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by

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1. INTRODUCTION

The possibility of hedging provides an opportunity for producers to reduce the risk associated with price and production uncertainty. McKinnon [17] was one of the first to study the problem, assuming normal distributions for price and harvest, and the objective of minimizing the variance of income. Anderson and Danthine [3] considered the problem where production was certain and producers had a mean-variance criterion; their later papers, [1] and [2], generalized this to include stochastic production. The papers by Rolfo [20] and Hildreth [11] study optimal hedging with production uncertainty using utility functions other than mean-variance. Most of these papers view the decision problem as static; the exception is Anderson and Danthine [1] who treat a two period problem. Other papers on optimal hedging include those by Danthine [7]; Feder, Just, and Schmitz [8]; Holthausen [12] and Batlin [4]. A recent paper by Marcus and Modest [16] studies dynamic hedging by a public firm with stochastic production. Their results are not applicable to privately held firms which are unable to make their total return free of all systematic risk. The decision-maker in this paper is taken to be the owner of a private firm.

The general abstraction from the dynamic nature of the hedging problem misrepresents the producer's situation. At the beginning of the production period (planting), the farmer chooses inputs and his position in the futures market. The knowledge that he will be able to revise his hedge in subsequent stages may affect both his initial production and hedging decisions. This point has been largely neglected because of the difficulty of characterizing the solution to the dynamic problem.

Choice of the constant absolute risk aversion (CARA) utility function permits a closed-form solution to the dynamic problem. This gives the hedge at any point in time as a function of the current futures price and the parameters of the harvest forecast and price equation. One result is that, if the current futures price is an unbiased predictor of the cash price at harvest, the optimal initial hedge is myopic; that is, the same solution is obtained from the corresponding static problem. This does not hold when the expectation at t of cash price at harvest differs from the futures price at t. A second result is that an expected increase or decrease in the amount hedged, over the production period, is consistent with the current futures price being either an upwardly or downwardly biased estimator of cash price at harvest. The sign of the expected change in the hedge depends on the magnitude of the bias relative to the degree of risk aversion.

These results hold in the limiting case where there is no basis risk (defined below) and the interest rate is 0. A small increase in basis risk leads to an increase in the optimal level of futures sales provided that the absolute level of risk aversion is small. A small increase in the interest rate reduces the level of sales if the futures price is expected to decline; otherwise, an increase in the interest rate increases the level of futures sales. These results hold for small levels of absolute risk aversion.

It is helpful to bear in mind the relationship between the hedging and standard portfolio problems. If production were nonstochastic, the farmer could sell the entire crop at the current futures price at planting and regard the proceeds as his initial wealth. Hedging either more or less than his known production is equivalent to investing in the risky asset in portfolio theory. Thus, in the case of nonstochastic production, the farmer's dynamic hedging problem can be treated as a special case of the class of problems discussed by Mossin [19]. With the CARA utility function, the solution to that problem requires "limited foresight": the investor distributes his assets between the risky and safe investment in order to maximize the expected utility of next period's wealth, compounded at the rate of the safe asset, over the remaining horizon. The myopic hedging result alluded to above is clearly related to this result from portfolio theory. When production is stochastic, the hedging problem is analogous to a portfolio problem in which initial wealth is unknown. For a more general discussion of myopia in dynamic problems see Tesfatsion [22, 23].

2. PROBLEM FORMULATION AND SOLUTION

The following formulation is a dynamic generalization of a static problem used by Bray [5]. To derive the model of continuous trading, suppose first that there are $n + 1$ trading dates which, for notational convenience, occur at regular intervals of ε. Futures are first traded at time 0; at $n\varepsilon = T$ the farmer's futures position is closed and he sells his crop on the cash market. At each trading date the farmer decides the number of futures contracts to hold, based on his current information about prices and his (future) harvest.

It is typically the case that the time of harvest, T, does not coincide with the time of maturity of the futures contract. This is referred to as an imperfect time hedge. It can be modeled by allowing the basis, defined as the difference between the futures price and cash price, to be a random variable.

Futures contracts are marked to market. That is, the purchase or sale of a contract involves no exchange of assets; any price change is debited or credited from the agents' account (Cox, Ingersoll, and Ross [6]). If the instantaneous interest rate is r, then the discount rate for a period of ε units of time is $\beta(\varepsilon) = e^{-r\varepsilon}$. Define p as the futures price, b as the basis, f as sales of futures contracts (f > 0 implies that the farmer takes a short position), and h_T as the harvest; recall $n\,\varepsilon = T$. The farmer's profits, discounted to time 0, are

$$\pi = -\sum_{i=0}^{n-1} \beta^{i+1} [p_{(i+1)\varepsilon} - p_{i\varepsilon}] f_{i\varepsilon} + \beta^n (p_{n\varepsilon} - b_{n\varepsilon}) h_T = \beta\, p_0\, f_0$$
$$+ \sum_{i=1}^{n-1} \beta^i p_{i\varepsilon}[\beta\, f_{i\varepsilon} - f_{(i-1)\varepsilon}] + \beta^n\{p_{n\varepsilon}[h_T - f_{(n-1)\varepsilon}] - b_{n\varepsilon}\, h_T\}. \quad (1)$$

