Department of Agricultural & Resource Economics, UCB
CUDARE Working Papers
(University of California, Berkeley)

Year 1980 *Paper* 85*R*

A Note on a One-sided, Chebychev-type Inequality

Peter Berck
University of California, Berkeley

Jarirus M. Hihn
University of California, Berkeley

Division of Agricultural Science
UNIVERSITY OF CALIFORNIA

Working Paper No. 85, Rev.

A NOTE ON A ONE-SIDED, CHEBYCHEV-TYPE INEQUALITY

by

Peter Berck and Jairus M. Hihn

California Agricultural Experiment Station
Giannini Foundation of Agricultural Economics
November 1980

A NOTE ON A ONE-SIDED, CHEBYCHEV-TYPE INEQUALITY

by Peter Berck and Jairus M. Hihn

Peter Berck is assistant professor and Jairus M. Hihn is assistant research economist, Department of Agricultural and Resource Economics, University of California, Berkeley.

A NOTE ON A ONE-SIDED, CHEBYCHEV-TYPE INEQUALITY

Given knowledge of a finite variance, semivariance, and mean, how tight of an upper bound can be placed on the probability of an outcome which is k standard deviations below the mean? With knowledge of a finite mean and variance only, it is a well-known result that the smallest upper limit is determined by Chebychev's inequality: $\Pr(X \leq E(x) - k\sigma) \leq 1/k^2$. In this note it will be shown that, by using the semivariance, a sharper bound can be placed on the area in the lower tail. This lower tail area is of interest in economics--and particularly agricultural and development economics--because of its relation to safety-first criteria. An example of safety first, drawn from agricultural economics, is the following: choose a cropping plan to maximize income subject to a predetermined probability of not making less than some critical amount. In development, Nakajima, Wharton, and Roumasset use safety-first models to analyze production decisions of low-income farmers. Wright provides an application to the 19th century cotton industry, Telser applies it to hedging, and Boussard and Petit use a safety-first model to explain cropping patterns. For a review of the theoretical discussions of the different safety-first rules, see Pyle and Turnovsky.

The alternative to using a Chebychev-type inequality with information on the semivariance is to actually specify a distribution, use Chebychev's inequality incorporating higher moments, or use an approximation function such as the Edgeworth series or Pearson system of distributions (see Walsh). Using Chebychev's inequality is a reasonable alternative if the process--like agricultural production (Day)--has considerable probability mass in the tails.

This note will derive the semivariance version of Chebychev's inequality, compare it to the variance version, and provide a brief example of its use drawn from California agriculture.

Let x be distributed as F with mean Ex. Its semivariance is defined as

$$\sigma^{*2} = \int_{-\infty}^{Ex} (x - Ex)^2 \, dF$$

and its semistandard deviation is $\sigma^* = \sqrt{\sigma^{*2}}$. The Chebychev inequality in terms of the semistandard deviation is

$$\text{(1)} \qquad \text{prob } (x \leq Ex - k\sigma^*) \leq \frac{1}{k^2} .$$

The proof follows the proof of Chebychev's inequality using the variance. Since

$$\int_{-\infty}^{Ex - k\sigma} (x - Ex)^2 dF + \int_{Ex - k\sigma}^{Ex} (x - Ex)^2 \, dF = \sigma^{*2}$$

then $\int_{-\infty}^{Ex - k\sigma^*} (x - Ex)^2 \, dF < \sigma^{*2}$. The function $(x - Ex)^2$ is at least as large as $k^2\sigma^{*2}$ on the interval $(-\infty, Ex - k\sigma^*)$ so

$$\sigma^{*2} > \int_{-\infty}^{Ex - k\sigma^*} (x - Ex)^2 \, dF \geq k^2 \sigma^{*2} \int_{-\infty}^{Ex - k\sigma^*} dF$$

or

$$\frac{1}{k^2} \geq \int_{-\infty}^{Ex - k\sigma^*} dF = \text{prob } (x \leq Ex - k\sigma^*).$$

To see that the semivariance inequality is tighter than the variance inequality, let $k = m\sigma/\sigma^*$ where σ is the standard deviation. Substituting for k in the semivariance inequality yields

$$\text{(2)} \qquad \text{prob } (x \leq Ex - m\sigma) \leq \frac{1}{m^2} \frac{\sigma^{*2}}{\sigma^2}.$$

Thus, the use of semivariance improves the Chebychev inequality by the factor σ^{*2}/σ^2. When the distribution is symmetric, this factor is one half, which is a dramatic improvement.

