b_{t+1})'$ . At time  $t$  the farmer makes his decision based on  $y_{t-1}$ . In matrix notation, the state system is

$$(2) \quad y_t = A y_{t-1} + B f_t + \Gamma \eta_t, \quad y_0 \text{ given}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and  $\eta_t = (\eta_{1,t}, \eta_{2,t}, \eta_{3,t})' \sim N(0, \Sigma)$ . Equation (1) can be rewritten as

$$(3) \quad \pi = \sum_{i=1}^T \frac{\beta^i y_t' Q_t y_t}{2}.$$

Current and future decisions do not depend on previous profits or the current level of wealth, due to the assumptions of CARA utility and a nonstochastic interest rate. Define  $J(t, y_{t-1})$  as the maximized expected value of the utility of revenue from current and future hedging and cash sales, discounted to time  $t$ ; the expectation is conditioned on  $y_{t-1}$ . Then  $J(t, y_{t-1})$  is the solution to

$$(4) \quad \max_{\{f_\tau\}_{\tau=t}^T} E_t \left\{ -\exp \left[ -\frac{k}{2} \left( \sum_{\tau=t}^T \beta^{\tau-t+1} y_\tau' Q_\tau y_\tau \right) \right] \right\}$$

subject to (2) with  $y_{t-1}$  given. The parameter  $k$  is the risk coefficient, and the notation  $E_t$  means the expectation conditioned on the information available at  $t$ . In particular,  $J(1, y_0)$  gives the expected utility, at the beginning of the season, of the revenue from hedging and cash sales.

The function  $J(t, y_{t-1})$  is of the form  $-s_t \exp(-y_{t-1}' W_t^* y_t/2)$ , and the optimal hedge is given by  $f_t = G_t y_{t-1}$ . The equations for calculating  $s_t$ ,  $W_t^*$ , and  $G_t$  are given in the Appendix. The problem differs from Jacobson's in several minor respects. The presence of discounting leads to a trivial modification. In addition,  $\pi$  is linear in  $f$ , so it is necessary to check that the first-order conditions do indeed imply a maximum and that the problem is well defined (the solution is bounded). These conditions are given in the Appendix. Finally, the difference equation (2) is written as a backward difference rather than a forward difference. This was done to make it more convenient to write  $\pi$ .



Now consider the production aspect of the problem. The entire problem is

$$\max_{h_1, \sigma_h, \left\{ f_\tau \right\}_{\tau=1}^T} E \left\{ -\exp - k \left[ \pi - c(h_1, \sigma_h) \right] \right\}$$

where  $\pi$  is given by (3), and  $c(h_1, \sigma_h)$  is the nonstochastic cost given as a function of expected harvest and the standard deviation of the innovation in the weekly harvest forecast. This type of cost function can be derived from a production function of the type considered by Just and Pope. At period 1, the farmer selects his expected harvest and the variance,  $(T + 1) \sigma_h^2$ .

Using the definition of  $J(\cdot)$  and the solution to the problem given by (4), the above problem is equivalent to

$$\max_{h_1, \sigma_h} - s_1 \exp \left[ k c(h_1, \sigma_h) - \frac{y_0' W_1^* y_0}{2} \right].$$

In the case of a single crop with no basis risk and continuous hedging, this problem can be solved explicitly. There, the optimal rule is to set  $c_h$  equal to the current price discounted to  $T$ . This result, which is analagous to the static results of Danthine; Feder, Just, and Schmitz; and Holthausen, holds at every instant, not just the initial time. The farmer may be able to intervene in the production process after the season has begun and thereby affect the current harvest forecast. The optimal hedging rule is closed loop; but the rule, "set marginal cost equal to discounted price," is open loop with revision. If at a given time the producer anticipates future intervention in the production process, then the dynamics for harvest forecast are no longer linear (unless the cost function is linear-quadratic in  $h$ ) and the joint

hedging-production problem does not fit the LEG mold. The introduction of basis risk is discussed with the numerical example below.

Since  $W_i^*$  is independent of the state, the first order condition for the choice of  $h$  can be easily solved. However,  $W_i^*$  is a function of  $\sigma_h$  so, for higher dimensional problems, it is necessary to use numerical derivatives to determine the optimal  $\sigma_h$ . Note that this parameter includes the uncertainty due to the intrinsic variability of harvest and also due to forecast errors. The farmer may alter  $\sigma_h$  by choice of production technology or by changing the resources devoted to sampling and forecasting.

There are several points to be made about the model. The problem was set up as if the farmer planned on entering the market each time the position was marked to market, i.e., every week. If the transaction costs are significant, the farmer may choose to enter the market less often, although he is still obliged to meet the marketing to market requirement. Suppose, for example, that  $T$  is an even number, and the farmer plans on entering the market every other week. Then equation (1) is replaced by

$$(1') \quad \pi = \sum_{i=1}^{T^*-2} \beta^{*(i-1)} \left[ \beta p_{(i-1)2+1} + \beta(\beta - 1) p_{(i-2)2+2} - \beta^2 p_{(i-1)2+3} \right] \cdot$$

$$f_{(i-1)2+1} + \beta^{*(T^*-2)} \left[ \left( \beta p_T + \beta(\beta - 1) p_T - \beta^2 p_{T+1} \right) f_{T-1} + \beta^2 p_{T+1} h_{T+1} \right],$$

where the definitions  $\beta^* = \beta^2$  and  $T^* = T/2$  are used. The state vector should now be written as  $y_i = [f_{2i-1}, h_{(i-1)2+3}, p_{(i-1)2+3}, p_{(i-1)2+2}, p_{(i-1)2+1}, b_{(i-1)2+3}]'$  so that (1') can be written as in (3). Corresponding changes in the difference equation (2) must be made.

