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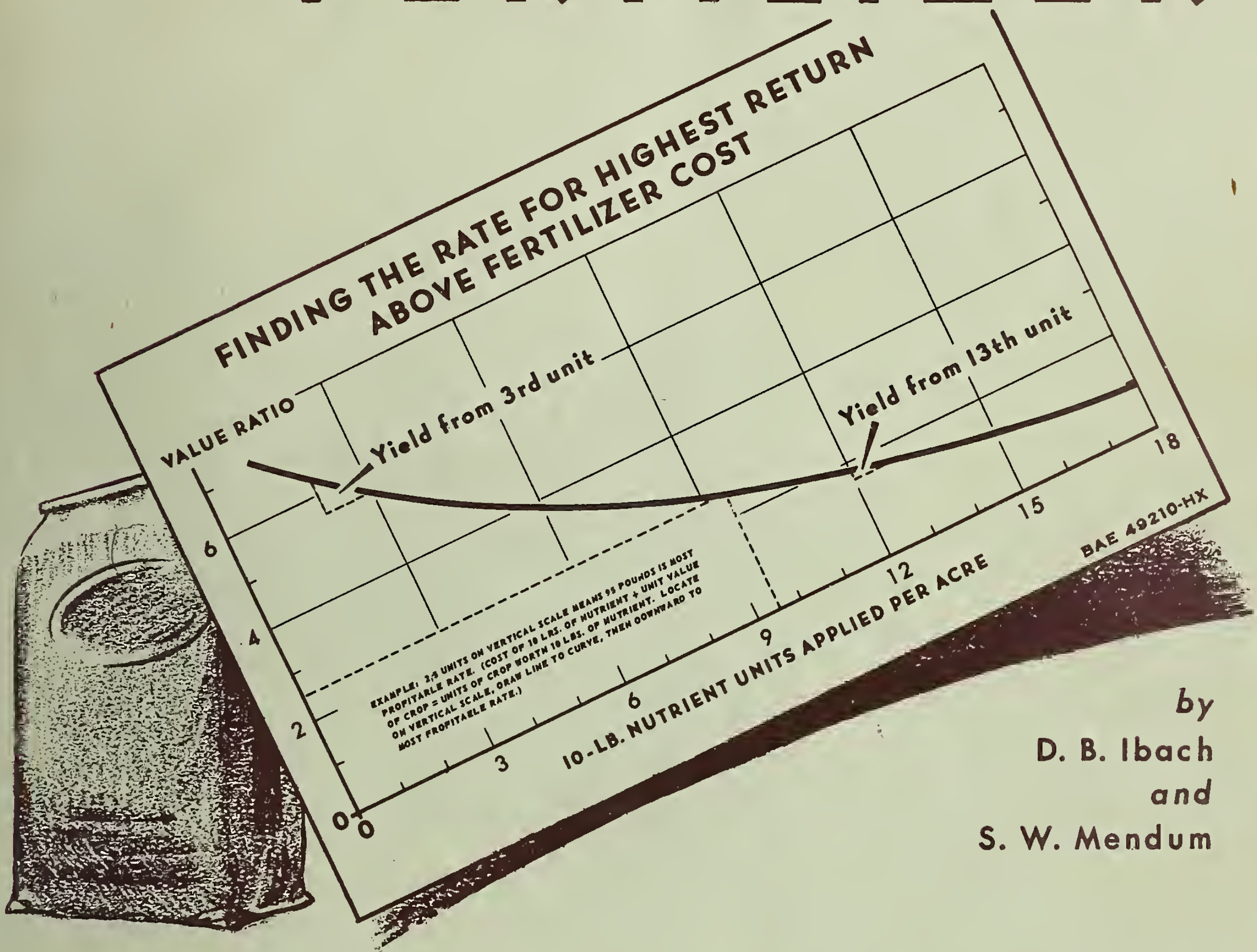
Determining Profitable Use of FERTILIZER



by
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UNITED STATES DEPARTMENT OF AGRICULTURE
BUREAU OF AGRICULTURAL ECONOMICS
Washington, D. C.
June 1953

FOREWORD

The substantial contribution of fertilizers in farm production is widely recognized. The amount of this contribution depends on the combination of fertilizer with many other yield influencing factors. These factors are soil characteristics (including the soil supply of plant nutrients), climate, and a wide variety of management practices. The contribution of fertilizer to added product also depends on the quantity used -- the rate applied per acre. Generally, throughout most of the range in response, the contribution of each added unit is smaller than the one that precedes it. But maximum dollar returns from the use of fertilizer occur when the last unit added just pays its way.

The problem is to: (1) determine the additional response associated with additional rates of application; (2) relate crop response to changes in prices and costs, thus determining most profitable use of fertilizers over a wide range in economic conditions. Procedures illustrating one method of handling these problems when one or more independent variables are involved, are presented in this report. Ways of preparing results for popular presentation to farmers are also illustrated.

This report is a "tool kit" which, together with a table of logarithms, permits those interested in the general subject, but not familiar with the mathematical aspects, to develop a thorough economic interpretation of adequate response data.

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DETERMINING PROFITABLE USE OF FERTILIZERS 1/

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INTRODUCTION

The chief aim of this report is to illustrate procedures for developing yield response curves and related steps involved in determining profitable use of fertilizers. However, considerable interpretive discussion is included. This report is technical and is designed for use of production economists and agronomists. Emphasis is placed on methods and procedures, including preparation of materials, that could be used by extension workers and other agricultural leaders in discussing economic aspects of use of fertilizer with farmers. In addition, suggestions are made as to experimental designs for development of response data that would be particularly useful for purposes of economic interpretation.

Profitable use of fertilizers depends on the physical response obtained and on prices of crops and costs of fertilizer. Crop response to all yield-influencing factors and returns per dollar cost from each would need to be taken into account in deciding upon the most profitable combination of all factors. Results from such a field of study would be particularly important to farmers who lack funds to carry each input to the point of its most profitable return. But such a study requires a knowledge of costs and returns from all factors used in producing the crop. Or, looking at the farm as a whole, it involves knowledge of comparative returns from all factors of production when they are devoted to different crop and livestock enterprises. Fertilizer response curves are only one of the keys to such information. Therefore, this report is limited to the problem of the rates and combinations of applied plant nutrients that will give the highest returns with respect to fertilizer.

For purposes of this report annual yield is used as the only measure of crop response. Some fertilizers have residual effects. Also, quality of the crop may be influenced by use of fertilizer. When data become available for variables such as these, economic interpretation of these aspects of response to fertilizers may also be rendered.

1/ Valuable assistance in handling certain statistical problems encountered in preparing this manuscript was rendered by D. D. Mason, Chief Biometrician, Bureau of Plant Industry, Soils, and Agricultural Engineering, and Glenn L. Burrows, Statistician in the Office of the Statistical Assistant, Bureau of Agricultural Economics. The individual plot data used in illustrating methods of analysis were supplied by J. F. Fudge, Department of Agronomy, Texas Agricultural Experiment Station, College Station, Tex.

Fertilizer has been important in increasing yields of crops and total farm output. Increases in yields brought about by use of more fertilizer have meant higher returns not only per acre, but also per hour of labor. For small farmers particularly, wise use of fertilizers means a substantial increase in size of business. Overhead costs per acre are usually higher on small farms than on large ones. Therefore, in order for small farms to obtain the same net return per acre as large farms they have to obtain high yields. A knowledge of yield response at different rates and combinations is necessary in order to decide upon the most profitable application of fertilizer. Widespread information of this type would guide farmers in making choices as to expenditures. In periods of declining prices, returns from all expenditures are reduced, but it is particularly important to maintain the practices that contribute most to net returns.

Farmers who have followed good practices, including profitable use of fertilizers, know about what yields represent the most profitable levels for their own farms, but they may not be aware of the effect of changes in prices and costs on rates that should be applied for maximum returns. Also, there are often different ways of reaching these yields. For example, different combinations of plant nutrients may result in the same yield, but one combination may cost less per acre than another. This aspect of the most profitable use of fertilizer has general implications as well as implications for individual farms. For example, if sulfur, which is used in the manufacture of phosphate fertilizer, should cost more in the future relative to other materials used in manufacturing fertilizers than it has in the past, it might influence recommendations that should be made as to plant nutrient ratios for specific purposes. If, for example, more nitrogen and less phosphate will result in the same yield of a given crop, recommendations should reflect this fact if reserves of sulfur should become relatively short. Such questions as this can be answered only by adequate information as to yield response at different rates and combinations of plant nutrients. Economic interpretation can then provide answers as to the most profitable rates and combinations for any crop price-fertilizer cost relationship. It can also determine the least-cost combination at which to obtain a stated yield.

Interest in fertilizers and in analysis of results from their use has increased in recent years along with favorable farm prices relative to costs of fertilizer. As a result of wider and more intensive use of fertilizer by farmers, experience has indicated that its use would be profitable at price-cost relationships less favorable than those experienced since the midforties. The question of most profitable use of fertilizers is therefore important to individual farmers and to the economy as a whole, whatever the future levels of prices and costs.

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METHODS OF ANALYSIS

Economic interpretation of crop response to fertilizers involves first the problem of describing the response curve from reported yields at rates of application included in the experiment. Obviously an experiment cannot include all possible rates and combinations. But as a basis for economic analysis it should include enough rates to permit describing the course of the curve through most of the range within which response occurs. If this is done, different equations may be used for interpolating between rates applied, and some may be safely used for limited extrapolation beyond the highest rate applied, subject to verification by field trial.

Any equation used in calculating yields at rates not included in the experiment is based on some general hypothesis. In a later section three different equations are compared with reference to results obtained when applied to the same experimental data. As experimental data covering a wide range in rates and combinations become available, it will be possible to determine which of different yield equations best reflects the response obtained. Perhaps no single equation will be found to be best for all occasions.

The equation used here to illustrate the problem of determining profitable use of fertilizers is based on the general principle of decreasing increments. As applied to fertilizer, this principle states that as fertilizer is added in units of uniform size, with other factors unchanged, yields increase at a decreasing rate in such a way that each increment in yield throughout the series is a constant percentage of the one that precedes it. Except for response to extremely small applications and special situations of little importance so far as economic analysis is concerned, this principle has a great deal of evidence to support it. It has therefore, gained considerable acceptance with reference to the effect of a single variable, other factors held constant.

This characteristic pattern of yield response to fertilizers was discovered independently by Mitscherlich 2/ in Germany and by Spillman 3/ in this country. The general form of yield equations developed by them have certain similarities, but the two equations are not precisely the same. A major difference in the thinking of these two men is reflected in the fact that Mitscherlich holds the view that the ratio of successive increments in yield for a stated size of a unit of application of a given growth factor is the same for all crops and all soils, when no other yield-limiting factors are present. Spillman's work reflects the view that this ratio of successive yield increments may differ under different circumstances. In other words, the constants of the yield equation take whatever values are dictated by the conditions under which the responses

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2/ Mitscherlich, E. A. Zum Gesetz vom Minimum. Eine antwort an TH. Pfeiffer und seine mitarbeiter. Landw. Vers. Sta. 77: [413]- 428, illus. 1912.

3/ Spillman, W. J. Use of the exponential yield curve in fertilizer experiments. U. S. Dept. Agr. Tech. Bul. 348, 67 pp., illus. 1933. Spillman, W. J., and Lang, E. The law of diminishing returns. 178 pages. Yonkers on Hudson. 1924.

occur. The two views may not necessarily be contradictory, but the Spillman approach appears more directly applicable to field conditions. Certainly, this approach does not require acceptance of the Mitscherlich hypothesis.

Spillman, however, developed an adaptation of his equation for use in estimating the effect on yield of two or more growth factors varied simultaneously. His equations for handling this problem rest largely on the work of the German mathematician Baule 4/ who was the first to point out that when two or more factors of growth are varied at the same time, the yield is a function of the product of their individual effects. Although it is not the purpose to analyze the various aspects of the work of these three men, this statement is included as a background to procedures illustrated here in determining profitable use of fertilizers.

The Yield Equation For One Variable

With reference to a single variable, the equation $y = M(1-R^x)$ is a form of an exponential function in which x, the variable, is an exponent of R, the constant ratio of successive increments in yield. In this form of the equation as applied to fertilizer, x represents the nutrient available to the plant, including the soil content equivalent in effect to that applied. The value of this ratio, R, is always less than 1.0 and it is determined by the nature of response, as shown by the reported yields which in turn reflect the conditions of the experiment. The numerical value of R is influenced by the size of unit used in making the calculations. The constant, M, is the theoretical maximum yield that may be obtained by adding fertilizer under conditions of the experiment. The calculated yield, y, is determined as a percentage of this theoretical maximum yield.

Figure 1 is a hypothetical illustration of the principle of decreasing increments, as expressed by the equation $y = M(1-R^x)$. Each increment in yield indicated in figure 1 is 75 percent of the one that precedes it, when calculated in terms of six units of equal size. In other words, R = 0.75. If the calculations were made using three units of fertilizer of equal size, the first increment would be 5,250 pounds, the second, 2,953.125 pounds, and the third, 1,661.132185 pounds.

4/ Baule, B. zu Mitscherlichs Gesetz der physiologischen Beziehungen. Landw. Jahrb. 51: [363]- 385, illus. 1918.

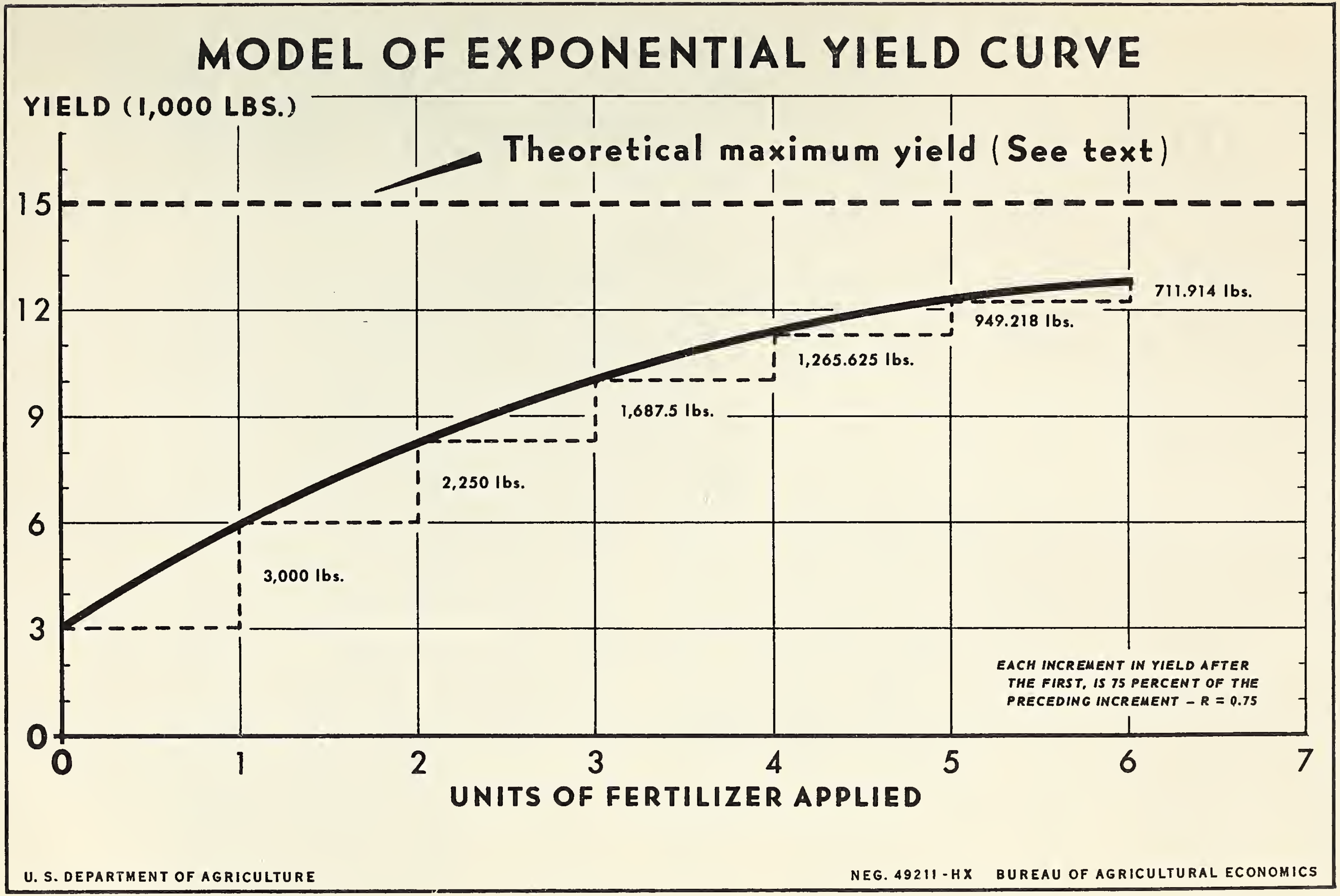
MODEL OF EXPONENTIAL YIELD CURVE
YIELD (1,000 LBS.)
Theoretical maximum yield (See text)
15
12
9
6
3
0
0
1
2
3
4
5
6
7
UNITS OF FERTILIZER APPLIED
3,000 lbs.
2,250 lbs.
1,687.5 lbs.
1,265.625 lbs.
949.218 lbs.
711.914 lbs.
EACH INCREMENT IN YIELD AFTER THE FIRST, IS 75 PERCENT OF THE PRECEDING INCREMENT – R = 0.75
U. S. DEPARTMENT OF AGRICULTURE
NEG. 49211-HX BUREAU OF AGRICULTURAL ECONOMICS

Figure 1

If after such grouping and carrying all decimals as indicated, the second increment is divided by the first, and the third is divided by the second, the value of R becomes exactly 0.5625. The point may also be illustrated by noting that reducing the number of units by 50 percent and doubling the size of each, has the same effect on R as squaring its original value. That is $0.75^2 = 0.5625$. But, in using the equation, the calculated yields would be the same whether R were determined by using six units of x of a given size or three units of twice that size. This is illustrated by the determination of M, the theoretical maximum yield that could be obtained by adding fertilizer.

Before calculating M, it is necessary to find the value of another constant, A, which is defined as the theoretical maximum increase in yield attainable through adding fertilizer. This constant, A, may be determined by dividing the first increment in yield by 1-R, When the total application is divided into six units of equal size, the first increment in yield is 3,000 pounds, R = 0.75 and 1-R = 0.25. Therefore A = 12,000 pounds. Also, when the application is divided into three units each twice as large as before, the first increment in yield is 5,250 pounds, R = 0.5625 and 1-R = 0.4375. As before, A = 12,000 pounds. Therefore M is the same in each instance as M = A, plus the yield at the first reading, which in this case is at the point of no application. As the yield at no application is 3,000 pounds, M is 15,000 pounds.

A great saving in subsequent calculations is accomplished through determination of the size of unit required to render R a specified value. This saving is made possible through use of a table of $1-R^x$ values in terms of some stated value of R. As pointed out later, the value R = 0.8 is used. Any other value could have been used. Calculation of the size of unit required to render R a specified value is solely for convenience and is not inconsistent with the fact that for any given size of unit R may have a different value for each experiment studied.

The Yield Equation For Two Or More Variables

As has been stated, the general concept of a constant rate of decreasing increments has gained considerable acceptance with reference to a single variable. The fitness of the multivariable form of the equation as developed by Spillman has not been thoroughly demonstrated. To do so requires its application to different sets of experimental data showing response to each nutrient in turn at different levels of the others. Such experiments need to include a number of rates encompassing at least the major part of the range in response. Whether interaction - the increase in yield obtained when two or more nutrients are added simultaneously, minus the sum of the increases obtained when each is added separately - is adequately reflected by the multivariable form of the exponential yield equation, has not been conclusively determined.

Some present-day factorial experiments afford a limited basis for such testing of the equation, but they do not include enough rates and combinations to render the tests conclusive. The experiment used later to illustrate methods of determining profitable use of fertilizer is a 4x3x3 factorial, and calculated yields are compared with reported yields as an indication of the adequacy of the equation as applied to this experiment. Also, a later section compares interactions developed from calculated yields with those read directly from reported yields. It appears that as the one-variable form of the equation reflects a constant ratio between successive increments in yield, the multivariable form reflects the hypothesis of a constant rate of interaction. If an accumulation of evidence as to interactions indicates that such a generalization is appropriate for a considerable range in rates and combinations, the multivariable form of the yield equation as developed by Spillman may have a substantial area of application.

When x represents a unit of application of N, of P_2O_5 and of K_2O in a three-variable fertilizer rate experiment, the yield equation takes the form of $y = M_3\ (1-R^N)(1-R^P)(1-R^K)$. For convenience in expression, P and K will be used hereafter when referring to P_2O_5 and K_2O. When three variables are involved, the constant that denotes the theoretical maximum yield is referred to as M_3. Details of calculating M_3 are left to a later section, but this value is determined by dividing the calculated yield at any point, on any one of the individual nutrient curves, by the product of the three $1-R^x$ values where x in each case represents the quantity of the nutrient which, in conjunction with stated quantities of the others, results in the calculated yield. The curve of response to each nutrient is calculated with quantities of the other nutrients held constant. The yield equation is reversible. Thus, $M_3 = y/ [(1-R^N)(1-R^P)(1-R^K)]$.

DATA USED IN ILLUSTRATING METHODS

As an illustration, the yield equation described and related procedures are applied to a 4x3x3 factorial experiment conducted in 1950 on pasture plots at Kirbyville, Tex. Yields were reported at no application of each nutrient and for three rates of N, two of P, and two of K. Each of the 36 treatment combinations was replicated four times, making a total of 144 plots. Analysis of variance conducted in terms of the logarithms of the yields is presented in a later section. All of the main effects were highly significant, but the main effect of N was most significant. The standard error of the yields reported for all replicated plots was 480 pounds of dry forage per acre, or about 10 percent of the average yield for all the plots.

The means of the reported yields for all replicated plots are shown later in comparison with the calculated yields. But table 1 shows the means of the reported yields for replicated plots for each nutrient varied in the presence of stated levels of the other 2 nutrients. Yields for all 36 treatment combinations are calculated from constants derived from the 8 combinations shown in table 1.

Table 1.- Reported yield per acre of dry forage and standard error of the mean at stated treatment combinations

Plant nutrients applied			Mean yield of 4 replicated plots 2/	Standard error
N	P 1/	K 1/		
Pounds	Pounds	Pounds	Pounds	Pounds
0	240	0	2,339	196
60	240	0	4,070	167
120	240	0	5,688	401
180	240	0	6,971	132
60	0	0	2,860	244
60	120	0	3,963	348
(60)	(240)	(0)	(4,070)	(167)
(60)	(240)	(0)	(4,070)	(167)
60	240	60	4,561	235
60	240	120	4,874	151

1/ For convenience in expression the letters P and K are in all instances used to denote P_2O_5 and K_2O, respectively.
2/ Total of 6 clippings during the year.

