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A Practical Algorithm for Multiple-Phase Control Systems in Agricultural and Natural Resource Economics

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Many important problems in agricultural and natural resource economics concern an intertemporal choice between alternate dynamic systems. This significance has motivated a theoretical literature generalizing the necessary conditions of Optimal Control Theory to multiple-phase problems. However, gaining detailed insight into their practical management is difficult because general numerical solution methods are not available. This paper resolves this deficiency through the development of a flexible and efficient computational algorithm based on a set of necessary conditions derived for finite-time, multiple-phase systems. Its effectiveness is demonstrated in an application to a nontrivial crop rotation problem.

Key words: crop management, multiple-phase systems, optimal control

Introduction

The Maximum Principle of Optimal Control Theory (Seierstad and Sydsaeter, 1987) has been employed extensively in agricultural and resource economics given its intuitive economic interpretation and the significant methodological extensions to this theory developed in other fields of study, such as engineering. Key examples of its application are the seminal works of Clark (1976) in the analysis of renewable resource exploitation, Dasgupta and Heal (1979) in the analysis of nonrenewable resources, Gisser and Sanchez (1980) in the analysis of the optimal extraction of groundwater, and McConnell (1983) in the study of soil conservation.

Yet, despite this broad utilization, there has been limited treatment of multiple-phase systems. These consist of multiple alternate regimes, each characterized by its own dynamic system, of which only one may be active at each point in time. This is surprising since this is a natural and more precise means of describing and analyzing many important choices facing decision makers. Selecting between individual crops to plant on a given area of land is one example (Doole, 2008). Other examples include determining the optimal time to switch between alternative energy sources (e.g., Tomiyama and Rossana, 1989), production technologies (e.g., Amit, 1986), and environmental policies.

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The optimal switching times for different regimes may be determined through the standard Maximum Principle if individual stages are represented by piecewise-constant control variables (e.g., Goetz, 1997). However, the numerical optimization of these systems is difficult since problem size increases geometrically with the number of discrete phases considered, and the optimal control trajectories within each phase will change substantially with each sequence. Moreover, the inclusion of transition costs introduces significant computational difficulties, as it introduces a discontinuous process into gradient calculations (Teo, Goh, and Wong, 1991). These limitations have motivated the analysis of multiple-phase systems in which the sequence of stages is pre-assigned. This approach is, in fact, relevant to many important economic problems, such as the alternative crop, technology, or government policy examples outlined above. Generalization of the Maximum Principle is required if control variables are defined within independent phases. Such conditions have been derived for two-stage systems with costless transition (Tomiyama and Rossana, 1989) and switching costs (Amit, 1986). The latter framework has also been extended to include an infinite horizon (Makris, 2001). Furthermore, Mueller, Schilizzi, and Tran (1999) developed necessary conditions for a three-phase problem.

Though this theory is well established, the practical management of multiple-phase problems is difficult to study given a distinct lack of suitable optimization algorithms. Gradient-based methods are difficult to apply to a multiple-phase system incorporating control variables in each stage because the state and costate equations are piecewise defined and the performance index has, by definition, discontinuous derivative(s) with respect to the control variable(s) within each regime. Transition costs also introduce step discontinuities into the adjoint and Hamiltonian trajectories along an optimal path. Moreover, the efficient computation of optimal strategies for multiple-phase problems of realistic complexity through dynamic programming (Miranda and Fackler, 2002) is nontrivial in most instances. This is intuitive given the large state and policy spaces typically encountered within such applications.

Some numerical procedures have been specifically constructed for multiple-phase optimal control systems, but are limited in their applicability. Mueller, Schilizzi, and Tran (1999) used linear programming to identify switch points following the manipulation of a set of necessary conditions derived for a specific problem. However, general application of this method is constrained by the lack of explicit solutions for many types of differential equations and the inability of the algorithm to optimize the control variables defined within each phase. In addition, some frameworks (e.g., Kim, Moore, and Hancher, 1989; Koundouri and Christou, 2006) do not determine the switching times through the appropriate necessary conditions of optimal control (Amit, 1986; Makris, 2001). Rather, the switching times arise naturally from endogenous switching thresholds inherent in the control systems considered. For example, in a study conducted by Kim, Moore, and Hancher (1989), the crop selected under optimal management changes as the shadow price of irrigation water adjusts over time. This method is conceptually interesting, but is restricted to those types of switching problems containing such thresholds. In addition, multiple-phase control problems also may be studied in a financial-options framework (Dixit and Pindyck, 1994), but only if no control variables are exercised over the duration of a stage (e.g., Willassen, 2004).

This paper presents a computational algorithm for the solution of multiple-phase optimal control problems incorporating transition costs. This is the first general algorithm available to practitioners that allows them to study these systems in considerable detail. Its capacity to incorporate substantial complexity overcomes the need for the restrictive assumptions required in analytical investigations of multiple-phase systems and the algorithm of Mueller, Schilizzi, and Tran (1999). The algorithm involves the iterative improvement of switch points using a root-finding procedure, an approach inspired by the use of shooting methods to solve boundary-value problems (Ascher, Mattheij, and Russell, 1995). The algorithm is based on a set of necessary conditions derived for a finite-time, multiple-phase system with different endpoint constraints and n stages. This derivation is necessary because previous theoretical studies have ignored alternative endpoint constraints, consequently narrowing their applicability, and the prior analysis of finite-time systems has been limited to two (Amit, 1986; Tomiyama and Rossana, 1989) or three (Mueller, Schilizzi, and Tran, 1999) regimes. The effectiveness of the algorithm is demonstrated in an application to a control problem incorporating strong nonlinearities and stiff process equations. Its primary purpose is to demonstrate the reasonable complexity of problems that may be solved using this computational procedure. This land-use rotation example closely follows a more detailed application in Doole (2008).

The paper is structured as follows. First, the model and necessary conditions are presented. The numerical algorithm is then described, along with a discussion of its implementation. Next, an application of this algorithm to a land-use rotation problem is presented. The final section offers conclusions and recommendations for further research. The derivation of the necessary conditions is presented in an appendix.

Model and Necessary Conditions

This section formally defines a model of a multiple-phase dynamic system. A Maximum Principle for its solution is derived and its key results are presented in a theorem.

- DEFINITION 1. A general multiple-phase system is assumed to incorporate an m-dimensional state vector $\mathbf{x}(t) = \{x^1(t), x^2(t), \ldots, x^m(t)\}$ of continuous functions, piecewise continuously differentiable over the time interval $t = [t_0, \ldots, t_n]$ and belonging to $X \in R^m$. The state variables are assumed fixed at the initial time and are denoted $\mathbf{x}_0$. The state variables free at the terminal time are denoted x_n^i for $i = [1, 2, \ldots, d]$. Terminal state variables x_n^i, for $i = [d+1, \ldots, m]$, are fixed.

- DEFINITION 2. A multiple-phase switching system is defined as $\Xi = \{T, K, \vartheta\}$, where:

  a. T is a set of discrete controls known as switching times that dictate the termination of one phase and the start of the next.

  b. $K = \{k_1, k_2, \ldots, k_n\}$ is a finite, fixed, and exogenously determined sequence of discrete (integer) states that indexes individual continuous dynamic systems, $\vartheta = \{\vartheta_k\}_{k \in K}$, where $\vartheta_k = [X, \mathbf{f}_k, U_j]$. The ordinal ranking of sequences is defined over the closed interval $j = [1, 2, \ldots, n]$.

  c. X is a continuous state space where $X \in R^m$.

  d. $\mathbf{f}_k$ is a vector of state equations for each stage k.

e. U_j is a set of admissible controls for each j in $j = [1, 2, ..., n]$. Each set lies in R^{v_j}, where v_j is the dimensionality of the control vector for phase j.

■ DEFINITION 3. A control input for a multiple-phase switching system Ξ consists of a set of vectors $\chi_{\Xi} = \{t, \mathbf{u}_j\}$, where:

a. $t = \{t_1, t_2, ..., t_{n-1}\}$ is a sequence of real numbers denoting switching times, the moment t_j at which stage k_j is terminated and the stage k_{j+1} becomes active. It follows that regime k_j is active over the interval $[t_{j-1^+}, t_{j^-}]$, where t_{j-1^+} is the moment just after t_{j-1}, and t_{j^-} is the moment just before t_j.

b. $t = t_n$ is a freely determined terminal time.

c. $\mathbf{u}_j = \{u_j^1, u_j^2, ..., u_j^{v_j}\}$ is a v_j-dimensional vector of control functions continuous over the interval $[t_{j-1^+}, t_{j^-}]$ for each j and belonging to $U \in R^{v_j}$.