Given an initial observation,  $y_0$ , the farmer can repeatedly solve the dynamic hedging problem varying the number of times he plans to enter the market ( $T^*$ ). He can compare the expected utility and the moments of profits under the different scenarios and select the optimal strategy. Entering the market more often gives him greater flexibility and a higher expected revenue but also higher transactions costs. Suppose that at  $t = 1$  he decides to enter the market every other week. At  $t = 2$  this strategy calls for leaving his initial hedge,  $f_1$ , unchanged until  $t = 3$ . However,  $f_1$  is not optimal at  $t = 2$  since he now has more recent information. It is a simple matter, using the methods described in the next section, to recalculate the moments and expected utility of profits, conditional on information available at  $t = 2$ , under the assumption that (1) he adheres to his initial strategy or (2) he departs from that strategy and revises his hedge at  $t = 2$ . This gives him the information necessary to decide whether it pays to reenter the market.

The model was not written in its most general form. The matrices  $A$  and  $\Sigma$  can depend on time; the algorithm in the Appendix shows them as time dependent. Future prices may become either more or less volatile as harvest approaches (Anderson and Danthine 1983b). In addition, the innovation in the harvest forecast need not have constant variance. As previously suggested, the farmer may be regarded as choosing a production technology which determines output variance or he may choose a sampling strategy which determines the variance of the error of harvest forecast. With either interpretation,  $E(r_{2,t}^2)$  may be time dependent. Numerical analysis will indicate the relative importance of decreasing the variance of forecast error at different times in the season. Since this depends in a complicated way on all the parameters of the problem, it is unlikely that analytic results can be obtained.

Inclusion of a constant in the state vector  $y$  allows an intercept to be included in (2) and a constant and linear cost to be included in the profit function. The latter permits an affine transactions cost to be included in profits. No alteration in the algorithm is required. Under some circumstances, it may be desirable to include a cost which is quadratic in the controls. This may arise if, for example, there is a control "fertilizer application" which involves a quadratic adjustment cost and which changes the expected harvest. Since the state vector is already augmented to include the control(s), no change is needed in the algorithm.

Equation (2) indicates that only the current futures price contains information about next period's futures price. This is an unnecessary restriction and was adopted only for purposes of exposition. More generally, write

$$p_{t+1} = a' z_t + \eta_t$$

where  $z_t$  is a vector of explanatory variables which may include current and lagged prices;  $z$  becomes a component of the state  $y$ . The only necessary assumptions are linearity and normality. The error  $\eta_t$  can be replaced by a moving average term by suitable definition of  $z_t$ .

The above modifications to the simple problem allow for a great deal of flexibility and make the model useful.

#### Methods for Analyzing the Optimal Hedge

The previous section discussed the optimal hedging strategy conditional on the parameters  $k$ ,  $T^*$ , and  $\sigma_h$ , at least the first two of which are determined by the farmer. As a decision-making aid, this is incomplete. For example, the

farmer is unlikely to know what his risk aversion coefficient is. He is more likely to be concerned with the probability that his profits are less than a certain amount than he is with the expected utility of profits. One would, therefore, like to determine the distribution function for profits for a given set of optimal control rules  $\{G\}$ . Different values of  $k$  generate different sets of control rules. The normality assumption implies that profits are distributed as a linear combination of noncentral  $\chi^2$  random variables (Johnson and Kotz II, chapter 29), but this distribution cannot be written in closed form.

Since the distribution function cannot be obtained in a useful form, the obvious alternative is to calculate the moments of profits. There are a number of ways of calculating the moments of a random variable like (3) given a linear system like (2) (Karp, 1985). These computations become expensive for large problems. However, close approximation to the first several moments can be obtained by using numerical approximations to the derivatives of the moment-generating function. In brief, the procedure is as follows. Begin with a solution to the hedging problem, i.e., a set  $\{G\}$  of control rules. Replace (2) by

$$y_t = D_t y_{t-1} + \eta_t$$

where  $D_t = A + B G_t$ . Use equation (7) of Karp (1985) to obtain

$$(7) \quad E \exp(q \pi) = z(q) \exp[q y_0' S(q) y_0] \equiv M(y_0; q).$$

The function  $M(\cdot)$  is the moment-generating function. Define the difference operator  $\Delta$  as

$$\Delta M(y_0, q) = M(y_0, q + .5 r) - M(y_0, q - .5 r)$$

where  $r$  is a small positive number;  $\Delta^n M = \Delta \Delta^{n-1} M(\ )$ , etc. Then the  $n$ th derivative of  $M$  with respect to  $q$ , evaluated at  $q_0$ , is

$$\lim_{r \rightarrow 0} \frac{\Delta^n M(y_0, q_0)}{r^n}.$$

The  $n$ th moment of  $\pi$  is given by the  $n$ th derivative of the moment-generating function, evaluated at  $q = 0$ . Let  $q_0 = 0$ , and use the fact that  $M(y_0, 0) = 1$ ; approximate the  $n$ th derivative of  $M$  with respect to  $q$ , evaluated at  $q = 0$ , as  $[\Delta^n M(y_0, 0)]/r^n$ . For example, approximating the first two moments of  $\pi$  requires calculating the expression in (7) for two different values of  $q$ ; approximating the third and fourth moments requires making the calculation four times. It is useful to approximate each moment using several values of  $r$  to check for convergence. Exact calculation of the first moment is inexpensive; this can be checked against the approximation to gauge the latter's accuracy.

