Journal of Agricultural and Resource Economics: Preprint
ISSN: 1068-5502 (Print); 2327-8285 (Online)

doi: 10.22004/ag.econ.369143

Nitrogen Response Modeling with a Multi-Degree Spline Plateau Approach

Yuan Zhang, B. Wade Brorsen, and W. Hence Duncan

Nitrogen response models are often estimated as plateau type models. Existing plateau models are nonlinear in the parameters. The model introduced here is a multi-degree spline plateau model, which is linear in parameters. The linearity allows using approximation methods so that estimating a spatially varying coefficient model with Bayesian Kriging could be a commercially feasible approach to guide precision nitrogen applications. With proper knot selection, the spline plateau model can fit as well as the linear and quadratic plateau models. The computational efficiency of the spline plateau model was demonstrated using R-INLA and it provided fast and accurate estimation.

Key words: Corn, nitrogen, plateau, spline, wheat, yield

Introduction

Nitrogen, an essential nutrient for plant growth, profoundly influences crop yield and quality. Excessive or insufficient nitrogen can cause adverse environmental, economic, and agronomic consequences. Hence, estimation of crop yield response to determine the optimal nitrogen rate emerges as a pivotal endeavor. Over-application of nitrogen fertilizers can lead to water and soil acidification, groundwater contamination, and accelerated ozone depletion (Bashir et al., 2013). Excessive fertilization can also lead to the accumulation of nitrites in crops, which can cause serious risks to human health (Ahmed et al., 2017). Farmers do not closely follow the nitrogen recommendations made by universities and tend to apply higher amounts of nitrogen than recommended rates (Sellars, Schnitkey, and Gentry, 2020). The Data Intensive Farm Management (DIFM) project at the University of Illinois was created to demonstrate to farmers on their own fields the effects of nitrogen applications (Bullock et al., 2019; Bullock, Mieno, and Hwang, 2020). What has not been resolved is how to turn the immense data from on-farm experiments into profitable variable rate nitrogen recommendations. Two computing solutions are machine learning models and nitrogen response functions with spatially varying parameters. This article is concerned with the spatially varying parameter approach. Trevisan, Bullock and Martin (2021) estimated a spatially varying parameter model with a quadratic functional form using

---

Yuan Zhang is a postdoctoral fellow in the Department of Agricultural Economics and Agribusiness at the University of Arkansas, B. Wade Brorsen (corresponding author, wade.brorsen@okstate.edu) is a regents professor and A.J. & Susan Jacques Chair in the Department of Agricultural Economics at Oklahoma State University, and W. Hence Duncan is a PhD student in the Department of Agricultural Economics at Oklahoma State University.
The research was funded by the A.J. & Susan Jacques Chair, the Oklahoma Agricultural Experiment Station and USDA National Institute of Food and Agriculture, Hatch Project number OKL03170. The authors acknowledge the contributions of the numerous agronomists that collected the data used in this study including Brian Arnall, Curtis Ransom, and Emerson Nafziger. Taro Mieno is also thanked for suggesting generalized additive models, which led to recognizing the need for the model developed here. The authors appreciate the constructive comments and suggestions provided by the anonymous reviewers.

Review coordinated by Liang Lu.

Geographically Weighted Regression (GWR). Building on their work, Lambert and Cho (2022) estimated a linear plateau model via GWR while Mieno et al. (2024) and Zhang et al. (2025) have estimated GWR quadratic plateau models. Paccioretti et al. (2021) explored a different path by using Bayesian methods and a quadratic functional form. More recently, Park, Brorsen and Li (2024) and Poursina and Brorsen (2025) extended these ideas to spatially varying linear plateau models using Bayesian Kriging. One weakness of all these models as well as machine learning models is that they overfit the data and give optimal N levels that vary widely across a field. In an attempt to mitigate the overfitting problem, Patterson (2023) estimated the spatially varying linear plateau models using informative priors. Patterson's approach, however, is too computationally slow to be commercially feasible. This article seeks to solve the computational infeasibility of Patterson's approach.