The state variable is continuous at the switching times in this model; however, jumps in the state variable may be accommodated with manipulation of the necessary conditions (see Seierstad and Sydsaeter, 1987, theorem 7).

■ DEFINITION 4. A trajectory (Γ) for a multiple-phase switching system Ξ and control sequence χ_{Ξ} is admissible over the interval $t = [t_0, t_1, t_2, ..., t_{n-1}, t_n]$ if it satisfies definitions 1–3 and the continuous dynamics $\dot{\mathbf{x}} = \mathbf{f}_j(\mathbf{x}(t), \mathbf{u}_j(t), t)$, for $[t_{j-1^+}, t_{j^-}]$ and $j = [1, 2, ..., n]$, for a predefined switching sequence $K = \{k_1, k_2, ..., k_n\}$.

The above definitions permit the classification of a general multiple-phase optimal control problem.

■ PROBLEM MP. For a multiple-phase system Ξ, identify an admissible trajectory that maximizes the objective functional:

$$(1)\qquad J = e^{-rt_n}G\big(\mathbf{x}(t_n)\big) - \sum_{j=1}^{n-1} e^{-rt_j}C_j\big(\mathbf{x}(t_j)\big) + \sum_{j=1}^{n}\left[\int_{t_{j-1^+}}^{t_{j^-}}\left[e^{-rt}F_j\big(\mathbf{x}(t), \mathbf{u}_j(t)\big)\right]dt\right],$$

subject to:

$$(2)\quad \dot{\mathbf{x}} = \mathbf{f}_j\big(\mathbf{x}(t), \mathbf{u}_j(t), t\big), \text{ for } \left[t_{j-1^+}, t_{j^-}\right] \text{ and } j = [1, 2, ..., n], \text{ given } K = \{k_1, k_2, ..., k_n\},$$

$$(3)\qquad t_j \text{ free, for } j = [1, 2, ..., n],$$

$$(4)\qquad \mathbf{x}(t_j) \text{ free, for } j = [1, 2, ..., n-1],$$

$$(5)\qquad \mathbf{x}_0 \text{ fixed},$$

$$(6)\qquad x_n^i(t_n) \text{ free, for } i = [1, ..., d],$$

and

$$(7)\qquad x_n^i(t_n) \text{ fixed, for } i = [d+1, ..., m],$$

where r is an appropriate discount rate, $G(\mathbf{x}(t_n))$ is a terminal-reward function, $C_j(\mathbf{x}(t_j))$ is a switching-cost function for the jth phase, and $F_j(\mathbf{x}(t), \mathbf{u}_j(t))$ is a single-valued reward function on $X^m \times U^v$ for the jth phase. Functions $G(\cdot)$, $C(\cdot)$, and $F(\cdot)$ are all real-valued functions that are twice continuously differentiable in the relevant arguments. The terminal-value function G is defined for $x_n^i(t_n)$, where $i = [1, \ldots, d]$. The terminal-reward function $G(\mathbf{x}(t_n))$ is defined as a salvage value in economic applications of optimal control. The switching-cost function $C_j(\mathbf{x}(t_j))$ is a cost accruing to the termination of one stage and the start of another. These can be understood as terminal-value functions for individual regimes; they are a pertinent feature of many multiple-phase systems. For example, it can be costly to remove one crop and establish another or invest in the productive capacity required for the subsequent phase (Amit, 1986).

The following theorem outlines the necessary conditions required for an optimal solution to Problem MP. In it, the necessary conditions of the standard Maximum Principle of Optimal Control Theory (Seierstad and Sydsaeter, 1987) are generalized to hold for a multiple-phase control system. These conditions provide a basis to the algorithm presented in the next section.

■ THEOREM. Consider a multiple-phase system Ξ described by definitions 1–4. For $j = [1, 2, \ldots, n]$ and switching sequence $K = \{k_1, k_2, \ldots, k_n\}$, let $(\mathbf{x}^*(t), \mathbf{u}_j^*(t), t_j^*)$ denote the admissible trajectory that maximizes the value of J in Problem MP. This is the optimal trajectory Γ^*.

Define a Hamiltonian function for each regime k_j as:

(8) $$H_j\left(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t\right) = e^{-rt} F_j\left(\mathbf{x}(t), \mathbf{u}_j(t)\right) + \boldsymbol{\lambda}_j(t)\mathbf{f}_j\left(\mathbf{x}(t), \mathbf{u}_j(t), t\right)$$

across the interval $[t_{j-1^+}, t_{j^-}]$. An optimal trajectory Γ^* requires the following conditions:

i. the initial condition,

(9) $$\mathbf{x}_0 = \mathbf{x}(t_0),$$

for fixed initial state variable(s) $\mathbf{x}_0$;

ii. n m-dimensional vectors of real-valued, piecewise continuous adjoint functions $\boldsymbol{\lambda}_j(t) = \{\lambda_j^1(t), \lambda_j^2(t), \ldots, \lambda_j^m(t)\}$, defined across $j = [1, 2, \ldots, n]$ and piecewise continuously differentiable over the interval $[t_{j-1^+}, t_{j^-}]$, that satisfy

(10) $$\dot{\boldsymbol{\lambda}}_j^T(t) = -\frac{\partial H_j\left(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t\right)}{\partial \mathbf{x}(t)},$$

where $\boldsymbol{\lambda}_j^T(t)$ denotes the transpose of the n adjoint vectors;

iii. optimal control function(s) that satisfy

(11) $$\underset{\mathbf{u}_j(t)}{\text{Max}}\, H_j\left(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t\right) \text{ for all } t \in [t_{j-1^+}, t_{j^-}];$$

iv. a terminal adjoint vector $\boldsymbol{\lambda}_n(t_n)$ that satisfies

(12a) $$\boldsymbol{\lambda}_n^T(t_n) = \frac{\partial e^{-rt_n} G(\mathbf{x}(t_n), t_n)}{\partial \mathbf{x}(t_n)}$$

for state variables $x_n^i(t_n)$, where $i = [1, \ldots, d]$, free at the terminal time and defined in G,

(12b) $$\boldsymbol{\lambda}_n^T(t_n) = 0$$

replaces equation (12a) for those state variables $x_n^i(t_n)$, where $i = [1, \ldots, d]$, that are not defined in G,

(12c) $$x_n^i(t_n) = \mathbf{x}(t_n)$$

replaces equations (12a) and (12b) for fixed terminal state variables $x_n^i(t_n)$, where $i = [d + 1, \ldots, m]$;

v. a terminal time that satisfies

(13a) $$H_n(\mathbf{x}(t), \mathbf{u}_n(t), \boldsymbol{\lambda}_n(t), t)\big|_{t_n} + \frac{\partial e^{-rt_n} G(\mathbf{x}(t_n), t_n)}{\partial t_n} = 0,$$

but if no terminal value function is defined, then the equivalent of equation (13a) is

(13b) $$H_n(\mathbf{x}(t), \mathbf{u}_n(t), \boldsymbol{\lambda}_n(t), t)\big|_{t_n} = 0,$$

but if, instead, the terminal time is fixed, then

(13c) *no additional necessary condition is required*, as $t = t_n$;

vi. adjoint vectors that satisfy the boundary conditions

(14) $$\boldsymbol{\lambda}_j^T(t_{j^-}) + \frac{\partial e^{-rt_j} C_j(\mathbf{x}(t_j))}{\partial \mathbf{x}(t_j)} = \boldsymbol{\lambda}_{j+1}^T(t_{j^+})$$

at switching times $t = \{t_1, t_2, \ldots, t_{n-1}\}$ and $j = [1, 2, \ldots, n - 1]$;

vii. Hamiltonian functions that satisfy the switching conditions

(15) $$H_j(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t)\big|_{t_{j^-}} - \frac{\partial e^{-rt_j} C_j(\mathbf{x}(t_j))}{\partial t_j} = H_{j+1}(\mathbf{x}(t), \mathbf{u}_{j+1}(t), \boldsymbol{\lambda}_{j+1}(t), t)\big|_{t_{j^+}}$$

for those switching times in $t = \{t_1, t_2, \ldots, t_{n-1}\}$.

■ *Proof.* See the appendix.