Having obtained a set of a finite number of moments of  $\pi$ , each set of which corresponds to a different value of  $k$  (or  $T^*$  or  $\sigma_h$ ), it is possible to proceed. To avoid repetition, assume that the only question is to determine the  $k$  that most accurately reflects the farmer's preferences. This section suggests three methods of helping the farmer choose  $k$ . The first method involves parameterizing on  $k$  and graphing the resulting mean-variance trade-off. The second method uses a Chebychev-type inequality to obtain an upper bound on the probability of profits falling below a given level. The

third method uses the higher moments to obtain an approximation of the distribution function for profits. This can then be used to obtain the expected value of a given function of profits or the probability that profits are below a given level.

The first method involves parameterizing on  $k$  to obtain various sets of control rules and corresponding pairs of mean and variance of profits. The farmer chooses the preferred mean-variance trade-off. Parameterizing on  $k$  does not sweep out the mean-variance frontier because the maximand is not a mean-variance criterion. For small values of  $k$ , the two criteria are similar and give similar results. This is not so for large values of  $k$ . Numerical experiments show that for large  $k$  further increases in risk aversion lead to a decrease in expected profits and an increase in the variance. This is not surprising--for large  $k$ , the third and higher moments of profits assume increasing importance.

The farmer may be more interested in the probability that profits are less than some critical value; call this value  $\alpha$ .<sup>1</sup> Let  $\mu$  and  $\sigma^2$  be the mean and variance of profits under the control rules which are optimal for a given  $k$ ; let  $\mu_3' = E \pi^3$  and  $\mu_4' = E \pi^4$  be the third and fourth moments about the origin. Assume that  $\alpha < \mu$  and  $\alpha < \mu + \sigma(\mu_3' - \sqrt{\mu_3'^2 + 4})/2$ . The following formulas are taken from Walsh (page 90) (the cases where  $\alpha$  does satisfy either of the above inequalities are treated there):

$$(8a) \quad \Pr \{ \pi < \alpha \} < \frac{\sigma^2}{\sigma^2 + (\alpha - \mu)^2}$$

$$(8b) \quad \Pr \{ \pi < \alpha \} < \left\{ \left[ \frac{-(\alpha - \mu)^2}{\sigma^2} + \frac{(\alpha - \mu) \mu_3'}{\sigma} + 1 \right]^2 + \left[ 1 + \frac{(\alpha - \mu)^2}{\sigma^2} \varphi \right]^{-1} \right\} \varphi$$

where  $\varphi = \mu_4' - \mu_3'^2 - 1$ . These are the tightest bounds that can be achieved for the given level of information. The degree of improvement in the bound resulting from the use of the third and fourth moments depends in a complicated manner on  $\alpha$  and the moments. The example in the next section, which calculates the moments of profits resulting from an optimal control problem with reasonable parameter values, suggests that the gain in precision may be quite modest. Two-sided confidence intervals for  $\pi$  can be calculated using Chebychev's inequality or its higher moment analogs (Walsh).

The third method of evaluating the control rules that result from a given level of  $k$  involves the approximation of the distribution function of the random variable,  $\pi$ . For example, choose two levels of  $k$ ,  $k_1$ , and  $k_2$ . Solve the control problem for each  $k_i$  and obtain a finite number of moments for each random variable  $\pi(k_i)$ . Fit distribution functions to each set of moments; designate these functions as  $F(\pi; k_i)$ . The preferred value for  $k$  and, hence, the preferred set of control rules requires a comparison of the two distributions  $F(\pi; k_1)$  and  $F(\pi; k_2)$ . Meyer suggests one method of ranking distributions. A simpler approach is to use the distribution  $F(\pi; k_i)$  to calculate the probability of  $\pi$  being less than a critical level. For the same number of moments, this bound is smaller than that given in (8a) or (8b) since the latter gives the worst possible case.

The approximation of an unknown density or distribution function using known (or estimated) moments is a well-developed subject, but it appears to have found limited application in applied economics. The following discussion is drawn from Johnson and Kotz I (chapter 12); this book provides an excellent introduction to the subject.



It is convenient to standardize the random variable  $\pi$ . Replace  $\pi$  by  $\tilde{\pi} = (\pi - \mu)/\sigma$ , so  $\tilde{\pi}$  has 0 mean and unit variance. Write the  $i$ th central moment as  $\mu_i$ ,  $i \geq 2$ . The following quantities play an important role:  $\beta_1 = (\mu_3/\sigma^3)^2$  is a measure of skewness; and  $\beta_2 = \mu_4/\sigma^4$  is a measure of excess or kurtosis. Given the first four moments about the origin of the original random variable  $\pi$ , it is straightforward to obtain the central moments of the standardized variable  $\tilde{\pi}$  and, hence,  $\beta_1$  and  $\beta_2$ . Hereafter, it is understood that  $\beta_1$  and  $\beta_2$  give the skewness and kurtosis of  $\tilde{\pi}$ .