Pasture experiment, Kirbyville, Tex. - Texas Agr. Expt. Sta.-1950.

Any set of 8 treatment combinations similar to those in table 1 could be used to develop constants of the yield equations for the purpose of calculating yields at any combination. Two other sets of combinations were used with similar end results. The relative reliability of the yields, as determined by the standard errors of the means, is an appropriate guide in the choice of combinations to use in finding constants from which to calculate yields at all combinations. All of the different rates of application of each nutrient used in the experiment are indicated in table 1. But each nutrient was varied in the presence of every combination of the other two included in the experiment.

DEVELOPING THE RESPONSE CURVES

A response curve is developed for each nutrient at stated levels of the others as indicated in table 1. The problem is to develop a smooth curve, on which any point will represent the most probable yield for the indicated rate of application. For any given yield function employed only one curve truly represents the most probable yields. Such a curve is developed by the method of least squares. However, when only a few rates are included, the constants (R, M, and A described earlier, and others) may be developed by simpler methods that give results equivalent for practical purposes. When only 3 rates are involved, a carefully done graphic approximation necessarily gives precisely the same results as does the method of least squares. Graphic methods are used here to find the constants of the equation for the N curve. Later the constants are compared with those obtained by the method of least squares.

Before illustrating the graphic method of determining constants of the equation and calculating yields from them, it is desirable to identify and define all terms and symbols used in subsequent calculations.

Definition of Terms and Symbols

The Yield Equation

y = M ((fx)) ; or in terms here illustrated $y = M(1-R^x)$.
y = the calculated yield.
M = the theoretical maximum yield attainable from adding units of x, other yield-influencing factors constant.
R = the ratio between successive increments in yield - a constant less than 1.0.
x = the input variable, addition of a series of units of which results in the function represented by R^x. As used in the equation x consists of two parts (see below).
R^x = the value of R raised to the power indicated by x.
$1-R^x$ = the difference between 1.0 and the x power of R (R^x) at any stated level of the input. At zero x, R^x is always 1.0. Thus, if R as defined above is 0.8, when x = 1 unit $R^x = 0.8$ and $1-R^x = 0.2$. But, when x = 2 units R^x or $R^2 = 0.64$ and $1-R^x = 0.36$. As x increases R^x diminishes and $1-R^x$ more nearly approaches but never reaches 1.0. Thus, as the quantity of fertilizer (x) is increased, the calculated yield, $M(1-R^x)$ approaches, but never reaches M.

Other Constants Used in Developing the Yield Curve

i_1 = the increment in yield associated with the first unit of x.
i_2 = the second increment, etc.
A = the theoretical maximum increase that may be obtained from adding units of x, other yield-influencing factors constant.

Calculation: $A = i/(1-R)$; or $i_2 / [(1-R) - (1-R)^2]$, etc.

Y = the reported yield, e.g., Y_0 = reported yield at no application.
n, p, and k = calculated quantities of soil N, P, and K equivalent in effect to same quantities added at any points on the curves. As calculated, these quantities are expressed in terms of units of the size required to render R = 0.8. (See definition of u below.)

Calculation: (log M - log A) / log 0.8.

The logarithm of a decimal fraction is a negative number, but the sign is to be disregarded as the point of origin is arbitrarily placed. Log 0.8 = - 0.09691.

a, b, and c = the applied portions of N, P, and K, respectively; as used in the equation they are expressed in terms of units.

Thus, x for each nutrient is composed of two parts:

N becomes $n + a$ (soil N and applied N)
P becomes $p + b$ (soil P and applied P)
K becomes $k + c$ (soil K and applied K)

u = the size of a unit of x required to render R a specified value. From this point on, the term "unit" refers specifically to such a quantity. In these illustrations the calculations are performed after expressing x in terms of units of the size required to render R = 0.8. This conversion to a size of unit such that R will equal 0.8 greatly simplified the computations. Otherwise it is of no significance. Conversion to a size required to render R any specified value would give equivalent results.

Calculation: u = log 0.8 / log R, times the number of pounds per unit used in determining the value of R.

Thus, pounds applied (a, b, or c) divided by the determined respective values of u = units of a, b, or c. These added to the values of n, p, or k result in unit values of $n + a$, $p + b$ or $k + c$. These values are then located in table 15 and their corresponding $1-R^x$ values are read from that table.

The Yield Equation and M_3 - Three Variables

The equation for three variables: $y = M_3(1-R^{n+a})(1-R^{p+b})(1-R^{k+c})$

M_3 = the theoretical maximum yield for 3 variables under stated conditions.

Calculation: $M_3 = y / [(1-R^{n+a})(1-R^{p+b})(1-R^{k-c})]$

Constants for Determining the Most Profitable Rate or Combination

r', r", and r"' = cost of one unit of a, b, and c, respectively.

Calculation: u times costs per pound of the nutrients, including cost of application.

T is a fraction used in the course of computing the most profitable combination of a, b, and c at the costs per unit (r', r", and r"') of the nutrients. Its form depends on the number of variables.

Q is a fraction, 1/v M (-lnR), in which v is the value of a unit of the crop before harvest, and ln R is the natural logarithm of R, or of 0.8, as used here. Thus, (-ln 0.8) has a value of 0.2231435214, used in all computations of Q.

When the most profitable combination of nutrients associated with a specified value of v is to be reached, that value of T must be found which is equal to Q. Q can be calculated as soon as M has been determined. The work of finding the value of T associated with any specified value of Q, so that T - Q = 0, is demonstrated later as a series of clerical steps.

Finding Constants by Graphic Approximation

Referring to table 1, the combination 60-240-0 occurs in each of the nutrient series. Reported yields at each level may be plotted on graph paper, although when reported yields are available for only three rates at equal intervals, plotting may be omitted as the curve necessarily passes through all three points.

When yields are reported at 4 or more rates as for N in this experiment, the yields are plotted and a smooth curve is drawn freehand to fit as closely as possible. The method involves readings at three points, one of which is midway on the horizontal scale between the other two. In this instance the range used in the calculations is represented by the yields at 0 and 180 pounds of N. It is not necessary that the freehand curve, which is an estimate of a least-squares solution, pass through points represented by any of the reported yields. In this instance however, reported yields at the two rates just mentioned were thought to be satisfactory approximations of the location of the curve of least squares at those rates. The midpoint is read from the freehand curve where it crosses the ordinate that represents an estimated yield of 4,848 pounds at an application of 90 pounds (fig. 2).

Constants for the Nitrogen (N) Curve

The work of finding constants of the equation and calculating yields on the N curve is outlined in the steps indicated below. Reference to the list of definitions of terms and symbols may be helpful.

1.

x: a Pounds	x: a Graphic units	Y Pounds	i Pounds	R (i_2 / i)	1-R
0	0	2,339			
(90)	1	(4,848)	2,509		
180	2	6,971	2,123	0.84615	0.15385

2. A = i/(1-R) = 2,509/0.15385 = 16,308 pounds

3. M = A + Y_o = 16,308 + 2,339 = 18,647 pounds = 9.3235 tons.

4. Log R = log 0.84615 = 9.92744 -10 = -0.07256

5. u = log 0.8/log R = -0.09691/-0.07256 = 1.336 90-pound units
= 120.24 pounds

6. n = (log M - log A)/ log 0.8
= (4.27061 - 4.21240)/-0.09691 = 0.6007 units

7. Calculating yields on the N curve.

x: a Pounds	x: a Units	x: n Units	x: n + a Units	$1-R^{n+a}$ 1/	y Pounds 2/	Y Pounds
0	0.0000	0.6007	0.6007	0.125446	2,339	2,339
60	.4990	.6007	1.0997	.217601	4,058	4,070
120	.9980	.6007	1.5987	.300046	5,595	5,688
180	1.4970	.6007	2.0977	.373802	6,970	6,971

1/ From table 15. Values interpolated for fractional units not shown.
2/ y = M $(1-R^{n+a})$ = 18,647 $(1-R^{n+a})$.

Constants for the Phosphoric Oxide (P) Curve

Items in the calculation are similar to those just presented. Reported yields are taken from table 1. As yields are reported for only 3 rates having equal intervals, graphing is unnecessary.

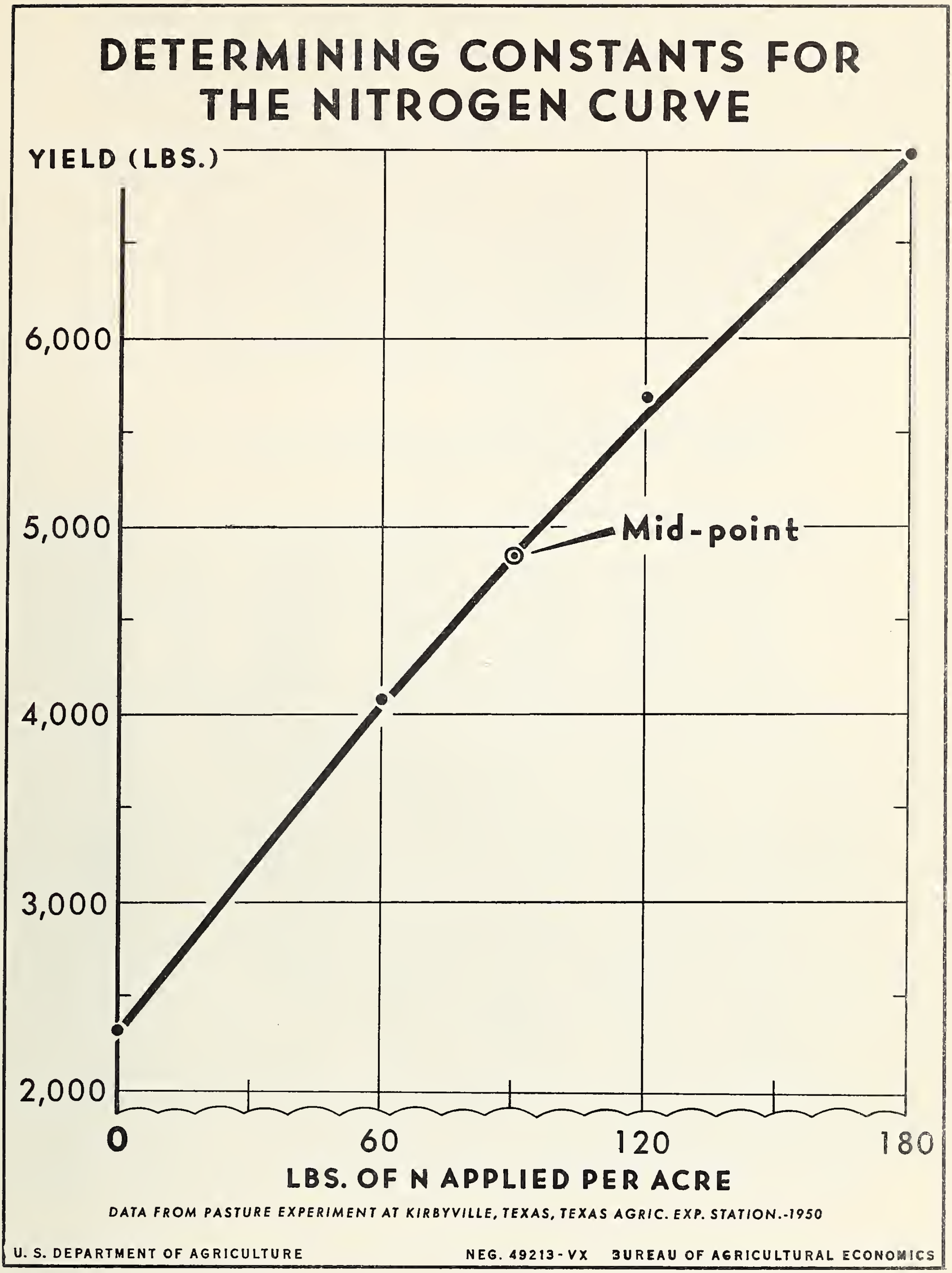
DETERMINING CONSTANTS FOR THE NITROGEN CURVE
YIELD (LBS.)
6,000
5,000
4,000
3,000
2,000
Mid-point
0
60
120
180
LBS. OF N APPLIED PER ACRE
DATA FROM PASTURE EXPERIMENT AT KIRBYVILLE, TEXAS, TEXAS AGRIC. EXP. STATION.-1950
U. S. DEPARTMENT OF AGRICULTURE
NEG. 49213-VX
BUREAU OF AGRICULTURAL ECONOMICS

Figure 2

1.

x		Y	i	R	1-R
b Pounds	b Graphic units	Pounds	Pounds	(i_2 / i)	
0	0	2,860			
120	1	3,963	1,103		
240	2	4,070	107	0.09701	0.90299

2. A = i/(1-R) = 1,103/.90299 = 1,221 pounds.
3. $M = A + Y_o$ = 1,221 + 2,860 = 4,081 pounds = 2.0405 tons
4. Log R = log 0.09701 = 8.98682 -10 = -1.01318
5. u = log 0.8 / log R = -0.09691 / -1.01318
   = 0.095649 120-pound units
   = 11.48 pounds
6. p = (log M - log A)/log 0.8
   = (3.61099 -3.08672)/-.09691 = 5.4076 units
7. Calculating yields on the P curve

x						
b Pounds	b Units	p Units	p + b Units	$1-R^{p+b}$	y Pounds 1/	Y Pounds
0	0.0000	5.4076	5.4076	0.700809	2,860	2,860
120	10.4549	5.4076	15.8625	.970975	3,963	3,973
240	20.9097	5.4076	26.3173	.997187	4,070	4,070

1/ $y = M (1-R^{p+b}) = 4{,}081 (1-R^{p+b})$.

Constants for the Potash (K) Curve

Items in the calculation similar to those for the N and P curves are:

1.

x		Y	i	R	1-R
c Pounds	c Graphic units	Pounds	Pounds	(i_2 / i)	
0	0	4,070			
60	1	4,561	491		
120	2	4,874	313	0.63747	0.36253

2. A = i/(1-R)= 491/.36253 = 1,354 pounds.
3. $M = A + Y_o$ = 1,354 + 4,070 = 5,424 pounds = 2.7120 tons.

4. Log R = log 0.63747 = 9.80446 -10 = -0.19554.

5. u = log 0.8 / log R = -0.09691/-0.19554 = 0.4956 60-pound units.
= 29.736

6. k = (log M - log A) / log .8
= (3.73432 - 3.13162) /-0.09691 = 6.2192 units

7. Calculating yields on the K curve

x						
c Pounds	c Units	k Units	k + c Units	$1-R^{k+c}$	y 1/ Pounds	Y Pounds
0	0.0000	6.2192	6.2192	0.750366	4,070	4,070
60	2.0178	6.2192	8.2370	.840802	4,561	4,561
120	4.0355	6.2192	10.2547	.898558	4,874	4,874

1/ y = M $(1-R^{k+c})$ = 5,424 $(1-R^{k+c})$.

THE THEORETICAL MAXIMUM YIELD - MORE THAN ONE VARIABLE

Determination of the theoretical maximum yield obtainable from adding a single variable with the others held constant has been illustrated. The theoretical maximum yield takes a different value when two or more nutrients are allowed to vary simultaneously. Such a maximum yield reflects the calculated effect of the physical optimum combination of the nutrients under conditions of the experiment.

When the theoretical maximum yield for one or more variables is calculated from constants that have necessarily been developed from reported yields covering only a portion of the range within which response would occur, it may reflect a quantity of output that departs substantially from the practical maximum yield. In the experiment studied here, rates of N applied were such that the response curve calculated from the four reported yields is extremely steep (fig. 2). If rates up to perhaps 500 or more pounds of N had been applied, all of the constants might have been changed and the theoretical maximum yield attainable from adding N in the presence of 240 pounds of P and 0 pounds of K might have been substantially different from the 9.3 tons calculated earlier. But this would not have affected materially the calculated yields if the curve is appropriate throughout the new range of application. Changes in the constants R, M, A, and i would have been accompanied by compensating changes in the size of a unit of application associated with these constants. This means that the constants developed from yields reported at the four rates applied are appropriate for calculating yields as long as additional fertilizer results in additional response. The yield equation does not apply beyond the range within which response occurs. Only experimental rates carried to or near this point can fix the range within which any yield equation is valid. Economic interpretation is not concerned beyond such a point, but it is concerned with the location of that point. From the evidence provided in the Texas Pasture experiment there is no way of knowing the extent to which extrapolations on the N curve represent yields that would have been obtained if higher rates of N had been applied. With the exponential yield equation some extrapolation is probably valid as is interpolation between rates, but unless rates are added until the curve becomes nearly flat, there is always the question as to the extent of additional response for the conditions of the experiment.

Finding M_3 - Three Variables

It is now possible by substituting the unit values of $n + a$, $p + b$, and $k + c$ in the yield equation for three variables, to find a value of M_3 for each of the eight treatment combinations used in finding constants of the equation for each nutrient. Thus, in the case of combination $a = 0$, $b = 240$, and $c = 0$, the yield equation becomes:

$$y = M_3\ (1-R^{0.6007})\ (1-R^{26.3173})\ (1-R^{6.2192})$$

$$= M_3\ (0.125446)\ (0.997187)\ (0.750366)$$

$$= M_3\ (0.093866)$$

In calculating M_3 for this combination, the process is reversed. For example, the calculated yield for this combination as determined in computing the constants for the N curve is 2,339 pounds. The equation then becomes:

$$M_3 = y/0.093866$$

$$= 2,339/0.093866$$

$$= 24,919 \text{ pounds}$$

Table 2 shows the calculations for M_3 from each of the treatment combinations used in finding constants of the equation $y = M(1-R^x)$ as applied to the response shown for each nutrient at stated levels of the other two nutrients. The combination common to all three curves is 60 - 240 - 0, with a reported yield of 4,070 pounds. The calculated yield for this combination on the N curve, however, is 4,058 pounds. As there are only three experimental readings on each of the other two curves, and the intermediate one is the mid-point, the graphic approximation curve passes through all reported observations. The slight discrepancy of 12 pounds between the reported yield at this combination, and the yield calculated at this point on the N curve may be considered negligible.

Table 2.- Determining the value of M_3 for the treatment combinations used in finding constants of the yield equation for each nutrient

Calculated from the N curve		$p+b$ and $k+c$ constant $b = 240$ lbs. $c = 0$ lbs.				Product:	Yield at	
$n+a$	$1-R^{n+a}$ 1/	$p+b$	$1-R^{p+b}$ 1/	$k+c$	$1-R^{k+c}$ 1/	$(1-R^{n+a})(1-R^{p+b})(1-R^{k+c})$	R^x $n+a$	M_3
Units		Units		Units			Pounds	Pounds
0.6007	0.125446	26.3173	0.997187	6.2192	0.750366	0.093866	2,339	24,919
1.0997	.217601	26.3173	.997187	6.2192	.750366	.162821	4,058	24,923
1.5987	.300046	26.3173	.997187	6.2192	.750366	.224511	5,595	24,921
2.0977	.373802	26.3173	.997187	6.2192	.750366	.279699	6,970	24,920

Calculated from the P curve		$n+a$ and $k+c$ constant $a = 60$ lbs. $c = 0$ lbs.				Product:	Yield at	
$p+b$	$1-R^{p+b}$ 1/	$n+a$	$1-R^{n+a}$ 1/	$k+c$	$1-R^{k+c}$ 1/	$(1-R^{n+a})(1-R^{p+b})(1-R^{k+c})$	R^x $p+b$	M_3
Units		Units		Units			Pounds	Pounds
5.4076	0.700809	1.0997	0.217601	6.2192	0.750366	0.114428	2,860	24,994
15.8625	.970975	1.0997	.217601	6.2192	.750366	.158541	3,963	24,997
26.3173	.997187	1.0997	.217601	6.2192	.750366	.162821	4,070	24,997

Calculated from the K curve		$n+a$ and $p+b$ constant $a = 60$ lbs. $b = 240$ lbs.				Product:	Yield at	
$k+c$	$1-R^{k+c}$ 1/	$n+a$	$1-R^{n+a}$ 1/	$p+b$	$1-R^{p+b}$ 1/	$(1-R^{n+a})(1-R^{p+b})(1-R^{k+c})$	R^x $k+c$	M_3
Units		Units		Units			Pounds	Pounds
6.2192	0.750366	1.0997	0.217601	26.3173	0.997187	0.162821	4,070	24,997
8.2370	.840802	1.0997	.217601	26.3173	.997187	.182445	4,561	24,999
10.2547	.898558	1.0997	.217601	26.3173	.997187	.194977	4,874	24,998
Average value used in subsequent calculations								24,968

1/ Read from table 15.
Pasture experiment, Kirbyville, Tex. - 1950.

Slight differences in the values of M_3 calculated from responses on any one of the three separate curves are caused by omission of decimals in the preceding computations. As the N curve has 4 observed points, a curve of best fit need not necessarily pass through any of them. In the graphic approximation method, the estimated curve was made to pass through the yields reported at 0 and 180 pounds of N, and the yield at the mid-point (90 pounds) was read from the freehand curve as 4,848 pounds. Using the resulting constants in the equation, 4,058 rather than 4,070 pounds is calculated at the 60 - 240 - 0 combination on the N curve.

In table 2, one of the factors $(1-R^x)$ used in obtaining each separate calculated value of M_3 has been derived from the responses to application of increasing quantities of a single nutrient with the other two held constant. But, except for considerations indicated above, the values of M_3 would be identical regardless of the curve from which they are calculated. The value of M_3 differs substantially from the value of M based on each curve regarded separately.

The definition of M should be kept in mind--it is a theoretical constant used in calculating the yield, and its size need not necessarily resemble the maximum yield that actually could be obtained. Where the experiment includes rates associated with near maximum yield for the conditions, the calculated value of M would be expected to approximate more closely the attainable yield. For N, the constants were necessarily developed from readings that all lie on a very steep segment of the curve.

Finding M_2 - Two Variables

Table 2 provides the necessary data to illustrate the calculation of M_2, the theoretical maximum yield when two variables are involved. For purposes of illustration here, $k + c$ may be left out of the calculations. It happens that in table 2, c was held at zero in calculating M_3 from the N and the P series. It is a simple matter therefore to calculate M_2 using values for $1-R^{n+a}$, $1-R^{p+b}$, and y as they appear in table 2. In table 3 the products of $(1-R^{n+a})(1-R^{p+b})$ are shown for the different combinations of these nutrients. These products serve as divisors in calculating M_2 from the yields appearing on the two response curves.

