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Technology in the Differential Input Demand Model

Mark G. Brown and Jonq-Ying Lee

This study considers incorporating changes in technology in the differential input demand system through effects on output and input marginal products. The effects of technology on input demand are related to Slutsky coefficients and input shares of marginal cost. Technology effects on marginal-product changes are viewed as price changes, and restrictions on technology are considered.

Key Words: differential demand system, inputs, technology

Based on production theory, a firm's demands for inputs used to produce some product depends on input prices and the position and shape of isoquants. Over time or across firms, changes in technology[1] can result in changes in isoquants, which in turn, would be expected to result in input demand changes that might be of interest (e.g., Jorgenson and Griliches). From a theoretical viewpoint, the role that technology plays in input demand is like that played by demographic variables and other demand shifters in consumer demand analysis.[2] Basmann, Tintner, and Ichimura have shown how demand-shifting variables affect utility and, in turn, consumer demand. The effects of such variables can be traced through the fundamental matrix equation of consumer demand (Barten 1964a, 1977), leading to demand system specifications that could be useful for empirical analysis (e.g., Brown and Lee; Duffy 1987, 1989; Theil 1980a).

In this study, we consider modeling technology variables in input demand systems through the approach used to model demand-shifting variables in consumer demand systems. Technology variables are included in the firm's production function, and cost is minimized subject to a given level of production, yielding input demand equations with prices, output, and the technology variables as arguments. We develop a differential input demand system (Laitinen; Theil 1980b) where a change in technology can be viewed as affecting input demands by inducing changes in "adjusted" input prices as in application of this modeling approach to consumer demand systems (e.g, Brown and Lee) and other proposed consumer demand models.[3] A change in an adjusted input price is the actual price change minus a change in the input's marginal

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[1] Although the results of this paper are for technology changes, the same results apply generally to other exogenous variables that might be added to the firm's production function, such as input quality variables, research and development, or dynamic variables.

[2] In addition to income and prices, various consumer characteristic variables (e.g., age, sex, education, etc.) lagged consumption or psychological stock variables reflecting habits, advertising, household composition, fixed levels of public or rationed goods, and product quality have been found to be important determinants of consumer demand (e.g., Deaton and Muellbauer; Hanemann 1982, 1984).

[3] Similar adjusted or corrected prices have been suggested by Barten (1964b) in the context of household composition effects on demand and by Fisher and Shell in the context of product quality effects.

product as the result of a change in technology. In addition, a change in technology affects input demands through its effect on output (this effect does not have a counterpart in consumer demand).

Model

Our specification of how technology affects input demand is based on the firm's version of the fundamental matrix equation of consumer demand (Laitinen; Theil 1980b). The approach begins with the traditional production problem of choosing that bundle of inputs that minimizes cost subject to a given level of output. In the traditional problem, output is specified as a function of input quantities; in our problem, output is further specified as a function of technology. One technology variable is considered. The results for this variable apply in a straightforward way to other such input demand–shifting variables. Formally, our problem can be written as the minimization of $x = p'q$ subject to $z = h(q, t)$, where z is output; $p' = (p_1, \ldots, p_n)$ and $q' = (q_1, \ldots, q_n)$ are price and quantity vectors, respectively, with p_i and q_i being the price and quantity of input i, respectively; x is cost; and t is the technology variable. The first-order conditions for this problem are the proportionality of marginal products and input prices, $p = \lambda\partial h/\partial q$, and the production function, $z = h(q, t)$, with λ being the Lagrange multiplier, which is equal to marginal cost, $\partial x/\partial z$. The solution to the first-order conditions is the set of input demand equations $q = q(p, t, z)$ and the Lagrange multiplier equation $\lambda = \lambda(p, t, z)$. The differential input demand system proposed by Theil (1980a, b) and Laitinen is an approximation of this set of demand equations, and the demand model developed in this paper is an extension of their approximation. Analysis by Barnett, Byron, and Mountain show the differential demand model is comparable to other popular flexible functional forms.