Define $fl_{i\varepsilon} = f_{(i-1)\varepsilon}$ the number of contracts held in the previous period. Define $u_{i\varepsilon}\,\varepsilon = \beta\, f_{i\varepsilon} - f_{(i-1)\varepsilon}$ so $u_{i\varepsilon}$ has the dimensions of a rate. If r = 0 so that $\beta = 1$, $u_{i\varepsilon}\,\varepsilon$ is the number of additional contracts sold at the beginning of the ith period. Equation (1) can be rewritten as

$$\pi = \beta\, p_0\, f_0 + \sum_{i=1}^{n-1} \beta^i\, p_{i\varepsilon}\, u_{i\varepsilon}\, \varepsilon + \beta^n[p(h - fl) - b\, h]_{|T}. \quad (2)$$

Define $h_{i\varepsilon}$ as the farmer's forecast at time $i\varepsilon$ of his harvest at time T. Suppose that h, p, and b obey the following stochastic difference equations

$$h_{(i+1)\varepsilon} = h_{i\varepsilon} + \Delta_{1,i\varepsilon} \quad (3a)$$

$$p_{(i+1)\varepsilon} = c(\varepsilon)\, p_{i\varepsilon} + \Delta_{2,i\varepsilon} \quad (3b)$$

$$b_{(i+1)\,\varepsilon} = b_{i\varepsilon} + \Delta_{3,i\varepsilon} \quad (3c)$$

where $\Delta_{i\varepsilon} = (\Delta_{1,i\varepsilon}, \Delta_{2,i\varepsilon}, \Delta_{3,i\varepsilon})'$ is identically and independently distributed with mean 0 and $E\,\Delta_{i\varepsilon}\,\Delta_{i\varepsilon}' = \Sigma\,\varepsilon$, Σ positive semidefinite; $c(\varepsilon) = e^{a\varepsilon}$. System (3) assumes that the disturbances are additive. This is unfortunate since it admits the possibility of negative prices and quantities. It is the price paid for a closed form solution. The difficulty that arises from using multiplicative disturbances, as is customary (e.g., Merton [18]), is discussed briefly below.

Equation (3a) is included because, at time $i\varepsilon$, the farmer uses his current prediction of his harvest in deciding on his level of futures sales. Equation (3b) allows the current futures price to be a biased estimator of the futures price in the next period. The current basis is assumed to be an unbiased estimator of the basis in the next period. This involves no loss of generality since the farmer is only interested in the basis at time T; $b_{i\varepsilon}$ can be reinterpreted as the forecast at $i\varepsilon$ of the basis at T. System (3) can be regarded as a reduced form system. If, for example, the farmer is located in a major production area, the change in his harvest forecast may be correlated with that of the aggregate of producers and, hence, with the change in the futures price. The definition of fl and u imply the additional equation

$$fl_{(i+1)\,\varepsilon} = \frac{(fl_{i\varepsilon} + u_{i\varepsilon}\,\varepsilon)}{\beta}\,. \quad (3d)$$

This completes the discrete time model.

The continuous trading model is obtained by taking the limit as $\varepsilon \to 0$. The result is (see Malliaris and Brock [15]).

$$\pi = p_0 f_0 + \int_0^T e^{-rt} p(t) u(t) dt + e^{-rT}[p(h - f) - bh]_{|T} \tag{2'}$$

$$dh = dz_1 \tag{3a'}$$

$$dp = a p dt + dz_2 \tag{3b'}$$

$$db = dz_3 \tag{3c'}$$

$$df = (rf + u) dt. \tag{3d'}$$

Here, $z = (z_1, z_2, z_3)'$ is the solution to a system of stochastic differential equations with 0 drift and infinitesimal variance Σ. Define $x = (f, h, p, b)'$, and rewrite (3) as

$$dx = (Ax + Bu) dt + \Gamma dz \tag{4}$$

where

$$A = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Choose units of h and p so that Σ can be written

$$\Sigma = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & \sigma \end{bmatrix}.$$

There are two points worth making about this formulation. The first regards the definition of $u_{i\varepsilon}\ \varepsilon$. It might, perhaps, seem more natural to use the alternative definition $u_{i\varepsilon} = f_{i\varepsilon} - f_{(i-1)\varepsilon}$ in which case $\int_0^T e^{-rt}\, p(t)\, [-rf(t) + u(t)]\, dt$ replaces the integral in (2') and $df = u\, dt$ replaces (3d'). A closed-form solution to the control problem posed below requires solving a matrix Ricatti equation. This can be easily obtained using the current formulation; the alternative definition of $u_{i\varepsilon}$ results in a more difficult Ricatti equation. This is explained below when the Ricatti equation is written down.

The second point concerns the interpretation of the integral $\int_0^T e^{-rt}\, p(t) \cdot u(t)\, dt$. For this discussion, let $r = 0$ to simplify the exposition. It is customary in the finance literature (e.g., Harrison and Pliska [10]) to take the limit of the expression after the first equality in (1) to obtain the integral $-\int_0^T f(t)\, dp(t)$. This is an Ito stochastic integral. Many of the results in finance are based on arbitrage arguments, and it has proven convenient to work with Ito integrals. With $r = 0$, the integral in (2) can be written $\int_0^T p(t) \cdot df(t)$. This is not an Ito integral; rather, it is a linear combination of Stratonivich [21] and Ito integrals. However, $\int_0^T p(t)\, u(t)\, dt$ is the standard form of the integral in stochastic control problems. The integrand is a function of the state and control (p and u, respectively) at a point in time. Note that the integral $\int_0^T f(t)\, dp(t)$ does not have this property, and it cannot be used with standard control methods.