COMPARING REPORTED AND CALCULATED YIELDS

This step is taken to check the calculated yields for all 36 combinations. Only 8 of these combinations were used in developing constants by means of which all yields are calculated. In order to reduce the size of the figures used in calculating the standard error of estimate, the difference between y and Y for each of the

Table 3.- Determining the value of M_2 for the treatment combinations used in finding constants of the yield equation for the N and P curves when no K is applied

N applied, in presence of 240 pounds applied P:	$(1-R^{n+a})$	Yield at R^{n+a}	M_2
Pounds		Pounds	Pounds
0	0.125093	2,339	18,698
60	.216989	4,058	18,701
120	.299202	5,595	18,700
180	.372750	6,970	18,699
P applied, in presence of 60 pounds applied N:	$(1-R^{p+b})$	Yield at R^{p+b}	
0	.152497	2,860	18,754
120	.211285	3,963	18,757
240	.216989	4,070	18,757
Average of M_2 for 6 different combinations			18,727

Pasture experiment, Kirbyville, Tex. - 1950.

36 combinations (table 4) was converted to tons. The sum of the squared deviations is 1.3653 tons. The standard error of estimate is $\sqrt{1.3653/28} = \sqrt{0.04876} = \pm 0.22081$ tons or 442 pounds. The denominator used is 28 as 8 degrees of freedon were used in finding the constants, but the squared deviations for all 36 combinations are included in the numerator. The standard error of the reported yields for all replicates (144 plots) in the experiment is approximately 480 pounds, or about 10 percent of the average yield of all plots. The fact that the standard error of estimate of 442 pounds for the calculated yields is less than the standard error of the reported yields is somewhat indicative of the applicability of the exponential yield equation for the purpose of estimating yields based on the data of this experiment.

Table 4.- Outline of steps in obtaining final calculated yields, and comparison of calculated with reported yields of dry forage per acre

Nutrients applied			Units: R = 0.8			$1-R^x$ values from table 15			Product of $1-R^x$ values	Calculated yields y 1/	Reported yield Y
N a	P b	K c	n+a	p+b	k+c	n+a	p+b	k+c			
Pounds	Pounds	Pounds	Units	Units	Units					Pounds	Pounds
0	0	0	0.6007	5.4076	6.2192	0.125446	0.700809	0.750366	0.065968	1,647	2,029
0	0	60	.6007	5.4076	8.2370	.125446	.700809	.840802	.073918	1,846	2,197
0	0	120	.6007	5.4076	10.2547	.125446	.700809	.898558	.078996	1,972	2,144
0	120	0	.6007	15.8625	6.2192	.125446	.970975	.750366	.091398	2,282	2,417
0	120	60	.6007	15.8625	8.2370	.125446	.970975	.840802	.102414	2,557	3,120
0	120	120	.6007	15.8625	10.2547	.125446	.970975	.898558	.109449	2,733	3,236
0	240	0	.6007	26.3173	6.2192	.125446	.997187	.750366	.093866	2,344	2,339
0	240	60	.6007	26.3173	8,2370	.125446	.997187	.840802	.105178	2,626	3,068
0	240	120	.6007	26.3173	10.2547	.125446	.997187	.898558	.112403	2,806	3,168
60	0	0	1.0997	5.4076	6.2192	.217601	.700809	.750366	.114429	2,857	2,860
60	0	60	1.0997	5.4076	8.2370	.217601	.700809	.840802	.128220	3,201	3,402
60	0	120	1.0997	5.4076	10.2547	.217601	.700809	.898558	.137027	3,421	3,272
60	120	0	1.0997	15.8625	6.2192	.217601	.970975	.750366	.158541	3,958	3,963
60	120	60	1.0997	15.8625	8.2370	.217601	.970975	.840802	.177649	4,436	4,658
60	120	120	1.0997	15.8625	10.2547	.217601	.970975	.898558	.189852	4,740	4,568
60	240	0	1.0997	26.3173	6.2192	.217601	.997187	.750366	.162821	4,065	4,070
60	240	60	1.0997	26.3173	8.2370	.217601	.997187	.840802	.182445	4,555	4,561
60	240	120	1.0997	26.3173	10.2547	.217601	.997187	.898558	.194977	4,868	4,874

- Continued -

Table 4.- Outline of steps in obtaining final calculated yields, and comparison of calculated with reported yields of dry forage per acre - continued

Nutrients applied			Units: R = 0.8			$1-R^x$ values from table 15			Product	Calculated	Reported
N	P	K	n+a	p+b	k+c	n+a	p+b	k+c	of $1-R^x$	yields	yield
a	b	c							values	y 1/	Y
Pounds	Pounds	Pounds	Units	Units	Units					Pounds	Pounds
120	0	0	1.5987	5.4076	6.2192	0.300046	0.700809	0.750366	0.157783	3,940	3,959
120	0	60	1.5987	5.4076	8.2370	.300046	.700809	.840802	.176800	4,414	4,260
120	0	120	1.5987	5.4076	10.2547	.300046	.700809	.898558	.188944	4,718	4,962
120	120	0	1.5987	15.8625	6.2192	.300046	.970975	.750366	.218609	5,458	5,875
120	120	60	1.5987	15.8625	8.2370	.300046	.970975	.840802	.244957	6,116	6,878
120	120	120	1.5987	15.8625	10.2547	.300046	.970975	.898558	.261783	6,536	6,916
120	240	0	1.5987	26.3173	6.2192	.300046	.997187	.750366	.224511	5,606	5,688
120	240	60	1.5987	26.3173	8.2370	.300046	.997187	.840802	.251570	6,281	6,372
120	240	120	1.5987	26.3173	10.2547	.300046	.997187	.898558	.268850	6,713	6,337
180	0	0	2.0977	5.4076	6.2192	.373802	.700809	.750366	.196569	4,908	4,719
180	0	60	2.0977	5.4076	8.2370	.373802	.700809	.840802	.220260	5,499	5,213
180	0	120	2.0977	5.4076	10.2547	.373802	.700809	.898558	.235390	5,877	4,888
180	120	0	2.0977	15.8625	6.2192	.373802	.970975	.750366	.272347	6,800	6,627
180	120	60	2.0977	15.8625	8.2370	.373802	.970975	.840802	.305171	7,620	8,204
180	120	120	2.0977	15.8625	10.2547	.373802	.970975	.898558	.326133	8,143	8,786
180	240	0	2.0977	26.3173	6.2192	.373802	.997187	.750366	.279699	6,984	6,971
180	240	60	2.0977	26.3173	8.2370	.373802	.997187	.840802	.313409	7,825	7,267
180	240	120	2.0977	26.3173	10.2547	.373802	.997187	.898558	.334937	8,363	8,892

1/ $y = M_3 (1-R^{n+a})(1-R^{p+b})(1-R^{k+c})$; M_3 = 25,968 from table 2.

Pasture experiment, Kirbyville, Tex. - 1950.

THE MOST PROFITABLE RATE OF APPLICATION - ONE VARIABLE

In many instances the problem of economic interpretation involves only one variable. Often the reason for this is that response data are available for only one variable. The practical problem, of course, is nearly always one of finding the most profitable combination of several variables.

The list of definitions presented earlier explains the terms used in the calculations. The values used for purposes of illustration are v = \$ 21 per ton and the costs of N, P, and K in that order are \$ 0.195, \$0.096 and \$ 0.079 per pound. The monetary value of u, the quantity of a nutrient required to render R = 0.8, is expressed as r' for N, r" for P and r"' for K. Consulting the number of pounds represented by u as shown in the calculations for each of the separate curves, the value of r' is found to be \$23.45; r" \$1.10, and r"' \$2.35, at the costs per pound indicated above.

Substituting in the equation Q = 1/v(M) (-ln 0.8) and expressing M in tons rather than pounds:

(a) For the N curve Q = 1/21 (9.3235)(0.2231435214)
= 0.02289

(b) For the P curve Q = 1/21 (2.0405)(0.2231435214)
= 0.10458

(c) For the K curve Q = 1/21 (2.7120)(0.2231435214)
= 0.07869

Based on constants previously determined, table 5 shows a method of calculating the most profitable rate of each of the three nutrients in the presence of stated levels of the other two. It also shows the yield at the most profitable rate, gross returns per acre, and returns above the cost of fertilizer applied.

Qr', Qr" and Qr"' are equivalent in value to R^{n+a}, R^{p+b}, and R^{k+c}, respectively, at the points that represent the most profitable levels on these curves. Subtracting each of these values from 1 results in the appropriate $1-R^x$ value. This value is found in table 15 and the corresponding unit value is recorded as n+a, p+b, or k+c. The values for n, p, and k are deducted to obtain the most profitable rates of application in terms of units. The unit values for a, b, and c are each multiplied by the appropriate value of u in pounds as determined in developing each nutrient curve. The result is the calculated most profitable rate in pounds.

The three most profitable rates as calculated above do not represent the most profitable combination. They are the most profitable individual rates when the quantities of the other two nutrients are held constant at the indicated levels. For example, 263 pounds is the most profitable rate of N only when 240 pounds of P are applied and when no potash is applied. But when the cost of the nutrient not varied is taken into account in each of the three columns of table 5, item 16 shows that combination 60 N, 49 P, and 0 K (second column) is more profitable than either of the other two shown.

Only the response to N at these levels of P and K, the value of the crop and the cost of N, are taken into account in determining the most profitable rate of N at the specified levels of the other nutrients. The cost of the applied nutrient(s) not varied is not a part of this determination.

Results of a fertilizer rate experiment that has only one variable, can quickly be put in a form that will permit extension workers and farmers to determine the most profitable rate of application for any crop price-fertilizer cost relationships that may occur. 5/ This is done by calculating the yields for each of a series of small units of application, such as 10 pounds. The difference between any two successive calculated yields is the increment in yield at that point. Table 6 shows such increments in yields for each of a series of 10-pound units of N applied when applications of P and K are held constant at 240 and 0 pounds, respectively. If the value of the crop per ton is $21 and the cost of N per pound is $0.195, it is calculated that 0.0929 ton of forage is equal in value to 10 pounds of N ($1.95 ÷ $21.00). Using table 6 to find the most profitable rate, this increase of 0.0929 ton in yield falls somewhere between the 260- and the 270-pound application. In table 5 the calculated most profitable rate was found to be 263 pounds. Similarly, if the value of the crop were only $15 per ton, cost of N unchanged, the most profitable rate would be between 80 and 90 pounds of N. But if the crop were valued at $25 per ton, the most profitable rate of N would be 350 pounds.

In using the method of table 6, calculation of the most profitable rate is necessarily in terms of whole units of the size chosen, but in table 5 the calculation is more precise. The smaller the size of unit used in the method of table 6, the more accurate will be the results.

Table 6 brings out the importance of the response curve. The average increase in yield per pound of fertilizer for the range often included in a fertilizer rate experiment is usually not a satisfactory guide in determining the most profitable rate. The total response in going from 0 to the highest rate of N applied in the experiment (180 pounds) is 2.3155 tons when 240 pounds of P and no K were applied. This is an average of 0.1286 ton per 10 pound unit. Reading from

5/ The cover chart is a hypothetical illustration of this method.

Table 5.- Most profitable rate of each nutrient to apply per acre, yield of dry forage at that rate, gross returns and returns above cost of fertilizer when other nutrients are held constant at indicated levels

Item	Unit	Nutrients applied - a varied (b = 240 lbs.; c = 0 lbs.)		b varied (a = 60 lbs.; c = 0 lbs.)		c varied (a = 60 lbs.; b = 240 lbs.)	
1. M	Ton		9.3235		2.0405		2.7120
2. Q			.02289		.10458		.07869
3. u	Pound		120.24		11.48		29.736
4. Cost of a unit(u) of fertilizer	Dollar	r'	23.45	r"	1.10	r"'	2.35
5. Qr',","'. Equivalent to R^x		Qr' or $R^{n/a}$	.53677	Qr" or $R^{p/b}$	.11504	Qr"' or $R^{k/c}$	.18492
6. $1-R^x$. 1 minus item 5		$1-R^{n/a}$	.46323	$1-R^{p/b}$	.88496	$1-R^{k/c}$	.81508
7. n/a; p/b; k/c. Read from table 15	Unit	n/a	2.79	p/b	9.69	k/c	7.56
8. n; p; k. Previously determined	Unit	n	.60	p	5.41	k	6.22
9. Most profitable rate	Unit	a	2.19	b	4.28	c	1.34
10. Most profitable rate. Item 3 times item 9	Pound	a	263		49		40
11. Yield. Item 1 times item 6	Ton		4.3189		1.8058		2.2105
12. Gross return. v ($21) times item 11	Dollar		90.70		37.92		46.42
13. Cost of variable nutrient. Item 4 times item 9	Dollar		51.36		4.71		3.15
14. Return above cost of variable nutrient	Dollar		39.34		33.21		43.37
15. Cost of fixed nutrient(s) applied	Dollar		23.04		11.70		34.74
16. Return above cost of all fertilizer	Dollar		16.30		21.51		8.63

Pasture experiment, Kirbyville, Tex. - 1950.

Table 6.- Calculated increments in yield of dry forage per acre associated with each additional 10 pounds of N, other yield influencing factors constant 1/

Application of N per acre	Yield increment	Application of N per acre	Yield increment
Pounds	Tons	Pounds	Tons
0	0.0000	180	0.1094
10	.1499	190	.1074
20	.1470	200	.1053
30	.1446	210	.1035
40	.1418	220	.1016
50	.1391	230	.0996
60	.1367	240	.0978
70	.1341	250	.0961
80	.1316	260	.0942
90	.1293	270	.0926
100	.1270	280	.0909
110	.1244	290	.0897
120	.1223	300	.0875
130	.1200	310	.0860
140	.1177	320	.0844
150	.1157	330	.0829
160	.1135	340	.0813
170	.1113	350	.0797

1/ To find the most profitable rate, divide the cost of 10 pounds of N by the estimated value per ton of dry forage before harvest. The result is the fraction of a ton of forage equal in value to 10 pounds of N. Locate this result in the "tons" column. The corresponding application of N is approximately the most profitable rate.

Pasture experiment, Kirbyville, Tex. - 1950.

table 6 this increment is associated with a rate of about 90 pounds per acre. Based on the response curve as indicated by the four rates used in the experiment, this would underestimate the most profitable rate by about 170 pounds per acre at approximately current prices and costs. As shown in figure 2 the response curve is still rising rapidly at the highest rate applied in the experiment.

THE MOST PROFITABLE COMBINATION - TWO VARIABLES

Steps in the calculations for two variables represent an extension of procedures similar to those involving only one variable. In solving for Q where two variables are involved, the value for M_2 in tons is substituted in the equation. As determined in table 3, M_2 has a value equivalent to 9.3635 tons. Therefore

$$Q = 1/21(9.3635)(0.2231435214)$$
$$= 0.02279$$

The two variables N and P are used here in illustrating the calculation of the most profitable combination, when no K is applied. The most profitable combination of these two nutrients occurs at the point where T, defined earlier, is equal to Q as calculated above. The contribution of N involves its cost per unit relative to the cost of a unit of P and the quantity of it needed at the most profitable combination of the two. Substituting for N its equivalent in terms of P to be applied at the most profitable combination, results in the expression

$$T = \frac{R^{p+b}}{r''} - \frac{R^{p+b}}{r''}\left[\frac{r'R^{p+b}}{r'\ R^{p+b} - r''\ R^{p+b} + r''}\right]$$

The undetermined quantity is p+b, the number of units of P at the most profitable combination. In terms of the illustration used here

$$T = \frac{R^{p+b}}{1.10} - \frac{R^{p+b}}{1.10}\left[\frac{23.45\ R^{p+b}}{23.45\ R^{p+b} - 1.10\ R^{p+b} + 1.10}\right] = 0.02279$$

This quadratic equation is solved as a series of clerical steps illustrated by table 7.

The size of a unit of a has been determined as 120.24 pounds and a unit of b is 11.48 pounds. Thus, the most profitable combination to apply is 233 pounds of N and 84 pounds of P. Items 15 and 10, respectively, are the $1-R^x$ values associated with units of these nutrients, soil content plus quantities applied at the most profitable combination. Substituting these values in the two-variable form of the equation:

$$y = 9.3635\ (0.43272)(0.94207$$
$$= 9.3635\ (0.40765)$$
$$= 3.82 \text{ tons.}$$

This is the calculated yield at the calculated most profitable combination of N and P when no K is applied.

Table 7.- The most profitable combination of N and P when no K is applied

Item	Description	Value	Item	Description	Value
1	Qr"	0.02507	10	1.0 minus item 9	.94207
2	- Qr'	- .53442	11	x value, table 15	12.77
3	Item 1 plus item 2	- .50935	12	p, already found	5.41
4	- 1.0 minus item 3	- .49065	13	b, in units	7.36
5	1/2 of item 4	- .24532	14	Item 2 / item 10 2/	.56728
6	Item 5 squared	.06018	15	1.0 minus item 14	.43272
7	Item 6 minus item 1	.03511	16	x value, table 15	2.54
8	Sq. root of item 7 1/	- .18739	17	n, already found	.60
9	Item 5 minus item 8 2/	.05793	18	a, in units	1.94

1/ Use the negative root. 2/ Drop minus sign.

Pasture experiment, Kirbyville, Tex. - 1950.

Reference to the calculations of constants for the K curve shows that the quantity of K (soil potash - no application) was 6.22 units, and that $1-R^{6.22}$ is 0.75041. Substituting this value, and the quantity M_3 in the three-variable form of this equation, results in the same yield.

$$y = 12.4840\ (0.43272)(0.94207)(0.75041)$$
$$= 12.4840\ (0.30590)$$
$$= 3.82 \text{ tons}$$

At \$21 per ton the gross return from 3.82 tons is \$80.22 per acre. The cost of 1.94 units of a and 7.36 units of b at the determined values of r' and r" respectively, is \$53.59. Thus, the return above the cost of the fertilizer at the most profitable combination of N and P when no K is added, is \$26.63 per acre. This is a higher return per acre than either of the two combinations shown in table 5 where no K is applied. In table 5, N and P in turn were fixed at stated levels, but in table 7 they are free to vary simultaneously.

THE MOST PROFITABLE COMBINATION - THREE VARIABLES

In determining the value of Q when 3 variables are involved, the quantity of M_3 (in tons) from table 2 is substituted in the equation.

For this experiment the equation then becomes:

$$Q = 1/\ 21\ (12.4840)(0.2231435214)$$
$$= 0.01709$$

The most profitable combination of nutrients occurs at the point where T, defined earlier, is equal to Q as calculated above. The contribution of N involves its cost per unit relative to the cost of a unit of P and the quantity of it needed at the most profitable combination of the two. The same is true for K. Substituting for the N and K factors their equivalents in terms of P to be applied at the most profitable combination, results in the expression:

$$T = \frac{r''\ (1-R^{p+b})\ (R^{p+b})\ \ (1-R^{p+b})}{[r'\ R^{p+b} + r''\ (1-R^{p+b})]\ [r'''\ R^{p+b} + r''\ (1-R^{p+b})]} = Q$$

In this formidable-looking equation the only undetermined quantity is p+b, the number of units of P at the most profitable combination. In terms of the illustration used here

$$T = \frac{1.10\ (1-R^{p+b})^2\ \ R^{p+b}}{[23.45\ R^{p+b} + 1.10\ (1-R^{p+b})][2.35\ R^{p+b} + 1.10\ (1-R^{p+b})]} = 0.01709$$

This reduces to

$$T = \frac{1.10\ (R^{p+b})^3 - 2.20\ (R^{p+b})^2 + 1.10\ R^{p+b}}{27.9375\ (R^{p+b})^2 + 25.960\ R^{p+b} + 1.21} = 0.01709$$

But, this is a cubic equation and ordinarily the answer is found only by a series of trials, which can be a tedious process. However, the number of units of p+b, used as the exponent of R when R = 0.8, is usually large enough so that R^3 will nearly always be a negligible quantity. Because this is the case, the problem can be met through solution of an imperfect quadratic equation. This work can be arranged as a series of clerical steps (table 8).

Recalling that the symbols a, b, and c refer to applied N, P, and K, respectively, table 8 shows that the most profitable rates of application of the three nutrients are 2.97, 9.35, and 5.32 units. Applying the unit values (u in pounds) for each nutrient as previously determined, the calculated most profitable combination is approximately 357 pounds N, 107 pounds P and 158 pounds K. This combination reflects the influence of the response to each nutrient, the relative nutrient costs, and the crop value - fertilizer cost relationship.

Table 8.- The most profitable combination of N, P, and K

Constants previously determined
r' = $23.45; r" = $1.10; r"' = $2.35; Q = 0.01709

Item	Description	Value	Item	Description	Value
1	-2 r"	- 2.20000	22	Neg. sq. root, item 21	- .08544
2	r' - r"	22.35000	23	Item 19 - item 22 1/	.03713
3	r"' - r"	1.25000	24	1.0 minus item 23	.96287
4	r" squared	1.21000	25	x value, table 15	14.76
5	Item 2 times r"	24.58500	26	p, already found	5.41
6	Item 3 times r"	1.37500	27	b, in units	9.35
7	Item 5 plus item 6	25.96000	28	r" times item 24	1.05916
8	Item 2 times item 3	27.93750	29	r' times item 23	.87070
9	Q times item 4	.02068	30	r"' times item 23	.08726
10	Q times item 7	.44366	31	Item 28 plus item 29	1.92986
11	Q times item 8	.47745	32	Item 28 plus item 30	1.14642
12	r" minus item 10	.65634	33	Item 29/ item 31	.45117
13	Item 1 minus item 11	- 2.67745	34	1.0 minus item 33	.54883
14	Item 13/r"	- 2.43405	35	x value, table 15	3.57
15	Item 12/r"	.59667	36	n, already found	.60
16	Item 9/r"	.01880	37	a, in units	2.97
17	Item 15/ item 14	- .24513	38	Item 30/ item 32	.07612
18	Item 16/ item 14	- .00772	39	1.0 minus item 38	.92388
19	1/2 of item 17	- .12257	40	x value, table 15	11.54
20	Item 19 squared	.01502	41	k, already found	6.22
21	Item 18 plus item 20	.00730	42	c, in units	5.32

1/ When item 19 is a negative number, drop minus sign.

Pasture experiment, Kirbyville, Tex. - 1950.

