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RISKY LOSS DISTRIBUTIONS AND MODELING THE LOSS RESERVE PAY-OUT TAIL

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Although an extensive literature has developed on modeling the loss reserve runoff triangle, the estimation of severity distributions applicable to claims settled in specific cells of the runoff triangle has received little attention in the literature. This paper proposes the use of a very flexible probability density function, the generalized beta of the 2nd kind (GB2) to model severity distributions in the cells of the runoff triangle and illustrates the use of the GB2 based on a sample of nearly 500,000 products liability paid claims. The results show that the GB2 provides a significantly better fit to the severity data than conventional distributions such as the Weibull, Burr 12, and generalized gamma and that modeling severity by cell is important to avoid errors in estimating the riskiness of liability claims payments, especially at the longer lags.

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INTRODUCTION

In many types of property-casualty coverages such as commercial liability insurance, coverage is provided for a fixed period such as one year, whereas claims arising from a given year's coverage are paid over a multi-year period extending over at least five years following the coverage year. The payout period following the coverage year is called the *runoff period* or *payout tail*, and lines of business with extended payout tails are called *long-tail lines*.[1] Because economic, insurance market, and legal conditions can change significantly during the runoff period, long-tail lines expose insurers to unusually high levels of risk. Therefore, accurately modeling the payout tail in long-tail lines is an important problem in actuarial and financial modeling for the insurance industry and a critical risk management competency for insurers.

Modeling the payout tail has received a significant amount of attention in the literature (e.g., Reid 1978; Wright 1990; Taylor 1985, 2000; Wiser, Cockley, and Gardner 2001). Modeling

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the payout tail is critically important in pricing, reserving, reinsurance decision making, solvency testing, dynamic financial analysis, and a host of other applications. A wide range of techniques has been developed to improve modeling accuracy and reliability. Most of the existing models focus on estimating the total claims payout in the cells of the loss runoff triangle, i.e., the variable analyzed is c_{ij}, where c_{ij} is defined as the amount of claims payment in runoff period j for accident year i. The value of c_{ij} is in turn determined by the frequency and severity of the losses in the ij-th cell of the triangle.

Although sophisticated models have been developed for estimating claim counts (frequency) and total expected payments by cell of the runoff triangle, less attention has been devoted to estimating loss severity distributions by cell. While theoretical models have been developed based on the *assumption* that claim severities by cell are gamma distributed (e.g., Mack 1991, Taylor 2000, p. 223), few empirical analyses have been conducted to determine the loss severity distributions that might be applicable to claims falling in specific cells of the runoff triangle. The objective of the present paper is to remedy this deficiency in the existing literature by conducting an extensive empirical analysis of U.S. products liability insurance paid claims. We propose the use of a flexible four-parameter distribution–the generalized beta distribution of the 2nd kind (*GB*2)–to model claim severities by runoff cell. This distribution is sufficiently flexible to model both heavy-tailed and light-tailed severity statistics and provides a convenient functional form for computing prices and reserve estimates. In addition, the *GB*2 nests most of the conventional distributions that have been used to model insurance claims, including the gamma, Weibull, Burr 12, and lognormal.

It is important to estimate loss distributions applicable to individual cells of the runoff triangle rather than to use a single distribution applicable to all observed claims or to discount claims to present value and then fit a distribution. If the characteristics of claims settled differ significantly by settlement lag, the use of a single severity distribution can lead to severe inaccuracies in estimating expected costs, risk, and other moments of the severity distribution. This problem is likely to be especially severe for liability insurance, where claims settled at longer lags tend to be larger and more volatile.

When distributions are fit to separate years in the payout tail, the aggregate loss distribution is a mixture distribution over the yearly distributions. To explore the economic implications associated with the alternative estimates of loss distributions, we compare a single aggregate fitted distribution based on all claims for a given accident year vs. the mixture distribution, using Monte Carlo simulations. In illustrating the differences between the two models, we innovate by comparing the distributions of the discounted or economic value of claim severities rather than using undiscounted values that do not reflect the timing of payment of individual claims, thus creating *discounted severity distributions*. Thus, we provide a model (the mixture model) that not only reflects the modeling of claim severities by runoff cell but also could be used in a system designed to obtain market values of liabilities for use in fair value accounting estimation and other financial applications. Ours is the first paper in the literature to compare the aggregate and mixed claim distributions and also the first to estimate discounted severity distributions.

The problem of estimating claim severity distributions by cell of the runoff triangle has been previously considered by the Insurance Services Office (ISO) (1994, 1998, 2002). In ISO

(1994) and (1998), a mixture of Pareto distributions was used to model loss severities by settlement lag period for products and completed operations liability losses. The two-parameter version of the Pareto was used, and the mixture consisted of two Pareto distributions. In ISO (2002), the mixed Pareto was replaced by a mixed exponential distribution, where the number of distributions in the mixture ranged from five to eight. The ISO models do not utilize discounting or any other technique to recognize the time value of money.

Although the ISO mixture approach clearly has the potential to provide a good fit to loss severity data, we believe that there are several advantages to using a single general distribution such as the *GB*2 rather than a discrete mixture to model loss severity distributions by payout lag cell. The *GB*2 is an extremely flexible distribution that has been shown to have excellent modeling capabilities in a wide range of applications, including models of security returns and insurance claims (e.g., Bookstaber and McDonald 1987, Cummins et al. 1990, Cummins, Lewis, and Phillips 1999). It is also more natural and convenient to conduct analytical work such as price estimation and the analysis of losses by layer utilizing a single distribution rather than a mixture. The *GB*2 also lends itself more readily to Monte-Carlo simulation than a mixture. And, finally, the *GB*2 and various members of the *GB*2 family can be obtained analytically as general mixtures of simpler underlying distributions, so that the *GB*2 is in this respect already more general than a discrete mixture of Paretos or exponentials.

In addition to proposing an alternative to the ISO method for estimating severity distributions by payout lag and introducing the idea of discounted severity distributions, this paper also contributes to the existing literature by providing the first major application of the GB2 distribution to the modeling of liability insurance losses. We demonstrate that fitting a separate distribution to each year of the payout tail can lead to large differences in estimating both expected losses and the variability of losses. These differences in estimation can have a significant impact on pricing, reserving, and risk management decisions, including asset/liability management and the calculation of value at risk (VaR). We also show that the four-parameter GB2 distribution is significantly more accurate in modeling risky claim distributions than traditional two or three-parameter distributions such as the lognormal, gamma, Weibull, or generalized gamma.

This paper builds on previous contributions in a number of excellent papers that have developed models of insurance claim severity distributions. Hogg and Klugman (1984) and Klugman, Panjer, and Willmot (2004) discuss a wide range of alternative models for loss distributions. Paulson and Faris (1985) applied the stable family of distributions, and Aiuppa (1988) considered the Pearson family as models for insurance losses. Ramlau-Hansen (1988) modeled fire, windstorm, and glass claims using the log-gamma and lognormal distributions. Cummins, *et al.* (1990) considered the four-parameter generalized beta of the second kind (GB2) distribution as a model for insured fire losses; and Cummins, Lewis, and Phillips (1999) use the lognormal, Burr 12, and *GB*2 distributions to model the severity of insured hurricane and earthquake losses. All of these papers show that the choice of distribution matters and that conventional distributions such as the lognormal and two-parameter gamma often underestimate the risk inherent in insurance claim distributions.

The paper is organized as follows: In section 2, we introduce the *GB2* family and discuss our estimation methodology. Section 3 describes the database and presents the estimated loss severity distribution results. The implications of these results are summarized in the concluding comments of section 4.