The necessary conditions defined in equations (8)–(13) are analogous to the standard Maximum Principle (Seierstad and Sydsaeter, 1987). This follows the definition of a multiple-phase problem as a set of n dynamic systems. In contrast, the switching conditions in equations (14) and (15) are not found in standard single-phase optimal control problems. These describe how individual systems are linked over time under optimal management, and have been derived previously for two- and three-stage problems in finite time by Amit (1986) and Mueller, Schilizzi, and Tran (1999), respectively. However, in contrast to these previous analyses, these conditions incorporate alternative endpoint constraints and allow any number of regimes to be defined.

Equation (14) determines the optimal level of the state variable(s) at each switching time $\mathbf{x}(t_j)$ (these are referred to as transition states in the following discussion). The shadow price variables, $\boldsymbol{\lambda}_j^T(t_j)$ and $\boldsymbol{\lambda}_{j+1}^T(t_j)$, represent the marginal adjustment in optimal value accruing to a change in the state variable within the corresponding stage when switching time t_j is approached from below or above, respectively. The second term in equation (14) represents the marginal transition cost for the active regime. Equation (14) thus states that it is optimal to switch when the marginal value of a change in the state variable is equivalent between stages.

Equation (15) describes the management of optimal switching times given the relative value of alternate stages. The value of a Hamiltonian function $H_j(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t)$ evaluated at a given time represents the shadow price of altering the length of this phase. The second term in equation (15) is the rate at which transition costs change over time within regime j. Equation (15) therefore states that it is optimal to switch to the subsequent regime at time t_j if the rate at which the capital value of each stage changes over time is equal at that point.

Equations (14) and (15) are not required if T is empty. In this instance, the theorem collapses to the standard Maximum Principle. The state variable(s) could be fixed for a given switching time t_j. In this case, equation (14) is no longer required for the determination of $\mathbf{x}(t_j)$. Alternatively, the control input χ_Ξ may contain fixed switching times; equation (15) is not required in this instance.

The boundary conditions are obviously affected if switching cost functions $e^{-rt_j}C_j(\mathbf{x}(t_j))$ and/or their relevant derivatives are not defined. If switching costs do not exist or are independent of the state vector, equation (14) requires equality between the adjoint variables of stages j and $j+1$; that is, $\boldsymbol{\lambda}_j^T(t_j) = \boldsymbol{\lambda}_{j+1}^T(t_j)$. Likewise, equation (15) simplifies to a requirement of equality between the total capital value of each regime at the switching time—specifically, $H_j(\cdot)|_{t_j} = H_{j+1}(\cdot)|_{t_j}$ if switching costs are not defined, or are independent of time. Note that switching costs will be a function of time in most economic problems because of discounting. These results are analogous to the Weierstrass-Erdmann corner conditions from variational calculus (Seierstad and Sydsaeter, 1987), which are also required when state and/or control variables are subject to inequality constraints. This equivalency highlights the close symmetry between multiple-phase problems with fixed- and free-stage sequencing, provided the latter is incorporated utilizing piecewise constant controls and transition costs do not exist.

An Algorithm for Multiple-Phase Control Systems

The theorem from the previous section may be used to identify analytical solutions to multiple-phase problems of low dimension. However, such solutions are extremely difficult to obtain, even in systems incorporating only weakly nonlinear differential equations. This section consequently describes an optimization algorithm suited to the study of more complex problems.

The following algorithm is motivated by the structure of the theorem, which infers decomposition into two distinct stages. The first concerns the solution of each phase as an independent control problem at each iteration. The second concerns the updating of the switch points using equation (14), equation (15), and a bisection technique (Judd, 1998). Bisection successively reduces the size of an interval where a root is bound between function values that are opposite in sign. Bisection is utilized here, as other root-finding methods—such as the Newton, Broyden, and secant methods—require continuity of the switching conditions, a condition that is invalidated by the presence of transition costs. Newton's method also requires derivative information that is not available in this instance. The existence of a solution to an interval bisection technique is guaranteed for a continuous function through the intermediate-value theorem, provided the initial function evaluations are opposite in sign. The step discontinuity that occurs at each switch point does not void this condition in computational application given its equivalence to a continuous function whose root is located between two floating-point numbers (Press et al., 1992).

Algorithm Initialization

- STEP 1. Determine a fixed-stage sequence K. Let i denote the number of iterations. Define the maximum number of permissible iterations ($\hat{\imath}$). Define the stopping tolerance ε. Define a set of initial conditions $\Lambda = \{t_0, \mathbf{x}_0\}$. Let numeric superscripts denote the iteration number for ease of reference. Provide estimates of the optimal switching times (t_j^i for $j = [1, 2, \ldots, n-1]$) and the transition states ($\mathbf{x}(t_j^i)$ for $j = [1, 2, \ldots, n-1]$) for $i = \{1, 2\}$. Ensure $t_j^1 < t_j^2$ and $\mathbf{x}(t_j^1) < \mathbf{x}(t_j^2)$.

- STEP 2. Optimize each phase k_j, for $j = [1, 2, \ldots, n-1]$, as a fixed-point control problem using equations (8)–(11), (12c), and (13c). Equations (12c) and (13c) are determined by the estimates of t_j^i and $\mathbf{x}(t_j^i)$ from step 1. Optimize the terminal stage using equations (8)–(11) and the relevant terminal conditions from equations (12) and (13). Obtain $\boldsymbol{\lambda}_j^T(t_j)$ and compute $H_j(t_j)$ for all j. Do for $i = \{1, 2\}$.

- STEP 3. Ensure $\big(H_j^1(t_j) - (e^{-rt_j}C_j(\cdot))_{t_j}^1 - H_{j+1}^1(t_j)\big)\big(H_j^2(t_j) - (e^{-rt_j}C_j(\cdot))_{t_j}^2 - H_{j+1}^2(t_j)\big) < 0$ and $\big(\boldsymbol{\lambda}_j^1(t_j) + (e^{-rt_j}C_j(\cdot))_{\mathbf{x}(t_j)}^1 - \boldsymbol{\lambda}_{j+1}^1(t_j)\big)\big(\boldsymbol{\lambda}_j^2(t_j) + (e^{-rt_j}C_j(\cdot))_{\mathbf{x}(t_j)}^2 - \boldsymbol{\lambda}_{j+1}^2(t_j)\big) < 0$ before starting the main computation.

Algorithm Main Computation (for $i = 3$ to $\hat{\imath}$)

- STEP 1. Form switch points using the midpoint formulas $t_j^i = t_j^{i-2} + (t_j^{i-1} - t_j^{i-2})/2$ and $\mathbf{x}(t_j^i) = \mathbf{x}(t_j^{i-2}) + (\mathbf{x}(t_j^{i-1}) - \mathbf{x}(t_j^{i-2}))/2$.

- STEP 2. Optimize each phase k_j for $j = [1, 2, \ldots, n-1]$ as a fixed-point control problem using equations (8)–(11), (12c), and (13c). Equations (12c) and (13c) are determined by the estimates of t_j^i and $\mathbf{x}(t_j^i)$ from step 1. Optimize the terminal stage using equations (8)–(11) and the relevant terminal conditions from equations (12) and (13). Obtain $\boldsymbol{\lambda}_j^T(t_j)$ and compute $H_j(t_j)$ for all j.

- STEP 3. If $(\lambda_j^i(t_j) + (e^{-rt_j}C_j(\cdot))^i_{\mathbf{x}(t_j)} - \lambda^i_{j+1}(t_j))(\lambda_j^{i-2}(t_j) + (e^{-rt_j}C_j(\cdot))^{i-2}_{\mathbf{x}(t_j)} - \lambda^{i-2}_{j+1}(t_j)) > 0$, then $\mathbf{x}(t_j^i) = \mathbf{x}(t_j^{i-2})$ and $\mathbf{x}(t_j^{i-1}) = \mathbf{x}(t_j^{i-1})$. Else, $\mathbf{x}(t_j^i) = \mathbf{x}(t_j^{i-1})$ and $\mathbf{x}(t_j^{i-2}) = \mathbf{x}(t_j^{i-2})$.

- STEP 4. If $(H_j^i(t_j) - (e^{-rt_j}C_j(\cdot))^i_{t_j} - H^i_{j+1}(t_j))(H_j^{i-2}(t_j) - (e^{-rt_j}C_j(\cdot))^{i-2}_{t_j} - H^{i-2}_{j+1}(t_j)) > 0$, then $t_j^i = t_j^{i-2}$ and $t_j^{i-1} = t_j^{i-1}$. Else, $t_j^i = t_j^{i-1}$ and $t_j^{i-2} = t_j^{i-2}$.

