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OPEN-LOOP AND FEEDBACK MODELS OF
DYNAMIC OLIGOPOLY

by

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Until recently, most theoretical and empirical dynamic oligopoly models
have been open loop, although such strategies are not subgame perfect. Sub-
game perfect feedback models are more difficult to use. By comparing the
paths and the steady-state equilibria of families of open-loop to subgame
perfect feedback dynamic oligopoly models, we show where open-loop models
are the same as feedback models and where they deviate substantially.

This paper makes seven contributions to the literature. First, a new
family of dynamic oligopoly models is presented that is related to a well-
known family of static models. Second, for certain members of these families,
the open-loop and feedback paths and steady states are shown to be identical
(or very close).[1] Third, the feedback model is shown to imply more com-
petitive behavior than does the open-loop model. Fourth, open-loop models
are shown to converge more rapidly. Fifth, the difference between discrete
time and continuous time models is demonstrated for the family of oligopoly
models. Sixth, the discrete time feedback model is used to show the depen-
dence of "consistent conjectures" on the length of the period of commitment.
Seventh, the effects of an increase in the number of firms or adjustment costs
is determined.

Fershtman and Kamien (1987) and Reynolds (1987) compare the steady states
for a Nash-Cournot model for a fixed number of firms.[2] Our results differ
since we study a family of oligopoly models, analyze both the adjustment paths
and the steady states, and examine the effects of an increase in the number of
firms.

Dynamic models must be used where there are substantial adjustment costs in prices, training, or in capital accumulation, or where there is learning over time. We use a variation of the well-known solution to the open-loop and feedback linear-quadratic model (Starr and Ho, 1969) to capture these adjustment lags. As we show in a related empirical paper that estimates both the open-loop and feedback models developed here (Karp and Perloff, 1988), these adjustment factors are important in real world oligopolistic markets. In the presence of adjustment lags, firms have more strategic opportunities than in static models.

The family of equilibria that we examine includes the collusive, price-taking, and Nash-Cournot models as special cases. The justification for including collusive and price-taking behavior are obvious. The Nash-Cournot model also may be reasonably motivated. If there are discrete time periods (such as growing seasons) during which firms cannot vary their output levels, it is reasonable for a firm to make the Cournot assumption that its competitors cannot respond to changes in its output level within a time period. Nonetheless, firms can respond over time.

We consider two possible ways in which firms may react to each other in the long run. First, in an open-loop model, firms choose an initial set of strategies (output levels) and stick to that path. Firms do not expect to revise their strategies after an unexpected shock (such as bad weather) affects the output levels of various firms. This failure to anticipate revision is generally irrational. Second, in a feedback model, firms react to each other over time; this model is subgame perfect.

The open-loop and feedback models are identical where firms collude or act as price takers. In other oligopolistic models, such as where firms make the

Nash-Cournot assumption within a period, the two models imply different adjustment paths and steady-state output levels, and the open-loop model is not subgame perfect.

In addition to the Nash-Cournot open-loop and feedback models, there are many other dynamic oligopolistic games that produce output paths that lie between those of collusion and price taking. Rather than trying to explicitly model each of these games, we generalize our model to allow for intermediate paths and steady-state output.

We use an index of behavioral assumptions by firms within a single time period to approximate these other games. This index is analogous to a conjectural variation in a static game. In dynamic feedback models, however, such an interpretation is inappropriate. While this index is not the explicit outcome of a game, it allows us to easily approximate a range of games. In particular, the collusive, price-taking, and Nash-Cournot models are obtained as special cases of this more general model. We treat this index as a single parameter but, more generally, it might be a function of exogenous variables.[3]

The first section describes the model. The second section demonstrates a method of nesting price-taking, collusion, Nash-Cournot, and other oligopoly models. The third section examines the steady states and trajectories. The following section uses the feedback model to examine static consistent conjecture models. Fifth, the effects of increasing the number of firms or the adjustment costs are analyzed. Finally, a summary and conclusions are presented.

1. <u>Definitions and the Model</u>

We start with a discrete time model in which the length of a period is ϵ. As $\epsilon \to 0$, the continuous time model is obtained. Most of the analysis is based on the continuous time model.

The industry consists of $n + 1$ firms where $n \geq 1$. At time t, firm i decides how much to produce in the current period $q_{i,t}$ or, equivalently, its change in output, $u_{i,t} \varepsilon = q_{i,t} - q_{i,t-\varepsilon}$. Since ε is in units of time, $u_{i,t}$ is a rate. Firm i incurs a quadratic cost of adjustment,

$$\left(\delta_{0,i} + \delta \frac{u_{i,t}}{2} \right) u_{i,t} \varepsilon,$$

and a quadratic cost of production,

$$\left(\theta_{0,i} + \theta \frac{q_{i,t}}{2} \right) q_{i,t} \varepsilon.$$

In period t, the demand curve facing firm i is

$$p_{i,t} = a_i - b \sum_j q_{j,t}. \tag{1}$$

Firm i's revenue in period t is $p_{i,t} q_{i,t} \varepsilon$. Given an instantaneous interest rate of r, the one-period discount rate is $e^{-r\varepsilon}$, and the objective of firm i is to maximize its discounted stream of profits,

$$\sum_{t=1}^{\infty} e^{-r(t-1)\varepsilon} \left[\left(p_{i,t} - \theta_{0,i} - \frac{\theta}{2} q_{i,t} \right) q_{i,t} - \left(\delta_{0,i} + \delta \frac{u_{i,t}}{2} \right) u_{i,t} \right] \varepsilon. \tag{2}$$

For simplicity, we set $a_i = a$ and assume $\theta_{0,i} = 0 = \delta_{0,i}$. The last equality implies that adjustment costs are minimized when there is no adjustment. As a result, the steady-state levels of output in the open-loop, collusive, noncooperative Nash-Cournot, and price-taking equilibria are equal to their static analogs. This equality holds for general cost and revenue functions and not simply the quadratic ones assumed here.[4]

The ith firm's objective (2) is written in matrix form as

$$\sum_{t=1}^{\infty} e^{-r(t-1)\epsilon} \left[a\ e_i'(q_{t-\epsilon} + u_t\ \epsilon) \right.$$

$$\left. - \frac{1}{2}(q_{t-\epsilon} + u_t\ \epsilon)'\ K_i(q_{t-\epsilon} + u_t\ \epsilon) - \frac{1}{2} u_t'\ S_i\ u_t \right]\epsilon, \qquad (2a)$$

where e_i is the ith unit vector and e is a column vector of 1's, $K_i = b(e\ e_i' + e_i\ e') + \theta\ e_i\ e_i'$ (so K_i is a matrix with b's on the ith row and column except for the (i, i) element which is $2b + \theta$; all other elements are 0), and $S_i = e_i\ e_i'\ \delta$. As $\epsilon \to 0$, this expression approaches

$$\int_0^{\infty} e^{-rt} \left[a\ e_i'\ q_t - \frac{1}{2} q_t'\ K_i\ q_t - \frac{1}{2} u_t'\ S_i\ u_t \right] dt. \qquad (2b)$$

We assume that $q_{i,t}$ is unconstrained so that negative sales are possible. Negative prices can be interpreted as very low prices. When prices fall below a certain level, firms would prefer to be buyers rather than sellers; they must bear the adjustment cost to make the transition.