STATISTICAL MODELS

This section reviews a family of flexible parametric probability density functions (pdf) that can be used to model insurance losses. We begin by defining a four-parameter generalized beta (GB2) probability density function, which includes many of the models considered in the prior literature as special cases. We then describe the *GB*2 distribution, its moments, interpretation of parameters, and issues of estimation. This paper applies several special cases of the GB2 to explore the distribution of an extensive database on product liability claims. Previously, the GB2 has been successfully used in insurance to model fire and catastrophic property losses (Cummins, *et al.* 1990, Cummins, Lewis, and Phillips 1999) and has been used by a few other researchers such as Ventor (1984). Other applications of the *GB*2 distribution, also known as the transformed beta distribution, arising in enterprise risk analysis for property-casualty insurance companies are included in Brehm, *et al.* (2007).

The Generalized Beta Distribution of the Second Kind (GB2)

The GB2 probability density function (pdf) is defined by

$$GB2(y;a,b,p,q) = \frac{|a|\, y^{a p-1}}{b^{a p} B(p,q)(1+(y/b)^{a})^{(p+q)}} \quad . \tag{1}$$

for $y > 0$ and zero otherwise, with b, p, and q positive, where $B(p,q)$ denotes the beta function defined by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{2}$$

and $\Gamma(\,)$ denotes the gamma function**.** *The* h^{th} *order moments of the GB2* are given by

$$E_{GB2}(y^{h}) = \frac{b^{h} B(p+h/a, q-h/a)}{B(p,q)} \quad . \tag{3}$$

The parameter a of the *GB*2 can be either positive or negative. It is interesting to note that $GB2(y;-a,b,p,q) = GB2(y;a,b,q,p)$, thus a *GB*2 pdf with a negative *a* parameter is equivalent to a corresponding *GB*2 with the parameter *a* being the absolute value of the negative value and with the parameters *p* and *q* being interchanged.

The *GB*2 distribution includes numerous other distributions as special or limiting cases. Each special case is obtained by constraining the parameters of the more general distributions. For example, an important special case of the generalized beta is the generalized gamma (*GG*)

$$GG(y;a,\beta,p) = \operatorname*{Lim}_{q\to\infty} GB2(y;a,b=\beta q^{1/a},p,q)$$
$$= \frac{|a|\, y^{a\,p-1} e^{-(y/b)^a}}{\beta^{a\,p}\Gamma(p)} \tag{4}$$

for $y > 0$ and zero otherwise. The moments of the generalized gamma can be expressed as

$$E_{GG}(y^h) = \beta^h \frac{\Gamma(p+h/a)}{\Gamma(p)} \quad . \tag{5}$$

In this special case of the *GB*2 distribution the parameter *b* has been constrained to increase with *q* in such a way that the *GB*2 approaches the *GG*.

Interpretation of Parameters

The parameters *a*, *b*, *p*, and *q* generally determine the shape and location of the density in a complex manner. The h^{th} order moments are defined for the *GG* if $0 < p + h/a$ and for the *GB*2 if $-p < h/a < q$. Thus we see that these models permit the analysis of situations characterized by infinite means, variances, and higher order moments. The parameter *b* is merely a scale parameter and depends upon the units of measurement.

Generally speaking, the larger the value of *a* or *q*, the “thinner” the tails of the density function. In fact, for “large” values of the parameter *a*, the probability mass of the corresponding density function becomes concentrated near the value of the parameter *b*. This can be verified by noting that as the parameter *a* increases in value the mean and variance approach *b* and zero, respectively. As mentioned, the definition of the generalized distributions permits negative values of the parameter *a*. This admits “inverse” distributions and in the case of the generalized gamma is called the inverse generalized gamma. Special cases of the inverse generalized gamma are used as mixing distributions in models for unobserved heterogeneity. Butler and McDonald (1987) used the *GB*2 as a mixture distribution.

The parameters *p* and *q* are important in determining shape. For example, for the *GB*2, the relative values of the parameters *p* and *q* determine the value of skewness and permit positive or negative skewness. This is in contrast to such distributions as the lognormal that are always positively skewed.

Relationships With Other Distributions

Special cases of the *GB*2 include the beta of the first and second kind (*B*1 and *B*2), Burr types 3 and 12 (*BR*3 and *BR*12), lognormal (*LN*), Weibull (*W*), gamma (*GA*), Lomax, uniform, Rayleigh, chi-square, and exponential distributions. These properties and interrelationships have been developed in other papers (e.g., McDonald, 1984, McDonald and Xu, 1995, Ventor, 1984, and Cummins *et al.*, 1990) and will not be replicated in this paper. However, since prior insurance applications have found the Burr distributions to provide excellent descriptive ability, we will formally define those pdf’s:

$$BR3(y;a,b,p)=GB2(y;a,b,p,q=1)$$
$$= \frac{|a|\, p y^{ap-1}}{b^{ap}(1+(y/b)^a)^{p+1}} \tag{6}$$

and

$$BR12(y;a,b,q) = GB2(y;a,b,p=1,q)$$
$$= \frac{|a| q y^{a-1}}{b^a(1+(y/b)^a)^{q+1}}. \tag{7}$$

Again, notice that the first lines of both (6) and (7) show the relationship between the *BR*3 or *BR*12 and the *GB*2 distribution. The *GB*2 distribution includes many distributions contained in the Pearson family (see Elderton and Johnson 1969, and Johnson and Kotz 1970), as well as distributions such as the *BR*3 and *BR*12 which are not members of the Pearson family. Neither the Pearson nor generalized beta family nests the other.

The selection of a statistical model should be based on flexibility and ease of estimation. In numerous applications of the *GB*2 and its special cases, the *GB*2 is the best fitting four-parameter model and the *BR*3 and *BR*12 the best fitting three-parameter models.

Parameter Estimation

The method of maximum likelihood can be applied to estimate the unknown parameters in the models discussed in the previous sections. This involves maximizing

$$l(\theta) = \sum_{t=1}^{N} \ln(f(y_t;\theta)) \tag{8}$$

over θ, where $f(y_t;\theta)$ denotes the pdf of independent and uncensored observations of the random variable *Y*, θ is a vector of the unknown distributional parameters, and *N* is the number of observations. E.g., If the pdf is the Burr 12, then we see from equation (7) that the uknown parameters are *a, b,* and *q*, and θ = *(a, b, q)*.

In the case of censored observations the log-likelihood function becomes

$$l(\theta) = \sum_{t=1}^{N} \left[I_t \ln(f(y_t;\theta)) + (1-I_t)\ln(1-F(y_t;\theta)) \right] \tag{9}$$

where $F(y_t;\theta)$ denotes the distribution function and I_t is an indicator function equal to 1 for uncensored observations and zero otherwise.[2] When I_t equals zero, i.e., a censored observation, $F(y_t;\theta)$ is evaluated at y_t equal to the policy limit plus loss adjustment expenses.[3]

ESTIMATION OF LIABILITY SEVERITY DISTRIBUTIONS

In this section, the methodologies described in section 2 are applied to the Insurance Services Office (ISO) closed claim paid severity data for products liability insurance. We not only fit distributions to aggregate loss data for each accident year but separate distributions are also fit to the claims in each cell of the payout triangle, by accident year and by settlement lag, for the years 1973 to 1986. Several distributions are used in this analysis. This section begins with a description of the database and a summary of the estimation of the severity distributions by cell of the payout triangle. The increase in risk over time and across lags is considered using means, variances, and medians. We then turn to a discussion of the estimation of the overall discounted severity distributions for each accident year using a single distribution for each year and a mixture of distributions based on the distributions applicable to the cells of the triangle.