While using a quadratic functional form would also be a solution, we seek a plateau function because it is more consistent with the way the world actually works and Mieno et al. (2024) showed that using a quadratic function can overestimate the true nitrogen rate. Building on von Liebig's law of the minimum, Cate and Nelson (1971) introduced a linear plateau model to illustrate the relationship between crop yield and the scarcest resource and has been used in subsequent studies (Bullock and Bullock, 1994; Bäckman, Vermeulen, and Taavitsainen, 1997; Monod et al., 2002). After Berck and Helfand (1990) first proposed the concept of stochastically varying linear plateau parameters, researchers extended their idea to estimate stochastic plateau models with time random effects using both classical (Makowski, Wallach and Meynard, 1999; Tembo et al., 2008; Biermacher et al., 2009; Tumusiime et al., 2011; Brorsen and Richter, 2012; Harmon et al., 2016; Nafziger and Rapp, 2020) and Bayesian methods (Pautsch, Babcock and Breidt ,1999; Ouedraogo and Brorsen, 2018; Cho et al., 2020; Panyi and Brorsen, 2024). Here, parameters are allowed to vary across site-year by estimating different models for each site-year.

In addition to the plateau models, generalized additive models (Hegedus et al., 2023) were also used in precision agriculture for nitrogen response estimation. However, the parameters of generalized additive models are difficult to interpret and do not impose concavity. Machine learning techniques have been employed for site-specific nitrogen response estimation. Random forest (Paccioretti et al., 2021; Peerlinck et al., 2022; Hegedus et al., 2023) and convolutional neural network (Barbosa et al., 2020; Morales et al., 2022) have demonstrated effectiveness in estimating nitrogen response functions. In addition to not imposing concavity, the main obstacle to commercializing machine learning models is that they require measured variables at each site within a field. While measures of slope, satellite sensing, and soil type often exist, other measures such as electrical conductivity are often costly to obtain. Machine learning models are limited to measuring spatial variability that is due to observed variables and usually restrict functional relationships to be the same across the field or across multiple fields (Hegedus et al., 2023) in order to have sufficient degrees of freedom.

The new model is a multi-degree spline plateau model. This model is linear in parameters, which leads to a Bayesian model that can be estimated using the inverse Laplacian approximation that is fast enough to be commercially viable (Rue, Martino, and Chopin, 2009). Spline models create flexible shapes by using smooth, continuous curves (Perperoglou et al., 2019). Multi-degree spline models have segments with polynomials of different degrees (Shen and Wang, 2010; Beccari et al., 2017; Toshniwal et al., 2020). The spline model is widely used in medicine, some examples include Largo et al. (1978), Gauthier et al. (2020), and Schuster et al. (2022). Traditional generalized additive models (GAMS) are also spline models.

Three datasets were used to assess the performance of the new functional form. The first dataset was corn data from the Performance and Refinement of Nitrogen Fertilization Tools (PRNT) project, which includes 49 site-year trials conducted between 2014 and 2016 across eight Midwestern U.S. states, with nitrogen application rates ranging from 0 to 280 lb/acre. The second dataset was corn data from Maximum Return to Nitrogen (MRTN) study, with experimental data from central Illinois from 2013 through 2023, with nitrogen rates up to 341 lb/acre for corn following soybean. The third dataset was wheat data of experiment E502 from Lahoma,