The farmer's problem is to choose f as a measurable function of $h(t)$, $p(t)$, $b(t)$, and t to maximize the expected utility of profits. We assume that the optimal f is continuous with probability 1. This is a weak assumption since $h(t)$, $p(t)$, and $b(t)$ are all continuous with probability 1. Continuity of f guarantees its admissibility (Fleming and Rishel [9], p. 156). The problem of choosing $f(h, p, b, t)$ can be replaced by the equivalent problem of choosing an initial value $f[h(t), p(t), b, t]_{|t=0}$ and subsequent variations, i.e., $u[h(t), p(t), b(t), t]$ for $t \geq 0$. The assumption that f is continuous guarantees that the optimal u is bounded when the initial f is chosen optimally. The boundedness of u guarantees its admissibility for the control problem specified below; boundedness is also important because it insures that there is no ambiguity in taking the expansion used to derive the dynamic programming equation.

Define $J(x, t)$ as the maximum of the expected utility of profits for arbitrary $h(t)$, $p(t)$, $b(t)$, t given that $f(t)$ has been chosen optimally. Let the farmer have constant absolute aversion to risk with parameter $k > 0$. Then,

<u>Problem *</u>:

$$J(x, t) = \max_{[u(\tau)]_{\tau=t}^{T}} - E_t\{\exp[-k\, e^{rt}(\int_t^T e^{-r\tau}\, p(\tau)\, u(\tau)\, d\tau$$

$$+ e^{-rT}[p(h - f) - bh]_{|T})]\}$$

subject to (4); $x(t)$ given [$f(t)$ chosen optimally]. The farmer's problem is:

<u>Problem **</u>

$$\max_{f(t)} \exp[-k\, p(t)\, f(t)] \cdot J[f(t), h(t), p(t), b(t), t].$$

Problem ** requires finding the optimal initial condition for problem *. Problem * is a variation of the Linear Exponential Gaussian (LEG) control problem solved by Jacobson. It differs from his problem in two minor respects. First, the dynamic programming equation is linear in the control.

This results in a singular solution; the sufficiency condition requires consideration of both problem * and problem ** and the assumption that the optimal f is continuous. Second, the integral and final payoff in the exponent is multiplied by ke^{rt} rather than a constant as in the LEG problem. At each point, future profits are discounted back to the current time. It is unnecessary to do this where utility is time additively separable. In the current problem, however, failure to apply the discount would imply that the farmer's risk aversion parameter is time dependent. In that case, letting the season run from time t_1 to $T + t_1$ would lead to a different solution than when the season runs from time 0 to T.

Assume that J(x, t) is continuously differentiable in t and twice continuously differentiable in x. The assumption that u is bounded justifies the expansion that leads to the dynamic programming equation

$$-J_t = \max_u \left[-kpu\, J - r \ln(-J)\, J + J_x'(Ax + Bu) + \frac{1}{2} \operatorname{tr} J_{xx} \Gamma \Sigma \Gamma'\right]. \quad (5)$$

Details are provided in the appendix. The assumption that the optimal u is bounded implies

$$-kpJ + J_x' B = 0 \quad (6)$$

which defines the singular arc. Following Jacobson, try the ansatz $J(x, t) = -F(t) \exp[-x' S(t) x/2]$. Substitution into (5) gives[1]

$$\dot{F} \exp\left(\frac{-x' S x}{2}\right) + \frac{J x' \dot{S} x}{2} = \max_u \left\{-J\, kpu + J \left[-r \ln F + \frac{r x' S x}{2}\right] - J x' S[Ax + Bu] + J \operatorname{tr}\left[\frac{-S \Gamma \Sigma \Gamma' + x' S \Gamma \Sigma \Gamma' S x}{2}\right]\right\}.$$

Equation (6) becomes

$$kp + x' SB = 0. \tag{7}$$

Equate coefficients in the dynamic programming equation and use (7) to obtain the system

$$\begin{aligned} \dot{S} &= r\,S - SA - AS + S\,\Gamma\,\Sigma\,\Gamma'\,S \\ S(T) &= k \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \end{aligned} \tag{8a}$$

$$\begin{aligned} \dot{F} &= \left(r \ln F + \frac{\mathrm{tr}\, S\,\Gamma\,\Sigma\,\Gamma'}{2} \right) F \\ F(T) &= 1. \end{aligned} \tag{8b}$$

Define $\tilde{A} = rI/2 - A$, and rewrite (8a) as

$$\dot{S} = S\,\tilde{A} + \tilde{A}\,S + S\,\Gamma\,\Sigma\,\Gamma'\,S. \tag{8a'}$$

Define $Q(t) = S(t)^{-1}$ and use $dQ(t)/dt = -S^{-1}\,\dot{S}\,S^{-1}$ together with (8a') and the boundary condition to write

$$\begin{aligned} -\dot{Q} &= \tilde{A}\,Q + Q\,\tilde{A} + \Gamma\,\Sigma\,\Gamma' \\ Q(T) &= \frac{1}{k} \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{9}$$

The reason for the particular definition of u is now apparent. The alternative definition would lead to a term in (8a) which is independent of S; in that case the solution to S(t) is no longer simply $Q^{-1}(t)$.