- STEP 5. Stop and print output if $t_j^i - t_j^{i-1} < \varepsilon$ and $\mathbf{x}(t_j^i) - \mathbf{x}(t_j^{i-1}) < \varepsilon$ for all j, or $(\lambda_j^i(t_j) + (e^{-rt_j}C_j(\cdot))^i_{\mathbf{x}(t_j)} - \lambda^i_{j+1}(t_j)) < \varepsilon$ and $(H_j^i(t_j) - (e^{-rt_j}C_j(\cdot))^i_{t_j} - H^i_{j+1}(t_j)) < \varepsilon$.

- STEP 6. If $i = \hat{\imath}$, then stop and report progress; else, go to step 1. □

The following application is programmed in MATLAB version 7.1 (Miranda and Fackler, 2002). The MATLAB code for the algorithm is appended to this article filed at the AgEcon Search repository (http://ageconsearch.umn.edu/). Each sub-problem (phase) is solved utilizing a variant of the MISER parameterization algorithm of Teo, Goh, and Wong (1991). This involves an approximation of control functions within each phase through interpolation with sets of linear basis functions and solution of the discretized problem using nonlinear programming (NLP). Adjoint and state equations are integrated explicitly over the length of a stage using a differential algebraic equation method (Ascher, Mattheij, and Russell, 1995) following the definition of an initial guess of the optimal control. These control histories are subsequently iteratively improved using NLP, with the integration of the process equations repeated at each step to calculate the required gradients, until an optimal solution is obtained.

Approximation of control trajectories with basis functions introduces some degree of suboptimality. However, this is greatly reduced through the use of a high number of knot points in each stage. Ten nodes are included in each phase in this application. This is motivated by the short length of each phase (which increases the level of accuracy obtained with this number of basis functions) and a need for computational efficiency in the iterative algorithm.

It is well known that the bisection technique employed in the algorithm will converge linearly to a root in $\log(\mu_0/\varepsilon)/\log(2)$ iterations, where μ_0 is the size of the initial interval and ε is the stopping tolerance (Press et al., 1992). A loose stopping criterion ($\varepsilon = 0.01$) is used in the outer iteration in the following application, so that numerical errors generated in the optimization phase do not detrimentally affect convergence. Moreover, there is little benefit in identifying the defined state variables and switching time to a greater accuracy.

Application of Algorithm

This section describes an application of the algorithm to a crop-rotation problem. This example contains realistic biological processes within the level of complexity that may be incorporated in an optimal control framework.

Background

Annual ryegrass (*Lolium rigidum* Gaud.) is the most economically important weed constraining crop production in Western Australia. Moreover, nearly half of the annual ryegrass populations in the primary grain-growing region of this state (the Western Australian wheat belt) are estimated to be resistant to regular selective herbicides (Pannell et al., 2004). This reduces producer profit through forcing substitution toward less cost-effective replacements, such as the mechanical collection of weed seeds at harvest. The greater profitability of cereals, relative to livestock activities, in many farming systems in this region motivates continuous cropping. However, the inclusion of regular pasture phases has the potential to delay or help to minimize the effects of herbicide resistance through permitting the use of a wide range of weed control strategies, such as grazing and the application of nonselective herbicides.

Although the management of ryegrass infestations in extensive cropping systems in this region has been studied previously (Gorddard, Pannell, and Hertzler, 1995; Pannell et al., 2004), these analyses ignored the optimal management of pasture-crop rotations and inputs available during a pasture phase given the computational difficulties associated with the numerical solution of multiple-phase control problems. Doole and Pannell (2008a, b) compared the profitability of a number of alternative land-use sequences, including some incorporating pasture, in the Western Australian wheat belt. A metaheuristic search procedure, compressed annealing, was used to identify the optimal portfolio of weed management strategies among each pre-defined rotation in a complex simulation model (Doole and Pannell, 2008a). However, this framework could not simultaneously optimize phase length and the choice of weed control methods. These limitations evident in earlier work may be overcome through the employment of the algorithm presented in the previous section. Moreover, output from the following application is useful to improve interpretation of the results of the more complex model studied by Doole and Pannell (2008a, b).

It is assumed that a producer wishes to determine the optimal management of a single field in the central wheat belt of Western Australia. The goal of the producer is to maximize the net present value of a rotation between lucerne (*Medicago sativa* L.) pasture and wheat (*Triticum aestivum* L.) cropping. This consists of the sum of the net present value of income earned during the lucerne and the cereal phase and the cost of switching between them. The model explicitly studies the management of weed control inputs and phase length across the steady-state cycle. This approach is adopted as a suitable terminal value function does not exist and an appropriate equation could not be estimated given a lack of data. Stationarity of the steady-state cycle is represented through requiring equality between the initial (x_α^0) and terminal (x_β^2) state vectors. There is one switching time (t_1), and the terminal time (t_2) is free. The free terminal time determines the length of the second phase in the rotation. Time notation is omitted where not required in the following discussion for notational parsimony.

Two state variables are used to represent the weed seed population because of herbicide resistance (Gorddard, Pannell, and Hertzler, 1995). First, $x^s(t)$ is the population of annual ryegrass seeds that following germination are susceptible to the selective Group A herbicide (diclofop-methyl). Second, $x^h(t)$ is the population of seeds that following germination is resistant to this herbicide.

Pasture System Equations

The producer's problem in the lucerne phase is to maximize net present value (F_1), defined by:

(16) $$\max_{u_\alpha^1, u_\alpha^2, t_1} F_1 = \int_{t_0}^{t_1} e^{-rt}\left(\varsigma t\left(1-\frac{t}{\tau}\right) a u_\alpha^1\left(1-\frac{u_\alpha^1}{b}\right) - c_{\alpha,dose}^2 u_\alpha^2 - c_{\alpha,appl}^2\right) dt - c_{lest},$$

subject to:

(17) $$\dot{x}_\alpha^s = x_\alpha^s\left(v_1 + v_2\left(1-\frac{u_\alpha^1}{u_\alpha^1 d + l}\right)e^{-su_\alpha^2}R\right),$$

(18) $$\dot{x}_\alpha^h = x_\alpha^h\left(v_1 + v_2\left(1-\frac{u_\alpha^1}{u_\alpha^1 d + l}\right)e^{-su_\alpha^2}R\right),$$

(19) $$x_\alpha^0 = \left\{x_\alpha^s(t_0), x_\alpha^h(t_0)\right\},$$

(20 $$x_\alpha^1 = \left\{x^s(t_1), x^h(t_1)\right\},$$

and

(21) $$t_1 \text{ fixed},$$

where u_α^1 is the sheep stocking rate; u_α^2 is nonselective herbicide application [measured in kilograms of active ingredient (glyphosate) per hectare]; ς is a parameter describing the intertemporal productivity of lucerne pasture; τ is the maximum productive length of a lucerne phase; a and b are parameters describing the relationship between stocking rate and profit; $c_{\alpha,dose}^2$ is the cost of active ingredient for the nonselective herbicide; $c_{\alpha,appl}^2$ is the application cost for the nonselective herbicide; c_{lest} is a fixed cost representing the establishment cost of lucerne; $v_1 = -g-(1-g)M_{seed}$, where g is the germination rate and M_{seed} is the natural mortality rate of ungerminated seeds; $v_2 = g(1-M_{plant})$, where M_{plant} is the natural mortality rate of germinated seeds; d and l are parameters describing the strength of the relationship between grazing rate and weed control; s describes the efficacy of the nonselective herbicide; and R is the number of seeds produced by a single weed. Equation (19) is the set of initial conditions, and the terminal conditions in equations (20) and (21) are determined by the estimated switch points in the algorithm.

Cereal System Equations

The terminal time is free in the cereal phase to endogenize its optimal length in the cycle. This is dealt with numerically through the introduction of an additional control parameter $u_\beta^3 = [0, \ldots, t_{max}]$, where the control variable may take any value in the set $U_\beta^3 = [0, \ldots, t_{max}]$, and t_{max} is the maximum length of the cereal phase.