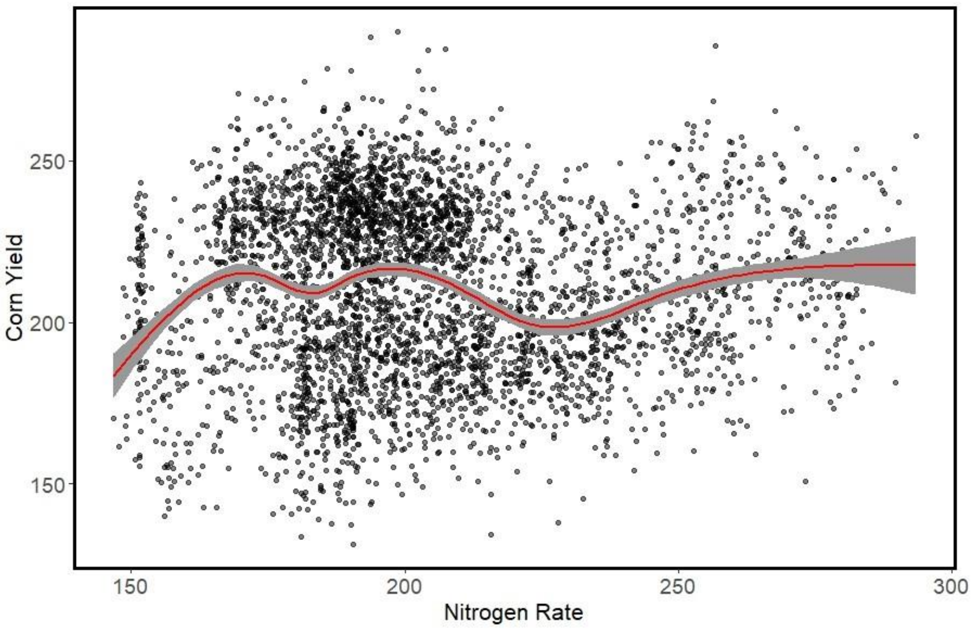

Figure 1. Scatter Plot with GAM Spline Curve. Source: Coleman, Brorsen, and Mieno (2023).

Oklahoma experimental station, includes 1304 observations of nitrogen application rates between 0 and 100 lb/acre, ensuring phosphorus and potassium were non-limiting factors. Two ways were used to estimate the models, site-year specific models (Nafziger and Rapp, 2020) and deterministic models for the whole dataset. The feasibility of utilizing the spline plateau model for nitrogen response estimation was assessed by comparing the mean squared error (MSE) and sum of squared error (SSE) with those form the linear plateau model, quadratic plateau model, and quadratic model [1]. To further evaluate the computational efficiency and accuracy of the spline plateau model, simulated data were used for estimation in the R-INLA software package (Rue et al., 2013).

Method

The regular spline model without restriction of the shape can produce a multimodal wave, which is unlikely to be the true model. Figure 1 is an example from Coleman, Brorsen, and Mineo (2023), based on on-farm experimental data collected through the DIFM project in Illinois over three years (2017, 2019, and 2021) from a single field. The figure illustrates how the regular GAMS model produces unrealistic fluctuations in the relationship between nitrogen rate and corn yield, deviating from the expected plateau-shaped pattern. These multimodal waves are likely the result of overfitting, capturing noise rather than the true agronomic relationship. As an alternative, each segment in the proposed spline model can be restricted to provide a plateau functional form. At lower nitrogen application rates, the model accommodates a linear response in crop yield,

[1] Machine learning models are often evaluated with cross validation (Tanaka et al., 2024) or another form of out-of-sample forecasting. Our models are not selected using the data and so our in-sample statistics are valid (one exception is the multi-degree spline that uses the plateau value from the quadratic plateau).

acknowledging the direct correlation between nitrogen availability and yield. The cubic shape provides flexibility over the middle part of the function and the third part is a plateau. The proposed multi-degree spline plateau model can be expressed in the following form:

(1)
$$y_{it} = \begin{cases} \beta_0 + \beta_1 N_{it} + \varepsilon_{it}, & \text{if } N_{it} \leq N_1^* \\ \beta_2 + \beta_3 N_{it} + \beta_4 N_{it}^2 + \beta_5 N_{it}^3 + \varepsilon_{it}, & \text{if } N_1^* < N_{it} \leq N_2^* \\ \beta_6 + \varepsilon_{it}, & \text{if } N_{it} > N_2^* \end{cases}$$

where y_{it} is the corn yield of observation i in year t, N_{it} is the nitrogen application rate of observation i in year t, N_1^* and N_2^* are the two knots, and ε_{it} is a random error term that is normally distributed. The two knots N_1^* and N_2^* were pre-selected based on observations of the mean plots. This model has four constraints, two constraints to make sure the model is continuous at the two knots N_1^* and N_2^*:

$$\beta_0 + \beta_1 N_1^* = \beta_2 + \beta_3 N_1^* + \beta_4 {N_1^*}^2 + \beta_5 {N_1^*}^3$$

$$\beta_2 + \beta_3 N_2^* + \beta_4 {N_2^*}^2 + \beta_5 {N_2^*}^3 = \beta_6$$

and two constraints to make sure the model is differentiable at the two knots:

$$\beta_1 = \beta_3 + 2\beta_4 N_1^* + 3\beta_5 {N_1^*}^2$$

$$\beta_3 + 2\beta_4 N_2^* + 3\beta_5 {N_2^*}^2 = 0$$

The proposed multi-degree spline plateau model can be defined using dummy variables as

(2)
$$y_{it} = D_{1it}(\beta_0 + \beta_1 N_{it}) + D_{2it}(\beta_2 + \beta_3 N_{it} + \beta_4 N_{it}^2 + \beta_5 N_{it}^3) + D_{3it}\beta_6 + \varepsilon_{it}$$

where D_{1it}, D_{2it} and D_{3it} are dummy variables, D_{1it} equals 1 when the nitrogen rate N_{it} is less than or equal to N_1^*, D_{2it} equals 1 when the nitrogen rate N_{it} is greater than N_1^* and less than or equal to N_2^*, D_{3it} equals 1 when the nitrogen rate N_{it} is greater than N_2^*. Given the four equal constraints β_2, β_3, β_4, and β_5 can be substituted by β_0, β_1, and β_6. Therefore, the model is

(3)
$$y_{it} = \delta_0\beta_0 + \delta_1\beta_1 + \delta_6\beta_6 + \varepsilon_{it}$$

where

$$\delta_0 = D_{1it} + \frac{D_{2it}}{(N_1^* - N_2^*)^3}\left((3N_1^* {N_2^*}^2 - {N_2^*}^3) - 6N_1^* N_2^* N_{it} + 3(N_1^* + N_2^*)N_{it}^2 - 2N_{it}^3\right)$$

$$\delta_1 = D_{1it}N_{it} + \frac{D_{2it}}{(N_1^* - N_2^*)^3}(2{N_1^*}^2 {N_2^*}^2 + (-4{N_1^*}^2 N_2^* - {N_2^*}^3 - {N_2^*}^2 N_1^*)N_{it} + 2({N_1^*}^2 + {N_2^*}^2 + N_1^* N_2^*)N_{it}^2 - (N_1^* + N_2^*)N_{it}^3)$$

$$\delta_6 = (\frac{D_{2it}}{(N_1^* - N_2^*)^3}\left({N_1^*}^3 - 3{N_1^*}^2 N_2^* + 6N_1^* N_2^* N_{it} - 3(N_1^* + N_2^*)N_{it}^2 + 2N_{it}^3\right) + D_{3it}$$

The detailed calculation process of equation (3) is shown in Appendix 1. It can be obtained through equation (3) that this model has three free parameters, β_0, β_1, and β_6. The form in equation (3) would be needed to use in a software package such as R-INLA. Since the purpose of constructing a spline plateau model is to get better predictions, a pretest or grid search might be used to select two knots to get a better fit. While such a grid search would make inference invalid, that is a minor concern since the ultimate goal is the prediction of optimal nitrogen. Concavity was not imposed. Enforcing concavity would require inequality constraints on the second derivative[2], which R-INLA does not support (Lindgren and Rue, 2015). Also, since the second derivative is a function of the nitrogen level, concavity can only be imposed or tested for specific values of nitrogen rather than globally. Instead, concavity was evaluated post-estimation. For the cubic part of the spline plateau model, the second derivative is given by

$$\frac{d^2 y_{it}}{dN_{it}^2} = 2\beta_4 + 6\beta_5 N_{it}$$

Rather than try to evaluate concavity over the entire cubic segment $[N_1^*, N_2^*]$, the second-order condition (SOC) was assessed at the knot where the second derivative reaches its maximum. When $\beta_5 < 0$, the SOC is decreasing over $[N_1^*, N_2^*]$, so the SOC was evaluated at N_1^*. When $\beta_5 > 0$, the SOC is increasing, and the SOC was evaluated at N_2^*. If the SOC at the point was less than or equal to zero, the cubic part was considered concave.

