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Uniform Substitutes When Group Preferences are Blockwise Dependent

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This paper extends the uniform substitutes model developed by Theil (1980) in a block independent framework to one derived in a blockwise dependent framework. The approach is developed in the case of the demand for different brands of the same good. The uniform substitutes preference structure is nested under weakly separable preferences and the restrictions can be tested statistically with a log likelihood ratio test. Conditional expenditure and price elasticities are derived for uniform substitutes.

Key Words: brands, differential demand, elasticities, multistage budgeting, uniform substitutes

In his book, *The System-Wide Approach to Microeconomics,* Henri Theil (1980, pp. 199–200, 209–10) derived the preference structure of uniform substitutes for a group of n_g goods contained in group S_g in a block independent framework. This preference structure implies that the marginal utility of a dollar spent on each good in group S_g will be affected negatively and symmetrically when an additional dollar is spent on any other good in S_g. Theil referred to these goods as uniform substitutes and suggested that the concept might be useful in market analysis for brands of the same commodity. The approach he developed is appropriate when the preference structure for the groups of goods is block independent, or what Theil referred to as block independence.

In practice, estimating the demand for brands of the same commodity might require one to use multistage budgeting (Barten 1977), where the consumer allocates income in different stages. For example, the consumer might first allocate her income among broad groups of goods such as food, clothing, transportation, and so on. Conditional on the expenditure for the group, the consumer then makes allocation decisions among the commodities in the group. Finally, conditional on the expenditure of the commodity, the consumer makes allocation decisions among the brands of the commodity. If this were indeed the consumer's allocation method, we might postulate that the preference structure in the first stage is block independent, but it is not realistic to postulate that the preferences among the different commodities in the same group are block independent.

A more realistic preference structure would allow for blockwise dependence in the second stage (Theil 1976, 1980). If this is the case, the preference structure for the brands under the hypothesis of uniform substitution should be developed in a blockwise-dependent framework. In this paper, we derive the preference structure for goods that are uniform substitutes when the group preferences are blockwise dependent. In the next section, the methodology is derived and explained. This is followed by the derivation of (conditional) income and

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price elasticities when brands are uniform substitutes. Finally, comments and conclusions are made.

Methodology

Let S_g represent a group of brands of good g, and let there be $g = 1, \ldots, G$ goods. The consumer's allocation problem is first to allocate total expenditure, E, among these G goods (first stage) and next to allocate total expenditure on good g, E_g, among all $i = 1, \ldots, n_g$ brands of good g (second stage).[1] Thus, E_i is the expenditure spent on brand i of good g. The preference structure in the first stage can be represented by either block independence or blockwise dependence (Theil 1976, 1980). Both preference structures enable one to estimate the demand for brand i conditional on E_g, the total expenditure spent on good g; however, in most cases such as this one, the latter preference structure is more appropriate.

Let $q_1, \ldots, q_{n_g}$ and $p_1, \ldots, p_{n_g}$ represent quantities and prices of the i brands of good g, respectively, and let $W_g = E_g/E$ and $w_i = E_i/E$ represent the budget shares of group S_g (i.e., good g) and brand i of good g, respectively. Define θ_{ij} such that $\theta_{ij} = (\mu/\phi E)p_i u^{ij} p_j$, where μ represents the marginal utility of income; u^{ij} is the (i, j)th element of $\mathbf{U}^{-1}$, the inverse of the Hessian matrix for the utility function (Theil 1980); and ϕ is the income flexibility or the reciprocal of the income elasticity of the marginal utility of income ($1/\phi = [d\mu/dE]E/\mu$). Additionally, let $\theta_i = \partial p_i q_i/\partial E$ represent the marginal share of brand i of good g, $\Theta_{gh} = \Sigma_{i\in S_g}\Sigma_{j\in S_h}\theta_{ij}$, and $\Theta_g = \Sigma_h \Theta_{gh}$ ($g, h = 1, \ldots, G$) represents the marginal share of group S_g. From $E_g = \Sigma_{i\in S_g} E_i$, it follows that $W_g = \Sigma_{i\in S_g} w_i$. Following Theil, Chung, and Seale (sec. 6.6), the conditional differential demand for a good $i \in S_g$ under weakly separable preferences is

$$(1) \quad w_i^* d(\log q_i) = \theta_i^* d(\log Q_g) + \sum_{j\in S_g} \pi_{ij}^* d(\log p_j),$$