The Database

The database consists of products liability losses covering accident years 1973 through 1986 obtained from the Insurance Services Office (ISO). Data are on an occurrence basis, i.e., the observations represent paid and/or reserved amounts aggregated by occurrence, where an occurrence is defined as an event that gives rise to a payment or reserve. Because claim amounts are aggregated within occurrences, a single occurrence loss amount may represent payments to multiple plaintiffs for a given loss event. Claim amounts represent the total of bodily injury and property damage liability payments arising out of an occurrence.[4] For purposes of statistical analysis, the loss amount for any given occurrence is the sum of the loss and loss adjustment expense. This is appropriate because liability policies cover adjustment expenses (such as legal fees) as well as loss payments. In the discussion to follow, the term loss is understood to refer to the sum of losses and adjustment expenses. We use data only through 1986 because of structural changes in the ISO databases that occurred at that time that makes construction of a continuous database of consistent loss measurement difficult. This data set is quite extensive and hence is sufficient to contrast the implications of the methodologies outlined in this paper.

It is important to emphasize that the database consists of paid claim amounts, mostly for closed claims. Hence, we do not need to worry about the problem of modeling inaccurate loss reserves. The use of paid claim data is consistent with most of the loss reserving literature (e.g., Taylor 2000) and is the same approach adopted by the ISO (1994, 1998, 2002).

In the data set, the date (year and quarter) of the occurrence is given, and, for purposes of our analysis, occurrences are classified by accident year of origin. For each accident year, ISO classifies claims by payout lag by aggregating all payments for a given claim across time and assigning as the payment time the dollar weighted average payment date, defined as the loss dollar weighted average of the partial payment dates. For example, if two equal payments were made for a given occurrence, the weighted average payment date would be the midpoint of the two payment dates. For closed occurrences,[5] payment amounts thus represent the total amount paid. For open claims, no payment date is provided. For open occurrences, the payment amount provided is the cumulative paid loss plus the outstanding reserve. The overall database consists of 470,319 claims.

The time between the occurrence and the weighted average payment date defines the number of lags. Lag 1 means that the weighted average payment date falls within the year of origin, lag 2 means that the weighted average payment date is in the year following the year of origin, etc. Open claims are denoted as lag 0.

By considering losses at varying lag lengths, it is possible to model the tail of the payout process for liability insurance claims. Modeling losses by payout lag is important because losses settled at the longer lags tend to be larger than those settled soon after the end of the accident period. And, as we will demonstrate, the distributions also tend to be riskier for longer lags.

Both censored and uncensored data are included in the database. Uncensored data represent occurrences for which the total payments did not exceed the policy limit. Censored data are those occurrences where payments did exceed the policy limit. For censored data, the reported loss amount is equal to the policy limit so the total payment is the policy limit plus the reported loss adjustment expense.[6] Because of the presence of censored data, the estimation was conducted using equation (9).

The numbers of occurrences by accident year and lag length are shown in Table 1.[7] The number of occurrences ranges from 17,406 for accident year 1973 to 49,290 for accident year 1983. Thirty-four per cent of the covered events for accident year 1973 were still unsettled 14 years later. Overall, about 28% of the claims during the period studied were censored.

Table 1
Products Liability Data Set—Numbers of Occurrences by Accident Year

	Payment Lag															
Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*Open*	*Total*
1973	5,011	2,876	743	528	439	395	318	151	144	201	215	195	207	66	5,917	17,406
1974	5,630	4,254	1,081	720	751	552	340	265	257	260	241	190	135		7,528	22,204
1975	8,124	4,919	1,128	1,003	805	661	419	320	259	491	235	201			7,782	26,347
1976	7,388	4,480	1,206	834	805	611	436	388	338	300	487				6,394	23,667
1977	7,616	5,023	1,359	965	796	661	607	424	286	228					4,756	22,721
1978	8,095	5,030	1,379	1,071	966	1,081	776	508	424						6,687	26,017
1979	9,529	8,658	2,196	1,939	1,525	1,281	857	961							9,516	36,462
1980	14,785	9,736	2,625	2,259	1,759	1,117	928								12,521	45,730
1981	16,439	10,599	2,850	2,191	1,629	1,333									11,596	46,637
1982	18,157	11,062	2,948	2,258	1,875										11,390	47,690
1983	19,198	12,870	3,360	2,512											11,350	49,290
1984	18,091	11,155	3,769												11,778	44,793
1985	16,671	10,525													9,876	37,072
1986	12,798														11,485	24,283

Sample mean, standard deviation, skewness and kurtosis statistics for the products liability data are presented in Table 2.[8] As expected, the sample means generally tend to increase with the lag length, indicating that larger claims tend to settle later. Exceptions to this pattern occur at some lag lengths, a result that may be due to sampling error since a few large occurrences can have a major effect on the sample statistics.[9] Standard deviations also have a tendency to increase with lag length, indicating greater variation in total losses for claims settling later. Again, exceptions to the general pattern are attributable primarily to sampling error. Means and standard deviations also increase by accident year of origin. Sample skewnesses tend to fall between 2 and 125, revealing significant departures from symmetry. The sample kurtosis estimates indicate that these distributions have quite thick tails.

Table 2
Summary Statistics

Mean

	Payment Lag														
Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*Open (lag 0)*
1973	419	1,209	7,151	20,610	15,960	24,840	29,060	33,320	45,320	34,740	46,460	74,760	64,180	70,510	2,630
1974	523	1,282	5,905	12,490	20,140	28,760	40,640	38,580	50,270	64,560	66,680	51,370	49,370		6,165
1975	625	1,520	6,918	12,320	22,890	27,950	29,040	50,810	48,660	34,460	58,920	96,170			5,195
1976	556	1,353	6,655	15,780	24,680	32,030	41,140	42,110	69,010	47,410	14,070				8,744
1977	630	1,646	8,919	20,670	34,760	38,910	43,140	48,810	51,750	40,710					8,354
1978	682	1,803	10,270	22,040	31,000	39,480	41,760	50,570	63,430						9,845
1979	796	2,140	10,970	21,310	39,780	50,300	45,380	41,940							9,872
1980	860	2,288	12,770	23,370	34,010	65,670	46,350								9,583
1981	892	2,732	11,410	26,430	43,430	49,620									13,660
1982	1,062	3,134	15,600	27,010	44,130										15,580
1983	1,090	3,297	15,930	35,570											19,350
1984	1,168	3,620	20,630												21,830
1985	1,335	5,977													19,630
1986	1,603														8,651

Standard Deviation

	Payment Lag														
Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*Open (lag 0)*
1973	1,226	7,741	28,501	192,614	37,815	52,058	60,249	69,065	140,000	45,935	111,803	177,482	716,938	264,953	16,146
1974	5,886	6,458	20,748	31,501	39,243	48,888	82,341	70,071	139,284	209,284	107,238	331,662	184,932		89,275
1975	3,493	6,890	28,794	34,205	64,653	71,344	62,290	146,969	109,545	80,436	155,563	785,493			33,166
1976	2,809	5,881	22,939	40,497	68,264	57,271	137,840	83,785	186,815	111,803	86,371				98,133
1977	2,148	8,959	35,214	60,249	82,401	88,826	111,355	134,536	123,693	103,923					53,759
1978	3,746	8,371	42,308	66,408	83,726	140,712	105,830	151,327	256,320						80,436
1979	4,183	14,625	46,690	65,879	147,309	124,097	122,474	175,784							77,330
1980	4,567	10,752	51,478	69,570	83,307	397,492	106,771								85,264
1981	3,059	16,736	69,642	72,526	124,097	130,767									80,623
1982	4,040	17,193	49,396	65,115	111,355										58,224
1983	16,465	20,496	57,359	107,703											59,161
1984	5,658	21,331	110,000												107,703
1985	7,828	198,494													115,758
1986	14,717														43,243

Note: Monetary-valued numbers are the thousands of U.S. dollars.