The solution to (9) is a symmetric matrix whose upper triangular part, $Q^u(t)$, is

$$Q^u(t) = \begin{bmatrix} 0 & 0 & \frac{-e^{a\tau}}{k} & \frac{-1}{k} \\ & \frac{e^{-r\tau} - 1}{r} & \frac{\rho_{12}}{r - a}[e^{(a-r)\tau} - 1] & \left(\frac{\rho_{13}}{r} - \frac{1}{k}\right) e^{-r\tau} - \frac{\rho_{13}}{r} \\ & & \frac{e^{-\phi\tau} - 1}{\phi} & \frac{\rho_{23}}{r - a}[e^{(a-r)\tau} - 1] \\ & & & \frac{\sigma(e^{-r\tau} - 1)}{r} \end{bmatrix} \quad (10)$$

which uses the definitions $\tau = t - T$ and $\phi = r - 2a$. Although Q can be easily inverted to obtain S, the result is rather complex and not very illuminating. Therefore, attention is focused on what is hereafter referred to as the "simple case," obtained by setting $\sigma = r = \rho_{ij} = 0$, i, j = 1, 2, 3. The introduction of basis risk ($\sigma > 0$) and discounting ($r > 0$) can be studied by looking at perturbations around the simple case.

Recall that equations (6) and (7) result from the boundedness of the optimal u which is implied by the assumption that the optimal level of futures sales is continuous. To verify that continuity holds, consider problem **. The first-order condition to that problem, using $J(\ ) = -F \exp(-x' S x/2)$, duplicates (7). That is, given that (7) holds over (t, T), (7) must also hold at t. Since p, h, and b are continuous with probability 1, so too is f. The second-order condition for problem ** evaluated on (7) is $F(t)\, S_{11}(t) < 0$, where subscripts indicate the element of S(t). In the simple case ($r = \sigma = \rho_{ij} = 0$), the upper triangular part of the symmetric matrix S(t) is $S^u(t)$;

$$k^2\ D(t)\ S^u(t) = \begin{bmatrix} -\theta & \theta & \frac{-e^{a\tau}}{k} & -\tau\ \theta\ k \\ & -\theta & \frac{e^{a\tau}}{k} & \frac{-e^{2a\tau}}{k} \\ & & \tau & -\tau\ e^{a\tau} \\ & & & \tau\ e^{2a\tau} \end{bmatrix} \tag{11}$$

where

$$D(t) \equiv \left(\tau\ \theta + \frac{e^{2a\tau}}{k^2} \right) / \ k^2$$

$$\theta(t) \equiv \frac{(1 - e^{2a\tau})}{2a} > 0.$$

$D(t)$ is the determinant of Q in the simple case. Since $D(T) > 0$, a necessary and sufficient condition for the existence of $S(t)$ over an interval $[t_1, T]$ is that $D(t) > 0$ over that interval. For finite τ and a, this condition can be insured by choosing k sufficiently small. In this case $S_{11}(t) < 0$ for $t < T$. Solving (8b) gives $F(t) = \exp[\ -1/2 \int_t^T (S_{22} + S_{33})\ d\tau] > 0$. Conclude that, for finite T and a and sufficiently small k, $F(t)\ S_{11}(t) < 0$ so that (7) does indeed solve the maximization problem; hereafter, it is assumed that k is such that $D(t) > 0$. For this reason, the analysis concentrates on the case where k is small. Note that a/k can be of any sign and magnitude and k chosen so that the second-order condition holds.

3. ANALYSIS OF THE OPTIMAL HEDGE

With the control rule (7) and the inverse of Q(t), the unique elements of which are given by (10), it is straightforward to study the dependence of the optimal hedge on the parameters of the problem. In the previous section, it was established for the simple case that, provided the farmer is not excessively risk averse, the optimal hedge is given by (7) for all values of a and T. Given the continuity of all elements of Q and, hence, S in all arguments, the second-order condition $F(t)\ S_{11}(t) < 0$ also holds for small values of σ, r, and ρ_{ij}. Equation (7) and inspection of (11) lead to the observation:

Remark 1. In the absence of basis risk, discounting, and correlation of the random elements, an increase in the basis (futures - cash price) at harvest increases the optimal hedge ($S_{11} < 0 < S_{14}$).