Alternatively, the model can be interpreted as a standard investment problem in which $q_{i,t}$ is firm i's capacity, and sales lie in the interval $[0, q_{i,t}]$. This interpretation requires additional assumptions. Provided that initial capacity lies within a certain range (an n + 1 dimensional set) that depends on the market structure, firms will produce at capacity (for an example of the Nash-Cournot market, see Reynolds, 1987). Given an initial condition in this range, the open-loop and feedback solutions are as shown below. The reader can either adopt the literal interpretation or regard the model as the standard investment problem in which the initial conditions are such that the capacity constraint is always binding.

2. Two Families of Equilibria

We consider two families of equilibria: open loop and feedback. Members of each family are indexed by a parameter v, which describes the behavioral assumption that determines the outcome. This parameter is defined by $v = \partial\mu_{j,t}/\partial\mu_{i,t}$ for $i \neq j$ and all t. In a static model, v is a constant conjectural variation. Since the open-loop game is equivalent to a static problem, the same interpretation can be adopted, but that interpretation is inapplicable in the feedback game. We adopt the neutral description of v as a player's behavioral assumption. This assumption is taken as primitive and not explained by strategic considerations.

This procedure is justified on pragmatic grounds: The model is useful in estimation (see Karp and Perloff, 1988). The leading cases where $v = -1/n$, 0, or 1 result in the price-taking, Nash-Cournot, and collusive (for identical firms) equilibria, respectively. The behavioral parameter v provides a measure of the closeness of the observed market to a particular ideal market. If $v = -1/n$, each firm acts as if it believes its rivals will exactly offset its own deviation from equilibrium. Since the good is homogeneous, the firm acts as a price taker. If $v = 1$ and firms are identical, each firm acts as if its rivals will punish it for deviating from the equilibrium by making equal changes in their own output. This assumption is equivalent to a market-sharing agreement and leads to the collusive outcome.

In the open-loop equilibrium, each player chooses a sequence of changes in output, $u_{i,t}$, using a particular behavioral assumption, v. The equilibrium levels can be expressed in feedback form; in this case, strategies are open loop with revision. Revisions are unanticipated. When players choose their

current levels, they act as if they were also making unconditional choices regarding future levels.

In the feedback equilibrium, players recognize that their future choices will be conditioned on the future state; players select control rules rather than levels. The feedback equilibrium is obtained by the simultaneous solution of the n + 1 dynamic programming equations[5]

$$J_i(q_{t-\epsilon}) = \max_{u_{i,t}} \left[\left(a\, e_i'\, q_t - \frac{1}{2} q_t'\, K_i\, q_t - \frac{1}{2} u_t'\, S_i\, u_t \right) \epsilon + e^{-r\epsilon}\, J_i(q_t) \right] \quad (3)$$

where $q_t = q_{t-\epsilon} + \mu_t \epsilon$. The particular behavioral assumption, v, determines the control rule for player i and his value function, $J_i()$.

When $v = 0$, the result is the feedback Nash-Cournot game. Fershtman and Kamien (1987) and Reynolds (1987) show that the open-loop and feedback equilibria differ in this case. When $v = -1/n$ or $v = 1$, the open-loop and feedback equilibria are identical since, if players either take price as given or share the market in each period, it does not matter whether they choose levels or control rules.

3. Steady States and Paths

The principal differences between the open-loop and feedback equilibria are summarized as:

Remark 1. For $v \in (-1/n, 1)$ and for given symmetric initial output level q_0, output at t is greater under the feedback equilibrium than under the open-loop equilibrium; convergence to steady state is faster in the latter case. For $v = -1/n$ or 1, the trajectories and

control rules are identical under feedback and open loop. Under both open-loop and feedback policies, output decreases in v.

This remark is based on a combination of analytic and simulation results described below. The implication is that, for $v \in (-1/n, 1)$, industry profits are higher (and social surplus lower) under the open-loop equilibria. That is, feedback strategies are relatively procompetitive.

Feedback policies require knowledge of the current state (output of all firms in the previous period), so a possible policy conclusion is that this information should be made available. However, this conclusion ignores the likelihood that the degree of collusion, measured by v, may increase as information is shared. Riordan (1985) models a dynamic oligopoly with stochastic demand where firms are unable to observe their rivals' output. He concludes that aggregate output is greater in this case than in the case where firms are able to observe their rivals' output. Riordan's model is quite different from the current one; nevertheless, the conflicting conclusions illustrate the difficulty of a general comparison of social welfare when firms do or do not know their rivals' output. (*on part 2*)

Under the Nash-Cournot assumption (v = 0), the open-loop equilibrium can be obtained as the solution to a control problem (Hansen, Epple, and Roberds, 1985). Not suprisingly, a control approach can be used for arbitrary v. We use this fact to simplify simulation of the open-loop game and to illustrate the relationship between the feedback game and the control problem. To this end, consider the standard control problem:

$$J(q) = \max_{u} \int_0^{\infty} e^{-rt} \left[ae' \ q - \frac{1}{2} q' \ Kq - \frac{\delta}{2} u' \ u \right] dt \qquad (4)$$

subject to

$$\dot{q} = u, \qquad q_0 \text{ given,}$$

where $K = k_1 ee' + (k_0 - k_1) I$; that is, K is an $(n + 1) \times (n + 1)$ matrix with the parameter k_0 on the principal diagonal and the parameter k_1 elsewhere. In addition, K is positive-semidefinite and $\delta > 0$.