Table 2: Summary Statistics (Continued)

Skewness

	Payment Lag														
Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*Open (lag 0)*
1973	11.5	31.4	12.0	20.9	5.1	5.2	5.9	4.9	8.8	2.4	9.5	6.9	14.2	4.5	17.8
1974	69.2	23.3	10.0	7.7	5.2	3.5	6.6	5.5	11.4	12.8	3.9	11.4	6.6		53.1
1975	19.4	14.7	12.5	7.5	8.0	8.3	4.8	11.8	8.2	5.0	4.8	13.3			35.8
1976	39.5	18.8	12.1	7.1	9.6	4.2	12.1	6.0	5.8	5.5	10.8				39.7
1977	16.5	24.7	11.2	7.9	5.2	5.6	9.1	7.9	6.6	6.0					25.1
1978	58.3	18.4	10.6	8.0	8.8	10.4	8.2	9.8	13.4						35.1
1979	49.0	25.8	19.7	11.3	15.0	7.2	8.7	16.9							52.6
1980	36.2	20.7	14.1	7.2	8.0	25.9	5.4								42.1
1981	15.9	25.2	19.6	7.2	8.5	6.8									28.8
1982	17.9	20.2	7.6	5.6	6.6										10.7
1983	125.0	24.3	13.2	9.9											8.4
1984	39.0	24.5	29.5												48.4
1985	33.9	100.6													66.5
1986	56.8														30.4

Kurtosis

	Payment Lag														
Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*Open (lag 0)*
1973	200.6	1267.0	195.9	461.4	32.7	39.3	54.7	31.3	90.4	9.3	116.6	58.0	203.5	22.5	417.6
1974	5032.0	838.0	130.9	87.3	43.0	18.0	70.5	45.3	156.0	186.5	22.1	140.7	49.0		3065.0
1975	450.3	303.6	192.1	80.2	94.0	109.2	32.8	174.7	92.7	37.8	28.4	184.8			1995.0
1976	2204.0	511.7	218.9	70.2	144.7	27.7	181.3	58.4	42.5	40.7	130.7				1847.0
1977	406.7	818.1	167.5	89.0	35.2	45.0	12.8	84.3	56.7	47.0					846.8
1978	4400.0	473.6	141.4	84.3	122.9	149.2	96.7	133.5	224.3						1517.0
1979	3407.0	806.5	569.6	193.4	302.0	80.9	106.9	384.2							3850.0
1980	1700.0	609.0	314.9	76.2	104.0	768.2	39.7								2204.0
1981	433.1	829.6	527.4	787.1	105.3	64.4									1227.0
1982	472.0	570.4	47.0	51.6	68.9										168.9
1983	16660.0	850.9	278.1	161.4											112.4
1984	2600.0	827.3	1233.0												3367.0
1985	1642.0	10240.0													5635.0
1986	4171.0														1328.0

Estimated Severity Distributions

The products liability occurrence severity data are modeled using the generalized beta of the second kind, or *GB*2, family of probability distributions. Based on the authors' past experience with insurance data and an analysis of the empirical distribution functions of the products liability data, several members of the *GB*2 family are selected as potential products liability severity models. We first discuss fitting separate distributions for each accident year and lag length of the runoff triangle. We then discuss and compare two alternative approaches for obtaining the overall severity of loss distribution–(1) the conventional approach, which involves fitting a single distribution to aggregate losses for each accident year; and (2) the use of the estimated distributions for the runoff triangle payment lag cells to construct a mixture distribution for each accident year. In each case, we propose the use of *discounted severity distributions* that reflect the time value of money. This is an extension of the traditional approach where loss severity distributions are fitted to loss data without recognizing the timing of the loss payments, i.e., conventionally all losses for a given accident year are treated as undiscounted values regardless of when they were actually settled. The use of discounted severity distributions is consistent with the increasing emphasis on financial concepts in the actuarial literature, and such distributions could be used in applications such as the market valuation of liabilities.

Estimated Loss Distributions By Cell

In the application considered in this paper, separate distributions were fitted to the data representing occurrences for each accident year and lag length. I.e., for 1973 the distributions were fitted to occurrences settled in lag lengths 0 (open) and 1 (settled in the policy year) to 14 (settled in the thirteenth year after the policy year); for 1974, lag lengths 0 to 13, etc. The relative fits of the Weibull, the generalized gamma, the Burr 3, the Burr 12, and the four-parameter *GB*2 are investigated. These distributions were chosen because they represent two, three, and four parameter members of the *GB*2 family that have been used in prior actuarial applications. The parameter estimates are obtained by the method of maximum likelihood. Convergence only presents a problem when estimating the generalized gamma.

The log-likelihood statistics confirm that the *GB*2 provides the closest fit to the severity data for most years and lag lengths. This is to be expected because the *GB*2, with four parameters, is more general than the nested members of the family having two or three parameters. To provide an indication of its goodness of fit of two, three, and four parameter members of the *GB*2 family, Table 3 reports likelihood ratio tests comparing the Burr 12 and the Weibull distributions and the Burr 12 and *GB*2 distributions. The likelihood ratio test helps to determine whether a given distribution provides a statistically significant improvement over an alternative distribution. Testing revealed that the Burr 12 provides a better fit than the generalized gamma and slightly better than the Burr 3. Thus, we chose the Burr 12 as the best three-parameter severity distribution.

The Weibull distribution is a limiting case of the Burr 12 distribution, as the parameter q grows indefinitely large. Likelihood ratio tests lead to the rejection of the observational equivalence of the Weibull and Burr 12 distributions at the one per cent confidence level in 95 per cent of the cases presented in Table 3. The null hypothesis in the Burr 12-*GB*2 comparisons

Table 3
Likelihood Ratio Tests

Burr 12 vs. Weibull

Lag	*1973*	*1974*	*1975*	*1976*	*1977*	*1978*	*1979*	*1980*	*1981*	*1982*	*1983*	*1984*	*1985*	*1986*
0	766.00	1210.00	966.00	1286.00	214.00	366.00	1272.00	2340.00	1540.00	1060.00	460.00	320.00	540.00	1460.00
1	1560.00	1850.00	3250.00	2330.00	2204.00	2390.00	2640.00	3920.00	4140.00	4820.00	10700.00	4860.00	5280.00	6082.00
2	1168.00	1316.00	1672.00	1268.00	1516.00	1368.00	2930.00	2462.00	2594.00	3116.00	4820.00	3184.00	3526.00	
3	221.00	209.60	248.00	248.00	290.00	326.00	448.00	618.00	692.00	640.00	702.00	1102.00		
4	164.40	109.80	108.00	157.40	189.20	184.00	276.00	490.00	258.00	384.00	440.00			
5	57.00	46.60	85.60	124.40	131.40	70.00	196.00	180.00	370.00	294.00				
6	86.60	39.80	34.00	54.20	45.20	2.00	112.00	228.00	164.00					
7	45.00	21.40	12.40	62.20	2.80	22.60	1076.00	56.00						
8	12.00	18.00	13.80	21.00	24.00	24.40	68.00							
9	26.00	35.40	10.40	38.40	5.00	19.20								
10	3.40	27.20	111.80	2.80	0.40									
11	11.60	11.00	7.60	349.20										
12	38.40	67.40	186.40											
13	191.40	89.40												
14	40.82													

GB2 vs. Burr 12

Lag	*1973*	*1974*	*1975*	*1976*	*1977*	*1978*	*1979*	*1980*	*1981*	*1982*	*1983*	*1984*	*1985*	*1986*
0	168.00	108.00	52.00	372.00	0.00	16.00	22.00	0.00	80.00	240.00	280.00	60.00	0.00	120.00
1	184.00	76318.00	262.00	246.00	244.00	210.00	226.00	400.00	520.00	440.00	260.00	420.00	280.00	190.00
2	54.00	126.00	114.00	98.00	102.00	134.00	160.00	140.00	64.00	190.00	60.00	140.00	114.00	
3	0.20	12.00	4.00	10.00	8.00	4.00	22.00	12.00	6.00	16.00	10.00	0.00		
4	0.20	3.20	0.20	0.40	0.40	0.00	2.00	2.00	2.00	10.00	0.00			
5	0.80	3.40	21.40	0.00	0.00	18.00	0.00	2.00	0.00	6.00				
6	10.80	10.60	12.20	5.00	0.80	48.00	0.00	0.00	0.00					
7	0.40	0.20	1.00	0.20	9.80	2.00	46.00	10.00						
8	5.40	3.40	0.40	6.60	1.00	2.00	2.00							
9	0.80	3.80	0.00	12.20	0.40	1.40								
10	0.00	0.20	72.20	2.60	2.60									
11	0.20	0.20	2.40	36.40										
12	1.60	1.20	8.20											
13	11.60	8.60												
14	0.00													