This also holds for small values of σ (the measure of basis uncertainty) or as the cash price and futures price become perfectly correlated. To determine the effect of a slight increase in basis risk, differentiate (7) with respect to σ and rearrange:

$$\frac{\partial f}{\partial \sigma} = -\left(\frac{\partial S_{11}}{\partial \sigma} f + \frac{\partial S_{12}}{\partial \sigma} h + \frac{\partial S_{13}}{\partial \sigma} p + \frac{\partial S_{14}}{\partial \sigma} b\right) / S_{11}.$$

The first row of $\partial S/\partial \sigma$, obtained using $\partial S/\partial \sigma = -S\ \partial Q/\partial \sigma\ S$, is

$$\frac{\tau^3\ \theta}{k^3\ D^2}\left(\theta k,\ \frac{e^{2a\tau}}{\tau k},\ e^{a\tau},\ -e^{2a\tau}\right)$$

where the partial derivative is evaluated at $\sigma = r = \rho_{ij} = 0$. The sign of $\partial f/\partial \sigma$ is ambiguous. The interesting case is when k is small since, from the previous section, this assures that the second-order condition holds. Using the control rule to solve for f and substituting into the expression for $\partial f/\partial \sigma$ gives

$$\frac{\partial f}{\partial \sigma} \sim \frac{\tau^2 e^{2a\tau}}{k^2 D} h > 0$$

as $k \to 0$. Hence:

Remark 2. If the farmer's absolute aversion to risk, k, is small, an increase in basis uncertainty leads to an increase in the optimal level of futures sales. This holds in some neighborhood of $\sigma = r = \rho_{ij} = 0$.

The remaining analysis assumes that $\sigma = 0$. The presence of a nonstochastic basis adds no information except that contained in Remark 1, so hereafter set $b = 0$. In places it is convenient to allow the covariance of harvest forecast and futures price, ρ_{12}, to be nonzero. To avoid notational clutter, define $\rho_{12} = \rho$. Since $\sigma = 0$, set $\rho_{13} = \rho_{23} = 0$. The (1,1) and (1,2) cofactors of Q differ only in sign for this case ($b = r = \sigma = \rho_{13} = \rho_{23} = 0$) so the control rule can be written more concisely as

$$S_{12}\, y + (k + S_{13})\, p = 0 \tag{7'}$$

where y is defined as $h - f$, the unhedged portion of expected harvest. This can be seen more directly by noticing that, in the absence of basis risk and discounting, the original problem can be reformulated in the states y, p rather than f, h, p.

The easiest case to analyze is $a = 0$, where the current futures price is an unbiased estimator of cash price at harvest. Equation 7' gives

$$y(t) = w(t)\, p(t), \; w(t) = (1 - \rho^2)\,(T - t)\, k - \rho.$$

The unhedged portion of expected harvest tends to decrease over time; but for $\rho \neq 0$, $E_t\, y(T) \neq 0$. If the harvest forecast and futures price errors are inversely related, the final position in futures is less than harvest; otherwise it is greater. If $\rho > 0$, the sign of y may change over time: The farmer may begin the season with a position in futures less than expected production, increase futures sales over time, and finish with the hedge greater than harvest. The previous equation also leads to:

Remark 3. When the futures price is an unbiased estimator of cash price at harvest and in the absence of basis risk and discounting, the optimal hedge is myopic.

To see this, use the solution to the static (1 period) problem (Bray [5])

$$y^m = \beta \, \bar{p} - \alpha \, p(0)$$

where

$$\alpha = \frac{[(1 + k\rho)^2 - k^2]}{k},$$

$$\beta = \frac{(1 + k\rho)}{k};$$

$\bar{p}$ [$= e^a \, p(0)$] is the expectation at planting of cash price at harvest, $p(0)$ is futures price at planting, and y^m is the unhedged expected production in the myopic problem. Bray's solution has been normalized by setting the variance of the harvest and price forecast equal to one. To make the units of measurement in the dynamic and static problems the same, set $T = 1$ in the dynamic problem. When $a = 0$, $\bar{p} = p(0)$; since $w(0) = \beta - \alpha$, conclude $y^m = y(0)$. The initial dynamic hedge and the myopic hedge are equal in this case; hence, Remark 3.

For $a \neq 0$, the initial dynamic hedge and the myopic hedge differ. This can be seen by comparing y^m and the general expression for $y(0)$ or from the special case $\rho = 0$ discussed below. This result is very intuitive. If price is a random walk, then, because there is no adjustment cost associated with changing the hedge and because the degree of risk aversion is independent of wealth, the farmer does not benefit from the recognition that he will be able to change his futures position at a later date. However, if the current futures price provides a biased estimate of cash price at harvest, it matters at what point the sale is made, and the dynamic problem is not vacuous. The question whether the current futures price provides a biased estimator of future cash price (abstracting from the time imperfection of the hedge) has not been resolved either empirically or theoretically. For example, Anderson and Danthine [1] point out the possibility of bias even in a rational expectations equilibrium.

It is instructive to compare the solutions to Bray's static problem and the problem in which the objective function is linear in the mean and variance of profits (hereafter, the MV problem). The MV problem emerges if π, rather than its arguments, is normally distributed and the farmer maximizes the expected value of the CARA utility function. Hence, the two problems are equivalent except for the different assumptions regarding the distributions of price and harvest; both versions permit the possibility of negative profits. For ease of comparison, suppose that the current futures price is an unbiased estimator of cash price at harvest.

Let each problem be resolved periodically as the season progresses. Using the previous results, the unhedged portion with Bray's problem is $y(t) = w(t)\, p(t)$. The unhedged portion with the MV problem is $y^{MV}(t) = -\text{cov}_t[p(T)\, h(T), p(T)]/\text{var}_t[p(T)]$ (Anderson and Danthine [1] eq. 11) where the subscript t indicates that the variance and covariance are conditioned on the information at t.