where $\theta_i^* = \theta_i/\Theta_g$ is the conditional marginal share for brand $i \in S_g$, and p_i is the price of brand $i \in S_g$ such that, letting x represent either p, q, or Q_g, $d(\log x) = dx/x$. The π_{ij}^* are conditional Slutsky price parameters, $d(\log Q_g) = \Sigma_{i\in S_g} w_i^* d(\log q_i)$ is the Divisia quantity index for S_g, and $w_i^* = w_i/W_g$. The adding-up condition requires $\Sigma_i \theta_i^* = 1$ ($i \in S_g$), whereas homogeneity and symmetry require that $\Sigma_{j\in S_g}\pi_{ij}^* = 0$ and $\pi_{ij}^* = \pi_{ji}^*$, $i, j \in S_g$, respectively. By assuming θ_i^* and π_{ij}^* are constants, we obtain the conditional absolute price version of the Rotterdam model (Theil 1965, 1980),

$$(2) \quad \bar{w}_{it}^* Dq_{it} = \theta_i^* DQ_{gt} + \sum_{j\in S_g} \pi_{ij}^* Dp_{jt} + \varepsilon_{it},$$

where $\bar{w}_{it}^* = (w_{it}^* + w_{i,t-1}^*)/2$ and $Dx_{it} = \log x_{it} - \log x_{i,t-1}$, where x represents q, p, or Q_g.[2]

To estimate the system of equations represented by Equation (2), omit one equation and estimate the system's remaining $n_g - 1$ equations. Parameter estimates are invariant to the equation omitted (Barten 1969), and the parameters of the omitted equation can be recovered from

$$\theta_{n_g}^* = 1 - \sum_{i\neq n_g}^{i\in S_g} \theta_i^*$$

(the adding-up condition) and from

$$\pi_{in_g}^* = -\sum_{i\neq n_g}^{i\in S_g} \pi_{ij}^*$$

(the homogeneity condition). With symmetry

[1] As discussed above, the consumer's demand for a brand of a good is best described as a three-stage budgeting problem, where the consumer first allocates total expenditure among broad groups of goods; next, she allocates group expenditure among the goods contained in the group; finally, she allocates expenditure for each good among the brands of the good. Such a presentation would require more complicated notation and would not add much substance to the discussion. As such, the presentation is based on two-stage budgeting where the first-stage preferences among the goods are blockwise dependent.

[2] The right-hand side of Equation (2) has the same functional form of the first-difference version of the Deaton–Muellbauer model (1980).

imposed, the $n_g - 1$ equations can be estimated jointly with maximum likelihood using an iterative seemingly unrelated regressions (SUR) technique or by the scoring method (Hardy).

The restriction that the different brands of good g are uniform substitutes can be imposed on Equation (2) (Theil 1980). With blockwise dependence in the first stage and weak separability in the second stage, the conditional Slutsky price parameters are

$$(3) \quad \pi_{ij}^* = (\phi_{gg})(\theta_{ij}^* - \theta_i^*\theta_j^*),$$

where $\theta_{ij}^* = \theta_{ij}/\Theta_{gg}$ (or $\theta_{ij} = \theta_{ij}^*\Theta_{gg}$), and $\phi_{gg} = \phi\Theta_{gg}/W_g$ is the Frisch own-price elasticity of the group S_g (Theil, Chung, and Seale). When we impose uniform substitution within group S_g, the $n_g \times n_g$ submatrix of the Hessian of the utility function multiplied by $\phi E/\mu$ is equal to

$$(4) \quad \frac{\phi E}{\mu}\left[\frac{\partial^2 u}{\partial(p_iq_i)\partial(p_jq_j)}\right] = \begin{bmatrix} \theta^{11} & k & \cdots & k \\ k & \theta^{22} & \cdots & k \\ \vdots & \vdots & & \vdots \\ k & k & \cdots & \theta^{n_gn_g} \end{bmatrix},$$

such that the off-diagonal elements (i.e., $\theta^{ij} = 1/\theta_{ij}$, $i \neq j$) are all equal to a constant and positive value k, and the diagonal elements are also positive. Because $\phi E/\mu$ is negative, this type of preference structure implies that the marginal utility of a dollar spent on each good in S_g ($\partial u/\partial p_iq_i$) is affected negatively and by the same amount ($k\mu/\phi E$) when an additional dollar is spent on any other good in the group. Thus, all goods in S_g are affected uniformly from the additional consumption of any other good in the group. The inverse of the expression above is equal to $[\theta_{ij}]$ and, as shown by Theil (1980),

$$(5) \quad [\theta_{ij}] = \mathbf{D} - \frac{k}{1 + k\iota'\mathbf{D}\iota}\mathbf{D}\iota\iota'\mathbf{D},$$