The economically optimal nitrogen rate is determined by the condition that marginal revenue equals marginal cost ($MR = MC$). On the first part (linear part) of the spline function, we know that the optimum can only be at zero or at the switch point like linear plateau models. The switch point could only be optimal if $P\beta_1 = r$. If we assume that β_1 is drawn from a continuous distribution, then we can eliminate the switch point with probability one. The marginal revenue on the plateau portion is zero, so the economic optimum cannot be along the plateau. Therefore, the optimum must either be at zero or on the interior of the cubic portion. On the cubic part, marginal cost equals marginal revenue when $\partial E\pi / \partial N_{it}$ equals 0, which is

(4) $$E\pi = P(\beta_2 + \beta_3 N_{it} + \beta_4 N_{it}^2 + \beta_5 N_{it}^3) - rN_{it}$$

$$\frac{\partial E\pi}{\partial N_{it}} = P(\beta_3 + 2\beta_4 N_{it} + 3\beta_5 N_{it}^2) - r$$

The point N^* where marginal cost equals marginal revenue is

$$\frac{\partial E\pi}{\partial N^*} = P(\beta_3 + 2\beta_4 N^* + 3\beta_5 N^{*2}) - r = 0$$

(5) $$N^* = \frac{-2P\beta_4 \pm \sqrt{4(P\beta_4)^2 - 4(3P\beta_5(P\beta_3 - r))}}{6P\beta_5}$$

Equation (5) provides two possible solutions for the economically optimally nitrogen rate, N^*. In most cases, one of the solutions will lie outside the region of interest ($N_1^* \leq N^* \leq N_2^*$), and can be eliminated. If both solutions meet this condition, then the optimal rate is determined by comparing expected profit at N_1^* and N_2^*. So, we suggest finding the optimal by calculating the profit at three points: zero and the two solutions to equation (5) and then picking the solution with the highest expected profit.

Linear plateau, quadratic plateau and quadratic models were also estimated as a comparison. The linear plateau model is

(6) $$y_{it} = \min(\beta_0 + \beta_1 N_{it}, \mu) + \varepsilon_{it}$$

The optimal nitrogen rate of the linear plateau function was either zero or the plateau nitrogen rate depending on where expected profit was larger.

The quadratic plateau model is

(7) $$y_{it} = \min\left(\beta_0 + \beta_1 N_{it} + \beta_2 N_{it}^2, \beta_0 + \beta_1 N^* + \beta_2 N^{*2}\right) + \varepsilon_{it}$$

[2] Concavity could also be imposed at chosen values of nitrogen using Bayesian priors that gave zero prior probability to the nonconcave region. Such an approach was not considered since it would greatly slow the calculation and it would bias the coefficient estimates.

subject to

$$\beta_1 + 2\beta_2 N^* = 0$$

where N^* is the plateau level nitrogen rate. Continuity is imposed via the min() function and the constraint imposes differentiability. The optimal nitrogen level was computed by comparing the expected profit at the point where marginal cost equals marginal revenue and the expected profit when nitrogen application rate is 0 lb/acre. The point marginal cost equals marginal revenue N^* is where $\partial E\pi/\partial N_{it}$ equals 0, which is

$$E\pi = P(\beta_0 + \beta_1 N_{it} + \beta_2 N_{it}^2) - rN_{it} \quad (8)$$

$$\frac{\partial E\pi}{\partial N_{it}} = P\beta_1 + 2P\beta_2 N_{it} - r$$

The point $N^*$where marginal cost equals marginal revenue is

$$\frac{\partial E\pi}{\partial N^*} = P\beta_1 + 2P\beta_2 N^* - r = 0$$

$$N^* = \frac{r - P\beta_1}{2P\beta_2}$$

The quadratic response production function is

$$y_{it} = \beta_0 + \beta_1 N_{it} + \beta_2 N_{it}^2 + \varepsilon_{it} \quad (9)$$

The optimal nitrogen rate of the quadratic model is computed the same way as in the quadratic plateau model. All models were estimated using the SAS program PROC NLMIXED (SAS Institute, 2013).