The reported likelihood ratio statistics are twice the difference in the log-likelihood function values. The likelihood ratio statistic in the reported tests has a chi-square distribution with one degree of freedom. The hypothesis being tested in the comparison of the Weibull and the Burr 12 is that q in the Burr 12 distribution is infinite. The hypothesis in the comparison of the GB2 and the Burr 12 is that $p = 1$ in the GB2. A value larger than 3.8 represents rejection of the hypothesis at the 95% confidence level and values larger than 6.5 represent rejection of the hypothesis at the 99% confidence level. McDonald and Xu (1992) note that rejection of a hypothesis of an infinite parameter value at the 95% confidence level using the traditional likelihood ratio test is approximately equal to rejecting at the 98% confidence level with an appropriately modified test statistic.

is that the parameter p of the $GB2$ is equal to 1. The likelihood ratio tests lead to rejection of this hypothesis at the 5 per cent level in 58 per cent of the cases presented in Table 3 and also reject the hypothesis at the 1 per cent level in 52 per cent of the cases. Thus, the $GB2$ generally provides a better model of severity.

The parameter estimates for the $GB2$ are presented in the appendix. The mean exists if $p < \frac{1}{a} < q$ and the variance if $p < \frac{2}{a} < q$. The mean is defined in 96 out of the 118 cases presented in the appendix, but the variance is defined in only 33 cases. Thus, the distributions tend to be heavy-tailed. Further, policy limits must be imposed to compute conditional expected values in many cases and to compute variances in more than 70 per cent of the cases.

Because the means do not exist for many cells of the payoff matrix, the median is used as a measure of central tendency for the estimated severity distributions. The medians for the fitted $GB2$ distributions are presented in Table 4. As with the sample means, the medians tend to increase by lag length and over time. The median for lag length 1 begins at $130 in 1973 and trends upward to $389 by 1986. The lag 2 medians are almost twice as large as the lag 1 medians, ending at $726 in 1985. The medians for the later settlement periods are considerably higher. For example, the lag 4 median begins at $2,858 and ends in 1983 at $8,038. Numbers along the diagonals in Table 4 represent occurrences settled in the same calendar year. For example, 1986, lag 1 represents occurrences from the 1986 accident year settled during 1986; while 1985, lag 2 represents occurrences that were settled in 1986.

To provide examples of the shape and movement over time of the estimated distributions, $GB2$ density functions for lag 1 are presented in Figure 1. The lag 1 densities are shown for 1973, 1977 and 1981. The densities have the familiar skewed shape usually observed in insurance loss severity distributions. A noteworthy pattern is that the height of the curves at the mode tends to decrease over time and the tail becomes thicker. Thus, the probability of large claims increased over the course of the sample period.

A similar pattern is observed in the density functions for claims settled in the second runoff year (lag 2), shown in Figure 2. Again, the tail becomes progressively heavier for more recent accident years. The density functions for lag 3 (Figure 3) begin to exhibit a different shape, with a less pronounced mode and much thicker tails. For the later lags, the curves tend to have a mode at zero or are virtually flat, with large values in the tail receiving nearly as much emphasis as the lower loss values. As with the curves shown in the figures, those at the longer lags have thicker tails. Thus, insurers faced especially heavy-tailed distributions when setting products liability prices in later years.

Estimated Aggregated Annual Loss Distributions

In the conventional methodology for fitting severity distributions, the entire data set for each accident year is used to estimate a single aggregate loss distribution. However, recall from the description of the data that claims for a given accident year are paid over many years subsequent to the inception of the accident year. To estimate a discounted severity distribution using the single aggregate loss distribution approach, we discount claims paid in lags 2 and higher back