Define Q_t^i as aggregate output at t, $i = c, 0, f$; the superscripts indicate the paths given by the solutions to the control problem and open-loop and feedback games, respectively. Assume that the initial output is the same for all firms, $q_0 = (e \, Q_0)/(n + 1)$, for each of the three cases. Then Q_t is the solution to

$$\dot{Q}_t = \gamma^i + \rho^i \, Q^i/\delta. \tag{5}$$

The proofs of the following three propositions are based on comparison of the systems of equations that define γ^i and ρ^i. The details are contained in Appendix I.

Proposition 1. A sufficient condition for the open-loop and feedback equilibria to be identical is $v = 1$ (cartel with symmetric firms) or $v = -1/n$ (price takers).

Based on simulation results reported below, this condition appears to be necessary.

Proposition 2. Under the Nash-Cournot assumption, output is smaller in the open-loop equilibrium and converges to its steady state more rapidly than in the feedback equilibrium.

This proposition implies that the feedback Nash-Cournot equilibrium is farther from the monopoly solution than is the open-loop Nash-Cournot equilibrium. In this context, the feedback solution is relatively procompetitive. Fershtman and Kamien (1987) and Reynolds (1987) compare steady-state values in open-loop and feedback equilibria. Proposition 2 generalizes their results by comparing the entire equilibrium path. The intuition is that, under the feedback assumption, capacity discourages rivals' investment. Therefore, firms have a greater incentive to invest today as a means of preempting their rivals' future investment. Thus, they develop larger capacities and hence larger output levels.

Since Proposition 1 provides a sufficient but not a necessary condition, we cannot prove that the comparison in Proposition 2 also holds for $v \neq 0$. Extensive simulation, however, supports the intuition that the result does hold for $v \neq 0$. Figure 1a graphs the feedback and open-loop steady-state output as a function of v (using "base parameters" $n = 1$, $a = 250$, $b = 10$, $\delta = 5$, $\beta = .95$, $\varepsilon = 1$, $\theta = 0$). Figure 1b shows the difference between open loop and feedback is largest when the market is nearly competitive ($v = -.7$ in the base case). These simulation results suggest that the assumption of open-loop strategies is less serious when the market lies between Nash-Cournot and collusive than when it lies between Nash-Cournot and competitive.

To illustrate the relative rates of adjustment, we use a Nash-Cournot duopoly model. With the base parameters, the steady state for the Nash-Cournot open-loop model is 8.492, and the corresponding feedback steady-state output is 9.087. That is, where $\theta = 0$ (marginal production cost, net of adjustment cost, is constant), the feedback steady state is 7 percent higher than the open-loop steady state. By choosing different parameter values for the open-loop and feedback models, the steady states can be made equal. For

Figure 1.a
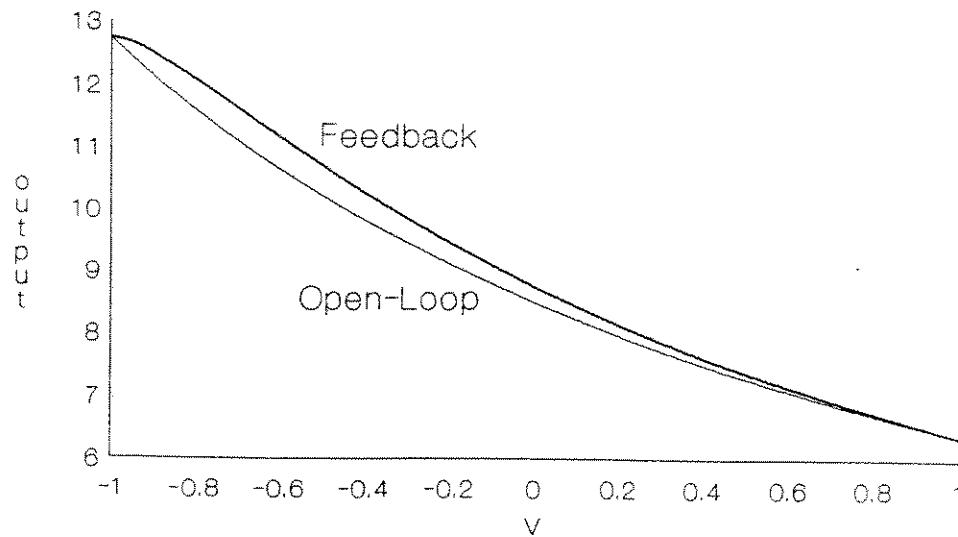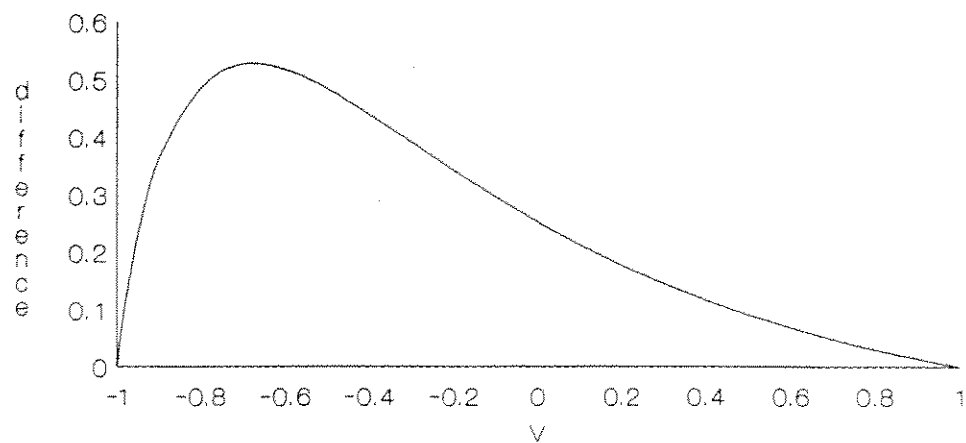## Feedback and Open-Loop
Steady-State Output

Figure 1.b
Difference between Feedback and
Open-Loop Steady-State Output

example, if we use the base parameters and hold $\theta = 0$ in the open-loop model but set $\theta = .7917$ in the feedback model, both steady states equal 8.492. Thus, with a slightly different short-run marginal cost curve, the two models can produce the same steady state.

These models can be distinguished empirically by determining the slope of the marginal cost curve, θ, or observing the adjustment paths. Table 1 shows that, although the open-loop with $\theta = 0$ and the feedback model with $\theta = .7917$ produce the same steady state, the feedback model adjusts more rapidly. In the first period, output in the feedback model is 1.6 percent higher than in the open-loop model. Note this result on rates of adjustment does not contradict Proposition 2 since θ differs between the two models here.