Following Theil and Laitinen, we examine the proportionality condition in the form $p_iq_i = \rho\partial(\log h)/\partial(\log q_i)$, where $\rho = \lambda z = \partial x/\partial(\log z)$. To determine the effects of prices, output, and technology on input demand, we totally differentiate this condition, along with the production function, to find

$$\text{(1a)}\quad p_iq_i[d(\log p_i) + d(\log q_i)] = [\partial(\log h)/\partial(\log q_i)]d\rho + \rho\sum_j [\partial^2\log h/\partial(\log q_i)\partial(\log q_j)]d(\log q_j) + \rho[\partial^2\log h/\partial(\log q_i)\partial(\log t)]d(\log t)$$

$$\text{(1b)}\quad d(\log z) = [\partial(\log h)/\partial(\log t)]d(\log t) + \sum_j [\partial(\log h)/\partial(\log q_j)]d(\log q_j),$$

where the result of Equation (1b) is based on $dz = (\partial h/\partial t)dt + \Sigma_j (\partial h/\partial q_j)dq_j$, or after dividing this result by z, we have $(dz/z) = (\partial h/\partial t)(t/z)(dt/t) + \Sigma_j (\partial h/\partial q_j)(q_j/z)(dq_j/q_j)$.[4]

From the first-order conditions, substitute p_iq_i/ρ for $\partial(\log h)/\partial(\log q_i)$ in Equations (1a) and (1b) and divide both sides of these equations by x. After rearranging terms, Equations (1a) and (1b) can be written as

$$\text{(2a)}\quad \sum_j [\Delta_{ij}w_i - \gamma\partial^2\log h/\partial(\log q_i)\partial(\log q_j)]d(\log q_j) = w_id(\log \rho) + \gamma[\partial^2\log h/\partial(\log q_i)\partial(\log t)] \times d(\log t) - w_id(\log p_i)$$

$$\text{(2b)}\quad \gamma d(\log z) = \delta d(\log t) + \sum_j w_jd(\log q_j),$$

where $w_i = p_iq_i/x$, the share of total cost from input i; $\gamma = \rho/x = (\partial x/\partial z)(z/x)$ or the elasticity of cost with respect to output; $\delta = \gamma\partial(\log h)/\partial(\log t) = (\partial x/\partial z)(z/x)(\partial z/\partial t)(t/z) = (\partial x/\partial t)(t/x)$ or the elasticity of cost with respect to the technology variable t; and Δ_{ij} is the Kronecker delta ($\Delta_{ij} = 1$ if $i = j$; otherwise, $\Delta_{ij} = 0$).

The result of Equation (2) can be written in terms of matrices as

$$\text{(3a)}\quad [\hat{\mathbf{w}} - \gamma\mathbf{H}]\mathbf{Dq} = \mathbf{w}D\rho + \gamma\mathbf{V}Dt - \hat{\mathbf{w}}\mathbf{Dp}$$

$$\text{(3b)}\quad \mathbf{w}'\mathbf{Dq} = \gamma Dz - \delta Dt,$$

where $\mathbf{w} = [w_i]$; $\hat{\mathbf{w}}$ is the diagonal matrix

[4] For some variables b and c, where $b = f(c)$, the log change $d(\log b) = db/b$, and the elasticity $\partial(\log b)/\partial(\log c) = (\partial b/\partial c)(c/b)$.

formed from $\mathbf{w}$ (diagonal elements equal to w_i; off-diagonal elements equal to zero); $\mathbf{Dq} = [d(\log q_i)]$; $\mathbf{Dp} = [d(\log p_i)]$; $D\rho = d(\log \rho)$; $Dt = d(\log t)$; $Dz = d(\log z)$; $\mathbf{H} = [\partial^2 \log h / \partial(\log q_i)\partial(\log q_j)]$; and $\mathbf{V} = [\partial^2 \log h / \partial(\log q_i)\partial(\log t)]$. $\mathbf{H}$ is the Hessian matrix, and $\mathbf{V}$ is a matrix indicating how technology variable t affects input marginal products. For cost minimization, it is sufficient that $[\hat{\mathbf{w}} - \gamma\mathbf{H}]$ is positive definite (Theil 1980b). The result of Equation (3) is the firm's fundamental matrix equation, the counterpart to the fundamental matrix equation for consumer demand.