where $[\theta_{ij}]$ is the $n_g \times n_g$ matrix of all θ_{ij} for $i, j \in S_g$, $\mathbf{D}$ is a diagonal matrix with positive diagonal elements, and ι is a vector of ones. Using $\Sigma\,\theta_{ij} = \theta_i^*\Theta_{gg}$ in the blockwise dependent case, $\theta_i^*\Theta_{gg} = d_i - (k\iota'\mathbf{D}\iota)d_i/(1 - k\iota'\mathbf{D}\iota)$ or $d_i = (1 + k\iota'\mathbf{D}\iota)\theta_i^*\Theta_{gg}$, and d_i is the ith diagonal element of $\mathbf{D}$. Furthermore, $\Sigma_{j\in S_g} d_i = \iota'\mathbf{D}\iota = (1 + k\iota'\mathbf{D}\iota)\theta_i^*\Theta_{gg}$, which, solving for $\iota'\mathbf{D}\iota$, gives the result $1 + k\iota'\mathbf{D}\iota = 1/(1 - k\Theta_{gg})$. Utilizing the above information, it is easy to show that with blockwise dependence among groups in the first stage and with uniform substitutes within a group,[3]

$$(6) \quad \theta_{ij} = \frac{\theta_i^*\Theta_{gg}(1 - k\theta_i^*\Theta_{gg})}{1 - k\Theta_{gg}} \qquad i = j$$

$$= -\frac{k\Theta_{gg}\theta_i^*\Theta_{gg}\theta_j^*}{1 - k\Theta_{gg}} \qquad i \neq j.$$

By summing over $j \in S_g$ and postmultiplying θ_{ij} by $d(\log p_j)$,

$$(7) \quad \sum_{j\in S_g} \theta_{ij}d(\log p_j)$$

$$= \frac{\theta_i^*\Theta_{gg}}{1 - k\Theta_{gg}}\left[d(\log p_i) - k\Theta_{gg}\sum_{j\in S_g}\theta_j^*d(\log p_j)\right]$$

$$= \frac{\theta_i^*\Theta_{gg}}{1 - k\Theta_{gg}}[d(\log p_i) - k\Theta_{gg}d(\log P_g')].$$

Next, subtract from above, $\Sigma_{j\in S_g}\theta_{ij}d(\log P_g') = \theta_i^*\Theta_{gg}d(\log P_g')$, where $d(\log P_g') = \Sigma_{j\in S_g}\theta_j^*d(\log p_j)$, giving

$$(8) \quad \sum_{j\in S_g}\theta_{ij}d\left(\log\frac{p_j}{P_g'}\right)$$

$$= \frac{\theta_i^*\Theta_{gg}}{1 - k\Theta_{gg}}[d(\log p_i)$$

$$- (k\Theta_{gg} + 1 - k\Theta_{gg})d(\log P_g')]$$

$$= \frac{\theta_i^*\Theta_{gg}}{1 - k\Theta_{gg}}d\left(\log\frac{p_i}{P_g'}\right).$$

This can be related back to the conditional π_{ij}^* (Equation 3) such that

$$(9) \quad \pi_{ij}^* = \left(\frac{\phi\Theta_{gg}}{W_g(1 - k\Theta_{gg})}\right)\theta_i^*(1 - \theta_i^*) \quad i = j$$

$$= -\left(\frac{\phi\Theta_{gg}}{W_g(1 - k\Theta_{gg})}\right)\theta_i^*\theta_j^* \quad i \neq j$$

[3] Theil (1980, pp. 209–10) showed that, under block independence of the upper group, $\theta_{ij} = [\theta_i(1 - k\theta_i)]/(1 - k\Theta_g)$ for $i = j$ and $\theta_{ij} = -[(k\theta_i\theta_j)/(1 - k\Theta_g)]$ for $i \neq j$.

Furthermore, letting $\phi^*_{gg} = (\phi\Theta_{gg})/[W_g(1 - k\Theta_{gg})]$, we can make the conditional uniform substitutes model estimable as

$$(10) \quad \bar{w}^*_{it} Dq_{it} = \theta^*_i DQ_{gt} + \phi^*_{gg}\theta^*_i D\left(\frac{p_{it}}{P'_{gt}}\right) + \varepsilon^*_{it},$$

where θ^*_i and ϕ^*_{gg} are treated as constants.

Equation (10) is nested in Equation (2), and, accordingly, one can test the uniform substitution restrictions with a log likelihood ratio test, where Equation (10) is the restricted model and Equation (2) is the unrestricted model. The test statistic is χ^2_r, where r represents the number of restrictions, and the actual test statistic is $-2(\text{LR}^* - \text{LR})$, where LR* represents the restricted log-likelihood value and LR represents the unrestricted log-likelihood value.