To show the trajectories more clearly for the four leading models, Figure 2 increases δ from the base level to 150. Figure 2 shows how both the paths and the steady states vary with v and the type of strategy used.

As noted above, the open-loop equilibrium can be obtained by solving a control problem:

Proposition 3. Aggregate output in the open-loop equilibrium and the control problem are the same if and only if $k_0 + nk_1 = [2 + n(v + 1)] b + \theta$. If $k_0 = (2 + nv) b + \theta$ and $k_1 = b$, the levels of output for the two problems are the same even if the n + 1 firms have different initial levels of output.

For the Nash-Cournot case (v = 0), Proposition 3 has been proved by Hansen, Epple, and Roberds (1985). For the price-taking case (v = -1/n), the integrand in (4) gives social surplus; this reproduces the well-known

Table 1

Trajectories for Nash-Cournot Models

Period	Open Loop $\theta = 0$	Feedback $\theta = .7917$
0	1.000	1.000
1	7.435	7.552
2	8.369	8.374
3	8.476	8.477
4	8.490	8.490
5	8.491	8.491
6	8.492	8.492

Figure 2
Trajectories for Four Models
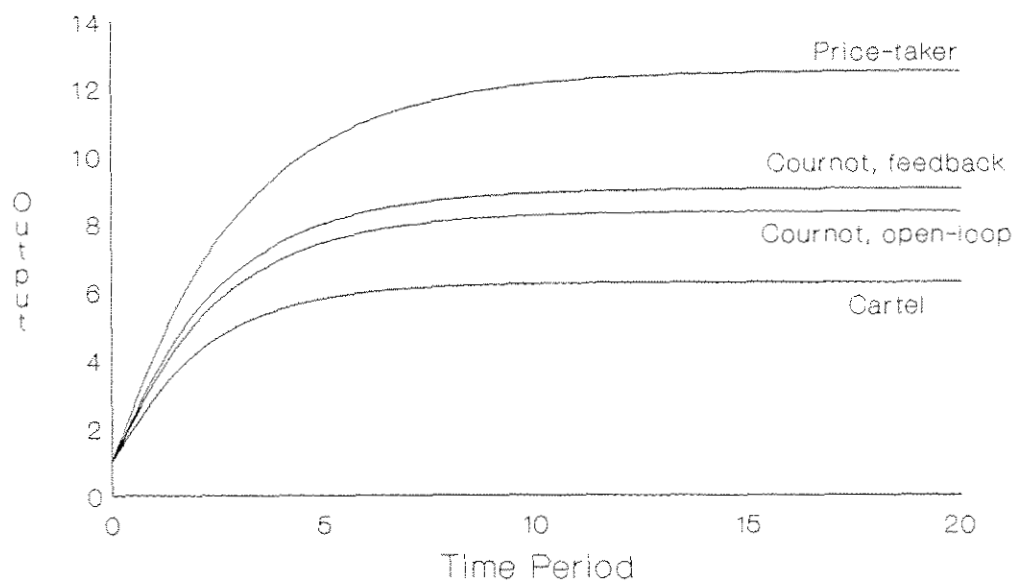
result that the competitive equilibrium can be obtained by solving the social planner's problem. For the collusive case ($v = 1$), the control problem that matches the open-loop game has $k_0 = (2 + n) b + \theta$, $k_1 = b$; whereas the control problem for the monopolist sets $k_0 = 2b + \theta$, $k_1 = 2b$. Therefore, the collusive game gives rise to the monopoly solution only if all firms produce equal quantities. Analogously, in static games, a conjectural variation of one produces the monopoly solution only in a symmetric equilibrium.

Replacing the feedback game with a control problem requires knowing the solution to the game rather than merely the parameters of the game. Although of no computational assistance, the device is of interest for two reasons. First, it leads to the recognition that the feedback equilibrium with a homogeneous good is observationally equivalent to a particular open-loop game with heterogeneous goods.[6] This equivalence has obvious econometric implications (Karp and Perloff, 1988).

Second, the device provides some intuition about why it is difficult to prove the feedback game is stable. Given the solution to the feedback game, it is possible to construct a K matrix such that the steady-state control rule of the resulting control problem is equal to the control rule of the game. However, the K matrix need not be positive definite, which violates one of the sufficiency conditions used to prove stability in the control problem. Details of this argument are provided in Appendix II.

4. Consistent Conjectures

We remarked above that the parameter v could be interpreted as a conjectural variation and that the more neutral description of "behavioral assumption" was adopted to emphasize that v is intended to describe, not

explain, an observed equilibrium. If, however, one wishes to interpret v as a conjectural variation, it is natural to ask if it can be made endogenous, as has been done in static games, by imposing consistency on the conjectures (Laitner, 1980; Bresnahan, 1981; Kamien and Schwartz, 1983; and Perry, 1982). In the static models, players with consistent beliefs are correct about both levels and the slopes of reaction functions in equilibrium. The same procedure can be applied directly to open-loop games because they are essentially static.

The interpretation in the feedback game is slightly different. At the beginning of a period of length ε, players anticipate an equilibrium in the current period that depends on lagged quantities; they expect that any deviation from this equilibrium will be met by an instantaneous response from their rivals. If the conjectural response is optimal (to a first-order approximation), then conjectures are said to be consistent.

The following proposition states the dependence of the consistent conjectural variation on ε, the length of each period.

Proposition 4. In the discrete time feedback game, the consistent conjecture depends on ε--the length of time between decisions. As $\varepsilon \to 0$, the consistent conjecture goes to 0. As $\varepsilon \to \infty$, the game becomes static; if $\theta = 0$, that model reduces to the case of linear demand and constant marginal cost discussed by Bresnahan (1981), where the consistent conjecture is -1/n.

The proof is in Appendix I.

The proposition shows that, even within the confines of this very restrictive model, any constant conjecture between 0 and -1/n is consistent depending on the length of time between adjustments. A static model favors the price-taking solution, while a model that permits continuous adjustment favors the Nash-Cournot solution.

The intuition for this result is based on the dynamic programming equation (2). A change in u_j (from equilibrium), say, Δu_j, results in a change in q_j of $\Delta u_j \epsilon$. Agent i's loss from a failure to respond to a change in u_j consists of two components: the reduction of his profits in the current period and the present value of the loss of finding himself with a suboptimal q_i in the subsequent period. Both of these components depend on $\Delta u_j \epsilon$ and on ϵ directly, since the current period's profits are a flow of profits times ϵ and next period's value function is discounted by $e^{-r\epsilon}$. When ϵ is large, for given Δu_j, the first component dominates; and it is clear that agent i can suffer a substantial loss from not responding to a change in u_j. That is, it pays to respond, so the slope of i's reaction function and, hence, the value of j's consistent conjecture should be large in absolute value. When ϵ is small, a given Δu_j has a negligible effect on i's payoff and can essentially be ignored, so the consistent conjecture is small.