To evaluate the performance and behavior of the models, we used different error metrics for deterministic and site-year-specific models. The MSE was calculated for each functional form to compare the overall performance of deterministic models. For site-year-specific models, we used the SSE for each functional form to assess how well the models captured variability within individual site-year combinations.

For the site-year specific models, a grid search was used to find the optimal nitrogen rate. For each site-year, the expected revenue for each nitrogen rate was calculated from 0 lb/acre up to the plateau nitrogen value, increments of 1 lb/acre. The average expected revenue was then determined for each nitrogen rate across all site-years, and the nitrogen rate with the highest average expected revenue was identified as the optimal nitrogen rate. Prices used for the two corn datasets were $0.40 per pound of nitrogen and $4.00 per bushel for corn grain following Nafziger and Rapp (2020). Price used for the wheat dataset was $0.68 per pound of nitrogen and $7.62 per bushel for wheat following Panyi and Brorsen (2024).

Data

Three different datasets were used. The first dataset was from PRNT project (Ransom et al., 2022), focusing on corn as the experimental crop. Between 2014 and 2016, a total of 49 site-year trials in eight Midwestern U.S. states were conducted. There were two sites per state per year, with three sites in Missouri in 2016. All site-years followed a standardized protocol (see Ransom et al. (2021) for the details of the experimental design). The experiment consisted of 16 treatments including eight different nitrogen fertilizer application rates (0 to 280 lb/acre) and two planting

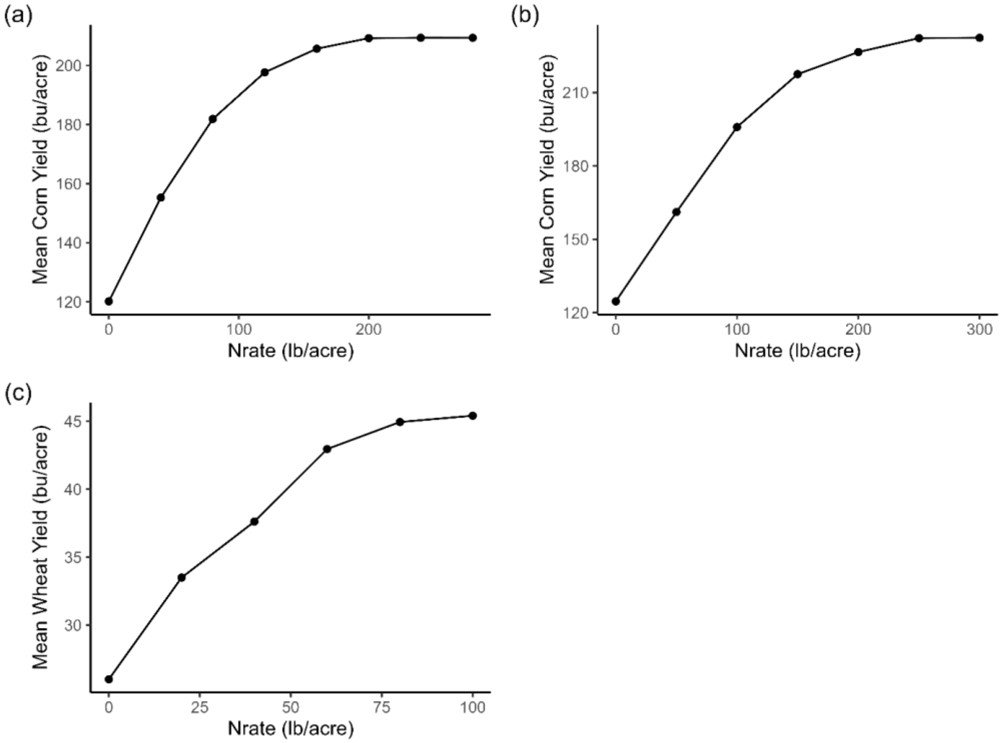

Figure 2. Mean Crop Yield (bu/acre) of Each Nitrogen Application Rate (lb/acre) of the Three Datasets.

timings (at planting or the majority sidedressed). Nafziger and Rapp (2020) argued that delayed application of nitrogen had little or no yield benefit for corn. Therefore, only the total nitrogen fertilizer application rate was considered, and planting timing was ignored.