Equations (3a) and (3b) can be solved to find log changes in input demand ($\mathbf{Dq}$) as a function of log changes in input prices ($\mathbf{Dp}$), log changes in output (Dz), and log changes in technology (Dt), as shown in the Appendix. From matrix Equation (A7), we can write the absolute price version of demand equation for input i as

$$\text{(4a)} \quad w_i d(\log q_i) = \theta_i[\gamma d(\log z) - \delta d(\log t)] + \sum_j \pi_{ij}[d(\log p_j) - \epsilon_j d(\log t)],$$

where $\theta_i = \partial[(p_i q_i)/\partial z]/[(\partial x/\partial z)]$ is the ith input's share of marginal cost; π_{ij} is the firm's Slutsky coefficient, and ϵ_j is the elasticity of the ith input's marginal product with respect to a change in technology (see the Appendix for details).

Alternatively, from matrix Equation (A8) we can write the relative price version of demand for input i as

$$\text{(4b)} \quad w_i d(\log q_i) = \theta_i[\gamma d(\log z) - \delta d(\log t)] - \psi \sum_j \theta_{ij}[d(\log p_j) - \epsilon_j d(\log t) - Dp']$$

where $Dp' = \Sigma_j \theta_j[d(\log p_j) - \epsilon_j d(\log t)]$ and can be viewed as a technology-adjusted Frisch price index (the usual Frisch price index is obtained when $d(\log t) = 0$), and ψ is a measure of the curvature of cost, as shown in the Appendix. ψ is positive and matrix $[\theta_{ij}]$ is positive definite given that the sufficient conditions for cost minimization mentioned above are fulfilled. Equations (4a) and (4b) are the Theil–Laitinen differential input demand model extended to include technology.

The general restrictions on input demand in Equations (4a) and (4b) are (e.g., Theil 1980b)

(5a) adding up:

$$\sum_i \theta_i = 1; \qquad \sum_i \pi_{ij} = 0; \qquad \sum_i \theta_{ij} = \theta_j,$$

(5b) homogeneity:

$$\sum_j \pi_{ij} = 0; \qquad \sum_j \theta_{ij} = \theta_i,$$

(5c) symmetry:

$$\pi_{ij} = \pi_{ji}; \qquad \theta_{ij} = \theta_{ji}.$$

Summing Equations (4a) or (4b) over i, we find Equation (3b). The left-hand side of Equation (3b) is referred to as the Divisia volume index, $d(\log Q) = \Sigma\, w_i d(\log q_i)$. Following Theil (1980b), we can view Equation (3b) as a total input decision and Equation (4) as an input allocation decision. The Divisia volume index indicates the total input change, given changes in output and technology. This decision does not involve input prices. Equations (4a) or (4b) indicate the change in demand for input i, given the change in the Divisia volume index and changes in input prices.

Input demand in Equation (4) can be made a function of the price of output z, along with input prices and technology, following Theil (1980b). From Equation (A2) and following definitions of terms in the Appendix, along with the definition that $\rho = (\partial x/\partial z)z$ and hence $\log \rho = \log(\partial x/\partial z) + \log z$, we can write the log change in marginal cost as

$$\text{(6)} \quad d[\log(\partial x/\partial z)] = (\gamma/\psi - 1)d(\log z) + \sum_j \theta_j[d(\log p_j) - \epsilon_j d(\log t)] - (\delta/\psi)d(\log t).$$

For profit maximization with output z sold at fixed price p_z, the firm will supply up to the point where marginal cost equals price (i.e., $\partial x/\partial z = p_z$). Hence, we set the left-hand side of Equation (6) to $d(\log p_z)$, solve for $d(\log z)$, and find

$$(7)\quad d(\log z) = [\psi/(\gamma - \psi)][d(\log p_z) - \sum_j \theta_j d(\log p_j)] + \left[\left(\delta + \psi \sum_j \theta_j \epsilon_j\right) \Big/ (\gamma - \psi)\right] d(\log t).$$