The producer's problem in the cereal phase is to maximize net present value (F_2), defined by:

(22) $$\max_{u_\beta^1, u_\beta^2, u_\beta^3} F_2 = \int_0^1 u_\beta^3 e^{-r(t_1+u_\beta^3 t)}\left(\begin{array}{c} py_0\left(1-\mu u_\beta^1\right)\left((1-z)+z\left(\frac{m}{m+kw(t)}\right)\right) \\ -c_{\beta,dose}^1 u_\beta^1(t) - c_{\beta,appl}^1 - c_\beta^2\left(\frac{u_\beta^2}{1-(u_\beta^2)}\right) - c_{cest}\end{array}\right) dt,$$

subject to:

$$\dot{x}_\beta^s = u_\beta^3 x_\beta^s \left(v_1 + v_2 e^{-qu_\beta^1}(1 - u_\beta^2)R\right), \tag{23}$$

$$\dot{x}_\beta^h = u_\beta^3 x_\beta^h \left(v_1 + v_2(1 - u_\beta^2)R\right), \tag{24}$$

$$x_\beta^1 = \left\{x^s(t_1), x^h(t_1)\right\}, \tag{25}$$

$$t_1 \text{ fixed}, \tag{26}$$

and

$$x_\beta^2 = \left\{x_\alpha^s(t_0), x_\alpha^h(t_0)\right\}, \tag{27}$$

where u_β^1 is selective herbicide application [measured in kilograms of active ingredient (diclofop methyl) per hectare], u_β^2 is the intensity of nonselective weed control utilized during the cropping phase, p is a constant price, y_0 is weed-free yield, η is the proportion of yield lost to phytotoxic damage for a given dosage of selective herbicide, z is the maximum proportion of grain yield lost at high weed density, m is a crop-dependent density parameter, k is a constant representing the competitiveness between the weed population and the wheat crop, $w(t)$ represents the total weed population, $c_{\beta,dose}^1$ is the cost of the selective herbicide dose, $c_{\beta,appl}^1$ is the cost of applying this dosage, c_β^2 is the cost of achieving 50% weed control using alternative weed control treatments (Gorddard, Pannell, and Hertzler, 1995), c_{cest} is a fixed cost representing the establishment cost of wheat, and q is a parameter designating the strength of the relationship between ryegrass mortality and selective herbicide dosage. The weed population is $w(t) = w^s(t) + w^h(t)$, where w^s is the susceptible weed population and w^h is the herbicide-resistant weed population. These are related to the susceptible and resistant seed populations through $w^s = x^s g(1 - M_{plant})e^{-qu_2^1}(1 - u_2^2)$ and $w^h = x^h g(1 - M_{plant})(1 - u_2^2)$. It is observable from equations (23) and (24) that the selective herbicide only affects the susceptible weed population. The initial conditions in equations (25) and (26) for the second phase are determined by the estimated switch points in the algorithm.

The effective removal of lucerne requires careful grazing management and the application of nonselective herbicides. A switching cost function for t_1 is therefore defined as $e^{-rt_1}c_{lrem}$, where c_{lrem} is the fixed cost of lucerne removal. This is obviously not a function of the state variables, so equation (14) will hold as $\lambda_1^T(t_1) = \lambda_2^T(t_1)$ at $x(t_1)$ in this example.

The decision problem faced by the producer is therefore to maximize the net present value of income earned during the steady-state rotation, defined as $J = F_1 + F_2 - e^{-rt_1}c_{lrem}$. The parameter values for this application are presented in table 1. All values are expressed in Australian dollars. Detailed information on the estimation of parameters may be obtained from the author on request.

Model Output

Base Model Output

This section presents the standard solution obtained using the algorithm presented in the previous section. The base model incorporates initial conditions of 50 susceptible

Table 1. Parameter Values for the Two-Phase Model

Parameter	Description	Value
r	Discount rate	$r = 0.05$
ς	Parameter describing lucerne productivity	$\varsigma = 0.78$
τ	Maximum productive length of a lucerne phase	$\tau = 4.96$
a, b	Pasture profit function parameters	$a = 25.32$, $b = 14.88$
$c^2_{\alpha,dose}$, $c^2_{\alpha,appl}$	Nonselective herbicide cost parameters	$c^2_{\alpha,dose} = \$12.50$, $c^2_{\alpha,appl} = \$2.50$
g	Germination rate	$g = 0.8$
M_{seed}	Rate of seed mortality	$M_{seed} = 0.55$
M_{plant}	Rate of plant mortality	$M_{plant} = 0.05$
d, l	Parameters describing weed control by grazing	$d = 1.11$, $l = 0.5$
s	Efficacy of nonselective herbicide	$s = 7.87$
R	Seed production per plant	$R = 100$
c_{lest}	Cost of lucerne establishment	$c_{lest} = \$88$
p	Price per ton of wheat	$p = \$185$
y_0	Weed-free yield	$y_0 = 1.82$ tons ha^{-1}
η	Parameter describing rate of phytotoxic damage	$\eta = 0.14$
z, m, k	Wheat yield function parameters	$z = 0.6$, $m = 105$, $k = 0.33$
c_{cest}	Cost of cereal establishment	$c_{cest} = \$82$
q	Efficacy of selective herbicide	$q = 7.45$
$c^1_{\beta,dose}$, $c^1_{\beta,appl}$, c^2_{β}	Control cost parameters for crop phase	$c^1_{\beta,dose} = 40$, $c^1_{\beta,appl} = \$2.50$, $c^2_{\beta} = \$1.09$

Note: All values are expressed in Australian dollars.

seeds per square meter (ss m^{-2}) and 25 herbicide-resistant seeds per square meter (rs m^{-2}). The algorithm uses 14 iterations to identify a solution for this problem. As shown in figure 1, the switching time denoting the length of the lucerne phase occurs at 3.6 years, while the cereal phase is defined for the four years between this switching time and the terminal time of 7.6 years. The optimal transition states are 45 ss m^{-2} and 23 rs m^{-2}.

Figure 1 illustrates a number of important results. First, both seed populations decline over the length of the lucerne phase (i.e., between [0, 3.6]). This decrease reflects the combined efficacy of grazing and nonselective herbicide application for weed control in a pasture phase. The value of integrated weed management during a pasture phase highlights the intuitive unsuitability of single-phase models (e.g., Gorddard, Pannell, and Hertzler, 1995) for resistance analysis. Second, the state trajectories within the first phase are clearly convex. The initial reduction in the state variables is caused directly by spray-topping with glyphosate, with an application rate that decreases from 0.67 kilograms of active ingredient per hectare (kg ai h^{-1}) to 0.29 kg ai ha^{-1} over the duration of this regime. This phase could be much shorter if effective weed control were the only determinant of value. However, it is extended to maintain grazing—at a constant rate of 7.46 DSE ha^{-1} under optimal management—over the most productive years of the lucerne stand. Third, no selective herbicide is applied in the cropping phase. Rather, intensive nonselective control is maintained at around 98%. This is necessary to minimize population growth in the ryegrass population due to its strong competitiveness with crops and its large seed production. Finally, the maximum length of the cereal crop (four years) is adopted following the effective restraint of the ryegrass seed population over the pasture phase.

In contrast to the algorithm developed by Mueller, Schilizzi, and Tran (1999), the procedure utilized here does not require the differential equations describing the transition of the state and adjoint variables to be explicitly solvable. This improves its ability to incorporate realistic functional forms and thereby provide relevant and practicable conclusions. For example, Mueller, Schilizzi, and Tran identified an optimal cropping phase of 25 years duration in their standard model, compared with four years in this application. The former is unrealistic since it will often be profitable to switch to pasture production after four years of continuous cropping following soil structure decline and the development of disease (Greenland, 1971). This is one example of the value accruing to the greater complexity that may be incorporated through the use of the algorithm presented above.

Important parameters in the decision problem are now perturbed to gain insight into their impacts on the optimal solution identified by the algorithm. A low (p_{low} = \$148) and high (p_{high} = \$222) wheat price are defined as a 20% decrease and increase, respectively, of the standard cereal price (p_{base} = \$185). The implications of alternative discount rates (r) of $\{0, 0.025, 0.075, 0.1\}$ and removing the switching cost are also explored. The switch points for the base model and these alternative scenarios are presented in table 2 to aid comparison. The characteristics of each solution are discussed in detail in the following.

Impacts of Changes in the Cereal Price

A 20% decrease in the price received for a ton of wheat increases the optimal switching time from 3.4 years to 4.8 years (figure 2). This result is logical given the subsequent lower profitability of the cropping phase, relative to the pasture enterprise. The price decrease also reduces the marginal value of weed management carried out during the lucerne phase, as the optimal transition states increase from 45 ss m^{-2} and 23 rs m^{-2} to 49 ss m^{-2} and 24 rs m^{-2}, respectively. The continuity of the state variables and the point of nondifferentiability (corner) in the time derivatives of their trajectories at the switching time (t_1) are both obvious in figure 2. These follow naturally from the definition of the discrete phases.