The Pearson system provides one method of using the moments to fit a distribution. For a random variable  $x$ , suppose that the probability density  $p(x)$  satisfies the differential equation

$$\frac{1}{p} \frac{dp}{dx} = - \frac{c_3 + x}{c_0 + c_1 x + c_2 x^2}.$$

The form of the solution to this equation depends on the values of the parameters  $c_0$ ,  $c_1$ ,  $c_2$ , and  $c_3$ . These parameters have a simple relation to the moments of the random variable and, hence, to  $\beta_1$  and  $\beta_2$ . Calculation of  $\beta_1$  and  $\beta_2$  and inspection of a chart (Johnson and Kotz I, Figure 1, p. 14) determine to which type of distribution the moments correspond. These types include the beta, gamma, and  $\tau$  distributions. Calculation of the parameters  $c_0$ ,  $c_1$ ,  $c_2$ , and  $c_3$  permits calculation of the parameters of that distribution. If the distribution is of a common type, it is possible to use tables to determine the probability that  $\tilde{\pi}$  (and, hence,  $\pi$ ) falls below a given level. For less common types of distributions, numerical integration may be used.

Various types of expansions provide alternatives to the Pearson system. The idea is to express the unknown density (that of  $\tilde{\pi}$ ) with known (or estimated) moments as a function of some tractable density,  $f(\cdot)$ . One of the most

common expansions, the Edgeworth expansion, chooses  $f(\cdot)$  as the standard normal density and uses a particular arrangement of the terms in the series. Designate the resulting representation of the unknown density as  $g(\tilde{\pi})$ . The moments given by  $g(\cdot)$  equal those of  $\tilde{\pi}$ , but  $g(\cdot)$  may not be a proper density in that  $g < 0$  is possible. However, the boundary of the region in  $(\beta_1, \beta_2)$  space for which  $g \geq 0$  is guaranteed is known. Even for  $\beta_1, \beta_2$  such that  $g < 0$  for some values of  $\tilde{\pi}$ , the Edgeworth expansion can provide a good approximation to the unknown distribution.

The Edgeworth expansion involves an infinite number of moments of  $\tilde{\pi}$ . In practice, only the leading terms of the expansion are used. Johnson and Kotz I suggest that the first four moments generally provide a sufficiently good approximation. One reason for not using fifth and higher moments is that, with observed data, these are often not estimated accurately. In the present case, the moments of  $\tilde{\pi}$  are not estimated but are approximated. Since approximation of the  $n$ th moment involves division by  $r^n$ , a very small number, the approximation of higher moments is likely to be poor. Therefore, only the first four moments, or equivalently,  $\beta_1$  and  $\beta_2$ , are used in the Edgeworth expansion reported in the next section. This uses equation (44)' of Johnson and Kotz I (p. 17).

This section has indicated how a parameter, such as  $k$ , of the hedging problem can be selected. Once that parameter is selected, the farmer can follow the hedging and production rules given in the previous section. Two distinct issues were considered. The first concerned the calculation of the moments of profits. A numerical approximation of the derivative of the moment-generating function was suggested. In a previous study (Karp, 1985), this approximation was compared to exact calculation of the first two moments;

the results were encouraging. The second issue concerned how the moments should be used. Three possibilities were suggested: (1) derive the mean-variance trade off, (2) obtain an upper bound on the probability of a low level of profits by means of a Chebychev-type inequality, and (3) approximate the unknown distribution. The third alternative can be accomplished using the Pearson system or an expansion such as the Edgeworth expansion. The next section applies the methods of this and the previous section.

#### Illustration of the Methods

A simple example illustrates the methods of the previous two sections. The parameter values used are of a reasonable order of magnitude but do not represent detailed statistical analysis. Wheat yield data (U. S. Economic Research Service) for the United States (1967-1983) suggest expected harvest of 32.32 bushels per acre with a sample variance of 8.2. The length of the season,  $T$ , was set at 16 weeks; and the variance of the weekly innovation in harvest forecast was taken to be  $8.2/16 = .51$ . Using 1983 weekly data (Chicago Board of Trade) (June 1-September 4), the sample variances of the weekly change in futures price and basis were, respectively, .021 and .004. The initial futures price and expectation of the harvest basis were taken to be 3.54 and .057 dollars per bushel. The weekly discount rate,  $\beta$ , was set at .998, implying an annual interest rate of 10 percent. The covariances of  $p$ ,  $b$ , and  $h$  were set at 0. These parameter values are referred to as base values. The regression of  $p_t$  on  $p_{t-1}$  using 1983 data implied a coefficient of  $a = .94$ ; choosing  $a$  to solve  $p_1 a^{16} = p_{17}$  gave  $a = 1.002$ . The results below, except where indicated otherwise, use the intermediate value  $a = .98$ . This implies that price is expected to fall.