Figure 1: GB2 Lag 1 Density Functions

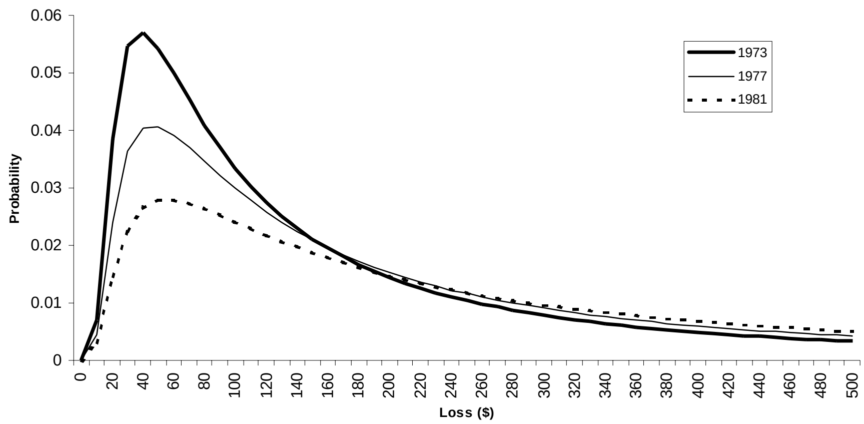


Figure 2: GB2 Lag 2 Density Functions

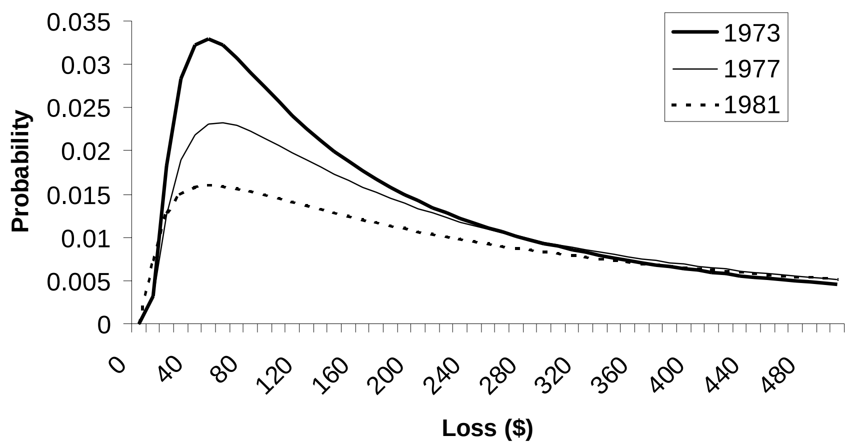


Figure 3: GB2 Lag 3 Density Functions

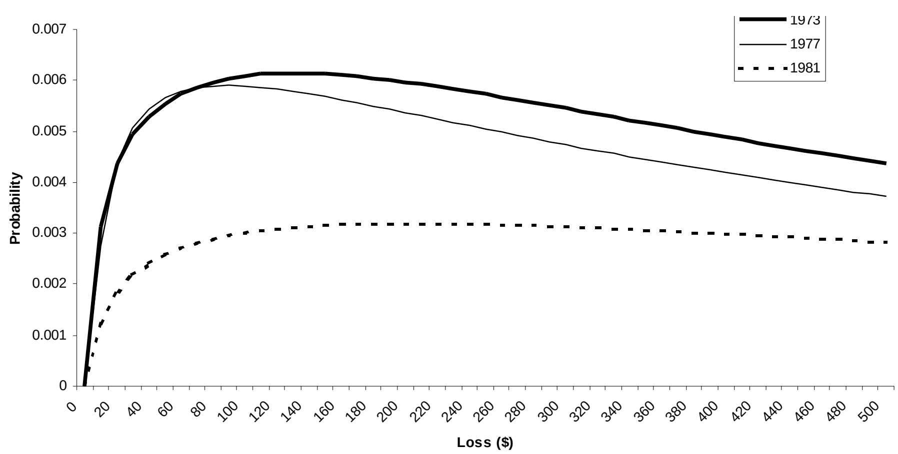


Table 4
Medians of Estimated GB2 Distributions

Year	*Lag 0*	*Lag 1*	*Lag 2*	*Lag 3*	*Lag 4*	*Lag 5*	*Lag 6*	*Lag 7*	*Lag 8*	*Lag 9*	*Lag 10*	*Lag 11*	*Lag 12*	*Lag 13*	*Lag 14*
1973	232	130	243	1,291	2,858	4,820	7,463	9,645	10,704	14,624	17,305	19,057	29,752	1,680	1,445
1974	294		251	1,362	3,372	6,827	10,948	14,183	15,864	19,098	24,464	30,137	2,619	3,026	
1975	394	146	290	1,473	3,147	4,560	6,621	8,161	16,908	17,484	4,103	7,150	2,460		
1976	812	161	328	1,354	4,477	6,545	11,993	12,235	14,000	12,293	9,877	1,535			
1977	643	183	345	1,613	4,617	9,594	11,103	10,967	12,260	15,436	5,982				
1978	659	207	377	1,989	4,964	8,274	6,585	11,986	13,208	12,332					
1979	1,040	247	9,050,370	2,140	5,237	9,264	13,102	13,912	7,507						
1980	1,262	247	1,468	2,366	5,533	9,602	1,245	13,845							
1981	1,336	273	528	2,632	6,060	11,199	11,409								
1982	1,622	304	584	3,099	7,394	11,093									
1983	2,456	283	557	3,135	8,038										
1984	3,622	300	701	3,626											
1985	4,609	210	726												
1986	1,796	389													

Table 5
Comparison of Mixture and Aggregated Distributions

Year	*Statistic*	*Mixture Distribution*	*Aggregated Distribution*
1973	Mean	\$6,266	Not Defined
	Std. Dev.	\$90,250	Not Defined
	95th Percentile	\$14,975	\$15,056
	99th Percentile	\$90,524	\$204,014
1977	Mean	\$6,896	Not Defined
	Std. Dev.	\$87,442	Not Defined
	95th Percentile	\$20,820	\$21,531
	99th Percentile	\$93,162	\$231,664
1981	Mean	\$8,383	Not Defined
	Std. Dev.	\$119,991	Not Defined
	95th Percentile	\$17,879	\$17,214
	99th Percentile	\$103,448	\$111,633
1985	Mean	\$9,599	\$13,791
	Std. Dev.	\$52,860	Not Defined
	95th Percentile	\$36,584	\$20,789
	99th Percentile	\$136,387	\$116,031

to the accident year. That is, the sample of claims used to estimate the distribution for accident year j is defined as $y_{itj}^{d} = y_{itj} / (1+r)^{t-1}$, where y_{itj} = the payment amount for claim i from accident year j settled at payout lag t, y_{ijt}^{d} = the discounted payment amount, and r = the discount rate. In the sample, $i = 1, 2, \ldots, N_{jt}$ and $t = 1, \ldots, T$, where N_{jt} = number of claims for accident year j settled at lag t, and T = number of settlement lags in the runoff triangle. Any economically justifiable discount rate could be used with this approach. In this paper, the methodology is illustrated using spot rates of interest from the U.S. Treasury security market as the discount rate.[10] We use the spot rate that has the same number of years to maturity as the lag length. Thus for a claim in year j that is settled at lag t we use a t-year spot rate from year j.

Based on the samples of discounted claims, maximum likelihood estimation provides parameter estimates of the $GB2$ distributions for each accident year. For example, of the 17,406 claims filed in 1973, 11,489 were closed during the next fourteen years and were discounted back from the time of closing to the policy year using spot rates of interest in force in the policy year. The 5,917 claims that remained open represent estimates of ultimate settlement amounts and were discounted back to the policy year assuming that they would settle in the year after the last lag year for the given policy year. Equation (9) was used for the parameter estimation, as there were claims that exceeded the policy limits and enter the estimation as censored observations. The appendix presents the $GB2$ parameter estimates.

The primary purpose for estimating the aggregated distributions is to compare it to the mixture distributions in the next section. Still, it is interesting to note that the $GB2$ distribution seems to provide the best model of the aggregated losses. We calculated likelihood ratio statistics for testing the hypotheses that the $GB2$ is observationally equivalent to the $BR3$ and $BR12$ distributions. In ten of the fourteen years, the $GB2$ provides a statistically significant improvement in fit relative to both the *BR*3 and *BR*12. In one year, 1978, the observed differences between the *GB*2 and the *BR*3 nor *BR*12 are not statistically significant at conventional levels of significance. In 1973, 1976, and 1977, the *GB*2 provides a significant improvement relative to the *BR*3, but not relative to the *BR*12. Thus, the estimated *GB*2 distribution generally appears to provide a more accurate characterization of annual losses than any of its special cases.

Estimated Mixture Distribution for Aggregate Loss Distributions

A more general formulation of an aggregate loss distribution can be constructed using a mixture distribution. In this section, we present the mixture distribution for the undiscounted case, followed by the discounted mixed severity distribution. The section concludes with a brief discussion of our Monte Carlo simulation methodology.

The undiscounted mixture distribution can be developed by first assuming that each year in the payout tail may be modeled by a possibly different distribution. It will be assumed that the distributions come from a common family ($f(y_t;\theta_t)$), with possibly different parameter values (θ_t) where the subscript "t" denotes t^{th} cell in the payout tail and y_t is the random variable loss severity in cell t for a given accident year.[11] The next step involves modeling the probability of a claim being settled in the t^{th} year, p_t, as a multinomial distribution. In the undiscounted case, the aggregate loss distribution is then obtained from the mixture distribution given by

$$f(y;\theta) = \sum_{t=1} \pi_t f(y_t;\theta_t) \tag{10}$$

Note that if θ_t is the same value for all cells, then the aggregate distribution would be the same as obtained by fitting an aggregate loss distribution $f(y;\theta)$ to the annual data. However, as mentioned, we find that the parameters differ significantly by cell within the payout triangle.

To obtain the discounted severity distribution for the mixture case, it is necessary to obtain the distributions of the discounted loss severity random variables, $y_t^d = y_t/(1+r)^{t-1}$. With the discount factor r treated as a constant, a straightforward application of the change of variable theorem reveals that discounting involves the replacement of the scale parameter of the *GB*2 distribution (equation (1)) by $b_t^d = b_t/(1+r)^{t-1}$, where b_t = the *GB*2 scale parameter for runoff cell t, and b_t^d = the scale parameter for the discounted distribution applicable to cell t.[12]

We estimate the parameters of the multinomial distribution, π_i, using the actual proportions of claims settled in each lag in our data set. The estimate of π_1 is the average of the proportions of claims actually settled in lag 1, the estimate of π_2 is the average of the proportions of claims actually settled in lag 2, etc. The estimate for p_0 is given by

$$\pi_0 = 1 - \sum_{i=1}^{n} \pi_i \ . \tag{11}$$

With the estimated cell severity distributions and the multinomial mixing distribution at hand, the mixture distribution was estimated using Monte Carlo simulation. The simulation was conducted by randomly drawing a lag from the multinomial distribution and then generating a random draw from the estimated severity distribution that corresponds to the accident year and lag. Each simulated claim thus generated is discounted back to the policy year in order to be consistent with the data used in the estimated aggregate distribution.[13] The estimated discounted mixture distribution is the empirical distribution generated by the 10,000 simulated claims for a given accident year, where each claim has been discounted to present value.