The second dataset was from the MRTN project (Nafziger et al., 2022). The experiments were conducted in Illinois and the sites were categorized as central, southern, and northern. The crop for this experiment was corn following soybean. This study used data from central Illinois from 2013 through 2023. Nitrogen fertilizer application rates ranged from 0 to 341 lb/acre. Since some plots had pre-applied fertilizers, the nitrogen rate for each site-year in these experiments varied (232 nitrogen fertilizer application rates in total).

The third dataset was for hard red winter wheat from Experiment 502 at Lahoma, Oklahoma experimental station (36°23'16.8"N 97°06'28.8"W). The experiment began in 1971 and is still ongoing. The data included observations from 1971 through 2018. A total of 13 different wheat varieties were planted in different years depending on the planting choices of local wheat farmers. The experiment included 14 different treatments of nitrogen, phosphorus, and potassium combinations. This study exclusively focused on plots where phosphorus and potassium would not be a limiting factor (Ouedraogo and Brorsen, 2018). Specifically, the dataset comprised plots that received 40 lb/acre of P_2O_5 and 60 lb/acre of K_2O. The consolidated data had 6 nitrogen levels (0 to 100 lb/acre) with a total of 1304 observations.

Figure 2 shows average crop yields for each nitrogen fertilizer application rate across the three datasets: (a) corn yield from the PRNT project, (b) corn yield from the MRTN project, and (c) wheat yield from Experiment 502. All three datasets show the same pattern, with crop yields showing a rapid rate of increase at low nitrogen rates and slowing down with increasing nitrogen rates. Note that each point in Figure 2(b) represents the average yield for that range (e.g., 50 for nitrogen rates from 0 lb/acre to 50 lb/acre and 100 for nitrogen rates from 50 lb/acre to 100 lb/acre). The PRNT data and MRTN data clearly reached the plateau, while with the Experiment 502 dataset, it is not as clear that the nitrogen levels applied were sufficient to reach the plateau.

Simulated data were used to evaluate the performance of the proposed spline plateau model in R-INLA, which uses the inverse Laplacian approximation. A stochastic plateau model was used to simulate corn yield responses over a 10-year period across a 10×10 spatial grid, resulting in 100 unique cells, each defined by distinct (x, y) coordinates. The simulation included five nitrogen rates (60, 110, 160, 210, and 260 lb/acre) to assess yield responses. These rates were selected to align with those commonly used in DIFM trials. The yield response model calculates yield based on nitrogen rate, plateau level, and spatial and temporal effects. The corn yield (Y_{it}) at location i in year t, given nitrogen rate N_{it} was modeled as

$$Y_{it} = \min\,(aN_{it} + b, P_{it}) \tag{10}$$

where $a = 0.69$ is the marginal yield gain per nitrogen unit, $b = 98.18$ is the base intercept, P_{it} is the plateau yield for location i in year t. Plateau yields incorporate spatially correlated base levels and time effects

$$P_{it} = S_i + \varepsilon_t \tag{11}$$

where S_i is the spatial varying plateau base, modeled using a Gaussian random field with a spatial covariance matrix and $\varepsilon_t \sim N(0, \tau^2)$ representing yearly random effects. The spatial component of P_{it} was modeled using a covariance structure of

$$\mathrm{Cov}(S) = sill * \exp\left(-\frac{d}{range}\right) + nugget \tag{12}$$

where d is the distance between cells, the sill is equal to 500, and the range is equal to 3. Yearly effects were simulated as random draws from a normal distribution of $\varepsilon_{it} \sim N\left(0, \sqrt{235.95}\right)$.