The optimal switching time decreases to three years and the optimal transition states decrease from 45 ss m^{-2} and 23 rs m^{-2} to 39 ss m^{-2} and 19 rs m^{-2}, respectively, at a cereal price that is 20% higher than in the base model. The higher wheat price magnifies the marginal value of weed control conducted during the lucerne phase. The lucerne stand is retained over its most productive period (lucerne production peaks at $\tau \cdot 2^{-1}$ = 2.48 years), despite the enhanced value of the cereal crop. This finding demonstrates the importance of grazing to recoup the significant establishment cost of lucerne and the direct relationship between livestock and pasture production.

The optimal stocking rate does not change following any of these modifications of the cereal price. This result is intuitive since the (logistic) profit function will decline past a given grazing rate ($b \cdot 2^{-1}$ = 7.15 DSE ha^{-1}) and the marginal contribution of an additional grazing unit to weed control is rapidly diminishing at this point. In contrast,

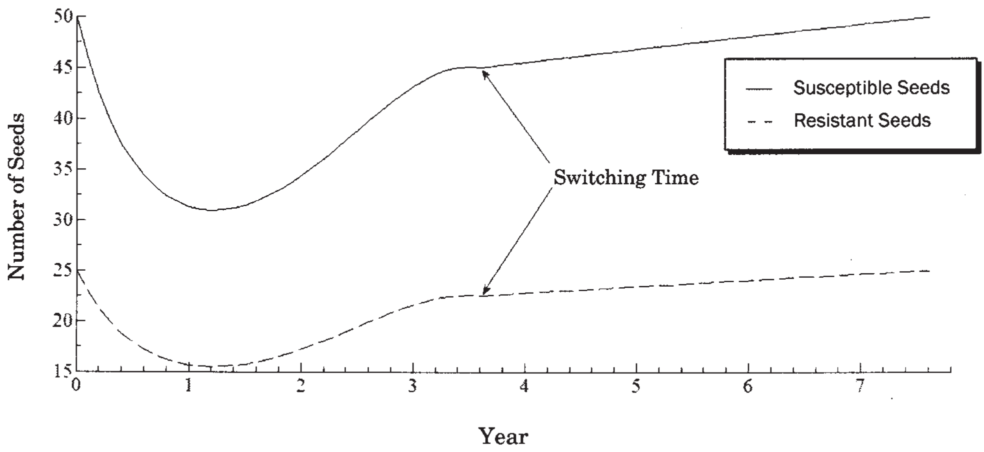

Figure 1. The optimal seed trajectories for a lucerne and cereal rotation with initial seed populations of 50 ss m^{-2} and 25 rs m^{-2}

Table 2. Optimal Switch Points for Different Model Scenarios

Model Scenario	Switching Time in Years (t_1)	Susceptible Seed Population at t_1 ($x^s(t_1)$)	Resistant Seed Population at t_1 ($x^h(t_1)$)
Base model (p = \$185, r = 0.05)	3.6	45	23
p_{low} = \$148	4.8	49	24
p_{high} = \$222	3.0	39	19
$r = 0$	5.2	40	20
$r = 0.025$	4.7	42	21
$r = 0.075$	3.1	48	24
$r = 0.1$	2.7	53	27
No switching cost	2.8	41	21

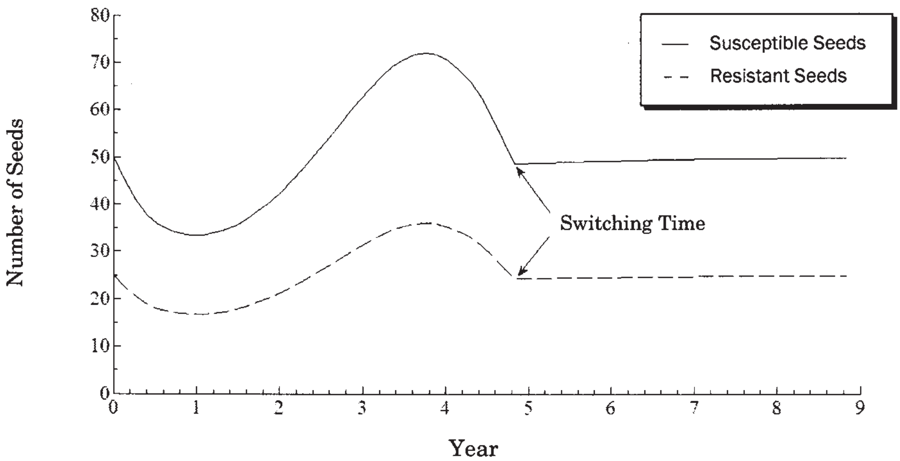

Figure 2. The optimal seed trajectories across a lucerne and cereal rotation for the low cereal price

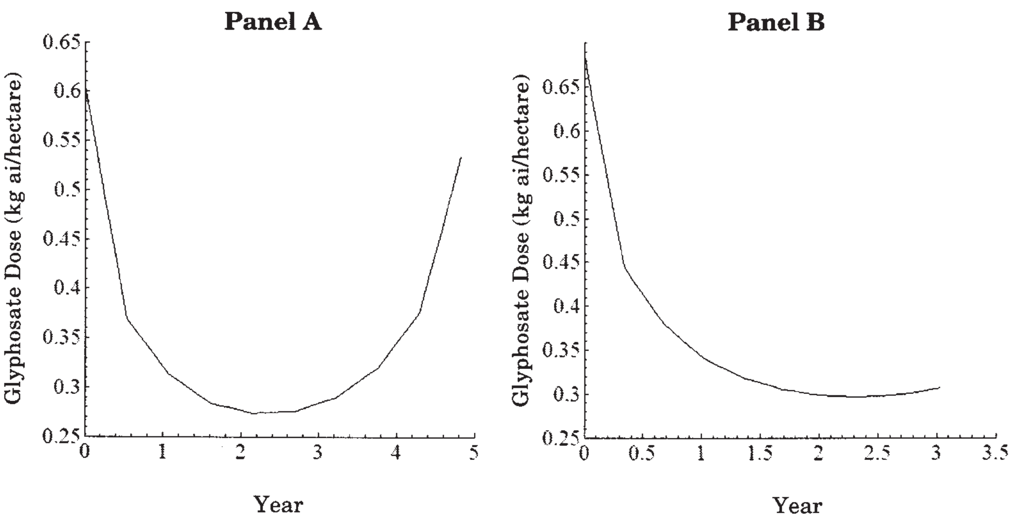

Figure 3. Kilograms of glyphosate applied during the pasture phase for the low cereal price (panel A) and the high cereal price (panel B)

the pattern of glyphosate application changes markedly. The reductions in the weed populations at the beginning and the end of the pasture phase when the wheat price is low (figure 2) are achieved through high rates of glyphosate application (figure 3, panel A). In comparison, the shorter pasture phase motivated by the increased cereal price encourages the producer to maintain low seed populations throughout its duration. This is achieved with a primarily declining dose rate (figure 3, panel B), very similar to that employed in the base model. The use of linear basis functions to approximate the continuous control functions is evident in figure 3, where the close examination of each trajectory reveals its piecewise definition.

Implications of Alternative Discount Rates

An increase in the discount rate decreases the profitability of each phase, consequently reducing the optimal switching time (table 2). This effect offsets the positive impact that a higher discount rate has on the relative value of the first phase through the switching-cost function. In contrast, the susceptible and resistant seed densities at the switching time increase with the discount rate under optimal management. This result is intuitive because discounting directly reduces the future cost of weed infestation, motivating a decrease in the mean rate of glyphosate application from 0.376 kg ai ha^{-1} to 0.358 kg ai ha^{-1} over the length of the lucerne stage as the discount rate increases from 0% to 10%.

Effects of Removing Transition Costs

The costless removal of lucerne ($c_{lrem} = 0$) decreases the optimal switching time from 3.6 to 2.8 years and the optimal transition states from 45 ss m^{-2} and 23 rs m^{-2} to 41 ss m^{-2} and 21 rs m^{-2}, respectively. The derivative of the switching-cost term (i.e., $re^{-rt}c_{lrem}$) is positive in equation (15), the switching condition that determines the optimal switching time, since extending the length of the pasture phase will reduce switching costs through discounting, increasing its dynamic profitability. Costless removal eliminates this benefit, thus reducing the value of the lucerne enterprise, and thereby motivating a shorter pasture phase under optimal management.