To illustrate the extent of the improvement that can be expected in the Chebychev bound, consider the case of crop yields and revenues in Kern County California (table 1). The yields (revenues) of alfalfa, cotton lint, potatoes, and sugar beets were predicted from regression of yield (revenue) on a constant term, and the dependent variable lagged one and two time periods. The variance and semivariance of these predictions have as their major components the variance and semivariance of the error terms of these regressions, so that statements about likelihood of getting yields (revenues) m standard deviations from the predicted value can be constructed from these statistics. Table 1 gives the predicted yields and revenues as well as their standard deviation and the ratio of semivariance to variance. As an example of how to use the table, the probability of getting a cotton yield of 1.59 bales per acre (which is two standard deviations below the mean) is less than 1/4 using the upper bound from the variance inequality and less than .14 (which is $1/2^2 \cdot \sigma^{*2}/\sigma^2$) using the semivariance inequality in form (2). In general, the column labeled "Ratio of Prediction Semivariance to Variance" gives the improvement of the semivariance inequality over the variance inequality. As was expected from Day's earlier work on agricultural yields in a Pearson system, the improvement is not the 1/2 that would result from a symmetric distribution.

One final note on the table. The revenues (and apparently incomes) of sugar beets were stabilized by the government over much of the sample period, and this stabilization is reflected in the miniscule probability of having revenues below their mean for each crop.

Table 1. Means, Standard Deviations, and Ratios of Semivariance to Variance For Predicted Yields and Revenues of Four California Crops

Activity	Mean of Prediction (per acre)	Standard Deviation of Prediction, σ	Ratio of Prediction Semivariance to Variance, σ^{*2}/σ^2
Alfalfa			
Yield	7.22 tons	.50	.40
Revenue	$365	53.	.41
Cotton Lint			
Yield	2.17 bales	.29	.56
Revenue	$451	65.	.48
Potatoes			
Yield	2.36 tons	1.54	.44
Revenue	$1746	311.	.43
Sugar Beets			
Yield	12.88 tons	3.59	.52
Revenue	$493	128.	.18

Source: Data from the Kern County Agricultural Commissioner's Report, various years. For method of computation, see text.

The preceding discussion clearly demonstrates that the upper bound on the probability of an outcome which is k standard deviations below the mean calculated from the semivariance is a dramatic improvement over an upper bound based on variance. Given the growing interest in safety-first rules in agricultural and development economics, this result should prove to be useful.

What prevents more widespread use of semivariance and semivariance inequalities is computational difficulty when considering more complicated decision problems. The difficulty is that one often wants to know the semivariance of the sum of two or more variables, and that sum cannot be expressed as a simple sum of appropriate semivariances and covariances.

REFERENCES

Agricultural Commissioner, Kern County, California. Agricultural Crop Reports, various years.

Boussard, J., and M. Petit. "Representation of Farmers' Behavior Under Uncertainty with a Focus-Loss Constraint." J. Farm Econ. 49(1967):869-880.

Day, Richard H. "Probability Distribution of Field Crop Yields." J. Farm Econ. 47(1965):713-741.

Nakajima, C. "Subsistence and Commercial Family Farms: Some Theoretical Models of Subjective Equilibrium." Subsistence Agriculture and Economic Development, ed. C. R. Wharton, Jr., pp. 165-185. Chicago: Aldine Publishing Company, 1969.

Pyle, David H. and Stephen J. Turnovsky. "Safety First and Expected Utility Maximization in Mean-Standard Deviation Portfolio Analyses." Rev. Econ. and Stat. 52(1970):75-81.

Roumasset, J. Rice and Risk: Decision Making Among Low-Income Farmers. Amsterdam: North-Holland Publishing Company, 1976.

Telser, L. G. "Safety First and Hedging." Rev. of Econ. Stud. 23(1955):1-16.

Walsh, J. Handbook of Nonparametric Statistics. Princeton: D. von Nostrand Company, Inc., 1962.

Wharton, C. R., Jr. "Risk, Uncertainty, and the Subsistence Farmer: Technical Innovation and Resistance to Change in the Context of Survival." Joint Session of the American Economics Association and Association for Comparative Economics, Chicago, 1968.

Wright, G. "Cotton Competition and the Post-Bellum Recovery of the American South." J. Econ. Hist. 35(1974):610-635.