In view of Proposition 4, it is not surprising that, for given $v \in (-1/n, 1)$, the equilibrium output depends on ϵ in the feedback game; output is independent of ϵ for the open-loop game. Consider, for example, the Nash-Cournot case where $v = 0$. Under the feedback model, a firm expects its rivals to react to its current decision only after an interval of ϵ. Its current decision, therefore, depends on ϵ. For the open-loop model, a firm expects no response on the part of its rivals (for $v = 0$), and the

equilibrium is, therefore, independent of ϵ. For the base parameters with $v = 0$, when ϵ goes from 1 (yearly adjustment) to .25 (quarterly adjustment), steady-state feedback output increases 5 percent. When ϵ decreases from 1 to .083 (monthly adjustment), steady-state feedback output increases by 7 percent. The tendency for steady-state output to decrease in ϵ (for $v = 0$) held throughout the simulations. This tendency is consistent with Reinganum and Stokey's (1985) observation on the importance of the period of commitment in dynamic games.

5. Number of Firms and Adjustment Costs

As the number of firms, $n + 1$, increases, the equilibrium trajectories change. By setting $\theta = 0$ and normalizing so that $\delta = (n + 1) c$ where $c > 0$ is constant, the price-taking and collusive equilibria are invariant to n. As n becomes large, the adjustment cost for each firm becomes infinite so each firm makes only infinitesimal adjustments and thus captures only an infinitesimal share of the market. Thus:

Proposition 5. Given $\theta = 0$ and the normalization $\delta = (n + 1) c$, the open-loop and feedback Nash-Cournot converge to the competitive equilibrium as $n \to \infty$.

This proposition can be proven by examining the equations that determine the stationary control rules. The following argument provides a heuristic proof. From Proposition 2, the open-loop and feedback equilibria are identical for $v = -1/n$ which goes to 0 as $n \to \infty$. The open-loop model is a static game for which it is well known that the Nash-Cournot equilibrium converges to the competitive equilibrium as $n \to \infty$.

Table 2 shows the effects of an increase in the number of firms for the base case on the steady-state output in the Nash-Cournot model using the normalization that $\delta = c(n + 1)$. The Nash feedback output increases more rapidly than does the Nash open-loop output as the number of firms increases.

Reynolds (1987) shows that, as $\delta \to 0$, the open-loop and feedback Nash-Cournot models do not converge. Proposition 5 shows that, as both δ and the number of firms go to infinity, the two steady states do converge. Simulation results show that increasing δ while holding n constant causes the steady-state feedback output to increase. As mentioned above, the steady-state output under the open-loop model is independent of δ due to the assumption that adjustment costs are minimized when adjustment is 0. Recall the intuition for Proposition 2: In the feedback game, current investment serves as a deterrent to rivals' future investment but increases output. This deterrence is enhanced the greater the adjustment cost. A larger adjustment cost moves the feedback steady state farther from the open-loop steady state. A larger value of δ also makes it more costly to invest in deterrence (if current output lies below the steady state), which works against the first effect. The net effect on the Nash-Cournot feedback steady state in this example is small: Increasing δ by 400 percent in the base case causes the steady-state output to increase by 2.5 percent.

6. Summary and Conclusions

Since feedback models are difficult to use, many researchers have relied on open-loop models. Our analysis indicates that if behavior is "close" to price taking or collusive, then the trajectories and steady states will be very similar to those of the feedback model. When oligopolists are not

Table 2

The Effect of Increasing the Number of
Firms on Industry Output[a]

Number of	Nash-Cournot	
firms (n + 1)	Open Loop	Feedback
1 (monopoly)	12.74	12.74
2	16.98	17.48
3	19.11	19.92
4	20.38	21.35
5	21.23	22.27
10	23.16	24.16
15	23.88	24.74
∞ (price takers)	25.48	25.48

[a]Parameters: a = 250, b = 10, β = .95, and
δ = 2.5 (n + 1).

cooperative or price takers, both the paths and the steady states differ. The difference between the two models, in our simulations, were greater for more competitive behavior than for more collusive behavior. The feedback model implies more competitive behavior and slower adjustment in general.

Based on the subgame perfect feedback model, constant marginal (production) costs, and linear demand, the consistent conjectures model depends on the length of time periods. As these become shorter, the consistent conjecture is the Nash-Cournot. As they become longer and the game becomes static, the consistent conjectures approach price taking as predicted in the static model literature.

Increasing the number of firms, of course, leads to more competitive behavior. Increasing the adjustment costs increases the Nash-Cournot steady-state output slightly.

Appendix I

Proof of Propositions

We show that ρ^i, $i = c, o, f$, is the negative root of

$$(\rho^c)^2 - r\delta \rho^c - \delta(k_0 + n\, k_1) = 0 \tag{AI.1a}$$

$$(\rho^0)^2 - r\delta \rho^0 - \delta\{b[2 + n(v + 1)] + \theta\} = 0 \tag{AI.1b}$$

$$(\rho^f)^2 - r\delta \rho^f - \delta\{b[2 + n(v + 1)] + \theta\} = \eta(v) \tag{AI.1c}$$

where

$$\eta(v) \equiv -[n\rho_1^2(1 - v + nv - nv^2) + n^2 \rho_1 z(1 + nv - v^2 - nv^3)$$

$$+ n^3 z^2(v + v^2 + nv^2 - 2v^2 - nv^3)];$$

ρ_1 and z are defined below. The parameters γ^i are given by

$$\gamma^c = \frac{(n + 1)\, a}{(r\delta - \rho^c)}, \tag{AI.2a}$$

$$\gamma^0 = \frac{(n + 1)\, a}{(r\delta - \rho^0)}, \tag{AI.2b}$$

and

$$\gamma^f = \frac{(n + 1)\, [a + \phi(v)]}{(r\delta - \rho^f)} \tag{AI.2c}$$

where

$$\phi(v) \equiv \frac{n(\rho_1) + nvz)\, \hat{h}\, (1 - v)(1 + nv)}{\delta},$$

and $\hat{h}$ is defined below.