In general, $\text{cov}_t[p(T)\,h(T),\,p(T)] \neq 0$ even if $\text{cov}_t[p(T),\,h(T)] = 0$; the two covariances may have opposite signs. This means that the two problems may prescribe qualitatively different behavior at some or all points in time: One may recommend that the farmer hedge more than expected production and the other, that he hedge less. This lack of robustness is not particularly surprising but is worth keeping in mind when evaluating the models.

Now consider the case where $a \gtreqless 0$. Define $y^* \equiv E_0 \left[\lim_{t\to T} y(t)\right]$, the expectation at $t = 0$ of unhedged production at harvest, given that the dynamic rule is followed. Use $E_0\, p(T) \equiv \overline{p} = e^a\, p(0)$ and L'Hospital's rule with (7') to obtain $y^* = e^a(a/k - \rho)\, p(0)$. An expected upward drift in price or a negative correlation between price and harvest forecast errors ($a > 0 > \rho$) tends to discourage the farmer from finishing the season in a short position. When $\rho = 0$, $a \neq 0$, unhedged expected production at $t = 0$ is greater in the myopic problem than in the dynamic problem $[y^m > y(0)]$; y^m may be greater or less than y^*. For $\rho = 0$, the three quantities are:

$$y^m = \left(\frac{e^a - 1}{k} + k\right) p(0)$$

$$y(0) = \left[\frac{2a(e^{-a} - e^{-2a})}{k(1 - e^{-2a})} + k\right] p(0)$$

$$y^* = \left(\frac{a\,e^a}{k}\right) p(0)\ .$$

These expressions imply

$$y^m - y(0) = \frac{e^a - 1}{1 - e^{-2a}}\left[1 - (2a + 1)\,e^{-2a}\right]\frac{p(0)}{k}, \qquad (12)$$

$$y^* - y^m = \left[1 - (1 - a)\, e^a\right] \frac{p(0)}{k} - k\, p(0) \; . \qquad (13)$$

Equation (12) implies $y^m \geq y(0)$, with equality holding only at $a = 0$. Equation (13) is ambiguous; however, $1 - (1 - a)\, e^a \geq 0$, with equality holding only at $a = 0$. This implies that for any $a = 0$, there exists $k^* > 0$ such that $y^* - y^m > 0$ for all $k < k^*$; also, for any $k > 0$, there exists $a^* > 0$ such that $y^* - y^m < 0$ for all $|a| < a^*$. This follows from the fact that all equations are continuous in their parameters. From the previous analysis of the case $a = 0$, it follows that for $|a|$ sufficiently small (and thus, for $|a|/k$ sufficiently small) $y(0) > y^*$. Define f^m, $f(0)$, f^* as (expected) futures sales, in correspondence with y^m, $y(0)$, y^*. The conclusion can be summarized as

Remark 4. (i) For large $|a|/k$, the dynamic futures position tends to decrease;[2] and the myopic futures position lies between the initial and expected final dynamic position $[f(0) \geq f^m > f^*]$. (ii) For small $|a|/k$, the dynamic futures sales tends to increase; and the myopic position lies below the initial dynamic hedge $[f^* > f(0) \geq f^m]$.[3] Strict inequality holds except at $a = 0$.

A tendency for the hedge to rise or fall during the growing season is consistent with either normal backwardation $(a > 0)$ or contango $(a < 0)$. The direction of the tendency is determined by the size of $|a|/k$. The explanation for this result lies in the fact that the farmer considers both the expected gains (or losses) from his activity in the futures market over $(0, T)$ and from closing his position at T. Consider the certainty equivalent paths under two different values of a, $a_1 > 0$ and $a_2 < 0$; suppose $|a_i|/k$, $i = 1, 2$ is large, so that f_i is expected to fall over the season. [The notation f_i means $f(a_i)$.] Since $y_1^* > 0$, the farmer expects to end the season with futures sales less than his harvest. He begins the season with a small (relative to the case $a < 0$) level of sales and then proceeds to buy back contracts as price increases. He expects to make a profit on the sale of these at T. The initial sale with the expectation of subsequent purchases may appear perverse, but it is simply a hedge against an unexpected drop in price. The second case, with $a = a_2 < 0$, is more obvious. Since $y_2^* < 0$ and $f_2(0) > f_2^*$, conclude that $f_2(0) > h(0)$: The farmer begins by hedging more than expected harvest. As price falls, he makes profits buying contracts. The situation where $|a|/k$ is small has a similar interpretation. There, the case $a > 0$ has the more obvious interpretation.