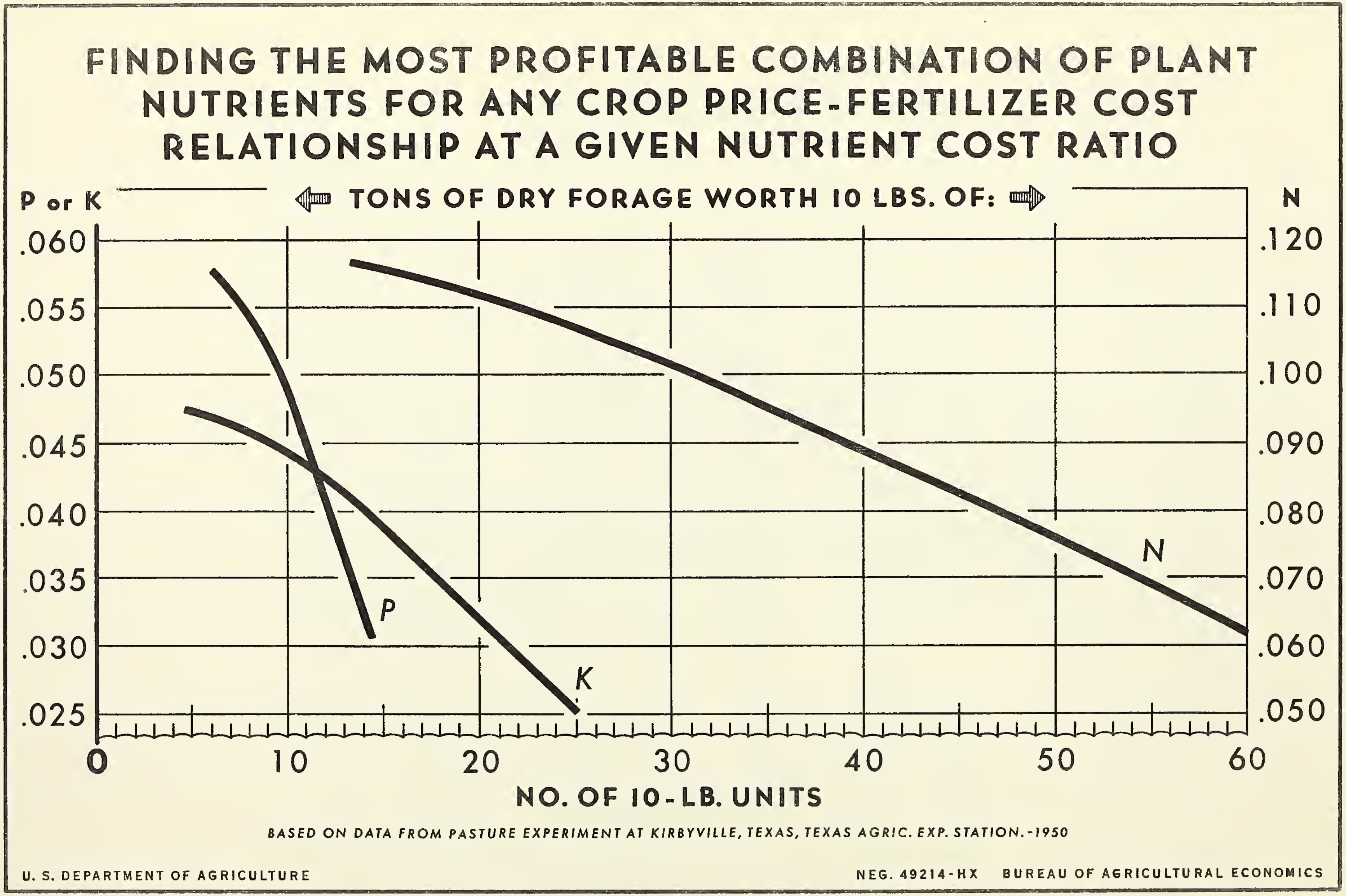
FINDING THE MOST PROFITABLE COMBINATION OF PLANT NUTRIENTS FOR ANY CROP PRICE-FERTILIZER COST RELATIONSHIP AT A GIVEN NUTRIENT COST RATIO
P or K
TONS OF DRY FORAGE WORTH 10 LBS. OF:
N
.060
.055
.050
.045
.040
.035
.030
.025
.120
.110
.100
.090
.080
.070
.060
.050
P
K
N
0
10
20
30
40
50
60
NO. OF 10-LB. UNITS
BASED ON DATA FROM PASTURE EXPERIMENT AT KIRBYVILLE, TEXAS, TEXAS AGRIC. EXP. STATION.-1950
U. S. DEPARTMENT OF AGRICULTURE
NEG. 49214-HX
BUREAU OF AGRICULTURAL ECONOMICS

Figure 3

The yield is the function of M_3 and the product of the $1-R^x$ values for N, P, and K. Items 34, 24, and 39, table 8, are the respective $1-R^x$ values of these nutrients at the point of highest return per acre with respect to fertilizer. Substituting these values in the three-variable form of the yield equation:

$$y = 12.4840\ (0.54883)(0.96287)(0.92388)$$
$$= 12.4840\ (0.48823)$$
$$= 6.0951 \text{ tons which at \$21.00 per ton are worth \$128.00}$$

The unit values of a, b, and c, and the costs per unit, r', r", and r"' are shown in table 8. From these it is determined that the total cost of the most profitable combination of all three nutrients is approximately $92.00. Thus, the return above the cost of fertilizer at the most profitable combination is approximately $36.00 per acre.

Optimum Rates Over a Range in Price-Cost Ratios

Table 6 and the front cover illustration show how results from a single variable fertilizer rate experiment can be prepared to serve as a practical guide for extension workers and farmers in determining the most profitable rate of application. A similar practical guide is presented for use when three variables are involved.

A few values of Q are calculated, each reflecting a substantially different unit price of the crop (v). The steps shown in table 8 are then worked out for each of these values of Q. A chart, figure 3, is prepared, the abscissa of which is scaled to show small units, in this case 10 pounds of a nutrient. The ordinate is scaled to indicate the quantity of crop equal in value to 10 pounds of a nutrient.

In table 8, Q is based on a value of dry forage of $21 per ton. The values of r', r", and r"' are based on costs per pound of $0.195, $0.096, and $0.079 for N, P, and K, respectively. Thus, at $21 per ton, 0.0929 ton of forage is equal in value to 10 pounds of N. Table 8 shows that the most profitable rate, at this price of forage and these costs of fertilizer, is 357 pounds of N. In constructing the N curve of figure 3, a dot is placed at the point of intersection of 357 pounds on the abscissa and 0.0929 ton on the ordinate scale for that nutrient. Points of intersection for the P and the K curves were located in a similar manner. This process was repeated, using four widely different prices per ton of forage in order to result in points of intersection that would provide a basis for drawing the freehand curves.

In figure 3 the higher the reading on the vertical scale the lower the value of the crop in relation to the cost of fertilizer. Approximately the lowest value per ton at which it would pay to use any fertilizer is represented by the reading on the vertical scale that is associated with the upper end of each curve. The most profitable combination is indicated by the relative shapes of the response curves and the relative cost of the nutrients, as well as by the value of the crop. For example, as the curve of response to P flattens earlier than those for N and K, the effect of increasing the ratio of crop value to the cost of phosphate fertilizer on the rate of application that would be most profitable, is much less marked. For this reason, in figure 3, which reflects the influence of changing crop value-fertilizer cost ratios on rates that would be most profitable, the P curve turns abruptly downward almost at the start. This is in marked contrast particularly with the N curve which reflects exceptional elasticity in response to changes in value of the crop relative to the cost of fertilizer.

The return above the cost of fertilizer is that portion of gross returns available to meet other costs, including management. When a crop value-fertilizer cost relationship is such that it would not pay to use any fertilizer, item 21 of table 8 will be a negative number. By trial methods, a value of Q (which is dependent on v, or the unit price of the crop) may soon be found that will render item 21 essentially zero, though still on the positive side. When such a value of Q is found, completion of table 8 will result in units of a, b, and c that represent rates of application associated with practically the lowest unit value of the crop at which it would pay to apply any fertilizer. In this illustration, a price per ton of $16.675 resulted in Q = 0.021528, and item 21, table 8 = 0.000030. Completion of table 8 then resulted in units of a, b, and c equivalent to 135, 61, and 46 pounds per acre, respectively. These quantities may be read on the abscissa of figure 3 by dropping lines vertically from the upper ends of the three curves. Under the conditions of this experiment, if the value per ton of dry forage falls below $16.675, the application of any fertilizer would produce a smaller return above the cost of fertilizer than the gross value of the calculated yield when no fertilizer is added. At the above-mentioned price per ton, the return per acre above the cost of the most profitable combination of nutrients (135, 61, and 46 pounds of N, P, and K) is $13.88. This is only 15 cents more than the value of the calculated yield when no fertilizer is added. Thus, for all practical purposes, to apply any fertilizer when the price per ton of dry forage falls much below $17 per ton, cost of fertilizer unchanged, would represent an encroachment on other costs.

In an earlier illustration it was pointed out that a price of $15 per toh would justify a rate of between 80 and 90 pounds of N in the presence of 240 pounds of applied P. Also it was pointed out that the cost of an applied nonvarying nutrient is not taken into account in determining the most profitable rate of a single variable. But when 2 or more nutrients are varied simultaneously, the problem is one of determining the most profitable combination with none of them fixed at any specified level.

All of the above conclusions are based on fertilizer nutrient costs indicated in table 8, equivalent to $0.195, $0.096, and $0.079 per pound of N, P, and K, respectively. But, as long as costs of these nutrients bear this relationship to each other, figure 3 applies irrespective of the actual costs per pound of the nutrients or the price per unit of the crop.

Effect of Changes in Nutrient Costs and Cost Ratios

Costs per pound of each of the three nutrients have been indicated. The ratios of these costs per pound ($0.195 for N, $0.096 for P, and $0.079 for K) to the cost of K per pound are 2.47, 1.22, and 1.00. To illustrate the effect of changes in nutrient costs and cost ratios, the steps of table 8 were conducted using current retail costs per pound of plant nutrients as they occur in specified straight materials. In nitrate of soda, the cost per pound of N is $0.171. (N can be obtained at much lower costs per pound if purchased in other forms, particularly solution.) Using ordinary superphosphate the cost per pound of P is $0.0855. For K, the cost per pound is $0.0571 if bought in 50 percent muriate of potash. At these unit costs, based on United States average costs per ton of these materials, the nutrient cost ratio is 2.99, 1.50, and 1.00.

At $21 per ton of dry forage, the return above cost of fertilizer at the level and ratio of nutrient costs when obtained from the straight materials indicated above, would be about $51 per acre when the most profitable combination is applied. At the level and ratio of nutrient costs when obtained from mixed fertilizer, the return above fertilizer cost as previously calculated, was found to be about $36 per acre, at the most profitable combination. If the fertilizer were applied as three separate materials, there would be some additional cost of application. Or, if the farmer attempted to mix the materials before application, additional costs would be involved so that the final difference in return per acre would not be as great as is indicated by the above figures.

As pointed out earlier, if only the level of nutrient costs is changed, only one graph like figure 3 is needed to estimate the most profitable combination for any crop price-fertilizer cost relationship. But, if farmers can choose nutrients from different sources so that there will be substantial differences in cost ratios of nutrients, as well as differences in actual costs per unit, additional graphs similar in form to figure 3 can be prepared to serve as practical guides for extension workers and farmers in deciding on the most profitable combination for each situation.

SUBSTITUTION OF NUTRIENTS 6/

Within limits yet to be defined by experimental evidence, there are possibilities of substituting one nutrient for another in order to obtain a specified yield. The practical limit on yield may be found for a given field on a farm, when all yield-limiting factors have been removed as far as possible. In practice, the most profitable yield will usually be somewhat short of the calculated most profitable yield, particularly when the element of risk is taken into account. When dealing with inputs such as fertilizer, that may be varied in intensity, a practical problem is to obtain a given level of yield at the lowest possible cost. Ways of finding the most profitable combination of two or more nutrients at any price-cost relationship have been illustrated. The problem in this section is to find the lowest cost combination that will give a stated yield.

This problem has two aspects--first, to what extent is it physically or agronomically possible to substitute one nutrient for another and at the same time maintain a given yield? The answer to this question can be supplied only by experimental evidence of yields obtained over a considerable range of rates and combinations.

It is recognized that even though substitution may be both possible and profitable in the short run, there are basic physiological factors that may restrict substitution possibilities to a very narrow range over any considerable period of time. The degree of feasible substitution is influenced, among other things, by the nutrient status of the soil. The effect of substitution of nutrients on quality of crop, even for a short period, might be undesirable in certain situations. These problems should be explored. But the idea of substitution of applied nutrients within limits as to degree and time, poses no problems not already present in situations where the crop is adversely affected by an unbalanced plant-nutrient condition.

6/ Equations for equal product combinations and for marginal rates of substitution used in this section were supplied by Glenn L. Burrows, Bureau of Agricultural Economics.

The second part of the problem is, what must be the cost relationship between nutrients to make a particular combination lowest in cost? This may also be stated as: "How much of one nutrient substitutes for another at the lowest cost combination?" This quantity of one nutrient that substitutes for another at the lowest cost combination that would result in a given yield is usually referred to as the marginal rate of substitution. The solution is best approached in this way where only two nutrients are involved.

When three nutrients are involved the question is more complicated and the equations set up to answer it are in terms of "At specified nutrient cost ratios, what is the lowest cost combination of nutrients at which to obtain a given yield?"

The Case of Two Variables

Different combinations of N and P with no K applied, are used to illustrate the marginal rates of substitution of two variables. It has been shown in the earlier illustration that the most profitable combination (no K applied) is 233 pounds of N and 84 pounds of P at the fertilizer cost rates and forage prices assumed earlier and that the yield at this combination is 3.82 tons. In the following illustration, it is assumed that under field conditions, and considering the element of risk, the most profitable yield is probably about 3 tons. What are some different combinations of N and P that are calculated to produce a yield of 3 tons? The equation for this is

$$(y/M_2)\ /(1-R^{n+a}) = 1-R^{p+b} \quad \text{or}$$
$$(y/M_2)\ /(1-R^{p+b}) = 1-R^{n+a}$$

where y is the desired yield (constant product). The definition of M_2 has been given, and as shown earlier it represents 9.3635 tons. The value of $1-R^{n+a}$ (or $1-R^{p+b}$) is easily derived for any rate of application of a, or of b from constants already derived. For example, u for N has been determined as 120.24 pounds, therefore 300 pounds of a is equivalent to 2.50 units. As n in these illustrations = 0.60 unit, n+a is 3.10 units and $1-R^{3.10}$ is 0.49930. If y = 3 tons, y/M_2 = 0.32039. Thus, $1-R^{p+b}$ when a = 300 pounds is 0.32039/0.49930 or 0.64168 and this value is associated with 4.60 units of p+b. It has been determined that p is equivalent to 5.41 units or 62 pounds. In other words, this calculated quantity of p is 0.81 unit or 9.3 pounds in excess of the quantity required to yield 3 tons of forage when 300 pounds of N are applied. This calculated quantity of p bears some relation to the quantity utilized by the growing plant. The 9.3 pounds may be regarded as the portion of the calculated soil supply of about 62 pounds that is unavailable or unused by the plant in the presence of 300 pounds of applied nitrogen.

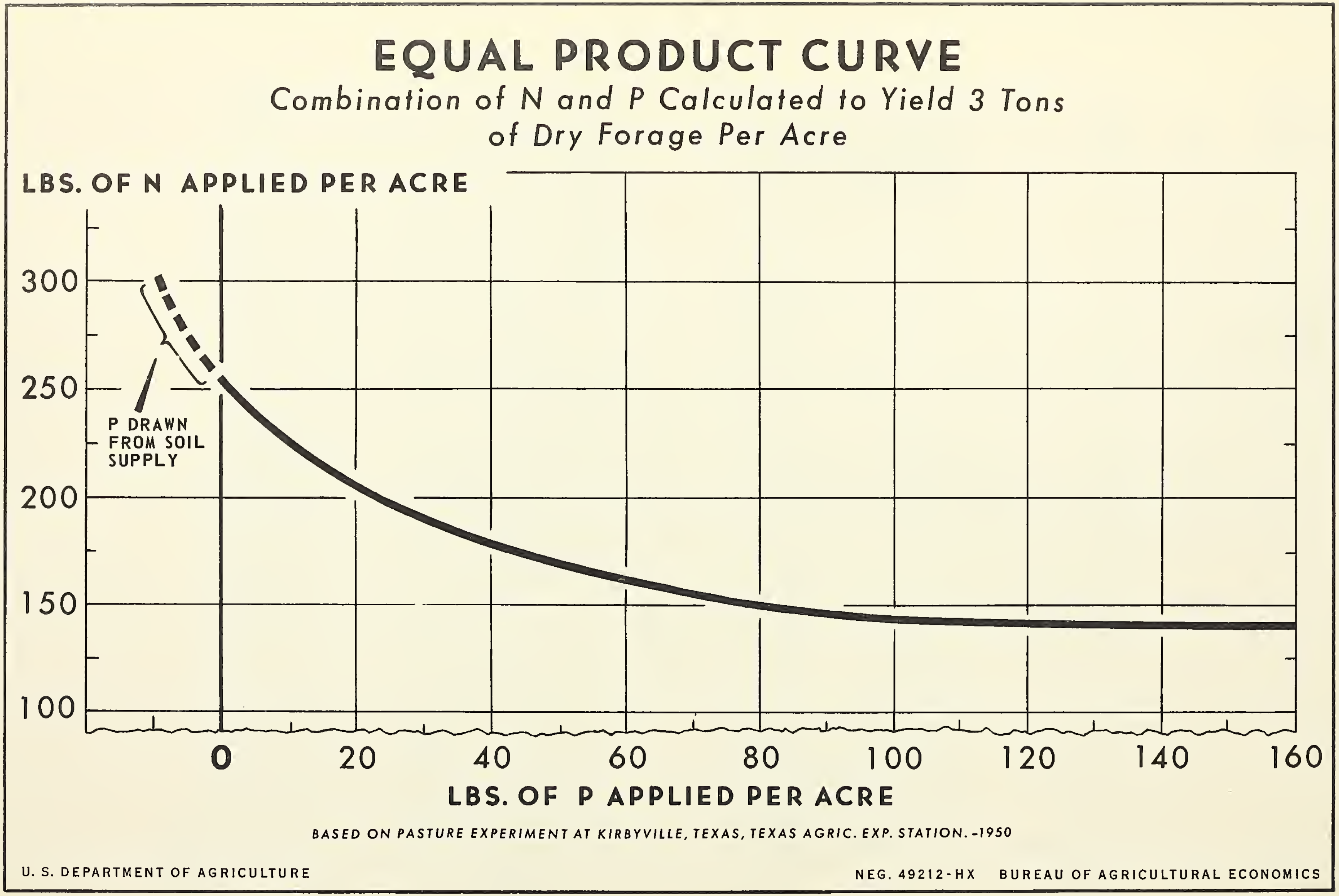
EQUAL PRODUCT CURVE
Combination of N and P Calculated to Yield 3 Tons of Dry Forage Per Acre
LBS. OF N APPLIED PER ACRE
300
250
200
150
100
P DRAWN FROM SOIL SUPPLY
0
20
40
60
80
100
120
140
160
LBS. OF P APPLIED PER ACRE
BASED ON PASTURE EXPERIMENT AT KIRBYVILLE, TEXAS, TEXAS AGRIC. EXP. STATION. -1950
U. S. DEPARTMENT OF AGRICULTURE
NEG. 49212-HX
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Figure 4

The marginal rates of substitution for the combinations shown in table 9 are calculated as

$$\text{MRS in units} = \frac{R^{p+b}}{R^{n+a}} \cdot \frac{1-R^{n+a}}{1-R^{p+b}}$$

$$\text{MRS in pounds} = \text{MRS in units times } \frac{u_a}{u_b}$$

As table 9 shows the values for $1-R^{n+a}$ and $1-R^{p+b}$ for the various combinations, the values of R^{n+a} and R^{p+b} are derived by subtraction. Substituting the values in finding u_a/u_b, 120.24/11.48 = 10.47387. This represents the number of pounds in a unit of N required to render R = 0.8, divided by the corresponding number of pounds in a unit of P.

If values associated with the combination of 160 pounds of N and 65 pounds of P are substituted in the above equation, the marginal rate of substitution is calculated as

$$\frac{0.08439}{0.65008} \cdot \frac{0.34992}{0.91561} = \frac{0.02953}{0.59522} = 0.04961 \text{ units times } 10.47387,$$
or 0.52 pounds.

This means that at the combination 160 N and 65 P, 0.52 pounds of N substitutes for one pound of P in producing a yield of three tons. Or, if the cost per pound of P is 0.52 times the cost per pound of N, the above mentioned combination will yield highest net return. The marginal rate of substitution calculated in this manner for each combination is shown in the last column of table 9.

Each marginal rate of substitution is associated with a particular point on the equal-product curve (fig. 4). The marginal rate of substitution is not to be confused with the average rate over any considerable segment of the curve. At certain positions on the curve the two may be nearly the same when the average rate is calculated for very small intervals. For example, in going from combination 10 to combination 11 (table 9) 6 pounds of P substitutes for 10 pounds of N. Thus, on the average over this short range, 1.67 pounds of N are required to substitute for 1 pound of P. The marginal rate at the point of combination 11 is 1.62, as shown. But toward the extremes of the range shown in table 9, the average rate of substitution departs substantially from the marginal rate for each 10-pound change in the application of N. And, of course, the marginal rate at any point departs widely from the average rate over any considerable range from that point.