The Control Problem

The maximized value of (3) is

$$\gamma + he' \, q + \left(\frac{1}{2}\right) q'Hq$$

where γ and h are scalars and H is a matrix with ρ_0 on the principal diagonal and ρ_1 elsewhere. The control rule is:

$$u = \frac{1}{\delta}(he + Hq). \tag{AI.3}$$

The algebraic Ricatti system is:

$$-r\delta H = \delta K - HH$$

$$r\delta he = \delta ae + Heh.$$

Writing these in terms of the three unknowns, ρ_0, ρ_1, and h gives

$$-r\delta\rho_0 = \delta k_0 - \rho_0^2 - n\rho_1^2 \tag{AI.4a}$$

$$-r\delta\rho_1 = \delta k_1 - 2\rho_0 \rho_1 - (n - 1) \rho_1^2 \tag{AI.4b}$$

$$h = \frac{a\delta}{(r\delta - \rho_0 - n\rho_1)}. \tag{AI.4c}$$

Define $\rho^C = \rho_0 + n \rho_1$, $\gamma^C = (n + 1) h$. Multiply (AI.4b) by n and add to (AI.4a) to obtain (AI.1a); (AI.4c) implies (AI.2a).

The Open-Loop Game

Define $\bar{v}_i \in R^{n+1}$ as a column vector with 1 in the _i_th position and v in every other position; player i behaves as if $\partial u / \partial u_i = \bar{v}_i$ and his objective is to maximize (1b). The open-loop current value Hamiltonian, H_i, is:

$$H_i = a\, e_i'\, q - \frac{1}{2}\, q'\, K_i\, q - \frac{1}{2}\, u'\, S_i\, u + \lambda_i'\, u$$

where λ_i is i's shadow value of the state q. The necessary conditions for an interior solution are:

$$\frac{\partial H_i}{\partial u_i} = -\delta u_i + \bar{v}_i\, \lambda_i = 0 \tag{AI.5a}$$

$$-\frac{\partial H_i}{\partial q_i} = \bar{v}_i'(\dot{\lambda}_i - r\lambda_i) = \bar{v}_i'(-ae_i + K_i q). \tag{AI.5b}$$

The first-order conditions (AI.5a, b) are obtained as the limiting form, as $\varepsilon \to 0$, of the first-order conditions to the discrete open-loop problem. Try a solution of the form, $\lambda_i = h_i + H_i q$; for a steady-state control rule, $\dot{h}_i = \dot{H}_i = 0$ for all i. "Guess" that H_i is a matrix with $H_i(i, i) = \rho_0$, $H_i(i, j) = \rho_1$, $H_i(j, i) = \hat{\rho}_1$ for all $j \neq i$, and $H_i(\ell, s) = z$ for all ℓ, $s \neq i$; h_i is an n + 1 column vector with $h_i(i) = h$, $h_i(j) = \hat{h}$ for all $j \neq i$. Certain symbols, such as ρ_0, ρ_1, and h, were used to describe the control problem. The duplication is intended to emphasize the similarity of the various problems. Note that H_i is asymmetric.

Solve (AI.5a) to obtain

$$u_i = \frac{h + nv\hat{h}}{\delta} + \frac{1}{\delta}\left[(\rho_0 + nv\hat{\rho}_1)\, q_i + (\rho_1 + nvz) \sum_{j \neq i} q_j \right]. \tag{AI.5c}$$

Stack up the $n + 1$ necessary conditions of the form (AI.5c) to write $u = (h* + H*q)/\delta$ where $h* = (h + nv\hat{h})$ e and $H*(i, i) = \rho_0 + nv\hat{\rho}_1$, $H*(\ell, s) = \rho_1 + nvz$, for all $\ell \neq s$. Substitute this solution into (AI.5b) together with $H_i = 0 = h_i$ to obtain the system

$$-r\delta\bar{v}_i' H_i = \bar{v}_i'(\delta K_i - H_i H*)$$

and

$$r\delta\bar{v}_i' h_i = \bar{v}_i'(a\delta e_i + H_i h*).$$

These two systems give three equations for the three unknown functions, $\rho_0 + nv\hat{\rho}_1$, $\rho_1 + nvz$, and $h + nv\hat{h}$:

$$-r\delta(\rho_0 + nv\hat{\rho}_1) = \delta[(2 + nv) b + \theta] - (\rho_0 + nv\hat{\rho}_1)^2 - n(\rho_1 + nvz)^2, \quad \text{(AI.6a)}$$

$$-r\delta(\rho_1 + nvz) = \delta b - 2(\rho_0 + nv\hat{\rho}_1)(\rho_1 + nvz) - (n - 1)(\rho_1 + nvz)^2, \quad \text{(AI.6b)}$$

and

$$(h + nv\hat{h}) = \frac{a\delta}{(r\delta - \hat{\rho}_0 - nv\hat{\rho}_1 - n\hat{\rho}_1 - n^2vz)}. \quad \text{(AI.6c)}$$

Define $\rho^0 = \rho_0 + nv\hat{\rho}_1 + n\rho_1 + n^2vz$. Multiply (AI.6b) by n and add to (AI.6a) to obtain (AI.1b). Define $\gamma^0 = (n + 1)(h + nv\hat{h})$ to obtain (AI.2b). Proposition 3 follows from inspection of the control rules (AI.3) and (AI.5c) and comparison of (AI.4) and (AI.6).

The Feedback Game

The stationary dynamic programming equation for player i is

$$r\left(\gamma_i + h_i' q + \frac{1}{2} q' H_i q\right) = \max_{u_i}\left[a e_i' q - \frac{1}{2} q' K_i q - \frac{\delta}{2} u_i^2 + (h_i + H_i q)' u\right].$$

The parentheses on the left side gives $J_i(q)$, i's value function. The necessary condition for an interior maximum is

$$u_i = \frac{1}{\delta}(h_i + H_i \, q)' \, \bar{v}_i. \tag{AI.7}$$

Substitute (AI.7) into the dynamic programming equation and equate coefficients to obtain the system

$$0 = r\delta H_i + \delta K_i + H_i \, \bar{v}_i \, \bar{v}_i' \, H_i - H_i\left(\sum_{j=1}^{n+1} e_j \, \bar{v}_j' \, H_j\right) - \left(\sum_{j=1}^{n+1} H_j \, \bar{v}_j \, e_j'\right) H_i$$

$$0 = -r\delta h_i + \delta a e_i - H_i \, \bar{v}_i \, \bar{v}_i' \, h_i + \left(\sum_{j=1}^{n+1} H_j \, \bar{v}_j \, e_j'\right) h_i + H_i\left(\sum_{j=1}^{n+1} e_j \, \bar{v}_j' \, h_j\right).$$