In order to indicate the advantage of entering the market frequently, the optimal hedging problem was solved for 16 values of  $k$  in the range (.0025, .12) under the assumption that the farmer enters the market (1) every week, (2) every second week, and (3) every fourth week. The mean and variance of profits were calculated in each case, and the result is shown in figure 1. The three curves give the mean-variance trade-off for the three strategies. Since transactions costs were set at 0, the farmer does better for every level of risk aversion by entering the market 16 times. The inclusion of transactions costs may cause the relative positions of the curves to change, and the curves may cross. In the latter case farmers with different levels of risk aversion and the same transaction costs will not only make different hedges but will change their hedge with different frequency.

For the results shown, a lower level of risk aversion corresponds to a higher level of expected profits and a higher variance. As mentioned in the previous section, it is possible for the mean-variance graph to be decreasing. This occurs at large values of  $k$ . The graph indicates that the location of the mean-variance point is more sensitive to changing the number of times the farmer hedges when the level of risk aversion is small. This occurs because, for small levels of risk aversion, the farmer is more willing to take advantage of the opportunity for speculation.

The mean-variance trade-off offers the farmer some help in choosing his preferred levels of  $k$  and  $T^*$  and, thus, his optimal hedging and production rules. Supplementary information is obtained by considering the probability  $\delta$  (associated with each level of  $k$ ) that profits fall below a given level  $\alpha$ . The previous section discussed methods of putting a bound on  $\delta$  and of obtaining an approximation of the distribution of  $\pi$  and, hence, an approximation