The term $(\psi/[\gamma - \psi])$ is the elasticity of supply with respect to the price of the product deflated by the Frisch price index, $\Sigma_j \theta_j d(\log p_j)$. The term $[(\delta + \psi \Sigma_j \theta_j \epsilon_j)/(\gamma - \psi)]$ is the elasticity of supply with respect to technology. The elasticity of supply is positive based on the second-order conditions for profit maximization requiring $\gamma > \psi$ (Theil 1980b). The sign of the elasticity of supply with respect to technology is indeterminate because there are no conditions on δ or the ϵ. Substitution of the result of Equation (7) into Equation (4) yields input demand equations as functions of the output price, input prices, and technology.

Consider the input allocation decision, with $d(\log Q)$ replacing the term $\gamma d(\log z) - \delta d(\log t)$ in Equation (4a) based on the result of Equation (3b). Further consider replacing the technology term, $-\Sigma_j \pi_{ij}\epsilon_j d(\log t)$, with $\beta_i d(\log t)$, where $\beta_i = w_i \partial(\log q_i)/\partial(\log t)$ given $d(\log Q)$ or the total input decision. Given data with sufficient variability on input cost, quantities, prices, and technology, variables can be constructed for $w_i d(\log q_i)$, $d(\log Q)$, $d(\log p_i)$, and $d(\log t)$, as suggested by Theil (1980b), and the coefficients θ_i, π_{ij}, and β_i can be estimated under the assumption they are constants; however, the ϵ_i coefficients cannot. Note that $\beta = -\pi\epsilon$, where $\beta = [\beta_i]$; $\pi = [\pi_{ij}]$; and $\epsilon = [\epsilon_j]$. Given π is singular by the restrictions in Equations (5a) and (5b), ϵ cannot be identified from estimates of π and β. However, using Equations (5a) and (5b) to delete a row and column from π (in general, the resulting matrix is nonsingular), we can estimate the technology marginal-product elasticity difference term, $\epsilon_j - \epsilon_n, j = 1, \ldots, n$. Formally, imposition of the restrictions in Equations (5a) and (5b) on Equation (4a) results in the input allocation specification

$$(8)\quad w_i d(\log q_i) = \theta_i d(\log Q) + \sum_{j=1,\ldots,n-1} \pi_{ij}[d(\log p_j^*) - \epsilon_j^* d(\log t)], \quad i = 1, \ldots, n-1,$$

where $d(\log p_j^*) = d(\log p_j) - d(\log p_n)$ and $\epsilon_j^* = \epsilon_j - \epsilon_n$. The coefficients θ_i, π_{ij}, and ϵ_j^* $(i, j = 1, \ldots, n-1)$ can be estimated as constants, and the coefficients θ_n, π_{in}, and π_{nj} can be recovered from the conditions in Equations (5a) and (5b).

The specification in Equation (8) indicates that the input allocation decision depends on relative marginal-product elasticity changes because of technology, $\epsilon_j^* d(\log t)$, along with relative price changes, $d(\log p_j^*)$. In Equation (8), the term in the bracket following the Slutsky coefficient can be viewed as a relative adjusted price change—the jth product's actual price change less the effect of technology on the jth product's marginal product, relative to the nth product's price change less the effect of technology on the nth product's marginal product (these changes are percent changes, with the differential model specified in log differences).[5]

Comparison of technology coefficient estimates, using some statistical test, might yield information useful for describing the technological change and refining the model estimates. Zero restrictions on the ϵ_j^* would mean prices for these inputs are unadjusted. For a neutral change in technology, all ϵ_j^* would be zero. For an input-specific technology change, say for input one, with generic effects on all other inputs, ϵ_1^* would take some value while ϵ_j^* $(j \neq 1)$ would be zero (this type of outcome underlies the specification of advertising effects in a consumer demand system proposed by Theil (1980a) and used by Duffy (1987, 1989) and Brown and Lee. Other restrictions (e.g., some group of ϵ_j^* might have a common value) could also be considered. To the extent that restrictions on the ϵ_j^* are acceptable, their

[5] Equation (8) can be written as $w_i d(\log q_i) = \theta_i d(\log Q) + \Sigma_j \pi_{ij} d(\log p_j^*)$, where p_j^* is the relative adjusted price $(p_j/t^{\epsilon_j})/(p_n/t^{\epsilon_n})$.

imposition might improve the precision of the model estimates; for example, more precise estimates of Slutsky coefficients might be obtained (Theil 1980a), with variation in both prices and the technology variable contributing to the estimation of these coefficients.