The ratio $|a|/k$ is a measure of the opportunity for speculative profits relative to the degree of risk aversion. This measure depends on the magnitude rather than just the sign of a/k since the potential for speculative profits exists for $a \gtrless 0$. The myopic hedge can be regarded as an approximation to the optimal hedge. The above analysis shows that (for small ρ) the approximation is biased downward. The extent of the bias is positively related to $|a|/k$.[4]

The next question concerns the effect of discounting on the optimal hedge. Use $\partial S/\partial r = -S(\partial Q/\partial r)\, S$ to obtain

$$\partial \begin{pmatrix} S_{11} \\ S_{12} \\ S13 \end{pmatrix} / \partial r = \frac{1}{k^6\, D^2} \begin{bmatrix} \dfrac{3(\tau\, \theta\, k)^2}{2} - \eta\, e^{2a\tau} \\ e^{2a\tau}(\eta + \tau\, \theta) - \dfrac{(\tau\, \theta\, k)^2}{2} \\ e^{a\tau}\, \tau\, k \left(\dfrac{3\, \theta\, \tau}{2} + \eta\right) \end{bmatrix}$$

where

$$\eta \equiv \frac{e^{2a\tau}(2a\tau - 1) + 1}{4\ a^2} > 0.$$

The partials are evaluated at $\sigma = r = \rho_{ij} = 0$. Differentiation of the control rule gives

$$\frac{\partial f}{\partial r} = - \left(\frac{\partial S_{11}}{\partial r} f + \frac{\partial S_{12}}{\partial r} h + \frac{\partial S_{13}}{\partial r} p \right) / S_{11}.$$

Recall b = 0 by assumption. Using the control rule to eliminate f results in a complicated expression. As argued above, the sign of the partial is chiefly of interest for small k. Eliminating f gives

$$\frac{\partial f}{\partial r} \sim \frac{-1}{k^6\ D^2\ S_{11}} \left[\tau\ \theta\ e^{2a\tau}\ h + \frac{\eta\ e^{3a\tau}}{k\ \theta} (1 - e^{a\tau})\ p \right]$$

as $k \to 0$. The term outside the square brackets is positive. For $a \leq 0$, both terms in the brackets are negative so $\partial f/\partial r < 0$. For $a > 0$, the second term is positive and dominates the first term as $k \to 0$; in that case $\partial f/\partial r > 0$. This is summarized in:

Remark 5. If the futures price is an unbiased estimator of cash price or in the presence of contango ($a \leq 0$), an increase in the interest rate reduces the optimal hedge. Under normal backwardation, an increase in the interest rate increases the optimal hedge. This holds in a neighborhood of $r = \sigma = \rho_{ij} = 0$.

When the interest rate is 0, the optimal hedge can be written $f = h + c_1(t)\ p$. Two farmers with the same absolute aversion to risk and facing the same price, but with different expected harvests, would have the same amount of unhedged expected production. For $r \neq 0$, $\sigma = \rho_{13} = \rho_{12} = 0$, the control rule can be written

$$f(t) = e^{r\tau}\ h(t) + C_2(t)\ p(t),\ \tau = t - T. \tag{14}$$

[Compare the (1,1) and (1,2) cofactors of Q.] This does not hold for $\sigma \neq 0$. If two farmers with the same risk aversion have expected harvests of h_1 and h_2, then, letting $y_i = h_i - f_i$, i = 1, 2, (14) implies

$$y_1 - y_2 = (h_1 - h_2)\ (1 - e^{r\tau}).$$

Remark 6. When there is no basis risk, given two individuals with the same risk aversion and different expected harvest, the farmer with the larger expected harvest will have a greater gap between expected harvest and futures sales. This difference increases with r and approaches 0 as the season ends ($\tau \to 0$).

The ability, in the absence of basis risk, to write the control rule as in (14) is reminiscent of the limited foresight result in portfolio theory mentioned in the introduction. To make the analogy clearer, let $a = \rho = 0$ so that the futures price is an unbiased estimator of cash price at harvest, and there is no correlation between expected harvest and price. Then, $C_2(t)$ in (14) becomes $C_2(t) = -e^{r\tau}\ k(1 - e^{r\tau})/r$. Compare this to Bray's static problem, modified to include discounting; in that problem k should be replaced by $\hat{k} = e^{r\tau}\ k$ when the length of the season is $-\tau$. This suggests:

Remark 7. If the futures price is an unbiased estimator of cash price at harvest, there is no basis risk, and all correlations vanish, the dynamic strategy requires limited foresight. This means that the static rule can be used except that the current estimate of harvest should be replaced by the current estimate discounted to harvesttime, the risk aversion parameter should be replaced by $\hat{k}$, and the current futures price should be replaced by the discounted stream of expected futures price,

$$\frac{1 - e^{r\tau}}{r} p(t) = e^{rt} \int_t^T e^{-rs} ds\, p(t).$$

There are several other minor points to be made about the model. The farmer can be viewed as choosing h(0), expected harvest at planting, and possibly σ_1 which implies the variance of harvest. Let the cost function be $g[h(0), \sigma_1]$. In the static case with no production uncertainty, it is well known that output is chosen so that $\partial g/\partial h(0) = p(0)$. This also holds if harvest forecast errors are additive (Danthine [7], p. 83). In the simple case ($\sigma = \rho_{ij} = r = 0$) with the dynamic hedging model, output is chosen so that $\partial g/\partial h = p(0) - b(0)$ which collapses to the previous rule when the (deterministic) basis is 0. The farmer sets marginal cost equal to the current cash price; he ignores the expected change in the cash price, $p(0)\ (1 - e^{aT})$, but takes into consideration the certain difference between futures and cash price. When $a = \sigma = \rho_{ij} = 0$, $r \neq 0$, the first-order condition for h(0) is $g'(h) = [p(0) - b(0)] \cdot e^{-rT}$ which has an obvious interpretation. Such transparent rules do not emerge for the more general case where a, σ, and r are nonzero.