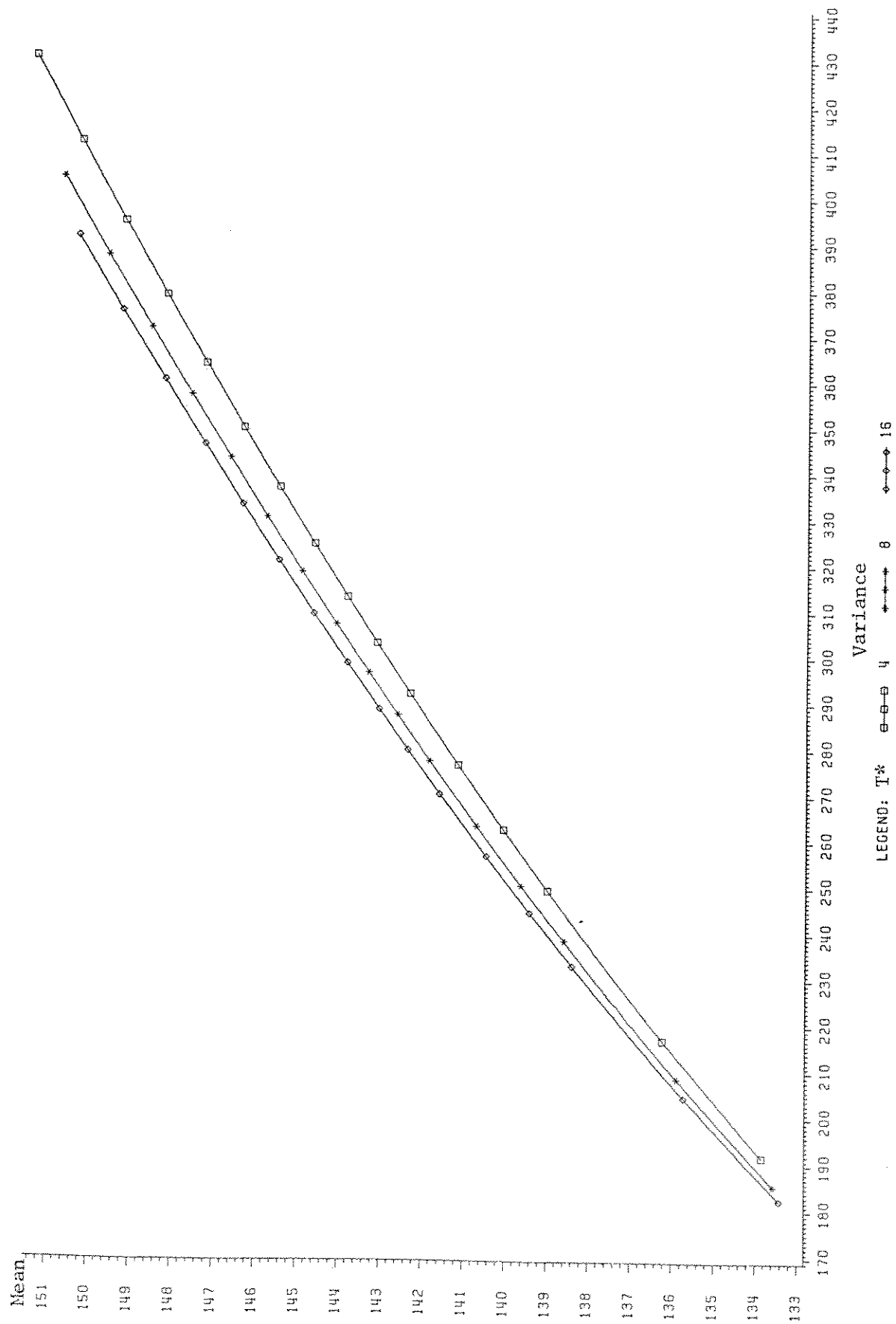


Figure 1. Expected profit versus variance

of  $\delta$ . To illustrate these methods, suppose that the farmer has decided to enter the market four times. He now chooses  $k$  on the basis of the mean and variance of profits and of the probability,  $\delta$ , of profits falling below  $\alpha = \$120$ . Table 1 provides him with this last piece of information. The first column gives  $k$ ; columns 2 and 3 give, respectively, the third and fourth moment of the standardized random variable,  $\tilde{\pi}$ . For all levels of  $k$ , the distribution of profits is skewed to the left. Since  $\mu_4 > 3$ , the value of the fourth moment for the standard normal, the tails are "somewhat thick." The fourth and fifth columns give  $\delta_1$  and  $\delta_2$ , the upper bounds on the probability that profits are less than \$120 per acre using, respectively, the first two and the first four moments [(8a) and (8b), respectively].

The coefficients of the Pearson system were calculated using  $\beta_1 = \mu_3^2$  and  $\beta_2 = \mu_4$ . For all values of  $k$ , the results indicate that  $\tilde{\pi}$  is distributed (approximately) as a beta. The parameters of the distribution were calculated from the coefficients of the Pearson system, and the SASS function PROBBETA was then used to obtain the probability that  $\pi < 120$ . This is reported as  $\delta_3$  in column 6. Column 7 gives  $\delta_4$ , the probability that  $\pi < 120$ , obtained from the Edgeworth expansion.

The table indicates that the use of the third and fourth moments in the Chebychev-type inequalities results in an improvement in the bound of only 1 percent to 3 percent (compare  $\delta_1$  and  $\delta_2$ ). However, the approximation techniques suggest that  $\delta$  is only 25 percent to 30 percent of the upper bound. It is encouraging that the two approximation techniques give comparable estimates. This suggests (but, of course, does not prove) that the approximations are close to the true density.

The upper bound of  $\delta$ , given by  $\delta_1$  or  $\delta_2$ , and the estimate of  $\delta$ , given by  $\delta_3$  or  $\delta_4$ , provide different pieces of information; their incorporation into

Table 1. Variation in the Distribution of Profits

k	$\mu_3$	$\mu_4$	$\delta_1$	$\delta_2$	$\delta_3$	$\delta_4$
.082	-.36679	3.1907	.311	.308	.0764	.0761
.09	-.36676	3.1926	.335	.324	.0867	.0864
.1	-.3659	3.194	.369	.366	.1012	.101

$\mu_3$  = skewness for standardized random variable.

$\mu_4$  = kurtosis for standardized random variable.

$\delta_1$  = upper bound on  $\Pr\{\pi < 120\}$  using (8a).

$\delta_2$  = upper bound on  $\Pr\{\pi < 120\}$  using (8b).

$\delta_3$  = estimate of  $\Pr\{\pi < 120\}$  using beta distribution.

$\delta_4$  = estimate of  $\Pr\{\pi < 120\}$  using Edgeworth expansion.

the decision-making process is a matter of judgment. If the true model for the stochastic processes were known and if the criterion were literally "safety first," then it would be appropriate to use the upper bound of  $\delta$  to evaluate the control rules. In practice, the parameters of the stochastic processes are estimated; and the decision-maker is likely to regard the inequality  $\delta \leq \bar{\delta}$ ,  $\bar{\delta}$  given, as a desirable characteristic rather than as a precise constraint. On both of these counts, it is more reasonable to seek a reliable estimate of  $\delta$  rather than its upper bound. Table 1 gives an indication of the extent to which the use of an upper bound rather than an estimate of  $\delta$  can lead to overly conservative behavior.

The table also shows that, as the hedger becomes more risk averse, the distribution becomes slightly less skewed. As he becomes more risk averse, the probability that profits are less than 120 increases. The decrease in expected profits more than offsets the decrease in variance. The farmer may find it strange that an increase in his risk aversion is associated with an increase in the probability of falling below some critical level of profits. Different choices of  $\alpha$  or  $k$  or different parameters in the control problem may reverse the result so that increases in risk aversion could lead to a decrease in the probability of profits being less than  $\alpha$ . The CARA utility function attaches a specific meaning to "risk aversion," whereas the same term means a host of things to most people.

The accuracy of the results in table 1 is conditioned on the accuracy of the approximations of the higher moments of profits. Recall that these are obtained as numerical derivatives to the moment-generating function. For each level of  $k$ , the first four moments were approximated using 10 values of the step size  $r$ , ranging from  $5 \times 10^{-4}$  to  $4 \times 10^{-6}$ ; the first moment was also



calculated exactly as a gauge of accuracy. As expected, the approximations for the lower moments are more stable than those of the higher moments. The approximation of the first moment is extremely stable. It varies by less than .01 percent and is within .01 percent of the true value. The approximations of the second and third moments vary by less than 1 percent. The approximation for the fourth moment is much less stable, varying by almost 10 percent. In addition, the resulting approximations of  $\delta$  using  $r$  close to  $4 \times 10^{-6}$  are nonsensical, falling outside the range  $(0, 1)$ . However, for larger  $r$ , e.g.,  $r \in (5 \times 10^{-4}, 4 \times 10^{-5})$ , the approximations of the fourth moment are very stable; and the resulting approximations  $\delta_2$  and  $\delta_3$  are even more so. Table 1 reports results using step size  $r = 5 \times 10^{-4}$ . This discussion indicates the importance of experimenting with different step sizes. If  $r$  is too large, the derivative is not approximated well; if  $r$  is too small, numerical problems arise because the approximation to the fourth moment requires  $r^4$ .