The current N to P cost ratio about corresponds to the marginal rate of substitution for combination 15. But if relative costs changed so that N cost only about one-third more per pound than P, the least-cost combination at which to produce a yield of three tons would be 190 pounds of a and 30 pounds of b (combination 12). If $0.096 is used as the cost per pound of P for

Table 9.- Some combinations of N and P calculated to yield three tons of dry forage per acre, and the marginal rates of substitution for each combination

Combination	N: a Pounds	N: a Units	N: $n+a$ (n= 0.60) units	N: $1-R^{n+a}$	$1-R^{p+b}$ 1/	P: $p+b$ (p= 5.41)	P: b Units	P: b Pounds 2/	R^{n+a}	R^{p+b}	Marginal rate of substitution of N for P: Units 3/	Marginal rate of substitution of N for P: Pounds 4/
1	300	2.50	3.10	0.49930	0.64168	4.60	- 0.81	- 9	0.50070	0.35832	0.55685	5.83
2	290	2.41	3.01	.48914	.65501	4.77	- .64	- 7	.51086	.34499	.50430	5.28
3	280	2.33	2.93	.47994	.66756	4.93	- .48	- 6	.52006	.33244	.45957	4.81
4	270	2.25	2.85	.47057	.68086	5.12	- .29	- 3	.52943	.31914	.41657	4.36
5	260	2.16	2.76	.45983	.69676	5.35	- .06	- 1	.54017	.30324	.37049	3.88
6	250	2.08	2.68	.45010	.71182	5.58	.17	2	.54990	.28818	.33137	3.47
7	240	2.00	2.60	.44020	.72783	5.85	.44	5	.55980	.27217	.29406	3.08
8	230	1.91	2.51	.42884	.74711	6.16	.75	9	.57116	.25289	.25415	2.66
9	220	1.83	2.43	.41855	.76548	6.50	1.09	13	.58145	.23452	.22054	2.31
10	210	1.75	2.35	.40808	.78512	6.89	1.48	17	.59192	.21488	.18869	1.98
11	200	1.66	2.26	.39607	.80892	7.42	2.01	23	.60393	.19108	.15491	1.62
12	190	1.58	2.18	.38520	.83175	7.99	2.58	30	.61480	.16825	.12674	1.33
13	180	1.50	2.10	.37412	.85638	8.70	3.29	38	.62588	.14362	.10024	1.05
14	170	1.41	2.01	.36143	.88645	9.75	4.34	50	.63857	.11355	.07250	0.76
15	160	1.33	1.93	.34992	.91561	11.08	5.67	65	.65008	.08439	.04961	0.52
16	150	1.25	1.85	.33822	.94728	13.19	7.78	89	.66178	.05272	.02844	0.30
17	140	1.16	1.76	.32479	.98645	19.28	13.87	159	.67521	.01355	.00661	0.07

1/ $(y/M_2)/(1-R^{n+a})$; y = 3 tons and M_2 = 9.3635 tons.

2/ Units b times u; u = 11.48 lbs. Figures in this column are rounded to whole pounds. As p is constant at 5.41 units (62 pounds) the minus quantities of b shown are interpreted as being unutilized by the plant at the respective levels of $n+a$. Beginning with combination 6, the soil supply of p is inadequate to maintain the yield at 3 tons at the reduced levels of $n+a$.

3/ $[R^{p+b}/R^{n+a}][(1-R^{n+a})/(1-R^{p+b})]$.

4/ Marginal rate in units times u_a/u_b, or 120.24/11.48, or 10.47387. Divide the cost per pound of P by the cost per pound of N. Locate result in this column. The associated rates of N and P represent approximately the least cost combination.

Pasture experiment, Kirbyville, Tex. - 1950.

both combinations, the total cost per acre of the fertilizer at combination 15 would be $37.44. But, if the cost of N declined from 1.98 to 1.33 times that of P, the cost of the latter remaining constant, the total cost of fertilizer at combination 12 would be $27.14. Thus, failure to make the change from combination 10 to combination 12 would reduce by $10.30 the returns per acre above the cost of the fertilizer.

The experiment on which these illustrations are based does not provide a means of checking on whether all of the different combinations indicated in table 9 and shown graphically in figure 4 would produce a yield of 3 tons. However, the indicated response to N was such at all combinations of P and K that the curve from 0 to 180 pounds of N almost resembled a straight line. At 180-0-0 the reported yield was 2.36 tons. At 120-120-0 it was 2.94 tons. As combination 17 is 140-159-0 pounds (table 9), the evidence supplied by the experiment makes the calculated yield of three tons for this combination appear reasonable. The P curve levels off rapidly after about 120 pounds, but the additional 20 pounds of N would increase the yield substantially according to the N curve. Furthermore, at 180 pounds of N and no applied K, the yield at 17 pounds of P would be 2.61 tons, according to the P curve. Therefore, considering the steepness of the N curve, combination 10 using 210 pounds of N and 17 pounds of P should result in the calculated yield of three tons.

Table 9, of course, includes several combinations that would not be selected in practice because the nutrient cost ratios required to render them economic would in all probability never occur. But, a farmer often applies a grade of fertilizer on the basis of cost per ton. For example, at current nutrient cost ratios, combinations 10 and 17 are about equal in cost per acre, but an N, P fertilizer representing about a 1 to 1 ratio costs a great deal less per ton than one having a plant nutrient ratio of about 12:1 such as combination 10. But the latter would require about 70 pounds less material per acre. This means a saving in handling cost. Also, combination 17 is uneconomical for the conditions of the Texas Pasture experiment, from a national point of view, because it represents a wastage of the sulphur required in the manufacture of the additional quantity of phosphate fertilizer.

The Case of Three Variables

When three variables are involved, marginal rates of substitution could be calculated for each one in turn at different combinations of the other two. However, this would involve a considerable number of combinations to cover the range associated with changes that might be expected in the ratios of the nutrient costs.

A more satisfactory approach is to assume a number of nutrient cost situations that represent a range wide enough to include any nutrient cost ratio that may be expected. The lowest-cost combination of nutrients that is associated with each of several such nutrient cost ratios may be calculated. Interpolations may then be made for other cost ratios if desired.

The problem is to find the minimum-cost combination of nutrients that will result in the desired specified yield at a particular nutrient cost ratio. The equation used in making this determination is:

$$(1 - y/M_3)m^3 - y/M_3(r' + r'' + r''')m^2$$
$$- y/M_3\ (r'r'' + r'r''' + r''r''')m - y/M_3(r'r''r''') = 0$$

In this equation, m is the critical factor associated with the minimum-cost combination. It is found by the method of trial and error. y is the specified yield desired. From previous work M_3 was found to be 12.4840 tons, r', \$23.45; r'', \$1.10; and r''', \$2.35. For purposes of illustration, the specified constant yield, y is 5 tons. Also, for purposes of illustration, the problem is to find the minimum-cost combination that would produce the desired yield of 5 tons if the cost of r' should be reduced by 25 percent and other nutrient costs remained unchanged. Thus, the value of r' becomes \$17.58 instead of \$23.45. Part of the work in solving the equation is handled as simple clerical steps based on values previously derived.

1. $y/M_3 = 5/12.4840$	=	0.400513
2. $1-(y/M_3)$	=	0.599487
3. Item 2/ item 1	=	1.49680
4. $r' + r'' + r''' = 17.58 + 1.10 + 2.35$	=	21.03
5. $(r'r'') + (r'r''') + (r''r''')$	=	63.236
6. $(r'r'')\ r'''$	=	45.44430
7. Item 4/ item 3	=	14.05
8. Item 5/ item 3	=	42.25
9. Item 6/ item 3	=	30.36

When the values for items 1 to 9 have been calculated, m is determined through a series of trials as indicated in table 10.

The first trial value of m may well be from 2.5 to 3.0 units larger than item 7 of the constants developed in the preceding paragraph. The calculations are rapidly done by machine and 5 or 6 trials will usually result in a value that will render item (e) of table 10 a close enough approximation to 0 for all practical purposes.

Table 10.- Trial values for m, at a stated nutrient cost ratio

	1st trial	2nd trial	3rd trial	4th trial	5th trial
m	16.80	16.60	16.69	16.68	
m^2	282.24	275.56	278.56	278.22	
m^3	4,741.63	4,574.30	4,649.17	4,640.71	
(a) Item 9	60.36	30.36	30.36	30.36	30.36
(b) m x item 8	709.80	701.35	705.15	704.73	
(c) m^2x item 7	3,965.47	3,871.62	3,913.77	3,908.99	
(d)(a)+(b)+(c)	4,705.63	4,603.33	4,649.28	4,644.08	
(e) m^3 - (d)	36.00	- 29.03	- .11	- 3.37	
m is:	too large	too small	satisfactory	too small	

Pasture experiment Kirbyville, Tex. - 1950.

In this illustration, r' = \$17.58, r" = \$1.10, and r"' = \$2.35. Table 11 shows the calculation of the minimum-cost combination of applied N, P, and K that would be used to obtain the desired yield, which in this case is 5 tons. The satisfactory value of m is 16.69 as determined above. The values of $1-R^x$ in table 11 are derived as m/(m + r'), m/(m + r"), and m/(m + r"'), respectively.

Table 11.- The minimum cost combination of applied N, P, and K to obtain a yield of 5 tons at a stated nutrient cost ratio

Item	Description	Value	Description	Value	Description	Value
1	m + r'	= \$34.27	m + r"	= \$17.79	m + r"'	= \$19.04
2	$1-R^{n+a}$	=1/ .48702	$1-R^{p+b}$	=1/ .93817	$1-R^{k+c}$	=1/ .87658
3	n + a	=2/ 2.99	p + b	=2/ 12.47	k + c	=2/ 9.38
4	n	=3/ .60	p	=3/ 5.41	k	=3/ 6.22
5	a	= 2.39	b	= 7.06	c	= 3.16
6	a in lbs.	=4/ 287	b, lbs.	= 4/ 81	c in lbs.	=4/ 94

1/ m/(m + r', m/(m+r"), and m/(m+r"'), respectively.
2/ Read from table 15.
3/ From constants previously developed.
4/ Item 5 times u for the nutrient concerned.

Pasture experiment, Kirbyville, Tex. - 1950.

Table 12.- Selected nutrient cost ratios, associated minimum cost combinations calculated to yield 5 tons of dry forage per acre, and fertilizer costs per acre at each combination

Ratio no.	Nutrient cost ratios (Relative costs per pound)			Least cost combinations			Total quantity of NPK	Fertilizer cost per acre 1/	Plant nutrient ratio		
	N	P	K	N	P	K			N	P	K
				Pounds	Pounds	Pounds	Pounds	Dollars			
1	2.47	1.22	1.0	269	103	120	492	71.85	1.0	0.38	0.45
2	1.85	1.22	1.0	287	81	94	462	57.18	1.0	.28	.33
3	1.25	1.22	1.0	353	68	62	483	46.32	1.0	.19	.18
4	1.25	1.00	1.0	325	78	65	468	43.32	1.0	.24	.20

1/ Based on estimated current cost of K of $0.079 per pound. Cost ratio 1 reflects approximately the current situation. Ratio no. 2 would reflect a 25 percent reduction in the unit cost of N.

Pasture experiment, Kirbyville, Tex. - 1950.

As a check on the accuracy of the above calculations, and substituting in the yield equation:

$$y = M\,(1-R^{n+a})\,(1-R^{p+b})\,(1-R^{k+c})$$
$$= 12.4840\ (0.48702)\ (0.93817)\ (0.87658)$$
$$= 5.0000 \text{ tons, the specified yield.}$$

Minimum cost combinations associated with stated cost ratios of the three principal plant nutrients have been calculated according to the procedure just illustrated, and are shown in table 12 along with the cost of fertilizer per acre at the unit costs indicated. There is a range of only 30 pounds in the total quantity of plant nutrients to be applied per acre to obtain the specified yield of 5 tons. But there is a considerable range in the nutrient ratio. Relative to N, twice as much P and 2.5 times as much K are contained in combination 1 as in combination 3. At current unit costs, combination 3 would cost $8.52 more per acre than combination 1. The latter is the least-cost combination at current cost ratios.

More information is needed as to substitution possibilities and the physical factors involved in instances where there is evidence that one nutrient may be substituted for another in obtaining a specified yield. The effect on quality of crop, and on removal from the soil of a nutrient for which another is substituted, are items of major importance. When substitution is possible, for a time at least, without harmful effects, the economic aspects may also be of major importance. Experimental designs along lines illustrated later should be helpful in answering these questions.

Experimental Design for Economic Interpretation 7/

Items of economic interpretation that have been presented are to be regarded primarily as illustrative rather than factual. Considerable extrapolation has been necessary. There is need for experimental evidence covering the major portion of the total response curve. Fertilizer rate experiments that have only one variable can easily be designed to provide a basis for a regression curve covering the entire range of response. It is only a matter of including enough rates properly distributed over the range to permit characterization of the response curve.

Where two variables are involved, the problem is increased. There is not only the question of characterizing each of the response curves but also the question of interaction. However, the problem is still manageable. Of course, to provide an adequate basis for characterizing each of the curves throughout the range in response, more rates are needed than have usually been included in

7/ Assistance in preparing illustrative experimental designs combining extended regression and factorial characteristics was rendered by D. D. Mason, Bureau of Plant Industry, Soils, and Agricultural Engineering.

the simple two-variable factorial experiments. Probably 6 rates usually should be regarded as the minimum for this purpose, with as many as 8 or 10 being more desirable. A 10 X 10 factorial would require 100 plots without replication. For purposes of economic interpretation, in many circumstances, at least, a 10 X 10 unreplicated factorial would be more desirable than a smaller number of rates with replications.

However, another type of design would be to establish a series of rates of each of the two variables with the other held constant, not necessarily replicated, and also set up a factorial series that would be replicated. Table 13 illustrates such a design.

Table 13.- Illustrative outline of data needed for establishing regression curves and determining interaction when two variables are involved 1/

Extended regression series		Factorial series	
N	P	N	P
Pounds	Pounds	Pounds	Pounds
0	60	0	0
10	60	0	20
20	60	0	100
40	60	20	0
80	60	20	20
120	60	20	100
160	60	200	0
180	60	200	20
200	60	200	100
80	0		
80	20		
80	40		
80	60		
80	80		
80	100		

1/ Outlined as data would be arranged for analysis. All plots of both regression and factorial series would be randomized. Actual rates used would vary as desired.

The extended regression series contains 9 rates of N and 6 rates of P. This series need not be replicated, but if facilities are available, replication would add to the value of the experiment. Probably, however, more value would be added by including another level in the factorial series. There are a total of 9 plots in the

Table 14.- Illustrative outline of data needed for establishing regression curves and determining interaction when three variables are involved 1/

Extended regression series			Factorial series		
N	P	K	N	P	K
Pounds	Pounds	Pounds	Pounds	Pounds	Pounds
0	60	40	0	0	0
10	60	40	0	0	20
20	60	40	0	0	100
40	60	40	0	20	0
80	60	40	0	20	20
120	60	40	0	20	100
160	60	40	0	100	0
180	60	40	0	100	20
200	60	40	0	100	100
80	0	40	20	0	0
80	20	40	20	0	20
80	40	40	20	0	100
80	60	40	20	20	0
80	80	40	20	20	20
80	100	40	20	20	100
			20	100	0
			20	100	20
			20	100	100
80	60	0	200	0	0
80	60	20	200	0	20
80	60	40	200	0	100
80	60	60	200	20	0
80	60	80	200	20	20
80	60	100	200	20	100
			200	100	0
			200	100	20
			200	100	100

1/ Outlined as data would be arranged for analysis. All plots of both regression and factorial series would be randomized. Actual rates used would vary as desired.

factorial series and these should be replicated, perhaps as many as 5 for the no treatment combination and at least 2 for the others. This would make 21 plots in the factorial series and 15 in the regression series, or a total of 36 plots. (If, in addition, the regression series were replicated, 51 plots would be required). A 4 X 4 factorial with 4 replicates, which for purposes of economic interpretation would not be as satisfactory as the type of experiment illustrated by table 13 even without replication of the regression series, would require 64 plots. The combination regression-factorial design provides an adequate basis for establishing a regression curve for each variable and for analysis of variance to determine whether interaction is significant. It also provides a basis for testing the adequacy of the multivariable form of the exponential yield equation in that yields for the combinations included in the factorial series may be used as checks on yields calculated wholly from constants derived from the regression series. In the regression series it may be desirable to have smaller intervals between rates at the low and high ends of the range. This would permit closer measurement of response in the "critical" segments of the curve.

When three variables are involved, and where a sufficient number of rates are included to provide an adequate basis for regression curves, the complete factorial would not be feasible because of the large number of plots required. The combination regression-factorial design is particularly suited to this situation. It is merely an extension of the design for two variables.

Where three variables including the number of rates shown in table 14 are involved, 5 replicates of the no-treatment combination and 2 for each of the others in the factorial series, with no replicates in the regression series, would require 78 plots. Replication of the regression series would add 21 to bring the total to 99 plots. The Texas Pasture experiment on which the illustrations of procedures were based was a 4 X 3 X 3 factorial with 4 replicates for each treatment combination, making a total of 144 plots. For purposes of economic interpretation the combination regression factorial design, even with only 78 plots, would be more desirable.

Extreme combinations are suggested in the factorial series to test the limits of the multivariable form of the exponential yield equation for calculating yields at combinations that depart widely from those included in the regression series. The factorial series also provides a basis for statistical measurement of interaction through analysis of variance. It will then be possible to test whether the presence or absence of significant interaction affects the adequacy of the multivariable form of the equation for calculating crop yields.

Table 15.- Values of $1-R^x$ when R = 0.8
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
0.00	------	000223	000446	000669	000892	001125	001338	001561	001784	002006
01	002229	002452	002674	002897	003119	003342	003564	003786	004009	004231
02	004453	004675	004897	005119	005341	005563	005785	006007	006228	006450
03	006672	006894	007115	007337	007558	007780	008001	008222	008444	008665
04	008886	009107	009328	009549	009770	009991	010212	010433	010654	010874
05	011095	011315	011536	011757	011977	012198	012418	012639	012859	013079
06	013299	013519	013740	013960	014180	014400	014620	014839	015059	015279
07	015499	015718	015938	016158	016377	016597	016816	017035	017254	017474
08	017693	017912	018131	018350	018569	018788	019007	019226	019445	019664
09	019883	020101	020320	020538	020757	020975	021194	021412	021631	021849
10	022067	022285	022503	022721	022940	023158	023376	023593	023811	024029
11	024247	024465	024683	024900	025118	025335	025553	025770	025987	026205
12	026422	026639	026856	027073	027290	027508	027725	027942	028158	028375
13	028592	028809	029026	029242	029459	029675	029892	030108	030325	030541
14	030757	030974	031190	031406	031622	031838	032054	032270	032486	032702
15	032918	033133	033349	033565	033780	033996	034212	034427	034642	034858
16	035073	035288	035504	035719	035934	036149	036364	036579	036794	037009
17	037224	037439	037654	037868	038083	038298	038512	038726	038941	039156
18	039370	039584	039799	040013	040227	040441	040655	040869	041083	041297
19	041511	041725	041939	042153	042366	042580	042794	043007	043220	043434
20	043648	043861	044074	044288	044501	044714	044927	045140	045353	045566
21	045779	045992	046205	046418	046631	046843	047055	047268	047481	047794
22	047906	048119	048331	048543	048755	048968	049180	049392	049604	049816
23	050028	050240	050452	050664	050876	051088	051298	051511	051723	051934
24	052145	052357	052569	052780	052991	053203	053414	053625	053836	054047
25	054258	054469	054680	054891	055102	055313	055524	055735	055945	056156
26	056366	056577	056787	056998	057208	057419	057629	057839	058049	058260
27	058470	058680	058890	059100	059310	059520	059729	059939	060149	060358
28	060568	060778	060987	061197	061406	061616	061825	062035	062244	062453
29	062662	062872	063081	063290	063499	063707	063916	064125	064334	064543
30	064752	064960	065169	065377	065586	065794	066003	066211	066420	066628
31	066836	067044	067253	067461	067669	067877	068085	068293	068500	068708
32	068916	069124	069332	069540	069747	069954	070162	070369	070577	070784
33	070992	071199	071406	071613	071820	072027	072234	072441	072648	072855
34	073062	073269	073476	073683	073889	074096	074302	074509	074715	074921
35	075128	075335	075541	075747	075953	076160	076366	076572	076778	076984
36	077190	077396	077602	077807	078013	078219	078424	078630	078836	079041
37	079247	079452	079658	079863	080068	080273	080478	080684	080889	081094
38	081299	081504	081709	081914	082119	082323	082528	082732	082937	083142
39	083347	083551	083756	083960	084165	084369	084573	084777	084982	085186
40	085390	085594	085798	086002	086206	086310	086613	086817	087021	087225
41	087429	087632	087836	088039	088243	088446	088649	088853	089056	089259
42	089463	089666	089869	090072	090275	090478	090681	090884	091086	091289
43	091492	091695	091898	092100	092303	092505	092708	092910	093113	093315
44	093517	093719	093922	094124	094326	094528	094730	094932	095134	095336
45	095538	095740	095941	096143	096345	096546	096748	096949	097151	097352
46	097554	097755	097956	098157	098359	098560	098761	098962	099163	099364
47	099565	099766	099967	100168	100369	100569	100770	100970	101171	101371
48	101572	101773	101973	102173	102374	102574	102774	102974	103175	103375
49	103575	103775	103975	104175	104374	104574	104774	104974	105173	105373
50	105573	105772	105972	106171	106371	106570	106770	106969	107168	107367
51	107567	107766	107965	108164	108363	108562	108760	108959	109158	109357
52	109556	109754	109953	110152	110350	110549	110747	110945	111143	111342
53	111540	111738	111937	112135	112333	112531	112729	112927	113125	113323
54	113521	113719	113916	114114	114312	114509	114707	114904	115102	115299
55	115497	115694	115891	116089	116286	116483	116680	116877	117074	117271
56	117468	117665	117862	118059	118256	118452	118649	118846	119042	119239
57	119435	119632	119828	120024	120220	120417	120614	120810	121006	121202
58	121398	121594	121790	121986	122182	122378	122574	122769	122965	123160
59	123356	123552	123748	123943	124138	124334	124529	124725	124920	125115