We also note that, in input allocation decision Equation (6), restrictions on technology coefficients, ϵ_j^*, do not directly involve the adding-up condition, whereas if the term $-\Sigma_j \pi_{ij}\epsilon_j^* d(\log t)$ were replaced by $\beta_i d(\log t)$, then the adding-up restriction on β_i requires $\Sigma_i \beta_i = 0$. Given this constraint, restrictions on β_i could be problematic (Bewley), with estimation of Equation (6) usually accomplished by omitting an arbitrary equation because of singularity of the demand system's covariance matrix as a result of adding-up conditions (Barten 1969). For β_j restricted to zero and unrestricted β_i, the set of explanatory variables for the input demands i and j would differ by technology variable t, and simple ordinary least squares estimates of these two equations need not satisfy adding-up. In contrast, restrictions on the ϵ_j^* do not present such problems.

Concluding Comments

Changes in technology and similar demand-shifting variables can be incorporated into the differential input demand model through their effects on output and input marginal products. The model suggests that technology affects the firm's implied total input and input allocation decisions. First, a change in technology results in a change in total input demand, as measured by the Divisia volume index. Then, the change in the Divisia volume index, along with changes in technology-adjusted prices (defined as actual price changes minus technology-induced changes in input marginal products), results in changes in individual input demands. Restrictions on the effects of technology are considered through restrictions on marginal-product elasticities with respect to technology.

Following Theil, we conclude by noting an area, input–output analysis, where the differential input demand system provides insight. Theil (1980a) has shown how input price changes can be incorporated into input–output analysis through the differential input demand model. Based on the results of this paper, Theil's work can be extended by allowing changes in technology, along with changes in input prices, to be incorporated into input–output analysis.

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Appendix

Solving Equation (3a) for $\mathbf{Dq}$, we find

$$\text{(A1)}\quad \mathbf{Dq} = \mathbf{A}^{-1}\mathbf{w}D\rho + \gamma\mathbf{A}^{-1}\mathbf{V}Dt - \mathbf{A}^{-1}\hat{\mathbf{w}}\mathbf{Dp},$$

where $\mathbf{A} = [\hat{\mathbf{w}} - \gamma\mathbf{H}]$. Premultiplying Equation (A1) by $\mathbf{w}'$ and solving for $D\rho$, we find

$$\text{(A2)}\quad D\rho = [\mathbf{w}'\mathbf{A}^{-1}\hat{\mathbf{w}}\mathbf{Dp} - (\delta + \gamma\mathbf{w}'\mathbf{A}^{-1}\mathbf{V})Dt + \gamma Dz]/\psi,$$

where $\psi = \mathbf{w}'\mathbf{A}^{-1}\mathbf{w}$, and the term $\mathbf{w}'\mathbf{Dq}$ has been replaced by the expression on the right-hand side of Equation (3b). The term $\psi = \gamma/\partial(\log \rho)/\partial(\log z) = \gamma/\partial^2\log x/\partial(\log z)\partial(\log z)$ is a measure of the curvature of cost.