Try a solution of the form $H_i(i, i) = \rho_0$, $H_i(i, j) = H_i(j, i) = \rho_1$ for $j \neq i$, and $H_i(s, \ell) = z$ for $s \neq i$, $\ell \neq i$; $h_i(i) = h$, $h_i(j) = \hat{h}$ for $j \neq i$. Substituting this trial solution into the above system results in

$$0 = r\delta\rho_0 + \delta(2b + \theta) - \rho_0^2 - [2n - (nv)^2] \rho_1^2 - 2n^2 \, v\rho_1 z, \tag{AI.8a}$$

$$0 = r\delta\rho_1 + \delta b - nv\rho_1^2 - (n - 1) \rho_1^2 - 2\rho_1 \rho_0 + [(nv)^2 \rho_1$$
$$- (n - 1) nv\rho_1 - n\rho_1 - n^2 zv] \, z, \tag{AI.8b}$$

$$0 = r\delta z - \rho_1^2 - 2z \, \rho_0 - 2(n - 1) \, z\rho_1 - 2nvz[\rho_1 + (n - 1) \, z] + (nvz)^2, \tag{AI.8c}$$

$$0 = -r\delta h + \delta a + (h + nv\hat{h}) \, n\rho_1(1 - v) + (\rho_0 + n\rho_1 v) \, h + n(\rho_1 + nzv) \, \hat{h}, \tag{AI.9a}$$

and

$$0 = -r\delta\hat{h} + (h + nv\hat{h}) \, nz(1 - v) + (\rho_0 + n\rho_1 v) \, \hat{h}$$

<div align="right">(AI.9b)</div>

$$+ (\rho_1 + nzv) \, [h + (n - 1) \, \hat{h}].$$

This derivation of system (AI.8) implicitly assumes that stationary control rules exist. This was not at issue for the three leading open-loop equilibria. In those cases the equilibrium trajectory could be generated by a control problem in which K, the metric on the state, was positive semidefinite (positive definite except where $\theta = 0$ and the market was either collusive or price taking, in which cases $k_0 = k_1$). Standard results ensure the stability of the Ricatti system and of the system given by (4) under these conditions. Similar results for the feedback game have, to our knowledge, not been obtained.

System (AI.8) sets the vector of time derivatives $(\dot{\rho}_0, \dot{\rho}_1, \dot{z})'$ equal to 0. A necessary and sufficient condition for the system to be locally stable, and thus a necessary condition for the existence of a stationary feedback control rule, is that the real parts of the characteristic roots of the Jacobian of (AI.8), evaluated at the steady state, be positive (recall that the system is solved backward in time).

To ensure that comparison between open-loop and feedback equilibria are meaningful, we make two further assumptions.

Assumption 1. Stationary feedback rules exist; i.e., the system, $-(\dot{\rho}_0, \dot{\rho}_1, \dot{z})'$, is stable.

Assumption 2. The vector q, generated by the equilibrium feedback rules, converges for arbitrary q_0.

Define $\rho^f = \rho_0 + n(1 + v) \rho_1 + n^2 vz$. Multiply (AI.8b) and (AI.8c) by $n(1 + v)$ and $n^2 v$, respectively, and add to (AI.8a) to obtain (AI.1c). Define $\gamma^f = (n + 1)(h + nv\hat{h})$. Multiply (AI.9b) by nv and add to (AI.9a) to obtain (AI.2c).

Proposition 1 follows immediately from comparison of (AI.1b) and (AI.1c), of (AI.2b) and (AI.2c), and from the fact that $\eta(v) = \phi(v) = 0$ for $v = -1/n$ or $v = 1$.

We show that

$$\eta(0) < 0 < \phi(0). \tag{AI.10}$$

This inequality, together with (AI.1b) and (AI.1c) and (AI.2b) and (AI.2c), and the definition of ρ^0 and ρ^f establish Proposition 2.

We first establish that $\rho_1(\rho_1 + nz) > 0$ so that $\eta(0) < 0$; then comparison of (AI.1b) and (AI.1c) implies $\rho^0 < \rho^f < 0$. First, note that $\rho_1 < 0$. To verify this inequality, use the facts that $\partial^2 J_i/(\partial q_i \, \partial q_j) = \rho_1$, $j \neq i$, and $\partial^2 J_i/\partial q_i^2 = \rho_0$ where $J_i(q)$ was defined as agent i's value function. Define $\hat{q}_i(q_j)$ as the optimal (for agent i) initial condition for q_i given q_j. Therefore, $d\hat{q}_i/dq_j = -\rho_1/\rho_0$. By Assumption 2 and equation (AI.7), $\rho_0 < 0$. Suppose $\rho_1 > 0$; this implies that an increase in the initial level of q_j would cause agent i to want to begin with a higher level of sales. However, for large enough q_j, price is negative and it is clear that agent i would prefer to begin with a lower level of q_i; hence, $\rho_1 < 0$ as stated.

It is now necessary to show that $\rho_1 + nz < 0$. Solve (AI.8c) with $v = 0$ to obtain $z = \rho_1^2/[r\delta - 2\rho_0 - 2(n - 1)\rho_1]$ so $\rho_1 + nz = \rho_1[r\delta - 2\rho_0 - (n - 2)\rho_1]/[r\delta - 2\rho_0 - 2(n - 1)\rho_1]$. The denominator is obviously positive, and for $n \geq 2$ the numerator is clearly negative. For $n = 1$, use Assumption 2 and

equation (AI.10) to obtain $\rho_0^2 - \rho_1^2 > 0$. Since ρ_1, $\rho_0 < 0$, this implies $\rho_0 < \rho_1$, which implies $r\delta - 2\rho_0 + \rho_1 > 0$. Therefore, $\rho_1 + nz < 0$ for all n, and $\eta(0) < 0$ as stated.

To complete the proof, rewrite (AI.9) for $v = 0$ as

$$\begin{bmatrix} r\delta - \rho_0 - n\rho_1 & -n\rho_1 \\ -(\rho_1 + nz) & r\delta - \rho_0 - (n-1)\rho_1 \end{bmatrix} \begin{bmatrix} h \\ \hat{h} \end{bmatrix} = \begin{pmatrix} \delta a \\ 0 \end{pmatrix} .$$

Using previous results, all elements of this matrix are positive so h and $\hat{h}$ must have the opposite sign.