A Comparison of Estimated Aggregate Loss Distributions

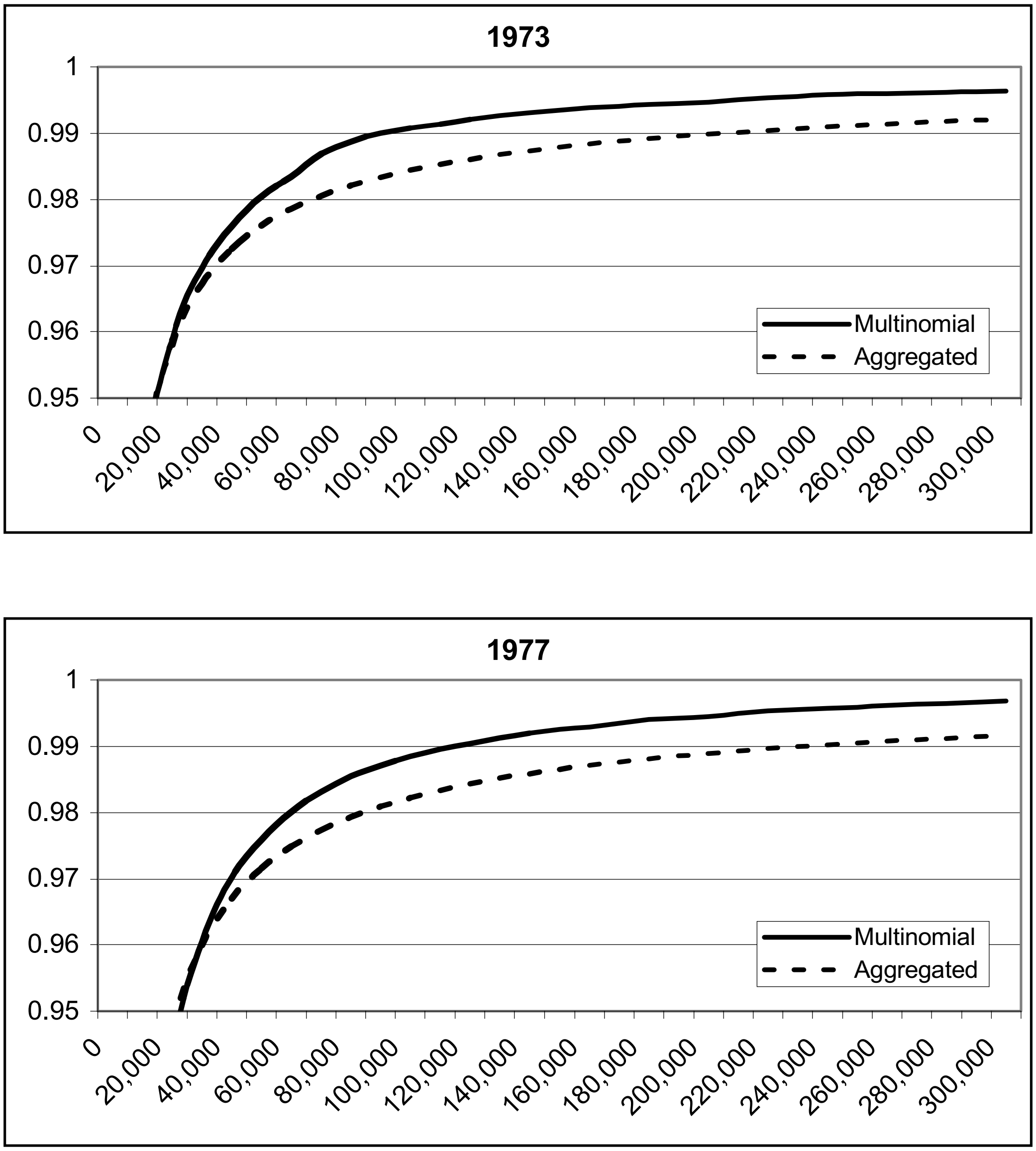
As stated above, the aggregate loss distribution can be estimated in two ways. In this section, we compare the discounted mixture distribution based on equation (10) to a single discounted loss distribution fitted to the present value of loss data aggregated over all lags. We will refer to these as the mixture distribution and the aggregated distribution, respectively. Both distributions were obtained using 10,000 simulated losses, from the mixture distribution and aggregated distribution, respectively.

With risk management in mind we illustrate the relationship between the mixture distribution and the aggregated distribution in Figure 4. Because the results are similar for the various accident years, we depict the comparison of the distributions for just four accident years – 1973, 1977, 1981, and 1985. These four years are representative of the typical relationship and are evenly spaced in time through the years covered by our data. The figure focuses on the right tail of the distribution in order to illustrate the type of errors that could be made when failing to use the mixture specification of the aggregate loss distribution.

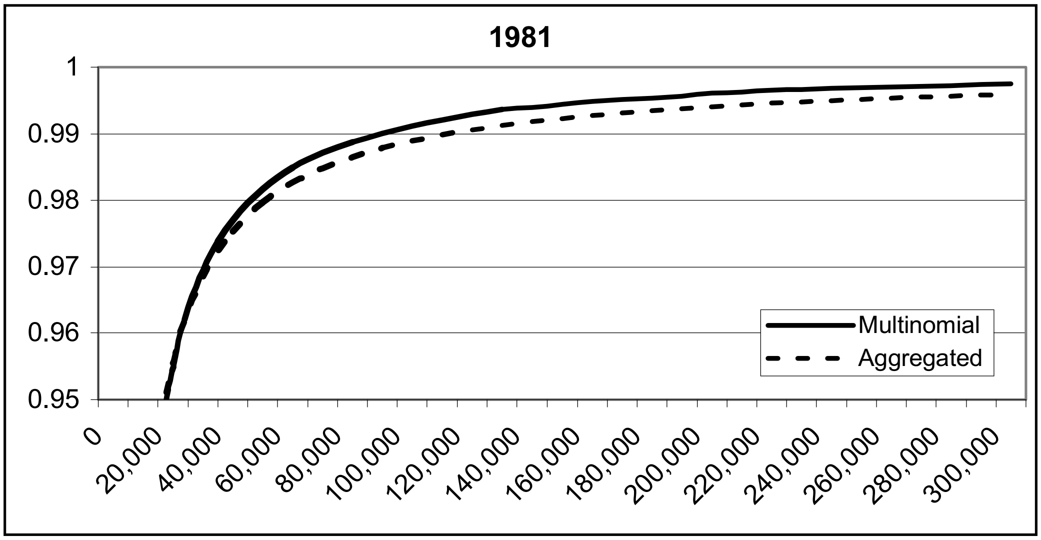
The most important conclusion based on Figure 4 is that the tails of the aggregated loss distributions are significantly heavier than the tails of the mixture distributions. Hence, the overall loss distribution applicable to the accident years shown in the figure appears to be riskier when the aggregate loss distribution is used than when the mixture distribution is used. We argue that the reason for this difference is that the aggregated distribution approach gives too much weight to the large claims occurring at the longer lags than does the mixture distribution approach. In the aggregate approach, all large claims are treated as equally likely, whereas in the mixture approach the large claims at the longer lags are given lower weights because of the lower probability of occurrence of claims at the longer lags based on the multinomial distribution. In addition, rather than fitting the relatively homogeneous claims occurring within each cell of the runoff triangle, the model fitted in the aggregate approach is trying to capture both the relatively frequent small claims from the shorter lags and the relatively frequent large claims from the longer lags, thus stretching the distribution and giving it a heavier tail. Thus, modeling the distributions by cell of the runoff triangle and then creating a mixture to represent the total severity distribution for an accident year is likely to be more accurate than fitting a single aggregate loss distribution (discounted or not) to the claims from an accident year regardless of the runoff cell in which they are closed.