A previous section mentioned that, with zero basis risk and continuous hedging, the marginal cost of expected harvest is set equal to the discounted price. With the discount rate used above, this gives  $p_1 - b_1 = 1.02979 c_h( )$ . Table 2 indicates how the rule is altered for moderate basis risk.<sup>2</sup> The interest rate still predominates. It is apparent that an increase in risk aversion can lead to an increase or decrease in planned harvest (since  $c$  is convex); however, the effect of risk aversion on planned harvest is small.

In the continuous time model with no discounting and no basis risk, the optimal hedge is expected to increase over time if the ratio of the absolute value of the percentage of expected change in price to the level of risk aversion is "small"; if the ratio is large, the hedge is expected to increase.<sup>3</sup>

Table 2. The Optimal Choice of Expected Harvest,  $h_1$

k	Equation to determine $h_1$
.082	$p_1 - b_1 = 1.02955 c_h( ) + 3.29 \times 10^{-4} h_1$
.09	$p_1 - b_1 = 1.02951 c_h( ) + 3.61 \times 10^{-4} h_1$
.1	$p_1 - b_1 = 1.02944 c_h( ) + 4.01 \times 10^{-4} h_1$

Numerical analysis indicates that this also holds in discrete time with discounting and basis risk. That is, the hedge is expected to rise (fall) if  $E|p_{t+1} - p_t|/p_t k = |1 - a|/k$  is small (large).

Figure 2 graphs the expected hedge for  $a = .98, 1, 1.02$ , and  $k = .0978$ . The graph is obtained using the optimal control rules with the equations of motion for  $p$ ,  $b$ , and  $h$  and setting all random variables equal to their expected values. For the first and last values,  $|a - 1|/k = .204$  is large, and the hedge is expected to fall. For the intermediate case,  $|a - 1|/k = 0$  is small, and the hedge is expected to rise. In the case where price is expected to fall ( $a = .98$ ), the hedger initially sells contracts and proceeds to buy them back over the season. He ends the season with a short position of approximately twice his expected cash crop. In the third case ( $a = 1.02$ ), price is expected to rise. The hedger initially sells a few contracts as a hedge against an unexpected drop in price. He then proceeds to buy contracts, finishing the season in a long position. For both of these cases, the hedger's behavior is motivated by speculation on expected price change; in neither case does his hedge approach expected harvest.

The important point is that, although the level of the hedge depends on whether price is expected to increase or decrease, the expected direction of change in the hedge depends on a comparison between the magnitude of the expected price change and the degree of risk aversion. When the price is not expected to change ( $a = 1$ ), the farmer hedges slightly less than expected production. As the season progresses, his hedge converges to expected production.

The example can also be used to indicate the value of entering the futures market. In the absence of hedging, with  $a = .98$  and the base value parameters, the expected value of revenue per acre is \$78.68; and the standard deviation