To establish the second inequality in (AI.10), we need only show $\hat{h} < 0$. Use (AI.9a) and (AI.9b) and the results of the previous paragraph to verify that h and $\hat{h}$ must have the opposite sign. Since $h = \partial J_i(0)/\partial q_i$, $\hat{h} = \partial J_i(0)/\partial q_j$, $j \neq i$, $h < 0 < \hat{h}$ would imply that, if all agents begin the game with 0 sales so that initial price is positive, agent i would prefer to begin with negative sales and have his rival(s) begin with positive sales. Since this must be false, we conclude $h > 0 > \hat{h}$, so $\phi(0) > 0$, completing the proof of Proposition 2.

To prove Proposition 4, write the first-order condition to player i's discrete time dynamic programming problem as

$$\sigma_i' \bar{v}_i + u' \Sigma_i v_i = 0$$

where

$$\sigma_i \equiv e^{-r\epsilon}(h_i + H_i q) + (ae_i - K_i q) \epsilon$$

$$\Sigma_i \equiv \epsilon H_i - \epsilon^2 K_i - S_i .$$

The consistent conjecture is obtained by differentiating i's first-order condition with respect to u_j using $\partial u_i / \partial u_j = v$ and setting the result to 0: $\overline{v}'_j \Sigma_i \overline{v}_i = 0$. When $\varepsilon = 0$, this expression reduces to $-v \delta_1 = 0$ which implies $v = 0$. For $\varepsilon = 0$, the condition for consistency of conjectures can be written as

$$\overline{v}'_j \; \varepsilon^2 \left[\frac{H_i}{\varepsilon} - \frac{S_i}{\varepsilon^2} - K_i \right] \overline{v}_i = 0,$$

which requires

$$\overline{v}'_j \left[\frac{H_i}{\varepsilon} - \frac{S_i}{\varepsilon^2} - K_i \right] \overline{v}_i = 0.$$

If this equation holds in the limit as $\varepsilon \to \infty$, it is necessary that $\overline{v}'_j K_i \overline{v}_i \equiv 1 + (n + 1) v + nv^2 = 0$, which requires that $v = -1/n$ or $v = -1$. Since $v = -1$ for $n < 1$ results in negative profits, it must be the case that $v = -1/n$. This result is identical to Bresnahan's (1981) static result.

Appendix II

Relation Between the Feedback Game and a Control Problem

We illustrate the relation between the feedback game and a control problem using a Nash-Cournot ($v = 0$) duopoly ($n = 1$). Let $\tilde{\rho}_1$ and $\tilde{z}$ solve (AI.8).

Define

$$b_1 \equiv b - \frac{\tilde{\rho}_1^2}{2\delta} < b$$

and

$$b_2 \equiv b - \frac{\tilde{\rho}_1 \tilde{z}}{\delta} > b.$$

Specializing the control problem given in (3) by setting $k_0 = 2b_1 + \theta$, $k_1 = b_2$ (and replacing a by $a + \rho_1 \hat{h}/\delta > a$) duplicates aggregate quantity in the Nash-Cournot feedback trajectory. This result is true even if the initial levels of output vary across firms. The feedback game with a homogeneous good is equivalent to an open-loop game with hetergeneous goods.

The static problem provides some intuition here. Consider the two static Nash-Cournot duopolies.

$$G1: \quad \max_{q_i}[a - b(q_i + q_j)] \, q_i$$

and

$$G2: \quad \max_{q_i}[a - b_1 \, q_i - b_2 \, q_j] \, q_i.$$

If the players are symmetric in both games and Q_i is aggregate output in game i, then $Q_1 < Q_2$ if and only if $3b > 2b_1 + b_2$. It can be shown that b_1 and b_2, defined above, satisfy this inequality.

Now consider two control problems with, respectively,

$$K_1 = \begin{pmatrix} 2b + \theta & b \\ b & 2b + \theta \end{pmatrix} \quad \text{and} \quad K_2 = \begin{pmatrix} 2b_1 + \theta & b_2 \\ b_2 & 2b_1 + \theta \end{pmatrix}$$

as metrics on the state q. Given the same price intercept, a, aggregate output in the second control problem is larger. (Charging a to $a + \rho_1 \hat{h}/\delta$ strengthens the conclusion.)

To see why the feedback game may not be stable, recall that the positive semidefiniteness of K is one of the set of sufficient conditions for the control problem given by (3) to be stable (i.e., for the Ricatti differential equations to converge to the stationary values). For $b > 0$, $\theta \geq 0$, K_1 (defined above) is always positive definite. However, there is apparently no guarantee that K_2 is positive semidefinite. As a result, sufficient conditions for the feedback game to be stable may or may not be fulfilled.

Footnotes

[1]Fudenberg, Levine, and Srivastava (1984) give other types of conditions under which the open-loop and feedback equilibria are approximately the same with many players.

[2]Both Fershtman and Kamien (1987) and Reynolds (1987) use dynamic linear-quadratic models. Hansen, Epple, and Roberds (1985) also use the dynamic linear quadratic model to study various open-loop models as well as the open-loop and feedback Stackelberg model. They do not compare the open-loop and feedback in symmetric firm markets, which is the focus of this paper. Van der Ploeg (1987) compares the steady states in a natural resources setting with general functional forms.

[3]An analogous approach is used by Gallop and Roberts (1979) in a static model.

[4]Treadway (1970) shows that the comparative statics of the steady state of cost-of-adjustment models differ from those of the "corresponding" static model. In a similar vein, Reynolds (1987) finds that the output under static Nash-Cournot and at the steady state of the open-loop dynamic Nash-Cournot models are different. However, under the assumption that adjustment costs are minimized when adjustment is 0 (i.e., at the steady state), these results no longer hold. This assumption seems reasonable if the objective is to compare the various dynamic models with their static analogs.

[5]It is well known that, for indefinite horizon games, there typically exist many equilibria even when these are required to be subgame perfect. We avoid the problem of nonuniqueness by considering the equilibrium strategies that result from the game with finite horizon T and letting $T \to \infty$.

[6]This statement is actually too strong. Suppose that the game were completely stationary and firms completely symmetric so that it was practical to impose the restrictions implied by the constant part of the control rules. In that case the slope coefficients of the control rules of a homogeneous firm game with feedback strategies would be the same as the slope coefficients of the rules of a heterogeneous (but symmetric) firm game with open-loop strategies; but the intercepts would be different so the two could still be distinguished. However, for the econometric work, we do not wish to impose the restrictions on the intercepts of the control rules, so that nonstationarity or firm-specific features may be included in the parameters a, θ_0, and δ_0.