Substituting the result from Equation (A2) into Equation (A1), we find

$$\text{(A3)}\quad \mathbf{Dq} = \mathbf{A}^{-1}\mathbf{w}[\mathbf{w}'\mathbf{A}^{-1}\hat{\mathbf{w}}\mathbf{Dp} - (\delta + \gamma\mathbf{w}'\mathbf{A}^{-1}\mathbf{V})Dt + \gamma Dz]/\psi + \gamma\mathbf{A}^{-1}\mathbf{V}Dt - \mathbf{A}^{-1}\hat{\mathbf{w}}\mathbf{Dp}.$$

Multiplying Equation (A3) by $\hat{\mathbf{w}}$ and rearranging terms, we obtain

$$\begin{aligned}\text{(A4)}\quad \hat{\mathbf{w}}\mathbf{Dq} = &-\psi[\hat{\mathbf{w}}\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi - (\hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}/\psi)(\mathbf{w}'\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi)]\mathbf{Dp}\\ &+ (1/\psi)[\psi\gamma\hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{V} - \hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}(\delta + \gamma\mathbf{w}'\mathbf{A}^{-1}\mathbf{V})]Dt + (\hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}/\psi)\gamma Dz,\end{aligned}$$

or, after premultiplying $\mathbf{V}$ by the identity matrix in the form $\hat{\mathbf{w}}\hat{\mathbf{w}}^{-1}$, we find

$$\begin{aligned}\text{(A5)}\quad \hat{\mathbf{w}}\mathbf{Dq} = &-\psi[\hat{\mathbf{w}}\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi - (\hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}/\psi)(\mathbf{w}'\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi)]\mathbf{Dp}\\ &+ (1/\psi)[\psi\hat{\mathbf{w}}\mathbf{A}^{-1}\hat{\mathbf{w}}\epsilon - \hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}(\delta + \mathbf{w}'\mathbf{A}^{-1}\hat{\mathbf{w}}\epsilon)]Dt + (\hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}/\psi)\gamma Dz,\end{aligned}$$

where $\epsilon = \gamma\hat{\mathbf{w}}^{-1}\mathbf{V} = [\partial(\log(\partial h/\partial q_i))/\partial(\log t)]$, the matrix of elasticities of input marginal products with respect to a change in technology.

After further rearrangement, we have

$$\begin{aligned}\text{(A6)}\quad \hat{\mathbf{w}}\mathbf{Dq} = &-\psi[\hat{\mathbf{w}}\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi - (\hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}/\psi)(\mathbf{w}'\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi)]\mathbf{Dp}\\ &+ \psi[\hat{\mathbf{w}}\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi - (\hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}/\psi)(\mathbf{w}'\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi)]\epsilon Dt + (\hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}/\psi)(\gamma Dz - \delta Dt),\quad \text{or}\end{aligned}$$

$$\text{(A7)}\quad \hat{\mathbf{w}}\mathbf{Dq} = \pi(\mathbf{Dp} - \epsilon Dt) + \theta(\gamma Dz - \delta Dt),$$

where $\pi = -\psi[\hat{\mathbf{w}}\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi - (\hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}/\psi)(\mathbf{w}'\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi)]$ (the Slutsky matrix for the firm) and $\theta = (\hat{\mathbf{w}}\mathbf{A}^{-1}\mathbf{w}/\psi)$ (the matrix of input shares of marginal cost). The result of Equation (A7) is the differential input demand model in absolute prices. A model in relative prices can also be specified by defining matrix $\Theta = [\theta_{ij}] = \hat{\mathbf{w}}\mathbf{A}^{-1}\hat{\mathbf{w}}/\psi$; that is,

(A8a) $\hat{\mathbf{w}}\mathbf{Dq} = -\psi(\Theta - \theta\theta')(\mathbf{Dp} - \epsilon Dt) + \theta(\gamma Dz - \delta Dt)$, or

(A8b) $\hat{\mathbf{w}}\mathbf{Dq} = -\psi\Theta(\mathbf{Dp} - \epsilon Dt)^* + \theta(\gamma Dz - \delta Dt)$

where $(\mathbf{Dp} - \epsilon Dt)^* = (\mathbf{Dp} - \epsilon Dt) - \iota\theta'(\mathbf{Dp} - \epsilon Dt)$. Notice that $\Theta\iota = \theta$ and $\iota'\Theta = \theta'$, where ι is a unit vector. The term $\theta'\mathbf{Dp}$ is referred to as θ, the Frisch price index, and the term $\theta'(\mathbf{Dp} - \epsilon Dt)$ is the Frisch price index adjusted to changes in technology.