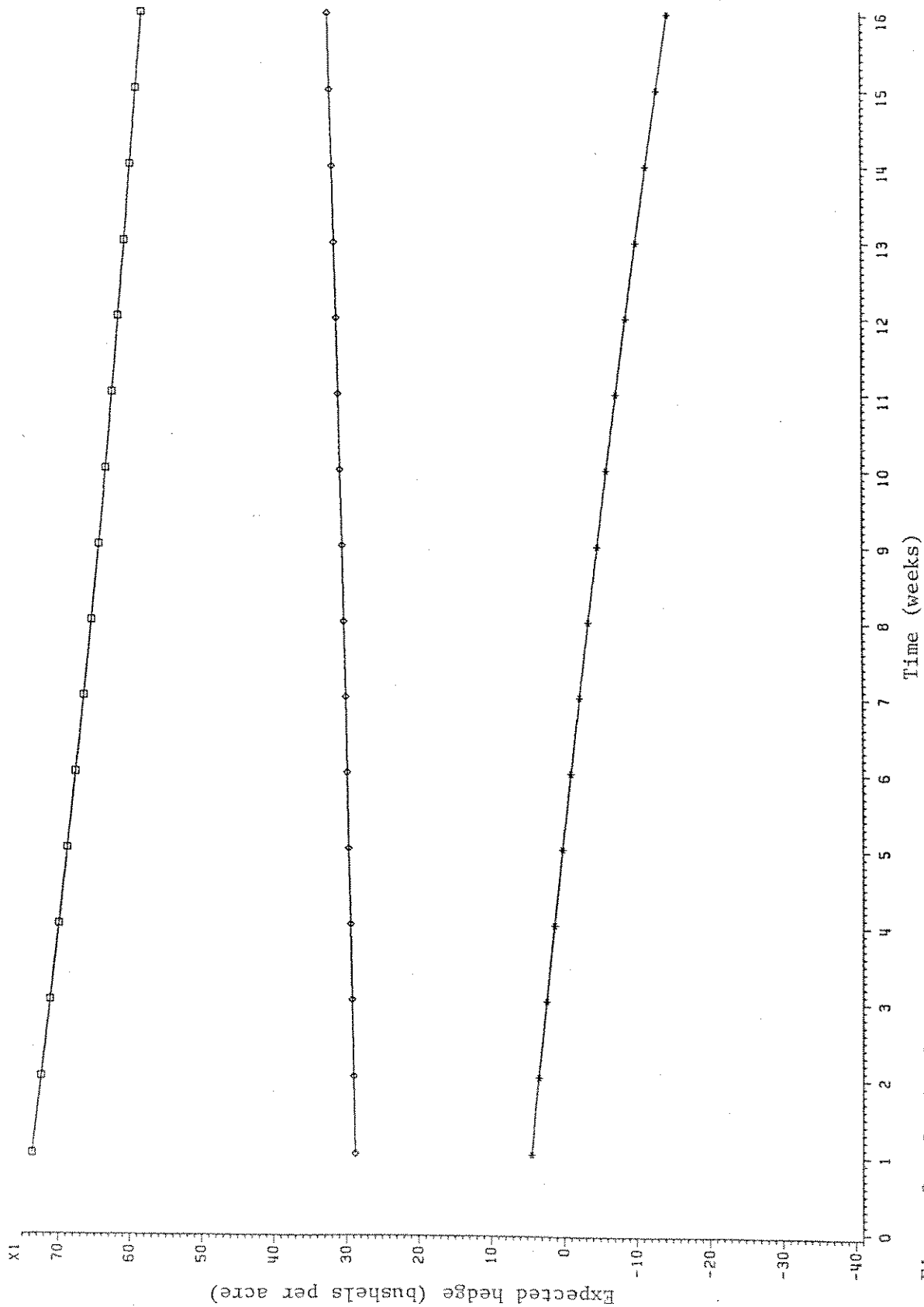


Figure 2. Graph of hedge for  $a = .98$  (square),  $a = 1$  (diamond), and  $a = 1.02$  (star)

is \$17.41. With the same parameter values and setting  $k = .0978$ , the optimal hedge results in expected profits of \$143.11 with a standard deviation of \$17.41. The advantage of hedging is so pronounced because  $a \neq 1$ , which implies the possibility for speculative profits.

### Conclusion

This paper has described and illustrated a practical technique for determining optimal hedges. The method uses the LEG control problem. It requires assumptions which are analogous to those underlying the more common mean-variance optimization problem which uses quadratic programming. Its advantage over the mean-variance approach is that it incorporates dynamics (anticipated revision of hedges).

The solution of the optimization problem is straightforward. Varying parameters, such as  $k$ , the degree of risk aversion, or  $T^*$ , the number of times the farmer enters the market, generates families of optimal control rules. The interesting practical problem is to determine which of these rules should be followed, i.e., to determine the best  $k$  or  $T^*$ . Several suggestions were made, all of which require calculation of the moments of profits. Perhaps the most promising involves approximating the distribution of profits either by using the Pearson system or by some type of expansion. The approximate distributions provide the decision-maker with more relevant information than do the Chebychev-type inequalities, which are extremely conservative.

The techniques discussed in this paper were motivated by the hedging and production problem. However, it is clear that the same methods can be used for any situation that conforms to the assumptions of the LEG control problem.

## Appendix

### The LEG Algorithm

The following definitions are used

$$G_t = -(B_t' \tilde{W}_t B_t)^{-1} B_t' \tilde{W}_t A_t$$

$$\tilde{W}_t = W_t - W_t \Gamma_t (\Sigma_t^{-1} + \Gamma_t' W_t \Gamma_t)^{-1} \Gamma_t' W_t$$

$$W_t^* = A_t' [\tilde{W}_t - \tilde{W}_t B_t (B_t' \tilde{W}_t B_t)^{-1} B_t' \tilde{W}_t] A_t.$$

The difference equations are

$$W_t = Q_t + \beta W_{t+1}^*; W_T = Q_T \text{ (i.e., } W_{T+1}^* = 0)$$

$$s_t = s_{t+1} |I + \Sigma_t \Gamma_t' W_t \Gamma_t|^{-1/2}; s_{T+1} = 1.$$

The second-order condition is that  $B_t' \tilde{W}_t B_t$  be negative definite for all  $t$ . In addition,  $I + \Sigma_t \Gamma_t' W_t \Gamma_t$  must be positive definite to insure that expected utility is bounded.

## Footnotes

<sup>1</sup>The confidence intervals discussed in this section use the moments of  $\pi$ . Other statistics, such as the semivariance, can also be used (Berck, 1982). For the problem at hand, the moments are easier to calculate.

<sup>2</sup>Table 2 is an approximation. It ignores the fact that the coefficients on  $p$  and  $b$  differed by at most .001 percent in absolute value. This difference is due to the fact that the futures price and basis follow different stochastic processes, so the current cash price is not exactly a sufficient statistic for the choice of  $h_1$ .

<sup>3</sup>This is a paraphrase of Remark 4 in Karp (1986). For some values of the above-mentioned ratio, the expected hedge is not monotonic in time; for very large values of the ratio and  $a > 1$ , the hedge is expected to increase.

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