- Continued -

Table 15.- Values of $1-R^x$ when $R = 0.8$ - Continued -
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
0.60	125310	125505	125701	125896	126091	126286	126481	126676	126870	127065
61	127260	127455	127649	127844	128039	128233	128428	128622	128817	129011
62	129206	129400	129594	129788	129982	130176	130370	130564	130758	130952
63	131146	131340	131534	131728	131921	132115	132308	132502	132696	132889
64	133083	133276	133470	133663	133856	134050	134243	134436	134629	134822
65	135015	135208	135401	135594	135787	135980	136172	136365	136558	136750
66	136943	137136	137328	137521	137713	137906	138098	138290	138483	138675
67	138867	139059	139251	139443	139635	139827	140019	140211	140403	140595
68	140786	140978	141170	141361	141553	151744	141936	142127	142319	142510
69	142701	142893	143084	143275	143466	143657	143848	144039	144230	144421
70	144612	144803	144994	145185	145376	145566	145757	145947	146138	146328
71	146519	146709	146900	147090	147280	147471	147661	147851	148041	148231
72	148421	148611	148801	148991	149181	149371	149561	149750	149940	150130
73	150319	150509	150698	150888	151077	151267	151456	151645	151835	152024
74	152213	152403	152592	152781	152970	153159	153348	153536	153725	153914
75	154103	154292	154480	154669	154858	155046	155235	155423	155612	155800
76	155988	156177	156365	156553	156741	156930	157118	157306	157494	157682
77	157870	158058	158245	158433	158621	158809	158997	159184	159372	159559
78	159747	159934	160122	160309	160497	160684	160871	161058	161245	161433
79	161620	161807	161994	162181	162368	162555	162741	162928	163115	163302
80	163488	163675	163862	164048	164235	164421	164607	164794	164980	165167
81	165353	165539	165725	165912	166098	166284	166470	166656	166841	167027
82	167213	167399	167585	167771	167956	168142	168327	168513	168698	168884
83	169070	169255	169440	169625	169811	169996	170181	170366	170552	170737
84	170922	171107	171292	171476	171661	171846	172031	172216	172400	172585
85	172770	172954	173139	173323	173508	173692	173876	174061	174245	174429
86	174614	174798	174982	175166	175350	175534	175718	175902	176085	176269
87	176453	176637	176821	177004	177188	177371	177555	177738	177922	178105
88	178289	178472	178655	178839	179022	179205	179388	179571	179754	179937
89	180120	180303	180486	180669	180852	181035	181217	181400	181583	181765
90	181948	182130	182313	182495	182678	182860	183042	183225	183407	183589
91	183771	183953	184135	184317	184499	184681	184863	185045	185227	185409
92	185591	185772	185954	186136	186317	186499	186680	186862	187043	187225
93	187406	187587	187768	187950	188131	188312	188493	188674	188855	189036
94	189217	189398	189579	189760	189940	190121	190302	190483	190663	190844
95	191024	191205	191385	191566	191746	191926	192107	192287	192467	192647
96	192827	193008	193188	193368	193548	193728	193907	194087	194267	194447
97	194627	194806	194986	195166	195345	195525	195704	195884	196063	196242
98	196422	196601	196780	196960	197139	197318	197497	197676	197855	198034
99	198213	198392	198571	198750	198928	199107	199286	199464	199643	199821
1.00	200000	200179	200357	200535	200714	200892	201070	201249	201427	201605
01	201783	201961	202139	202317	202495	202673	202851	203029	203207	203385
02	203563	203740	203918	204095	204273	204451	204628	204805	204983	205160
03	205338	205515	205692	205869	206047	206224	206401	206578	206755	206932
04	207109	207286	207463	207640	207816	207993	208170	208346	208523	208700
05	208876	209053	209229	209406	209582	209758	209935	210111	210287	210463
06	210640	210816	210992	211168	211344	211520	211696	211872	212047	212223
07	212399	212575	212750	212926	213102	213277	213453	213628	213804	213979
08	214154	214330	214505	214680	214856	215031	215206	215381	215556	215731
09	215906	216081	216256	216431	216606	216780	216955	217130	217304	217479
1.10	217654	217828	218003	218177	218352	218526	218700	218875	219049	219223
11	219398	219572	219746	219920	220094	220268	220442	220616	220790	220964
12	221138	221311	221485	221659	221832	222006	222180	222353	222526	222700
13	222874	223047	223220	223394	223567	223740	223913	224086	224260	224433
14	224606	224779	224952	225125	225298	225470	225643	225816	225989	226161
15	226334	226507	226679	226852	227024	227197	227369	227542	227714	227886
16	228059	228231	228403	228575	228747	228919	229091	229263	229435	229607
17	229779	229951	230123	230295	230466	230638	230810	230981	231153	231324
18	231496	231667	231839	232010	232182	232353	232524	232695	232867	233038
19	233209	233380	233551	233722	233893	234064	234235	234406	234577	234747

- Continued -

Table 15.- Values of $1-R^x$ when R = 0.8 - Continued
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
1.20	234918	235089	235260	235430	235601	235771	235942	236112	236282	236453
21	236623	236794	236964	237134	237304	237475	237645	237815	237985	238155
22	238325	238495	238665	238835	239004	239174	239344	239513	239683	239853
23	240023	240192	240362	240531	240701	240870	241039	241209	241378	241547
24	241717	241886	242055	242224	242393	242562	242731	242900	243069	243238
25	243407	243576	243744	243913	244082	244250	244419	244588	244756	244925
26	245093	245262	245430	245598	245767	245935	246103	246271	246440	246608
27	246776	246944	247112	247280	247448	247616	247784	247951	248119	248287
28	248455	248622	248790	248958	249125	249293	249460	249628	249795	249963
29	250130	250297	250464	250632	250799	250966	251133	251300	251467	251634
30	251801	251968	252135	252302	252469	252635	252802	252969	253135	253302
31	253469	253635	253802	253968	254135	254301	254468	254634	254800	254967
32	255133	255299	255465	255631	255797	255963	256130	256296	256461	256627
33	256793	256959	257125	257291	257456	257622	257788	257953	258119	258284
34	258450	258615	258781	258946	259111	259277	259442	259607	259772	259938
35	260103	260268	260433	260598	260763	260928	261093	261257	261422	261587
36	261752	261917	262081	262246	262410	262575	262740	262904	263068	263233
37	263397	263562	263726	263890	264055	264219	264383	264547	264711	264875
38	265039	265203	265367	265531	265695	265859	266022	266186	266350	266514
39	266677	266841	267005	267168	267332	267495	267659	267822	267985	268149
40	268312	268475	268638	268801	268964	269128	269291	269454	269617	269780
41	269943	270106	270269	270431	270594	270757	270920	271082	271245	271407
42	271570	271733	271895	272057	272220	272382	272545	272707	272869	273031
43	273194	273356	273518	273680	273842	274004	274166	274328	274490	274652
44	274814	274976	275137	275299	275461	275622	275784	275945	276107	276269
45	276430	276592	276753	276914	277076	277237	277398	277559	277721	277882
46	278043	278204	278365	278526	278687	278848	279009	279170	279331	279491
47	279652	279813	279973	280134	280295	280455	280616	280776	280937	281097
48	281258	281418	281578	281739	281899	282059	282219	282380	282540	282700
49	282860	283020	283180	283340	283500	283660	283819	283979	284139	284299
50	284458	284618	284777	284937	285097	285256	285415	285575	285734	285894
51	286053	286212	286372	286531	286690	286849	287008	287167	287326	287486
52	287644	287803	287962	288121	288280	288439	288598	288756	288915	289074
53	289232	289391	289550	289708	289867	290025	290183	290342	290500	290658
54	290816	290975	291133	291291	291449	291607	291765	291923	292081	292239
55	292397	292555	292713	292871	293029	293186	293344	293502	293659	293817
56	293975	294132	294290	294447	294604	294762	294919	295077	295234	295391
57	295548	295706	295863	296020	296177	296334	296491	296648	296805	296962
58	297118	297275	297432	297589	297746	297902	298059	298216	298372	298528
59	298685	298841	298998	299154	299311	299467	299623	299780	299936	300093
60	300249	300405	300561	300717	300873	301029	301185	301341	301496	301652
61	301808	301964	302119	302275	302431	302587	302742	302898	303053	303209
62	303364	303520	303675	303830	303986	304141	304296	304452	304607	304762
63	304917	305072	305227	305382	305537	305692	305847	306002	306157	306312
64	306466	306621	306776	306931	307085	307240	307394	307549	307703	307858
65	308012	208166	308321	308475	308629	308784	308938	309092	309246	309401
66	309555	309709	309863	310017	310171	310325	310478	310632	310786	310940
67	311093	311247	311401	311555	311708	311862	312015	312169	312322	312476
68	312629	312782	312936	313089	313242	313396	313549	313702	313855	314008
69	314161	314314	314467	314620	314773	214926	315079	315232	315385	315537
70	315690	315843	315995	316148	316300	316453	316605	316758	316910	317063
71	317215	317367	317520	317672	317824	317977	318129	318281	318433	318585
72	318737	318889	319041	319193	319345	319497	319648	319800	319952	320104
73	320256	320407	320559	320711	320862	321014	321165	321317	321468	321619
74	321771	321923	322074	322225	322376	322527	322678	322829	322980	323131
75	323282	323433	323584	323735	323886	324037	324188	324339	324489	324640
76	324791	324941	325092	325243	325393	325544	325694	325845	325995	326145
77	326296	326446	326596	326746	326897	327047	327197	327347	327497	327648
78	327798	327947	328097	328247	328397	328547	328697	328847	328997	329146
79	329296	329445	329595	329744	329894	330044	330193	330343	330492	330641

- Continued -

Table 15.- Values of $1-R^x$ when R = 0.8 - Continued -
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
1.80	330791	330940	331089	331238	331388	331537	331686	331835	331984	332133
81	332282	332431	332580	332729	332878	333027	333176	333325	333474	333622
82	333771	333919	334068	334217	334365	334514	334762	334810	334959	335107
83	335256	335404	335552	335701	335849	335997	336145	336293	336441	336589
84	336737	336885	337033	337181	337329	337477	337625	337773	337920	338068
85	338216	338363	338511	338659	338806	338954	339101	339249	339396	339543
86	339691	339838	339985	340133	340280	340427	340574	340721	340868	341015
87	341163	341310	341457	341604	341750	341897	342044	342191	342338	342484
88	342631	342778	342924	343071	343218	343364	343511	343657	343803	343950
89	344096	344243	344389	344535	344681	344828	344974	345120	345266	345412
90	345558	345704	345850	345996	346142	346288	346434	346580	346726	346872
91	347017	347163	347308	347454	347600	347745	347891	348036	348182	348327
92	348472	348618	348763	348909	349054	349199	349344	349489	349634	349780
93	349925	350070	350215	350360	350505	350650	350794	350939	351084	351229
94	351374	351518	351663	351808	351952	352097	352241	352385	322530	352675
95	352820	352964	353108	353252	353397	353541	353685	353829	353974	354118
96	354262	354406	354550	354694	354838	354982	355126	355270	355414	355558
97	355701	355845	355989	356132	356276	356420	356563	356707	356851	356994
98	357137	357281	357424	357568	357711	357854	357998	358141	358284	358427
99	358570	358713	358856	359000	359143	359286	359428	359571	359714	359857
2.00	360000	360143	360285	360428	360571	360714	360856	360999	361141	361284
01	361426	361569	361711	361854	361996	362139	362281	362423	362565	362708
02	362850	362992	363134	363276	363418	363560	363702	363844	363986	364128
03	364270	364412	364554	364695	364837	364979	365121	365262	365404	365545
04	365687	365829	365970	366112	366253	366394	366536	366677	366818	366960
05	367101	367242	367383	367525	367666	367807	367948	368089	368230	368371
06	368512	368652	368793	368934	369075	369216	369357	369497	369638	369778
07	369919	370060	370200	370341	370481	370622	370762	370903	371043	371183
08	371323	371464	371604	371744	371884	372025	372165	372305	372445	372585
09	372725	372865	373005	373145	373285	373424	373564	373704	373844	373983
10	374123	374263	374402	374542	374681	374821	374960	375100	375240	375379
11	375518	375657	375797	375936	376075	376214	376354	376493	376632	376771
12	376910	377049	377188	377327	377466	377605	377744	377882	378021	378160
13	378299	378438	378576	378715	378854	378992	379131	379269	379408	379546
14	379685	379823	379961	380100	380238	380376	380515	380653	380791	380929
15	381067	381205	381343	381481	381619	381757	381895	382033	382171	382309
16	382447	382585	382722	382860	382998	383135	383273	383411	383548	383686
17	383823	383961	384098	384236	384373	384510	384648	384785	384922	385060
18	385197	385334	385471	385608	385745	385882	386019	386156	386293	386430
19	386567	386704	386841	386978	387114	387251	387388	387524	387661	387798
20	387934	388071	388207	388344	388480	388617	388753	388889	389026	389162
21	389299	389435	389571	389707	389843	389980	390116	390252	390388	390524
22	390660	390796	390932	391068	391204	391339	391475	391611	391747	391883
23	392018	392154	392289	392425	392561	392696	392831	392967	393102	393238
24	393373	393509	393644	393779	393914	394050	394185	394320	394455	394590
25	394725	394860	394995	395130	395265	395400	395535	395670	395805	395940
26	396075	396209	396344	396479	396614	396748	396883	397017	397151	397286
27	397421	397555	397689	397824	397958	398093	398227	398361	398495	398630
28	398764	398898	399032	399166	399300	399434	399568	399702	399836	399970
29	400104	400238	400372	400505	400639	400773	400907	401040	401174	401307
30	401441	401574	401708	401842	401975	402108	402242	402375	402509	402642
31	402775	402908	403042	403175	403308	403441	403574	403707	403840	403973
32	404106	404239	404372	404505	404638	404771	404904	405036	405169	405302
33	405435	405567	405700	405832	405964	406097	406230	406362	406495	406627
34	406760	406892	407025	407157	407289	407421	407553	407686	407818	407950
35	408082	408214	408346	408478	408610	408742	408874	409006	409138	409270
36	409401	409533	409665	409797	409928	410060	410192	410323	410455	410586
37	410718	410849	410981	411112	411244	411375	311506	411638	411769	411900
38	412031	412163	412294	412425	412556	412687	412818	412949	413080	413211
39	413342	413473	413604	413734	413865	413996	414127	414258	414388	414519

- Continued -

Table 15.- Values of $1-R^x$ when R = 0.8 - Continued -
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
2.40	414650	414780	414911	415041	415172	415302	415433	415563	415694	415824
41	415954	416085	416215	416345	416475	416606	416736	416866	416996	417126
42	417256	417386	417516	417646	417776	417906	418036	418166	418295	418425
43	418555	418685	418814	418944	419074	419203	419333	419463	419592	419721
44	419851	419980	420110	420239	420369	420498	420627	420756	420886	421015
45	421144	421273	421402	421531	421660	421790	421919	422048	422177	422306
46	422434	422563	422692	422821	422950	423078	423207	423336	423465	423593
47	423722	423850	423979	424107	424236	424364	424493	424621	424750	424878
48	425006	425134	425263	425391	425519	425647	425775	425904	426032	426160
49	426288	426416	426544	426672	426800	426928	427055	427183	427311	427439
50	427567	427694	427822	427950	428077	428205	428332	428460	428588	428715
51	428843	428970	429097	429225	429352	429479	429607	429734	429861	429988
52	430116	430243	430370	430497	430624	430751	430878	431005	431132	431259
53	431386	431513	431640	431766	431893	432020	432147	432273	432400	432527
54	432653	432780	432906	433033	433159	433286	433412	433539	433665	433792
55	433918	434044	434170	434297	434423	434549	434675	434801	434928	435054
56	435180	435306	435432	435558	435684	435809	435935	436061	436187	436313
57	436439	436564	436670	436816	436941	437067	437193	437318	437444	437569
58	437695	437820	437946	438071	438197	438322	438447	438572	438698	438823
59	438948	439073	439198	439323	439448	439574	439699	439824	439949	440074
60	440199	440323	440448	440573	440698	440823	440948	441072	441197	441322
61	441446	441571	441696	441820	441945	442069	442194	442318	442443	442567
62	442691	442816	442940	443064	443189	443313	443437	443561	443685	443809
63	443933	444058	444182	444306	444430	444554	444678	444801	444925	445049
64	445173	445297	445420	445544	445668	445792	445915	446039	446163	446286
65	446410	446533	446657	446780	446904	447027	447150	447274	447397	447520
66	447644	447767	447890	448013	448136	448260	448383	448506	448629	448752
67	448875	448998	449121	449244	449367	449489	449612	449735	449858	449981
68	450103	450226	450349	450471	450594	450716	450839	450962	451084	451207
69	451329	451451	451574	451696	451818	451941	452063	452175	452308	452430
70	452552	452674	452796	452918	453040	453162	453284	453406	453528	453650
71	453772	453894	454016	454138	454259	454381	454503	454625	454746	454868
72	454990	455111	455233	455354	455476	455597	455719	455840	455962	456083
73	456204	456326	456447	456568	456690	456811	456932	457053	457174	457295
74	457416	457538	457659	457780	457901	458022	458142	458263	458384	458505
75	458626	458747	458867	458988	459109	459230	459350	459471	459592	459712
76	459833	459953	460074	460194	460315	460435	460555	460676	460796	460916
77	461037	461157	461277	461397	461517	461638	461758	461878	461998	462118
78	462238	462358	462478	462598	462718	462838	462958	463077	463197	463317
79	463437	463556	463676	463795	463915	464035	464155	464274	464394	464513
80	464633	464752	464871	464991	465110	465230	465349	465468	465587	465707
81	465826	465945	466064	466183	466302	466422	466541	466660	466779	466898
82	467017	467135	467254	467373	467492	467611	467730	467848	467967	468086
83	468205	468323	468442	468560	468679	468798	468916	469035	469153	469272
84	469390	469508	469627	469745	469863	469981	470100	470218	470336	470454
85	470573	470691	470809	470927	471045	471163	471281	471399	471517	471635
86	471753	471871	471988	472106	472224	472342	472460	472577	472695	472812
87	472930	473048	473165	473283	473400	473518	473635	473753	473870	473988
88	474105	474222	474339	474457	474574	474681	474808	474926	475043	475160
89	475277	475394	475511	475628	475745	475862	475979	476096	476213	476330
90	476447	476563	476680	476797	476914	477030	477147	477264	477380	477497
91	477614	477730	477847	477963	478080	478196	478312	478429	478545	478662
92	478778	478894	479011	479127	479243	479359	479475	479592	479708	479824
93	479940	480056	480172	480288	480404	480520	480636	480751	480867	480983
94	481099	481215	481331	481446	481562	481678	481793	481909	482025	482140
95	482256	482371	482487	482602	482717	482833	482948	483064	483179	483294
96	483410	483525	483640	483755	483871	483986	484101	484216	484331	484446
97	484561	484676	484791	484906	485021	485136	485251	485365	485480	485595
98	485710	485825	485939	486054	486169	486284	486398	486513	486627	486742
99	486856	486971	487085	487200	487314	487428	487543	487657	487772	487886

- Continued -

Table 15.- Values of $1-R^x$ when R = 0.8 - Continued
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
3.00	488000	488114	489228	488343	488457	488571	488685	488799	488913	489027
01	489141	489255	489369	489483	489597	489711	489825	489939	490052	490166
02	490280	490394	490507	490621	490735	490848	490961	491075	491189	491303
03	491416	491530	491643	491757	491870	491983	492096	492210	492323	492436
04	492550	492663	492776	492889	493002	493115	493229	493342	493455	493568
05	493681	493794	493907	494020	494132	494245	494358	494471	494584	494697
06	494809	494922	495035	495147	495260	495373	495485	495598	495710	495823
07	495935	496048	496160	496273	496385	496497	496610	496722	496834	496947
08	497059	497171	497283	497395	497508	497620	497732	497844	497956	498068
09	498180	498292	498404	498516	498628	498740	498851	498963	499075	499187
10	499298	499410	499522	499633	499745	499857	499968	500080	500191	500303
11	500414	500526	500637	500749	500860	500971	501083	501194	501305	501417
12	501528	501639	501750	501862	501973	502084	502195	502306	502417	502528
13	502639	502750	502861	502972	503083	503194	503305	503415	503526	503637
14	503748	503858	503969	504080	504190	504301	504412	504522	504633	504743
15	504854	504964	505075	505185	505296	505406	505516	505627	505737	505847
16	505958	506068	506178	506288	506398	506508	506618	506729	506839	506949
17	507059	507169	507279	507389	507498	507608	507718	507828	507938	508048
18	508157	508267	508377	508486	508596	508706	508815	508925	509035	509144
19	509254	509363	509473	509582	509691	509801	509910	510020	510129	510238
20	510348	510457	510566	510675	510784	510893	511003	511112	511221	511330
21	511439	511548	511657	511766	511875	511984	512093	512202	512310	512419
22	512528	512637	512745	512854	512963	513072	513180	513289	513397	513506
23	513615	513723	513831	513940	514048	514157	514265	514374	514482	514590
24	514699	514807	514915	515024	515132	515240	515348	515456	515564	515672
25	515780	515888	515996	516104	516212	516320	516428	516536	516644	516752
26	516860	516967	517075	517183	517291	517398	517506	517614	517721	517829
27	517936	518044	518152	518259	518367	518474	518582	518689	518796	518904
28	519011	519118	519226	519333	519440	519547	519654	519762	519869	519976
29	520083	520190	520297	520404	520511	520618	520725	520832	520939	521046
30	521153	521260	521366	521473	521580	521687	521794	521900	522007	522114
31	522220	522327	522433	522540	522646	522753	522859	522966	523072	523178
32	523285	523392	523498	523604	523710	523817	523923	524029	524135	524242
33	524348	524454	524560	524666	524772	524878	524984	525090	525196	525302
34	525408	525514	525620	525725	525831	525937	526043	526148	526254	526360
35	526466	526572	526677	526783	526888	526994	527099	527205	527310	527416
36	527521	527627	527732	527837	527943	528048	528154	528259	528364	528469
37	528574	528680	528785	528890	528995	529100	529205	529310	529415	529520
38	529625	529730	529835	529940	530045	530150	530255	530359	530464	530569
39	530674	530778	530883	530987	531092	531197	531302	531406	531511	531615
40	531720	531824	531929	532033	532137	532242	532346	532451	532555	532659
41	532763	532868	532972	533076	533180	533284	533389	533493	533597	533701
42	533805	533909	534013	534117	534221	534325	534429	534532	534636	534740
43	534844	534948	535052	535155	535259	535363	535466	535570	535674	535777
44	535881	535984	536088	536191	536295	536398	536502	536605	536709	536812
45	536915	537019	537122	537225	537328	537432	537535	537638	537741	537844
46	537947	538050	538154	538257	538360	538463	538566	538669	538772	538874
47	538977	539080	539183	539286	539389	539491	539594	539697	539800	539902
48	540005	540108	540210	540313	540415	540518	540620	540723	540826	540928
49	541030	541133	541235	541337	541440	541542	541644	541747	541849	541951
50	542053	542155	542258	542360	542462	542564	542666	542768	542870	542972
51	543074	543176	543278	543380	543482	543584	543685	543787	543889	543991
52	544092	544194	544296	544398	544499	544601	544702	544804	544906	545007
53	545109	545210	545312	545413	545515	545616	545717	545819	545920	546021
54	546123	546224	546325	546426	546528	546629	546730	546831	546932	547033
55	547134	547235	547336	547437	547538	547639	547740	547841	547942	548043
56	548144	548245	548345	548446	548547	548648	548748	548849	548950	549050
57	549151	549251	549352	549453	549553	549654	549754	549855	549955	550056
58	550156	550256	550357	550457	550557	550657	550758	550858	550958	551058
59	551158	551259	551359	551459	551559	551659	551759	551859	551959	552059