The CARA utility function implies that the degree of absolute risk aversion is independent of wealth, so it is not surprising that unhedged expected production, y, is independent of expected production in the absence of basis risk and discounting. This scale independence is not very plausible. However, if h(0) and σ_1 are jointly determined by the producer through his choice of inputs (i.e., g is not separable), then y(t) depends indirectly on h(t) through σ_1. An alternative choice of utility function, such as the isoelastic, would also eliminate the scale independence. Efforts to characterize the dynamic hedging problem with isoelastic (constant relative risk aversion) utility have not been successful.

4. CONCLUSION

A dynamic hedging problem with production uncertainty was solved and analyzed; the results were compared to the solution of the static analog. Several insights were obtained.

It was shown that the myopic hedge provides a downwardly biased approximation of the initial optimal dynamic hedge. The magnitude of this bias varies directly with the extent to which the current futures price is a biased estimator of cash price at harvest; it varies inversely with the degree of absolute risk aversion.

A second insight concerned the expected direction of change of the individual's position in futures. Normal backwardation, for example, is consistent with either an expected increase or decrease in futures sales over the production period. A systematic decrease in the level of futures sales suggests a high degree of bias relative to risk aversion; it does not suggest whether the current futures price is an upwardly or downwardly biased estimator of cash price at harvest.

Intuition might suggest that the myopic hedge would lie between the initial and expected final hedges in the dynamic problem. This intuition is correct only if there is a large degree of bias relative to risk aversion or, paradoxically, if the bias is 0. In the second case, the myopic hedge equals the initial dynamic hedge.

The myopic problem was also compared to a related problem in which the objective function is linear in mean and variance. The two models may suggest qualitatively different behavior. This lack of robustness is an argument for caution in interpreting the results of either model.

The sensitivity of the optimal hedge to the introduction of basis risk and discounting was studied. For low levels of risk aversion, introduction of basis risk leads to an increase in futures sales. Introduction of discounting leads to a decrease in the level of sales if the futures price is expected to fall. Under normal backwardation, discounting causes an increase in sales.

Finally, an analogy to the limited foresight result of portfolio theory was obtained.

APPENDIX: DERIVATION OF (5)

Write problem * as

$$J(x, t) = \max_u -E_t\{\exp[-k\, e^{rt} \int_t^{t+dt} e^{-r\tau}\, p(\tau)\, u(\tau)\, d\tau] \cdot$$

$$[\exp(-k\, e^{r(t+dt)} \{\int_{t+dt}^{T} e^{-r\tau}\, p(\tau)\, u(\tau)\, d\,\tau + e^{-rT}[p(h - f)$$

$$- b\, h]_{|T}\})]^{e^{-rdt}}\Big\}.$$

Rewrite this, using the principle of optimality, as

$$J(x, t) = \max_u -E_t\{\exp[-k\, e^{rt} \int_t^{t+dt} e^{-r\tau}\, p(\tau)\, u(\tau)\, d\tau] \cdot$$

$$[-J(x + dx, t + dt)]^{e^{-rdt}}\Big\}.$$

Expand this and take expectations using

$$E_t\, dx = (Ax + Bu)\, dt$$

$$E_t\, dx\, dx' = \Sigma\, dt$$

$$E_t(dx_i\, dx_j\, dx_n) = o(dt) \qquad i, j, n = 1, 2, 3, 4.$$

Simplifications result in (5). Karlin and Taylor [14], page 202, give examples of similar manipulations.

FOOTNOTES

[1]At this point, it is apparent why replacing the assumption of additive noise with multiplicative noise (i.e., replacing Brownian motion with geometric Brownian motion) leads to difficulties. No ansatz of the form of an exponential of a polynomial of degree n will be successful since the term $E_t(dx)' J_{xx}(dx)$ will involve elements to the order of x_i^{n+2}.

[2]This does not suggest monotonicity, only that $f(0) > f^*$.

[3]Simulation was used to examine the effect of ρ on the path of the expected hedge. The ratio a/k varied over 16 values and ρ took five values. For each value of ρ, there were numbers $r_1(\rho) < 0 < r_2(\rho)$ such that expected futures sales rose if $r_1 < a/k < r_2$ and fell otherwise. The simulation indicates that $dr_i/d\rho > 0$, $i = 1, 2$.

[4]Define

$$c(a) = \frac{e^a - 1}{1 - e^{-2a}} [1 - (2a + 1) e^{-2a}]$$

so

$$y^m - y(0) = \frac{c(a)\, p(0)}{k}.$$

Since $c > 0$ for $a \neq 0$, decreasing k increases $y^m - y(0)$. Since c reaches its minimum at $a = 0$ and dc/da is continuous, conclude that sgn dc/da = sgn a for small a. It is easy to establish that $dc/da > 0$ for all $a > 0$ and that $dc/da < 0$ for sufficiently small a($a < 0$, $|a|$ large), but monotonicity of c in a over all $a < 0$ has not been established.

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