One should also note that preference independence (additive preferences) is a special case of uniform substitutes where $k = 0$. Because k and Θ_{gg} are both positive, goods that are uniform substitutes are more price responsive to changes in the price of other goods in the group than under preference independence within the group.

Conditional Expenditure and Price Elasticities

Calculating conditional expenditure and price elasticities is relatively easy for the uniform substitutes case. Conditional expenditure elasticities are simply the conditional marginal shares divided by the conditional average shares, or $\eta^*_i = \theta^*_i/w^*_i$. Three types of conditional own-price elasticities can be calculated: Frisch, Slutsky, and Cournot. The conditional Frisch own-price elasticity is

$$(11) \quad F^*_{ii} = \frac{\phi^*_{gg}\theta^*_i}{w^*_i},$$

the conditional Slutsky own-price elasticity is

$$(12) \quad S^*_{ii} = \frac{\phi^*_{gg}\theta^*_i(1 - \theta^*_i)}{w^*_i},$$

and the conditional Cournot own-price elasticity is

$$(13) \quad C^*_{ii} = \frac{\phi^*_{gg}\theta^*_i(1 - \theta^*_i)}{w^*_i} - \theta^*_i.$$

It is also possible to calculate conditional Slutsky and Cournot cross-price elasticities for uniform substitutes. However, the Frisch cross-price elasticities vanish under the assumption of uniform substitutes. The conditional Slutsky cross-price elasticity is

$$(14) \quad S^*_{ij} = -\frac{\phi^*_{gg}\theta^*_i\theta^*_j}{w^*_i},$$

and the conditional Cournot cross-price elasticity is

$$(15) \quad C^*_{ij} = -\frac{\phi^*_{gg}\theta^*_i\theta^*_j - \theta^*_j w^*_j}{w^*_i} = -\frac{\phi^*_{gg}\theta^*_j(\theta^*_i - w^*_j)}{w^*_i}.$$

The conditional Slutsky cross-price elasticities are always positive, whereas the Cournot cross-price elasticities can be positive or negative, depending on how large the income effect of a price change is when nominal income is held constant or on the relative sizes of θ^*_i and w^*_j.

Conclusions

In this paper, a conditional differential demand system with preferences described by uniform substitution is developed when group preferences are blockwise dependent. This model is an extension of Theil's (1980) uniform substitutes model that he developed in a block independent framework. Whether within-group preferences are weakly separable or of uniform substitution can be statistically tested with a log likelihood ratio test. Furthermore, conditional expenditure and price elasticities are derived for uniform substitutes.

The discussion centered on the conditional demand for brands of the same good with multistage budgeting. The uniform substitutes preference structure would also be appropriate in estimating the import demand for the same good from different sources of production or the so-called Armington assumption. Although both the uniform substitutes model and the Armington model are parsimonious with respect to parameters and degrees of freedom, the uniform substitutes model has several advantages over the Armington model. For example, the uniform substitutes model is theoretically consistent, whereas the Armington model is not (Alston et al.; Davis and Kruse). Also, the preference structure of the Armington model must be maintained, whereas the uniform substitution assumption can be statistically tested.

References

Alston, J.M., C.A. Carter, R. Green, and D. Pick. "Whither Armington Trade Models." *American Journal of Agricultural Economics* 72(1990): 455–67.

Armington, P.S. *A Theory of Demand for Products Distinguished by Place of Production.* International Monetary Fund Staff Paper 16, 1969.

Barten, A.P. "Maximum Likelihood Estimation of a Complete System of Demand Equations." *European Economic Review* 1(1969):7–73.

———. "The Systems of Consumer Demand Functions Approach: A Review." *Econometrica* 45(1977):23–51.

Davis, G.C., and N.C. Kruse. "Consistent Estimation of Armington Demand Models." *American Journal of Agricultural Economic* 75(1994): 719–23.

Deaton, A., and J. Muellbauer. "An Almost Ideal Demand System." *American Economic Review* 70(1980):312–26.

Hardy, A. *The Econometric Analysis of Time Series,* 2nd ed. Cambridge, MA: The MIT Press, 1990.

Theil, H. "The Information Approach to Demand Analysis." *Econometrica* 33(1965):67–87.

———. *Theory and Measurement of Consumer Demand,* Volume 2. Amsterdam: North-Holland Publishing Company, 1976.

———. *The System-Wide Approach to Microeconomics.* Chicago: University of Chicago Press, 1980.

Theil, H., C.-F. Chung, and J.L. Seale, Jr. *International Evidence on Consumption Patterns.* Greenwich, CT: JAI Press Inc., 1989.