Conclusions

There is no general framework for the numerical optimization of multiple-phase systems in which control variables are defined in each stage. This is a significant limitation because these systems arise in many important economic problems, such as determining the optimal time to switch between production technologies, energy sources, and land uses. The computational algorithm presented in this paper offers a flexible and efficient platform for the solution of multiple-phase problems in which the number and sequence of phases is pre-assigned.

The reasonable complexity of problems that may be handled using this algorithm is demonstrated in an investigation of the optimal management of herbicide-resistant weed populations in a pasture-cereal rotation in the Western Australian wheat belt. Model output shows that integrated weed management implemented in a pasture phase is important when the efficacy of selective herbicides is limited by resistance. In fact, use of intensive nonselective control is necessary in both phases of the rotation, motivated by a strong economic incentive to maintain a low number of seeding plants given the competitiveness and large seed production of annual ryegrass. A higher wheat price, higher discount rate, or lower switching cost decreases the optimal length of the perennial pasture phase. Thus, even though the establishment of perennial pasture requires substantial investment, it is pertinent for producers to adjust the length of these phases in response to temporal changes in economic conditions and available technology.

Furthermore, this application provides insight into the operation of the multiple-phase control algorithm. The bisection procedure used therein requires switching conditions that alternate in sign to effectively begin the search for optimal switch points. Identification of these initial values may be time-consuming in some models; thus, coding of a search algorithm to automate this process may be of benefit. However, it would require careful construction to avoid infeasibilities from interrupting its operation. In addition, the identification of starting points through experimentation is valuable for increasing a practitioner's knowledge of the framework in question.

The use of an efficient computational procedure to solve the individual control problem posed by each phase is also important if the algorithm is to solve in a reasonable amount of time. The employment of control parameterization in this application is valuable given the complexity of the problem and the difficulty involved with other methods of solution, particularly those—such as multiple shooting—that require the analytical derivation of adjoint trajectories and accurate guesses of their evolution across time.

An interesting extension of this work would be an examination of the sufficiency of the necessary conditions derived in this paper, perhaps using viscosity solutions. Preliminary experiments also indicate that the efficiency of solution may be improved through solving the control problem posed by each individual phase using parallel processing.

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References

Amit, R. "Petroleum Reservoir Exploitation: Switching from Primary to Secondary Recovery." *Operations Res.* 34(1986):534–549.

Ascher, U. M., R. M. M. Mattheij, and R. D. Russell. *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations.* Philadelphia: SIAM, 1995.

Clark, C. W. *Mathematical Bioeconomics: The Optimal Management of Renewable Resources.* New York: John Wiley, 1976.

Dasgupta, P. S., and G. M. Heal. *Economic Theory and Exhaustible Resources.* Cambridge: Cambridge University Press, 1979.

Dixit, A., and R. Pindyck. *Investment Under Uncertainty.* Princeton, NJ: Princeton University Press, 1994.

Doole, G. J. "Optimal Management of Annual Ryegrass (*Lolium rigidum* Gaud.) in Phase Rotations in the Western Australian Wheatbelt." *Austral. J. Agr. and Resour. Econ.* 52(2008):339–362.

Doole, G. J., and D. J. Pannell. "Optimization of a Large, Constrained Simulation Model Using Compressed Annealing." *J. Agr. Econ.* 59(2008a):188–206.

———. "Role and Value of Including Lucerne (*Medicago sativa* L.) Phases in Crop Rotations for the Management of Herbicide-Resistant *Lolium rigidum* in Western Australia." *Crop Protection* 27(2008b):497–504.

Gisser, M., and D. A. Sanchez. "Competition versus Optimal Control in Groundwater Pumping." *Water Resour. Res.* 16(1980):638–642.
Goetz, R. U. "Diversification in Agricultural Production: A Dynamic Model of Optimal Cropping to Manage Soil Erosion." *Amer. J. Agr. Econ.* 79(1997):341–356.
Gorddard, R. J., D. J. Pannell, and G. Hertzler. "An Optimal Control Model for Integrated Weed Management Under Herbicide Resistance." *Austral. J. Agr. Econ.* 39(1995):71–87.
Greenland, D. J. "Changes in the Nitrogen Status and Physical Conditions of Soils Under Pasture with Special Reference to the Maintenance of the Fertility of Australian Soils Used for Wheat Growing." *Soils and Fertilizers* 34(1971):237–251.
Hull, D. G. *Optimal Control Theory for Applications.* New York: Springer Publishers, 2003.
Judd, K. L. *Numerical Methods in Economics.* Cambridge, MA: MIT Press, 1998.
Kim, C. S., M. R. Moore, and J. J. Hanchar. "A Dynamic Model of Adaptation to Resource Depletion: Theory and an Application to Groundwater Mining." *J. Environ. Econ. and Mgmt.* 17(1989):66–82.
Koundouri, P., and C. Christou. "Dynamic Adaptation to Resource Scarcity and Backstop Availability: Theory and Application to Groundwater." *Austral. J. Agr. and Resour. Econ.* 50(2006):227–245.
Makris, M. "Necessary Conditions for Infinite-Horizon Discounted Two-Stage Optimal Control Problems." *J. Econ. Dynamics and Control* 25(2001):1935–1950.
McConnell, K. E. "An Economic Model of Soil Conservation." *Amer. J. Agr. Econ.* 65(1983):83–89.
Miranda, M. J., and P. L. Fackler. *Applied Computational Economics and Finance.* Cambridge, MA: MIT Press, 2002.
Mueller, U., S. Schilizzi, and T. Tran. "The Dynamics of Phase Farming in Dryland Salinity Abatement." *Austral. J. Agr. and Resour. Econ.* 43(1999):51–67.
Pannell, D. J., V. Stewart, A. Bennett, M. Monjardino, C. Schmidt, and S. B. Powles. "RIM: A Bioeconomic Model for Integrated Weed Management of *Lolium rigidum* in Western Australia." *Agr. Systems* 79(2004):305–325.
Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. D. Flannery. *Numerical Recipes in C: The Art of Scientific Computing.* Cambridge: Cambridge University Press, 1992.
Seierstad, A., and K. Sydsaeter. *Optimal Control Theory with Economic Applications.* Amsterdam: North-Holland Publishers, 1987.
Teo, K. L., C. J. Goh, and K. H. Wong. *A Unified Computational Approach for Optimal Control Problems.* New York: Longman Scientific and Technical Publishers, 1991.
Tomiyama, K., and R. Rossana. "Two-Stage Optimal Control Problems with an Explicit Switch Point Dependence." *J. Econ. Dynamics and Control* 13(1989):319–337.
Willassen, Y. "On the Economics of the Optimal Fallow-Cultivation Cycle." *J. Econ. Dynamics and Control* 28(2004):1541–1556.

Appendix: Derivation of Necessary Conditions

This appendix outlines a proof of the theorem from the second section of the text using both weak (Amit, 1986) and strong (Seierstad and Sydsaeter, 1987; Hull, 2003) variations.

First, adjoin the n constraint(s) from equation (2) to the objective functional [equation (1)] using n m-dimensional vectors of adjoint multipliers $\boldsymbol{\lambda}_j(t)$ to form the augmented functional ϕ:

$$\text{(A1)} \qquad \phi = e^{-rt_n}G(\mathbf{x}(t_n)) - \sum_{j=1}^{n-1} e^{-rt_j}C_j(\mathbf{x}(t_j)) + \sum_{j=1}^{n} \int_{t_{j-1^+}}^{t_j^-} \left[e^{-rt}F_j(\mathbf{x}(t), \mathbf{u}_j(t)) + \boldsymbol{\lambda}_j^T(t)\left(\mathbf{f}_j(\mathbf{x}(t), \mathbf{u}_j(t), t) - \dot{\mathbf{x}}(t)\right)\right] dt.$$

Define a Hamiltonian function for each stage, $H_j(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t) = e^{-rt}F_j(\cdot) + \boldsymbol{\lambda}_j^T(t)\mathbf{f}_j(\cdot)$, and substitute into equation (A1):

$$\text{(A2)} \qquad \phi = e^{-rt_n}G(\mathbf{x}(t_n)) - \sum_{j=1}^{n-1} e^{-rt_j}C_j(\mathbf{x}(t_j)) + \sum_{j=1}^{n} \int_{t_{j-1^+}}^{t_j^-} \left[H_j(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t) - \boldsymbol{\lambda}_j^T(t)\dot{\mathbf{x}}(t)\right] dt.$$