- Continued -

Table 15.- Values of $1-R^x$ when R = 0.8 - Continued
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
3.60	552159	552259	552359	552459	552558	552658	552758	552858	552958	553057
61	553157	553257	553356	553456	553556	553655	553755	553855	553954	554054
62	554153	554253	554352	554451	554551	554650	554750	554849	554948	555047
63	555147	555246	555345	555445	555544	555643	555742	555841	555940	556039
64	556138	556237	556336	556435	556534	556633	556732	556831	556930	557029
65	557128	557227	557325	557424	557523	557622	557720	557819	557917	558016
66	558115	558214	558313	558411	558509	558608	558706	558805	558903	559001
67	559100	559198	559297	559395	559493	559591	559690	559788	559886	559984
68	560082	560181	560279	560377	560475	560573	560671	560769	560867	560965
69	561063	561161	561259	561357	561455	561553	561650	561748	561846	561944
70	562041	562139	562237	562335	562432	562530	562627	562725	562823	562920
71	563018	563115	563213	563310	563408	563505	563602	563700	563797	563894
72	563992	564089	564186	564284	564381	564478	564575	564672	564769	564866
73	564964	565061	556158	565255	565352	565449	565546	565643	565739	565836
74	565933	566030	566127	566224	566320	566417	566514	566611	566707	566804
75	566901	566997	567094	567191	567287	567384	567480	567577	567673	567770
76	567866	567963	568059	568155	568252	568348	568444	568541	568637	568733
77	568829	568925	569022	569118	569214	569310	569406	569502	569598	569694
78	569790	569886	569982	570078	570174	570270	570366	570462	570558	570654
79	570749	570845	570941	571037	571132	571228	571324	571419	571515	571610
80	571706	571802	571897	571993	572088	572184	572279	572374	572470	572565
81	572661	572756	572851	572947	573042	573137	573232	573328	573423	573518
82	573613	573708	573803	573899	573994	574089	574184	574279	574374	574469
83	574564	574659	574753	574848	574943	575038	575133	575228	575322	575417
84	575512	575607	575701	575796	575891	575985	576080	576174	576269	576363
85	576458	576552	576647	576741	576836	576931	577025	577119	577213	577308
86	577402	577496	577591	577685	577779	577873	577967	578062	578156	578250
87	578344	578438	578532	578626	578720	578814	578908	579002	579096	579190
88	579284	579378	579472	579565	579659	579753	579847	579941	580034	580128
89	580222	580315	580409	580502	580596	580690	580783	580877	580970	581064
90	581157	581251	581344	581438	581531	581624	581718	581811	581904	581998
91	582091	582184	582277	582370	582464	582557	582650	582743	582836	582929
92	583022	583115	583208	583301	583394	583487	583580	583673	583766	583859
93	593952	584045	584137	584230	584323	584416	584508	584601	584694	584787
94	584879	584972	585064	585157	585249	585342	585435	585527	585619	585712
95	585804	585897	585989	586082	586174	586266	586359	586451	586543	586635
96	586728	586820	586912	587004	587096	587189	587281	587373	587465	587557
97	587649	587741	587833	587925	588017	588109	588201	588292	588384	588476
98	588568	588660	588751	588843	588935	589027	589118	589210	589302	589393
99	589485	589577	589668	589760	589851	589943	590034	590126	590217	590309

Table 15.- Values of $1-R^x$ when R = 0.8 - Continued
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
4.0	59040	59131	59222	59313	59404	59494	59585	59675	59765	59854
1	59944	60033	60122	50211	60300	60388	60477	60565	60653	60740
2	60828	60915	61002	61089	61176	61262	61349	61435	61521	61607
3	61692	61778	61863	61948	62033	62117	62202	62286	62370	62454
4	62538	62621	62704	62787	62870	62953	63036	63118	63200	63282
5	63364	63446	63527	63609	63690	63771	63851	63932	64012	64093
6	64173	64253	64332	64412	64491	64570	64649	64728	64807	64885
7	64963	65041	65119	65197	65275	65352	65429	65506	65583	65660
8	65736	65813	65889	65965	66041	66117	66192	66268	66343	66418
9	66493	66567	66642	66716	66790	66864	66938	67012	67085	67159
5.0	67232	67305	67378	67451	67523	67596	67668	67740	67812	67883
1	67955	68027	68098	68169	68240	68311	68381	68451	68522	68592
2	68662	68732	68802	68871	68941	69010	69079	69148	69217	69285
3	69354	69422	69490	69558	69626	69694	69761	69829	69896	69963
4	70030	70097	70164	70230	70296	70363	70429	70495	70560	70626
5	70691	70757	70822	70887	70952	71016	71081	71146	71210	71274
6	71338	71402	71466	71529	71592	71656	71719	71782	71845	71908
7	71971	72033	72095	72158	72220	72282	72344	72406	72467	72528
8	72589	72650	72711	72772	72833	72893	72954	73014	73074	73134
9	73194	73254	73313	73373	73432	73491	73551	73610	73668	73727
6.0	73786	73844	73902	73960	74018	74076	74134	74192	74249	74307
1	74364	74421	74478	74535	74592	74649	74705	74761	74818	74874
2	74930	74986	75041	75097	75153	75208	75263	75318	75373	75428
3	75483	75538	75592	75647	75701	75755	75809	75863	75917	75970
4	76024	76077	76131	76184	76237	76290	76343	76396	76448	76501
5	76553	76605	76658	76710	76761	76813	76865	76917	76968	77019
6	77071	77122	77173	77224	77274	77325	77375	77426	77476	77526
7	77576	77626	77676	77726	77776	77825	77875	77924	77973	78022
8	78071	78120	78169	78218	78266	78315	78363	78411	78459	78507
9	78555	78603	78651	78698	78746	78793	78840	78888	78935	78982
7.0	79028	79075	79122	79168	79215	79261	79307	79354	79400	79445
1	79491	79537	79583	79628	79674	79719	79764	79809	79854	79899
2	79944	79989	80033	80078	80122	80166	80211	80255	80299	80343
3	80386	80430	80474	80517	80561	80604	80647	80690	80733	80776
4	80819	80862	80905	80947	80990	81032	81074	81117	81159	81201
5	81242	81284	81326	81367	81409	81451	81492	81533	81574	81615
6	81656	81697	81738	81779	81819	81860	81900	81941	81981	82021
7	82061	82101	82141	82181	82221	82260	82300	82339	82379	82418
8	82457	82496	82535	82574	82613	82652	82690	82729	82768	82806
9	82844	82882	82920	82959	82997	83035	83072	83110	83148	83185
8.0	83223	83260	83297	83335	83372	83409	83446	83483	83520	83557
1	83593	83630	83666	83702	83739	83775	83811	83847	83883	83919
2	83955	83991	84027	84062	84098	84133	84168	84204	84239	84274
3	84309	84344	84379	84414	84449	84483	84518	84552	84587	84621
4	84655	84690	84724	84758	84792	84826	84859	84893	84927	84960
5	84994	85027	85061	85094	85127	85161	85194	85227	85259	85292
6	85325	85358	85391	85423	85456	85488	85520	85552	85585	85617
7	85649	85681	85713	85745	85776	85808	85840	85872	85903	85934
8	85966	85997	86028	86059	86090	86121	86152	86183	86214	86245
9	86274	86305	86336	86367	86397	86428	86458	86488	86518	86548
9.0	86578	86608	86638	86668	86697	86727	86757	86786	86816	86845
1	86875	86904	86933	86962	86991	87020	87049	87078	87107	87136
2	87164	87193	87221	87250	87278	87306	87335	87363	87391	87419
3	87447	87475	87503	87531	87559	87587	87614	87642	87669	87697
4	87724	87752	87779	87806	87833	87860	87887	87915	87942	87968
5	87995	88022	88049	88075	88102	88128	88155	88181	88208	88234
6	88260	88286	88312	88338	88364	88390	88416	88442	88468	88494
7	88519	88545	88570	88596	88621	88647	88672	88697	88722	88747
8	88773	88798	88823	88847	88872	88897	88922	88947	88971	88996
9	89020	89045	89069	89094	89118	89142	89166	89190	89215	89239

- Continued -

Table 15.- Values of $1-R^x$ when R = 0.8 - Continued
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
10.0	89263	89287	89310	89334	89358	89382	89405	89429	89453	89476
.1	89500	89523	89546	89570	89593	89616	89639	89662	89685	89708
2	89731	89754	89777	89800	89822	89845	89868	89890	89913	89936
3	89958	89980	90003	90025	90047	90069	90091	90113	90135	90157
4	90179	90201	90223	90245	90267	90288	90310	90332	90353	90375
5	90396	90418	90439	90460	90482	90503	90524	90545	90566	90587
6	90608	90629	90650	90671	90692	90712	90733	90754	90774	90795
7	90815	90836	90856	90877	90897	90917	90937	90958	90978	90998
8	91018	91038	91058	91078	91098	91118	91138	91157	91177	91197
9	91216	91236	91255	91275	91294	91314	91333	91352	91372	91391
11.0	91410	91429	91448	91467	91486	91505	91524	91543	91562	91581
1	91600	91618	91637	91656	91674	91693	91711	91730	91748	91767
2	91785	91803	91822	91840	91858	91876	91894	91912	91930	91948
3	91966	91984	92002	92020	92038	92055	92073	92091	92108	92126
4	92144	92161	92179	92196	92213	92231	92248	92265	92283	92300
5	92317	92334	92351	92368	92385	92402	92419	92436	92453	92470
6	92487	92503	92520	92537	92553	92570	92586	92603	92619	92636
7	92652	92669	92685	92701	92718	92734	92750	92766	92782	92798
8	92814	92830	92846	92862	92878	92894	92910	92926	92942	92957
9	92973	92989	93004	93020	93035	93051	93066	93082	93097	93113
12.0	93128	93143	93159	93174	93189	93204	93219	93235	93250	93265
1	93280	93295	93310	93324	93339	93354	93369	93384	93399	93413
2	93428	93443	93457	93472	93486	93501	93515	93530	93544	93559
3	93573	93587	93602	93616	93630	93644	93658	93673	93687	93701
4	93715	93729	93743	93757	93771	93785	93798	93812	93826	93840
5	93853	93867	93881	93895	93908	93922	93935	93949	93962	93976
6	93989	94003	94016	94029	94043	94056	94069	94082	94095	94109
7	94122	94135	94148	94161	94174	94187	94200	94213	94226	94239
8	94252	94264	94277	94290	94303	94315	94328	94341	94353	94366
9	94378	94391	94403	94416	94428	94441	94453	94465	94478	94490
13.0	94502	94515	94527	94539	94551	94563	94575	94588	94600	94612
1	94624	94636	94648	94660	94671	94683	94695	94707	94719	94731
2	94742	94754	94766	94777	94789	94801	94812	94824	94835	94847
3	94858	94870	94881	94893	94904	94915	94927	94938	94949	94961
4	94972	94983	94994	95005	95017	95028	95039	95050	95061	95072
5	95083	95094	95105	95116	95126	95137	95148	95159	95170	95181
6	95191	95202	95213	95223	95234	95245	95255	95266	95276	95287
7	95297	95308	95318	95329	95339	95349	95360	95370	95381	95391
8	95401	95411	95422	95432	95442	95452	95462	95472	95483	95493
9	95503	95513	95523	95533	95543	95553	95563	95572	95582	95592
14.0	95602	95612	95621	95631	95641	95651	95660	95670	95680	95689
1	95699	95709	95718	95728	95737	95747	95756	95766	95775	95784
2	95794	95803	95813	95822	95831	95841	95850	95859	95868	95878
3	95887	95896	95905	95914	95923	95932	95941	95951	95960	95969
4	95978	95987	95995	96004	96013	96022	96031	96040	96049	96058
5	96066	96075	96084	96092	96101	96110	96119	96127	96136	96144
6	96153	96162	96170	96179	96187	96196	96204	96213	96221	96230
7	96238	96246	96255	96263	96271	96280	96288	96296	96304	96313
8	96321	96329	96337	96346	96354	96362	96370	96378	96386	96394
9	96402	96410	96418	96426	96434	96442	96450	96458	96466	96474
15.0	96482	96489	96497	96505	96513	96521	96528	96536	96544	96551
1	96559	96567	96574	96582	96590	96597	96605	96612	96620	96628
2	96635	96643	96650	96658	96665	96672	96680	96687	96695	96702
3	96709	96717	96724	96731	96739	96746	96753	96760	96768	96775
4	96782	96789	96796	96803	96811	96818	96825	96832	96839	96846
5	96853	96860	96867	96874	96881	96888	96895	96902	96909	96916
6	96923	96929	96936	96943	96950	96957	96963	96970	96977	96984
7	96991	96998	97004	97010	97017	97024	97030	97037	97044	97050
8	97057	97063	97070	97076	97083	97089	97096	97102	97109	97115
9	97122	97128	97135	97141	97147	97154	97160	97166	97173	97179

- Continued -

Table 15.- Values of $1-R^x$ when R = 0.8-Continued
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
16.0	97185	97192	97198	97204	97210	97216	97223	97229	97235	97241
1	97247	97253	97260	97266	97272	97278	97284	97290	97296	97302
2	97308	97314	97320	97326	97332	97338	97344	97350	97356	97362
3	97368	97373	97379	97385	97391	97397	97403	97408	97414	97420
4	97426	97431	97437	97443	97448	97454	97460	97465	97471	97477
5	97482	97488	97494	97499	97505	97510	97516	97521	97527	97532
6	97538	97544	97549	97554	97560	97565	97571	97576	97581	97587
7	97592	97598	97603	97608	97614	97619	97624	97630	97635	97640
8	97645	97651	97656	97661	97666	97671	97677	97682	97687	97692
9	97697	97702	97708	97713	97718	97723	97728	97733	97738	97743
17.0	97748	97753	97758	97763	97768	97773	97778	97783	97788	97793
1	97798	97803	97808	97813	97817	97822	97827	97832	97837	97842
2	97846	97851	97856	97861	97866	97870	97875	97880	97885	97889
3	97894	97899	97903	97908	97913	97917	97922	97927	97931	97936
4	97940	97945	97950	97954	97959	97963	97968	97972	97977	97981
5	97986	97990	97995	97999	98004	98008	98013	98017	98022	98026
6	98030	98035	98039	98043	98048	98052	98056	98061	98065	98069
7	98074	98078	98082	98087	98091	98095	98099	98104	98108	98112
8	98116	98120	98125	98129	98133	98137	98141	98146	98150	98154
9	98158	98162	98166	98170	98174	98178	98182	98186	98190	98194
18.0	98199	98203	98207	98211	98215	98219	98223	98226	98230	98234
1	98238	98242	98246	98250	98254	98258	98262	98266	98269	98273
2	98277	98281	98285	98289	98292	98296	98300	98304	98308	98311
3	98315	98319	98323	98326	98330	98334	98338	98341	98345	98349
4	98352	98356	98360	98363	98367	98371	98374	98378	98382	98385
5	98389	98392	98396	98399	98403	98406	98410	98414	98417	98421
6	98424	98428	98431	98435	98438	98442	98445	98449	98452	98456
7	98459	98462	98466	98469	98472	98476	98480	98483	98486	98490
8	98494	98497	98500	98503	98506	98510	98513	98516	98520	98523
9	98526	98530	98533	98536	98539	98543	98546	98549	98552	98556
19.0	98559	98562	98565	98568	98572	98575	98578	98581	98584	98587
1	98591	98594	98597	98600	98603	98606	98609	98612	98616	98619
2	98622	98625	98628	98631	98634	98637	98640	98643	98646	98649
3	98652	98655	98658	98661	98664	98667	98670	98673	98676	98679
4	98682	98685	98688	98691	98694	98697	98699	98702	98705	98708
5	98711	98714	98717	98720	98722	98725	98728	98731	98734	98737
6	98739	98742	98745	98748	98751	98753	98756	98759	98762	98765
7	98767	98770	98773	98775	98778	98781	98784	98786	98789	98792
8	98794	98797	98800	98802	98805	98808	98810	98813	98816	98819
9	98821	98824	98826	98829	98832	98834	98837	98839	98842	98845
20.0	98847	98850	98852	98855	98857	98860	98862	98865	98867	98870
1	98873	98876	98878	98880	98883	98885	98888	98890	98892	98895
2	98897	98900	98902	98905	98907	98910	98912	98914	98917	98919
3	98922	98924	98927	98929	98931	98934	98936	98938	98941	98943
4	98945	98948	98950	98953	98955	98957	98960	98962	98964	98966
5	98969	98971	98973	98976	98978	98980	98982	98985	98988	98990
6	98992	98994	98996	98998	99001	99003	99005	99007	99009	99012
7	99014	99016	99018	99020	99023	99025	99027	99029	99031	99034
8	99036	99038	99040	99042	99044	99046	99048	99051	99053	99055
9	99057	99059	99061	99063	99065	99067	99069	99071	99073	99075
21.0	99077	99080	99082	99084	99086	99088	99090	99092	99094	99096
1	99098	99100	99102	99104	99106	99108	99110	99112	99114	99116
2	99118	99120	99122	99124	99126	99128	99130	99132	99134	99136
3	99137	99139	99141	99143	99145	99147	99149	99151	99153	99155
4	99157	99158	99160	99162	99164	99166	99168	99170	99171	99173
5	99175	99177	99179	99181	99182	99184	99186	99188	99190	99192
6	99193	99195	99197	99199	99201	99202	99204	99206	99208	99209
7	99211	99213	99215	99216	99218	99220	99222	99223	99225	99227
8	99229	99230	99232	99234	99235	99237	99239	99241	99242	99244
9	99246	99247	99249	99251	99252	99254	99256	99257	99259	99261

- Continued -

Table 15.- Values of $1-R^x$ when R = 0.8 - Continued
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
22.0	99262	99264	99266	99267	99269	99270	99272	99274	99275	99277
1	99279	99280	99282	99283	99285	99286	99288	99290	99291	99293
2	99294	99296	99298	99299	99300	99302	99304	99305	99307	99308
3	99310	99312	99313	99315	99316	99318	99319	99321	99322	99324
4	99325	99327	99328	99330	99331	99333	99334	99336	99338	99339
5	99340	99341	99343	99344	99346	99347	99349	99350	99352	99353
6	99355	99356	99358	99359	99360	99362	99363	99365	99366	99368
7	99369	99370	99372	99373	99374	99376	99377	99379	99380	99381
8	99383	99384	99386	99387	99388	99390	99391	99392	99394	99395
9	99396	99398	99399	99401	99402	99403	99405	99406	99407	99408
23.0	99410	99411	99412	99414	99415	99416	99418	99419	99420	99421
1	99423	99424	99425	99427	99428	99429	99431	99432	99433	99434
2	99436	99437	99438	99439	99440	99442	99443	99445	99446	99447
3	99448	99449	99450	99452	99453	99454	99455	99456	99458	99459
4	99460	99461	99463	99464	99465	99466	99467	99468	99470	99471
5	99472	99473	99474	99476	99477	99478	99479	99480	99481	99483
6	99484	99485	99486	99487	99488	99490	99491	99492	99493	99494
7	99495	99496	99497	99498	99500	99501	99502	99503	99504	99505
8	99506	99507	99508	99509	99511	99512	99513	99514	99515	99516
9	99517	99518	99519	99520	99521	99522	99524	99525	99526	99527
24.0	99528	99529	99530	99531	99532	99533	99534	99535	99536	99537
1	99538	99539	99540	99541	99542	99543	99544	99545	99546	99547
2	99548	99549	99550	99551	99552	99553	99554	99555	99556	99557
3	99558	99559	99560	99561	99562	99563	99564	99565	99566	99567
4	99568	99569	99570	99571	99572	99573	99574	99575	99576	99577
5	99578	99579	99580	99581	99581	99582	99583	99584	99585	99586
6	99587	99588	99589	99590	99591	99592	99593	99593	99594	99595
7	99596	99597	99598	99599	99600	99601	99602	99602	99603	99604
8	99605	99606	99607	99608	99609	99609	99610	99611	99612	99613
9	99614	99615	99615	99616	99617	99618	99619	99620	99621	99621
25.0	99622	99623	99624	99625	99626	99626	99627	99628	99628	99629
1	99630	99631	99632	99633	99634	99635	99636	99636	99637	99638
2	99639	99640	99640	99641	99642	99643	99644	99644	99645	99646
3	99647	99648	99648	99649	99650	99651	99651	99652	99653	99654
4	99655	99655	99656	99657	99658	99659	99659	99660	99661	99661
5	99662	99663	99664	99664	99665	99666	99667	99668	99668	99669
6	99670	99670	99671	99672	99672	99673	99674	99675	99675	99676
7	99677	99678	99678	99679	99680	99681	99681	99682	99683	99683
8	99684	99685	99685	99686	99687	99687	99688	99689	99690	99690
9	99691	99692	99692	99693	99694	99694	99695	99696	99696	99697
26.0	99698	99699	99699	99700	99700	99701	99702	99703	99703	99704
1	99704	99705	99706	99706	99707	99707	99708	99708	99709	99710
2	99711	99712	99712	99713	99714	99714	99715	99716	99716	99717
3	99717	99718	99719	99719	99720	99720	99721	99722	99722	99723
4	99724	99724	99725	99726	99726	99727	99727	99728	99728	99729
5	99730	99730	99731	99731	99732	99733	99733	99734	99734	99735
6	99736	99736	99737	99738	99738	99739	99739	99740	99740	99741
7	99741	99742	99743	99743	99744	99744	99745	99746	99746	99747
8	99747	99748	99749	99749	99750	99750	99751	99751	99752	99752
9	99753	99753	99754	99755	99755	99756	99756	99757	99757	99758
27.0	99759	99759	99759	99760	99760	99761	99761	99762	99762	99763
1	99764	99764	99765	99765	99766	99766	99767	99767	99768	99768
2	99769	99769	99770	99770	99771	99771	99772	99772	99773	99773
3	99774	99774	99775	99775	99776	99776	99777	99777	99778	99778
4	99779	99779	99780	99780	99781	99781	99782	99782	99783	99783
5	99784	99784	99785	99785	99786	99786	99787	99787	99788	99788
6	99789	99789	99789	99790	99790	99791	99791	99792	99792	99793
7	99793	99794	99794	99795	99795	99796	99796	99796	99797	99797
8	99798	99798	99799	99799	99800	99800	99800	99801	99801	99802
9	99802	99803	99803	99804	99804	99805	99805	99805	99806	99806

- Continued -

Table 15.- Values of $1-R^x$ when R = 0.8 - Continued
(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
28.0	99807	99807	99807	99808	99808	99809	99809	99810	99810	99810
1	99811	99811	99812	99812	99813	99813	99813	99814	99814	99815
2	99815	99816	99816	99816	99817	99817	99817	99818	99818	99819
3	99819	99820	99820	99820	99821	99821	99821	99822	99822	99823
4	99823	99823	99824	99824	99825	99825	99825	99826	99826	99827
5	99827	99827	99828	99828	99829	99829	99829	99830	99830	99830
6	99831	99831	99832	99832	99832	99833	99833	99833	99834	99834
7	99835	99835	99835	99836	99836	99836	99836	99837	99837	99838
8	99838	99839	99839	99839	99840	99840	99840	99841	99841	99841
9	99842	99842	99842	99843	99843	99844	99844	99844	99845	99845
29.0	99845	99846	99846	99846	99847	99847	99847	99848	99848	99848
1	99849	99849	99849	99850	99850	99850	99851	99851	99851	99852
2	99852	99852	99853	99853	99853	99854	99854	99854	99855	99855
3	99855	99856	99856	99856	99857	99857	99857	99858	99858	99858
4	99858	99859	99859	99859	99860	99860	99860	99861	99861	99861
5	99862	99862	99862	99862	99863	99863	99863	99864	99864	99864
6	99865	99865	99865	99866	99866	99866	99866	99867	99867	99867
7	99868	99868	99868	99869	99869	99869	99869	99870	99870	99870
8	99871	99871	99871	99871	99872	99872	99872	99873	99873	99873
9	99873	99874	99874	99874	99875	99875	99875	99875	99876	99876

APPENDIX

Discussion of interaction, use of different yield equations, analysis of variance, and a demonstration of the work for finding the most probable values of the constants of the form of the exponential yield equation used, are included at this point. These subjects are important to the problem presented in this report.