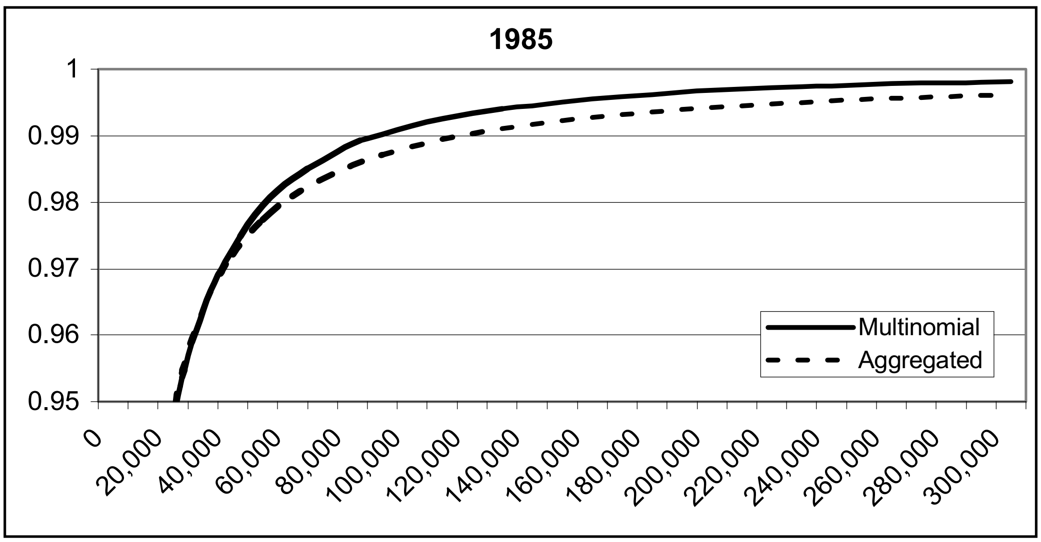
In our application, the aggregate approach overestimates the tail of the accident year severity distributions, but it is also possible that it would underestimate the tail in other applications with different patterns of closed claims. Of course, in many applications, such as reserving, it would be appropriate to work with the estimated distributions by payout cell rather than using the overall accident year distribution; but the overall discounted mixture distribution is also potentially useful in applications such as value-at-risk modeling based on market values of liabilities.

Figure 4: Aggregate GB2 Loss Distributions Using a Mixture form and an Aggregated Form



The solid line represents the cumulative loss distribution estimated by using a multinomial distribution to simulate losses drawn from the individual years of the payout tail. The dashed line represents an estimate of the cumulative loss distribution where all years of the tail are discounted to the time of pricing and a single *GB*2 distribution is fit.





SUMMARY AND CONCLUSIONS

This paper estimates loss severity distributions in the payout cells of the loss runoff triangle and uses the estimated distributions to obtain a mixture severity distribution describing total claims from an accident year. We propose the use of discounted severity distributions, which would be more appropriate for financial applications than distributions that do not recognize the timing of claims payments. We estimate severity of loss distributions for a sample of 476,107 products liability paid claims covering accident years 1973 through 1986. The claims consist of closed and open claims for occurrence based product liability policies. An innovation we introduce is to estimate distributions within each accident year/payment lag cell of the claims runoff triangle using a very general and flexible distribution, the generalized beta of the 2nd kind (GB2). Estimating distributions by cell is important because the magnitude and riskiness of liability loss distributions is a function both of the accident year of claim origin and the time lag between the occurrence of an event and the payment of the claim. Using a general severity distribution is important because conventional distributions such as the lognormal and gamma can significantly underestimate the tails of liability claims distributions.

The generalized beta family of distributions provides an excellent model for our products liability severity data. The estimated liability severity distributions have very thick tails. In fact, based on the *GB*2 distribution, the means of the distributions are defined for 81% of runoff triangle cells, and the variances are defined for only 28% of the cells. Thus, the imposition of policy limits is required in many cases to yield distributions with finite moments. The estimated severity distributions became more risky (heavy-tailed) during the sample period and the scale parameter for the early lags grew more rapidly than inflation. The results show that the gamma distribution, which has been adopted for theoretical modeling of claims by payout cell (e.g., Taylor 2000) would not be appropriate when dealing with the ISO products liability claims considered in this paper. Thus, it is appropriate to test the severity distributions that are to be used in any given application rather than making an assumption that the losses follow some conventional distribution.

Finally, we show that economically significant mistakes can be made if the payout tail is not accurately modeled. The mixture specification of the aggregate loss distribution leads to significantly different estimates of tail probabilities than does the aggregated form of the aggregate loss distribution. In our application, the aggregate loss distribution tends to give too much weight to the relatively large claims from the longer lags and hence tends to overestimate the right tail of the accident year severity distribution. Such errors could create serious inaccuracies in applications such as dynamic financial analysis, reinsurance decision making, and other risk management decisions. Thus, the results imply that the mixture distribution and the distributions applicable to specific cells of the runoff triangle should be used in actuarial and financial analysis rather than the more conventional aggregate distribution approach.

NOTES

1. Liability policies typically include a coverage period or accident period (usually one year) during which specified events (occurrences) are covered by the insurer. After the end of the coverage period, no new events become eligible for payment. However, the payment date for a covered event is not limited to the coverage period and may occur at any time after the date of the event. Because of the operation of the legal liability system, payments for covered events from any given accident period extend over a long period of time after the end of the coverage period. The payout period following the coverage period is often called the runoff period or payout tail. This description applies to occurrence-based liability policies, which are the type of policies used almost exclusively during the 1970s and early-to-mid 1980s and still used extensively at the present time. Currently, insurers also offer so-called “claims made” policies, which cover the insured for claims made during the policy period rather than negligent acts that later lead to claims as in the case of occurrence policies. Our data base applies to occurrence policies, but our analytical approach also could be applied to claims made policies, which also tend to have lengthy payout tails.

2. All optimizations considered in this paper were performed using the programs GQOPT, obtained from Richard Quandt at Princeton University, and Matlab.
3. It is also straightforward to generalize the likelihood function for truncated observations, i.e., cases where observations below a specified value are not reported (see Klugman, Panjer, and Willmot 2004). Truncated observations often arise in insurance applications due to the presence of deductibles in insurance policies. We do not observe truncated observations in our database.
4. Aggregating bodily injury and property damage liability claims is appropriate for ratemaking purposes because general liability policies cover both bodily injury and property damage.
5. Closed occurrences are those for which the insurer believes there will be no further loss payments. These are identified in the database as observations for which the loss reserve is zero.
6. Loss adjustment expenses traditionally have not been subject to the policy limit in liability insurance. More recently, some liability policies have capped both losses and adjustment expenses.
7. The term accident year refers to the coverage period in liability insurance, i.e., the 1973 accident year encompasses all events for which liability coverage was provided during 1973, regardless of when the claims were settled and payments were made.
8. It is to be emphasized that these are sample statistics corresponding to the data in nominal dollars. In many cases, the moments do not exist for the estimated probability distributions that are used to model the data.
9. The sample means are lower for 1985-1986 because these years are less mature than the earlier years.
10. We extracted spot rates of interest from the Federal Reserve H-15 series of U.S. Treasury constant maturity yields. This data series is available at *http://www.federalreserve.gov/releases/h15/data.htm*. For each claim that was discounted we used the spot rate of interest as of the policy year with maturity equal to the lagged delay until the claim was settled.
11. To simplify the notation, accident year subscripts are suppressed in this section. However, the y_t are understood to apply to a particular accident year.
12. Treating the discount factor as a constant would be appropriate if insurers can eliminate interest rate risk by adopting hedging strategies such as duration matching and the use of derivatives. Cummins, Phillips, and Smith (2001) show that insurers use interest rate derivatives extensively in risk management. Modeling interest as a stochastic variable is beyond the scope of the present paper.
13. Equivalently, the discounted losses could be simulated directly from the *GB*2 distributions applicable to each payout cell, using the adjusted scale parameters b_t^d.

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