Integrate the final term in the square brackets in equation (A2) by parts:

$$\text{(A3)}\qquad \phi = e^{-rt_n}G(\mathbf{x}(t_n)) - \sum_{j=1}^{n-1} e^{-rt_j}C_j(\mathbf{x}(t_j)) + \sum_{j=1}^{n}\left(-\boldsymbol{\lambda}_j^T(t_{j^-})\mathbf{x}(t_{j^-}) + \boldsymbol{\lambda}_j^T(t_{j-1^+})\mathbf{x}(t_{j-1^+})\right)$$
$$+ \sum_{j=1}^{n}\int_{t_{j-1^+}}^{t_{j^-}}\left[H_j(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t) - \dot{\boldsymbol{\lambda}}_j^T(t)\dot{\mathbf{x}}(t)\right]dt.$$

The differential of equation (A3) is:

$$\text{(A4)}\quad d\phi = \left(H_n(\cdot)\Big|_{t_n} + \frac{\partial e^{-rt_n}G(\mathbf{x}(t_n))}{\partial t_n}\right)dt_n + \sum_{j=1}^{n}\int_{t_{j-1^+}}^{t_{j^-}}\left\{\left(\frac{\partial H_j(\cdot)}{\partial \mathbf{x}(t)} + \frac{\partial \dot{\boldsymbol{\lambda}}_j^T(t)\mathbf{x}(t)}{\partial \mathbf{x}(t)}\right)\delta\mathbf{x} + \frac{\partial H_j(\cdot)}{\partial \mathbf{u}_j(t)}\delta\mathbf{u}\right\}dt$$
$$+ \sum_{j=0}^{n-1}\left(H_j(\cdot)\Big|_{t_{j^-}} - \frac{\partial e^{-rt_j}C_j(\mathbf{x}(t_j))}{\partial t_j} - H_{j+1}(\cdot)\Big|_{t_{j^+}}\right)dt_j + \left(\frac{\partial e^{-rt_n}G(\mathbf{x}(t_n))}{\partial \mathbf{x}(t_n)} - \boldsymbol{\lambda}_j^T(t_n)\right)d\mathbf{x}(t_n)$$
$$+ \sum_{j=1}^{n-1}\left(\boldsymbol{\lambda}_{j+1}^T(t_{j^+}) - \boldsymbol{\lambda}_j^T(t_{j^-}) - \frac{\partial e^{-rt_j}C_j(\mathbf{x}(t_j))}{\partial \mathbf{x}(t_j)}\right)d\mathbf{x}(t_j),$$

where $d\phi$, dt, and $d\mathbf{x}(t_n)$ are differential changes in the performance index, time, and the state variable at the final moment; $H_j(\cdot) = H_j(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t)$ for $j = \{1, 2, \ldots, n\}$; $\delta\mathbf{x}$ and $\delta\mathbf{u}$ represent variations in the state and control trajectories; and $H_0 \equiv 0$. The functional ϕ is extremal if it is stationary with respect to arbitrary perturbations. A stationary ϕ requires necessary conditions (i)–(vii) in the theorem in the second section of the text, except the control functions in (iii) are selected so $(H_j(\cdot))_{\mathbf{u}} = 0$ for $j = [1, 2, \ldots, n]$.

Condition (iii) is now generalized to hold for constrained control variables using strong variations. Assume that $(\mathbf{x}^*(t), \mathbf{u}_j^*(t), t_j^*)$ are optimal through satisfaction of conditions (i)–(vii) in the theorem. Form a comparison vector of control functions $\mathbf{u}_j(t)$ for each regime, admissible through definitions 1–4. Over the length of each phase ($[t_{j-1^+}, t_{j^-}]$), consider three intervals:

- $t_{j-1^+} \le t \le t_a$, where $\mathbf{u}_j^*(t) - \mathbf{u}_j(t) = 0$;
- $t_a \le t \le t_b$, where $\mathbf{u}_j^*(t) - \mathbf{u}_j(t) = \text{constant}$;

and

- $t_b \le t \le t_{j^-}$, where $\mathbf{u}_j^*(t) - \mathbf{u}_j(t)$ satisfies the terminal constraints on $\mathbf{x}(t)$ in (iv) in the theorem.

The relative performance of the comparison controls is defined through:

$$\text{(A5)}\quad \phi = \left(e^{-rt_n^*}G(\mathbf{x}^*(t_n^*)) - e^{-rt_n}G(\mathbf{x}(t_n))\right) - \left(\sum_{j=1}^{n-1} e^{-rt_j^*}C_j(\mathbf{x}^*(t_j^*)) - \sum_{j=1}^{n-1} e^{-rt_j}C_j(\mathbf{x}(t_j))\right)$$
$$+ \sum_{j=1}^{n}\int_{t_a}^{t_b}\left[H_j\left(\mathbf{x}^*(t), \mathbf{u}_j^*(t), \boldsymbol{\lambda}_j^*(t), t\right) - \boldsymbol{\lambda}_j^{T*}(t)\dot{\mathbf{x}}^*(t) - H_j\left(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t\right) + \boldsymbol{\lambda}_j^T(t)\dot{\mathbf{x}}(t)\right]dt$$
$$+ \sum_{j=1}^{n}\int_{t_b}^{t_{j^-}}\left[H_j\left(\mathbf{x}^*(t), \mathbf{u}_j^*(t), \boldsymbol{\lambda}_j^*(t), t\right) - \boldsymbol{\lambda}_j^{T*}(t)\dot{\mathbf{x}}^*(t) - H_j\left(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t\right) + \boldsymbol{\lambda}_j^T(t)\dot{\mathbf{x}}(t)\right]dt.$$

Terms across the interval $[t_b, t_{j^-}]$ for all j are expanded using a Taylor series, and the standard integration of parts is performed for $\boldsymbol{\lambda}_j^T(t)\delta\dot{\mathbf{x}}$. The first-order terms are retained. Simplifying this expression using the necessary conditions specified in (i)–(vii) in the theorem yields:

(A6)
$$d\phi = \sum_{j=1}^{n} \int_{t_a}^{t_b} \Big[H_j\big(\mathbf{x}^*(t), \mathbf{u}_j^*(t), \boldsymbol{\lambda}_j^*(t), t\big) - \boldsymbol{\lambda}_j^{T*}(t)\dot{\mathbf{x}}^*(t) - H_j\big(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t\big) + \boldsymbol{\lambda}_j^T(t)\dot{\mathbf{x}}(t) \Big] dt + \sum_{j=1}^{n} \boldsymbol{\lambda}_j^T(t_b)\delta\mathbf{x}(t_b).$$

The values of the state and adjoint vectors at time t_b are $\mathbf{x}(t_b) = \mathbf{x}(t_a) + \dot{\mathbf{x}}\Delta t$ and $\boldsymbol{\lambda}_j^T(t_b) = \boldsymbol{\lambda}_j^T(t_a) + \dot{\boldsymbol{\lambda}}_j^T\Delta t$, respectively. The *initial condition* for the optimal trajectory and the comparison path for the prior partition are the same; therefore, $(\mathbf{x}^*(t_a) - \mathbf{x}(t_a)) = 0$ in $\delta\mathbf{x}(t_b) \equiv (\mathbf{x}^*(t_a) - \mathbf{x}(t_a)) + (\dot{\mathbf{x}}^* - \dot{\mathbf{x}})\Delta t$, so $\delta\mathbf{x}(t_b) = (\dot{\mathbf{x}}^* - \dot{\mathbf{x}})\Delta t$. Substitution into equation (A6) yields:

(A7)
$$d\phi = \sum_{j=1}^{n} \Big[H_j\big(\mathbf{x}^*(t), \mathbf{u}_j^*(t), \boldsymbol{\lambda}_j^*(t), t\big) - H_j\big(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t\big) \Big] \Delta t.$$

The augmented functional will only be maximal in relation to the control vector for each phase if $d\phi > 0$ for all $u_j(t)$. The moment t_a is chosen arbitrarily, and $\Delta t > 0$. It follows that an optimal control vector $\mathbf{u}_j^*(t)$ for each phase j requires $H_j(\mathbf{x}^*(t), \mathbf{u}_j^*(t), \boldsymbol{\lambda}_j^*(t), t) > H_j(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t)$ across $[t_{j-1^+}, t_{j^-}]$, or, alternatively,

$$\operatorname*{Max}_{\mathbf{u}_j(t)} H_j\big(\mathbf{x}(t), \mathbf{u}_j(t), \boldsymbol{\lambda}_j(t), t\big) \text{ for all } t \in \big[t_{j-1^+}, t_{j^-}\big]. \qquad \Box$$