Interaction From Reported and From Calculated Yields

Mention was made of interaction in the section on methods of analysis. Interaction has been defined as the yield obtained when stated quantities of two or more nutrients are added simultaneously, minus the sum of the increases obtained when the nutrients are added one at a time. The hypothesis on which the exponential curve is based is that increases in yield diminish at a constant rate when one nutrient is added in units of equal size, other nutrients held constant. The multivariable form of the exponential yield equation results in calculated yields that pool the effects of two or three variables working together. These regularly reflect a diminishing rate of interaction between nutrients taken two at a time in the presence of constant levels of other yield-influencing factors. The purpose of this section is to demonstrate this effect, as applied to the Texas pasture experiment, in comparison with the interaction read directly from the reported yields.

Table 16 shows how interaction is determined directly from the reported yields at specified rates of application.

Table 16.- Interaction in terms of dry forage per acre at stated rates of application of N and P when no K was applied, as determined from the reported yields

N applied	No P applied	120 lbs. P applied	Increase from P
Pounds	Pounds	Pounds	Pounds
0	2,029	2,417	388
60	2,860	3,963	
Increase from N	831	Sum of increases	1,219
Increase: 60-120 over 0-0 (3,963-2,029)			1,934
Interaction			715

Pasture experiment, Kirbyville, Tex. - 1950.

The excess of the increase in yield from 60 pounds N in conjunction with 120 pounds P, over the sum of the separate increases (388 plus 831 = 1,219) from each of the nutrients added separately, is referred to as interaction. Interaction at all combinations in the experiment was worked out as in table 16. As this involves a large number of tables, only the end products, the interactions, are included in tables 17, 18, and 19.

Comparisons of interactions derived directly from reported yields, and those derived through yields calculated from the equation are of interest. However, they do not provide an adequate basis for judging the validity of the multi-variable form of the exponential yield equation for purposes of calculating interaction. Perhaps a principal reason for this is that interactions apparently were not significant in this experiment (table 21). That is, the interaction effects were small relative to the main effects and to the variation in yields of the replicate plots. Interaction based directly on yields from only three or four rates of application rather than on a regression curve fitted to yields obtained over a range in rates applied, reflect whatever erratic tendencies appear in the reported yields.

Table 17.- Reported and calculated N x P interactions in terms of dry forage per acre at different rates of K applied

N applied, pounds	K	0 to 120 pounds P		120 to 240 pounds P	
		Reported	Calculated	Reported	Calculated
	Pounds	Pounds	Pounds	Pounds	Pounds
0 to 60	0	715	456	185	45
60 to 120	0	815	410	- 294	39
120 to 180	0	- 8	381	531	38
0 to 60	60	333	524	- 45	50
60 to 120	60	1,362	458	- 409	49
120 to 180	60	373	428	- 431	41
0 to 60	120	1,104	558	374	55
60 to 120	120	658	490	- 760	48
120 to 180	120	1,944	457	560	44

Pasture experiment, Kirbyville, Tex. - 1950.

Interactions from reported yields at increasing rates of N and P exhibit aberrations at all 3 levels of K applied. For example, in table 17, comparison of the 715, 333, and the 1,104 pounds of dry forage that represent the interactions from reported yields for the same levels of N and P, but increasing levels of K, raises questions. Why should interaction for 0-60 N and 0-120 P at the intermediate level of K drop to about 1/2 of that reported at the low level, then at the high level rise to about 1.5 times that at the low level? Other similar examples can be detected from interactions based on reported yields.

Table 18.- Reported and calculated N x K interactions in terms of dry forage per acre at different rates of P applied

Pounds N applied	P	0 to 60 pounds K		60 to 120 pounds K	
		Reported	Calculated	Reported	Calculated
	Pounds	Pounds	Pounds	Pounds	Pounds
0 to 60	0	374	145	- 77	94
60 to 120	0	- 241	128	832	80
120 to 180	0	193	119	- 953	76
0 to 60	120	- 8	203	- 206	128
60 to 120	120	308	176	128	114
120 to 180	120	577	166	544	105
0 to 60	240	- 238	208	213	133
60 to 120	240	193	182	- 223	117
120 to 180	240	388	169	1,535	108

Pasture experiment, Kirbyville, Tex. - 1950.

The interactions derived through yields calculated from the equation follow a definite pattern. For example, at the same rates of N and P (table 17) interaction increases as the rate of K is increased. Similarly, in table 18, at the same rates of N and K, calculated interaction increases as the rate of P is increased. Also, at the same rates of P and K (table 19), calculated interaction increases as the rate of N is increased.

But at any stated level of the third nutrient, the calculated interaction between two nutrients decreases as the rate of either of the two nutrients is increased.

These comparisons are included here primarily to show that the multi-variable form of the exponential yield equation results in calculated yields which reflect a consistent pattern of interaction. Just as the single variable form does not permit calculation of lower yields associated with higher rates of fertilizer, the multi-variable form does not permit calculation of yields that reflect alternating higher and lower interaction, or "negative" interaction.

Table 19.- Reported and calculated P x K interactions in terms of dry forage per acre at different rates of N applied

Pounds P applied	N	0 to 60 pounds K		60 to 120 pounds K	
		Reported	Calculated	Reported	Calculated
	Pounds	Pounds	Pounds	Pounds	Pounds
0 to 120	0	535	76	116	50
120 to 240	0	26	7	- 16	4
0 to 120	60	153	134	40	84
120 to 240	60	-204	12	403	9
0 to 120	120	702	182	-664	116
120 to 240	120	-319	18	- 73	12
0 to 120	180	83	229	582	145
120 to 240	180	- 1,281	21	1,053	15

Pasture experiment, Kirbyville, Tex. - 1950.

As a basis for determining interaction, as well as for determining response to a single variable, the experiment should include rates high enough to indicate the point at which no further increases in yield may be expected. Once this point (or area) is located, economic interpretation is not concerned with what might happen if more fertilizer were added. It is concerned with most profitable rates and combinations within the range where response occurs. The form of exponential yield equation discussed here is useful as a basis for economic interpretation within that range.

Exponential and Parabolic Functions

Different yield functions may apply equally well when used with certain data for certain purposes. Yields calculated by different functions may differ only slightly within a given portion of the range in response. Even a straight line may appear to fit the data for a very short segment of the range in response. The usefulness of a yield function is determined by the way it performs throughout the range in response. The fact that one function may describe response a little more accurately than another for a part of the range does not mean that it is to be preferred unless it is adapted to the work of carrying out some of the more important aspects of economic interpretation. There is a good deal of inherent error in most fertilizer rate experiments. For example, an experiment carrying several replicates of each treatment is rated as good if the standard error of the individual plots is within 10 percent of the average yield of all the plots. Then, the element of risk and the uncertainty as to recurrence of other conditions such as weather, which would affect response if the experiment were repeated, makes the matter of pin-point accuracy more academic than practical. In addition, there is always the problem of interpreting results of an experiment for application to farm fields.

Some of the differences in the characteristics of the exponential yield equation and two forms of the parabola are noted as a result of applying each of them to an experiment involving 5 rates of 4-8-7 fertilizer used on potatoes in Maine. Yields were reported at rates of 0, 285, 380, 475, and 570 pounds of plant nutrients. The three functions compared are:

$y = M\,(1-R^x)$ (exponential - Spillman)
$y = a + b\,x + cx^2$ (polynomial parabola)
$y = ax^b$ (Cobb-Douglas function)

As described earlier, the hypothesis of the exponential yield equation is that as a yield-increasing variable is added in units of uniform size, yields increase at a constantly decreasing rate. As illustrated in table 20, the constants for this equation were determined by the method of least squares. The parabola used in this illustration was also fitted by the method of least squares.

The Cobb-Douglas function reflects a constant percentage change in yield associated with each percentage change in input. Logarithms of yields and of fertilizer rates are plotted on cross-section paper (or the natural numbers on double-log paper). As this function represents a straight line in the logarithms, a line of least squares may be fitted to the plotted logarithms by the equation $y = a + bx$. Yields can then be calculated using the logarithms as terms in the equation. Yields for the Cobb-Douglas function as shown in column 6, table 20, were thus calculated so that a comparison of all three functions could be made, each with the constants determined by the method of least squares.

But a principal advantage of the Cobb-Douglas function lies in the ease of obtaining an approximation to least squares by simply plotting the logarithms and connecting the points, thus forming a straight line. Logarithms of yields are then read from this line at selected rates of application. For example, the function, b, may be calculated as:

$$\frac{\log y_6 - \log y_0}{\log x_6 - \log x_0} = b$$

yields shown in column 7, table 20, were calculated in this way. Strictly speaking, the Cobb-Douglas function cannot be developed by using the yield at no application as one of the points because there is no logarithm for zero. However, such a reading may be approximated by using a very low application such as 1 pound as equivalent to no application. This was also done in calculating the yields shown in column 7, table 20, as a much closer fit to the reported yields was thus obtained than through selection of any other two points.

Table 20.- Reported yields of potatoes at different rates of application of plant nutrients, and yields calculated at these and other rates through use of different yield functions

x (1)	N, P, and K applied (2)	Reported yields Y (3)	Calculated yields - y: $y = M(1-R^x)$ 2/ (4)	$y = a + bx + cx^2$ 2/ (5)	$y = ax^b$ 1/ Least square form: 2/ (6)	$y = ax^b$ 1/ Graphic form 3/ (7)
	Pounds	Bushels	Bushels	Bushels	Bushels	Bushels
0	0	102	102	103	102	102
1	(95)	---	(237)	(222)	(328)	(335)
2	(190)	---	(337)	(323)	(391)	(402)
3	285	411	411	403	432	446
4	380	458	465	466	466	481
5	475	507	505	509	493	510
6	570	535	535	532	517	535
7	(665)	---	(557)	(537)	(540)	(557)
8	(760)	---	(573)	(523)	(558)	(577)
9	(885)	---	(585)	(500)	(581)	(600)
10	(950)	---	(594)	(437)	(591)	(611)
11	(1,045)	---	(600)	(366)	(606)	(626)
12	(1,140)	---	(605)	(276)	(617)	(641)
d^2			53	143	1,025	1,763
S_y (nearest bushel)			5	7	18	42

1/ Yield calculated for application of one pound of plant nutrients considered equivalent to yield at no application (see text).
2/ Constants determined by method of least squares (see text).
3/ b calculated as $\frac{\log y_6 - \log Y_0}{\log x_6 - \log x_1 \text{ lb.}}$

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Often fertilizer experiments do not carry enough rates to permit characterization of the entire range in response. However, many Maine potato growers are obtaining increasing yields throughout such a range as indicated in table 20. The highest rate used in the experiment was 570 pounds of plant nutrients. Clearly, the polynomial parabola is not suited for extrapolation. But both the exponential and Cobb-Douglas functions result in extrapolated yields that are known to be consistent with the experience of the better potato producers.

But the exponential function has theoretical advantage over the Cobb-Douglas function in the matter of extrapolation. It establishes a maximum yield that cannot be exceeded by any calculated yield. The Cobb-Douglas function established no maximum, hence calculated yields continue to rise with every estimated increase in fertilizer applied. For example, at 1,900 pounds of plant nutrients (10,000 lbs. of 19-percent material) the Cobb-Douglas function, when b in the equation is determined by the method of least squares, would estimate a yield of 703 bushels, in contrast with only 616 bushels calculated with the exponential equation. The last 95 pounds applied in the experiment resulted in an increase of only 28 bushels. While a 1,900-pound rate of application of plant nutrients would be impractical and might even cause serious damage to plant growth, the illustration tends to point out the advantage of the exponential-yield equation over either of the other two, in analyzing results of most fertilizer experiments. It establishes a maximum based on the responses shown, and each calculated yield represents a percentage of that maximum.

Where reported yields are available throughout the range in practical response that could be obtained, the three functions illustrated might give about the same results, so far as fitting a curve for a single variable is concerned. Even though it may be possible to develop ways in which the parabola or the Cobb-Douglas function may be used in calculating the combined effects of two or more variables acting simultaneously, the indicated advantages of the exponential yield equation remain. If an experiment includes a sufficient number of rates to characterize the curve, yields at higher rates may safely be calculated by means of the exponential equation.

Analysis of Variance

Analysis of variance shown in table 21 was conducted in terms of logarithms of yields so that measures of variance would reflect an assumption similar to that inherent in the equation used in calculating yields. Thus the standard error of estimate of 442 pounds of dry forage per acre for the calculated yields may be appraised with more meaning through comparison with the standard error of the yields reported for replicate plots. The standard error is 480 pounds, after taking out the effect of the various treatments. This standard error in the replicate plots is about 10 percent of the average yield for all the plots.

Most significant was the main effect of N, although the main effects of P and K were also highly significant. Both the linear and the quadratic variance ratios were highly significant for N and for P, as was the linear ratio for K. The quadratic ratio for K was significant. None of the interactions was significant when the analysis was conducted in terms of logarithms of the yields.

The analysis had been conducted earlier using the yields rather than the logarithms of the yields. When done in this way the analysis indicated that some of the interactions were significant; the main effects, of course, were highly significant. The variance ratios for main effects were 2,323 for N, 891 for P, and 184 for K. There was a variance ratio of 8 for N and K interactions when the analysis was conducted in terms of the reported yields, rather than the logarithms. As response to fertilizer is curvilinear rather than linear, use of the logarithms results in a more meaningful analysis of variance.

Finding the Most Probable Values of the Constants

The method of applying the principle of least-squares solution developed by W. J. Spillman is reproduced here with some documentation which should be helpful to those who are unfamiliar with the work. The illustration is limited to one of the variables (N) included earlier in the graphic method. When the least-squares solution is to be undertaken, the steps indicated in table 22 are carried out for each variable. Of course, if only three rates of application are involved, the method of graphic approximation when carefully done, yields precisely the same results as does a least-squares solution.

The problem is to determine the value of R that will render the same answer to two normal equations, one for A, the theoretical maximum increase attainable, and one for R, designated A'. Referring to totals of the indicated columns, table 22, and using n for number of observations and S instead of the more common Greek capital sigma.

$$A = \frac{n\ S\ YR^x - S\ Y\ SR^x}{(SR^x)^2 - n\ S(R^x)^2}$$

$$A' = \frac{n\ S\ YxR^x - SY\ SxR^x}{SR^x\ SxR^x - n\ SR^x xR^x}$$

Table 21.- Analysis of Variance 1/

Source	D.F.	SS.	M.S.
Nitrogen	3	3.53447	1.17816**
N_l	1	3.42176	3.42176**
N_q	1	.10976	.10976**
N_c	1	.00295	.00295
P_2O_5	2	.81527	.40764**
P_l	1	.57509	.57509**
P_q	1	.24018	.24018**
K_2O	2	.16876	.08438**
K_l	1	.14914	.14914**
K_q	1	.01962	.01962*
Rep	3	.05415	.01805**
NxP	6	.02471	.00412
NxK	6	.00743	.00124
NxR	9	.01048	.00116
PxK	4	.01637	.00409
PxR	6	.02184	.00364
KxR	6	.00230	.00038
NxPxK	12	.03373	.00281
NxPxR	18	.04342	.00241
NxKxR	18	.04489	.00249
PxKxR	12	.04913	.00409
NxPxKxR	36	.12847	.00357
Total	143	4.95542	

1/ Conducted in the Office of the Statistical Assistant, Bureau of Agricultural Economics, using logarithms of yields in pounds. Pasture experiment, Kirbyville, Tex. - 1950.

When the correct value of R has been reached through successive trials, $A - A' = 0$. The difference between A and A', when $R = 0.8677$ (table 22), amounting to 0.000077 is so small that this value of R may be considered satisfactory. The work of computing values for M, n, and u is also shown following table 22. In making the calculations, yields were expressed in tons to reduce the size of the figures handled.

Table 22 shows the computations for the value of R finally found to be satisfactory. The first trial value may be determined as:

$$\frac{Y_3 - Y_1}{Y_2 - Y_0} = R$$

Substituting the yield data from table 1, expressed in pounds

$$\frac{6,971 - 4,070}{5,688 - 2,339} = \frac{2,901}{3,349} = 0.86623$$

This value resulted in $E = + 0.7475$. When A exceeds A' (E positive) the trial value of R is too small; when A is smaller than A' the trial value is too large. After some experience has been gained, a satisfactory value of R may usually be found after three or four trials. The only purpose of finding $A-A' = 0$ is to arrive at a value of R and of the other constants that will permit calculation of the most probable yield according to the equation based on the reported yields. The solution described results in the most probable value of the constants, giving equal weight to each reported observation. A seemingly small deviation from 0 in the value of E may reflect enough error in the value of R and the other constants to affect the calculated yields materially. But, meticulous care in finding the constants may not necessarily result in calculated yields that reflect the true response, if statistical tests indicate substantial error in the reported yields. Therefore, when only four or perhaps even five or six rates are included and when there is substantial variation among the replicates of identical treatments, graphic approximation, or the method indicated above for finding the first trial value of R, may be regarded as satisfactory for practical purposes.

Table 22.- Arrangement of work for determining most probable values of constants for the responses to nitrogen 1/

x (60-lb. units)	Y response, yield, in tons	R^x (R, R^2, R^3)	xR^x (Col. 1 X Col. 3)	$(R^x)^2$ (Col. 3 squared)	$(R^x)(xR^x)$ (Col. 3 X Col. 4)	YR^x (Col. 2 X Col. 3)	$Y\ xR^x$ (Col. 2 X Col. 4)
1	2	3	4	5	6	7	8
0	1.170	1.000000	--	1.000000	--	1.170000	--
1	2.035	0.867700	0.867700	0.752903	0.752903	1.765769	1.765769
2	2.844	0.752903	1.505806	0.566863	1.133726	2.141256	4.282512
3	3.486	0.653294	1.959882	0.426793	1.280379	2.277383	6.832149
n=4	9.535	3.273897	4.333388	2.746559	3.167008	7.354408	12.880430

1/ When no observation is available at zero application or when the intervals between rates of application are not uniform, some modifications of the procedure are necessary. These modifications are not illustrated here.

Pasture experiment, Kirbyville, Tex. - 1950.

In solving equations 1, 2, and 3 below, column references refer to sums of columns. x is used as the sign of multiplication.

1. A : N (numerator) = (n x col. 7) - (col. 2 x col. 3)
= 29.417632 - 31.216608 = - 1.798976

D (denominator) = $(\text{col. } 3)^2$ - (n x col. 5)
= 10.718402 - 10.986236 = - 0.267834

A = - 1.798976/- 0.267834 = 6.716757

2. A': N'(numerator) = (- col. 2 x col. 4) + (n x col. 8)
= - 41.318855 + 51.521720 = 10.202865

D'(denominator) = (- n x col. 6) + (col. 3 x col. 4)
= - 12.668032 + 14.187066 = 1.519034

A' = 10.202865/1.519034 = 6.716680

E = A-A' = 6.716757 - 6.716680 = 0.000077 tons or 0.154 lb.

The value of E is so small that R = 0.8677 is acceptable.

3. M = 1/4 (col. 2 + A x col. 3)
= 7.8812 tons or 15,762 pounds

4. n (soil nitrogen) = (log M - log A)/ log 0.8 = 0.06942/-0.09691=0.7164

5. u = log 0.8 / log R = - 0.09691/ - 0.06163 = 1.572448 60-lb. units
= 94.35 lbs.

Yields in pounds are then calculated from these constants.

Table 23.- Comparison of reported yields and yields calculated at different levels of nitrogen 1/

	x						
a	a	n	n/a	$1-R^{n/a}$	Yield, y calculated 2/	Yield, Y reported	d^2
Pounds	Units	Units	Units		Pounds	Pounds	Pounds
0	--	0.7164	0.7164	0.147701	2,328	2,339	121
60	0.6359	0.7164	1.3523	.260482	4,106	4,070	1,296
120	1.2719	.7164	1.9883	.358324	5,648	5,688	1,600
180	1.9078	.7164	2.6242	.443214	6,986	6,971	225
Sum of squared deviations							3,242
Standard error of estimate					57		

1/ The constants for the equation were computed by the method of table 22.
2/ $y = 15{,}762\ (1-R^{n/a})$

Pasture experiment, Kirbyville, Tex. - 1950.

The standard error of estimate for yields calculated from the same data when the constants were determined by the method of graphic approximation was 94 pounds. The standard error of the means of the replicates of reported yields for the 0-, 60-, 120-, and 180-pound rates of application are ± 196, ± 167, ± 401, and ± 132 pounds, respectively (table 1). Therefore, as indicated earlier, the graphic-approximation method of finding constants may be preferred to a least-squares solution in instances of the type illustrated, in view of the additional work required by the latter. However, if several rates are included in the experiment and if they are distributed so that reported yields are available over a major portion of the range in which response occurs, a least-squares solution should be used in finding the constants for the yield curve.