Results

Deterministic Models

Four different deterministic spline plateau models were estimated for each dataset, with the knots varying across datasets. For each dataset, knot 1 was the same across all four models, while knot 2 varied: three were preselected, and the fourth corresponded to the plateau nitrogen level from the quadratic plateau model. The estimation results for the deterministic models are reported in Tables 1, 2, and 3. Concavity of the spline plateau models was evaluated post-estimation based on the SOC, as described in the methods section. The models that were not concave were mostly those where the second knot was set at higher nitrogen values, which caused the cubic polynomial to need to approximate the plateau. Figure 3 shows the estimated response curve for one of the non-concave cases. As shown in the figure, the nonconcavity occurs where the cubic segment is having to approximate a straight line (the plateau). In this case, the economic optimum can be expected to be at the lower level of nitrogen among the two solutions to equation (5).

For the corn data from the PRNT project, the first knot of the spline plateau model was set at 80 lb/acre, and second knots were set at 150 lb/acre, 200 lb/acre, 250 lb/acre, and the quadratic plateau nitrogen rate, 185 lb/acre, respectively. Among the models tested, the quadratic plateau model yielded the lowest MSE of 1590.8. In contrast, the linear plateau model had the highest MSE of 1604.1. Within the spline plateau models, the configuration with the second knot at 200 lb/acre achieved an MSE of 1593.3, outperforming the other three spline plateau models. The linear plateau model estimated a plateau nitrogen rate of 111.2 bu/acre, whereas the quadratic plateau model estimated a higher rate of 185 bu/acre. As the value of the second knot increased in the spline plateau models, a pattern emerged where higher knot values corresponded to

Table 1. Estimation Results of Deterministic Models: Data from the Performance and Refinement of Nitrogen Fertilization Tools (PRNT) Project.

	Linear Plateau Model	Quadratic Plateau Model	Quadratic Model	Spline Plateau Model Knot 2 at 150	Spline Plateau Model Knot 2 at 200	Spline Plateau Model Knot 2 at 250	Spline Plateau Model Knot 2 at Quadratic Plateau N Rate
β_0	121.70 (2.61)	121.10 (2.57)	125.20 (2.31)	122.82 (2.57)	123.50 (2.47)	124.44 (2.37)	123.25 (2.50)
β_1	0.76 (0.04)	0.95 (0.05)	0.81 (0.03)	0.72 (0.04)	0.70 (0.03)	0.68 (0.23)	0.71 (0.04)
β_2		-3.0×10^{-3} (2.0×10^{-4})	-2.0×10^{-3} (1.0×10^{-4})				
β_6				207.68 (0.88)	208.63 (0.99)	208.63 (1.15)	208.40 (0.95)
Plateau nitrogen rate	111.20 (4.12)	185.00		150.00	200.00	250.00	185.00
Optimal nitrogen rate	111.20 (103.16, 119.31)	165.51 (153.73, 177.29)	190.08 (183.64, 196.52)	141.89 (138.11, 145.68)	156.56 (143.86, 169.24)	163.20 (154.06, 172.37)	154.67 (142.39, 166.96)
MSE	1604.10	1590.80	1598.70	1596.00	1593.30	1594.10	1593.60

Notes: Numbers in parentheses are standard errors. For the optimal nitrogen rate, the numbers in parentheses represent the 95% confidence interval.

Table 2. Estimation Results of Deterministic Models: Data from the Maximum Return to N (MRTN) Project.

	Linear Plateau Model	Quadratic Plateau Model	Quadratic Model	Spline Plateau Model Knot 2 at 150	Spline Plateau Model Knot 2 at 200	Spline Plateau Model Knot 2 at 250	Spline Plateau Model Knot 2 at Quadratic Plateau N Rate
β_0	127.80 (2.42)	125.00 (2.54)	127.82 (2.31)	127.30 (2.52)	127.48 (2.43)	127.86 (2.36)	127.55 (2.41)
β_1	0.77 (0.03)	1.00 (0.05)	0.91 (0.04)	0.78 (0.04)	0.78 (0.04)	0.76 (0.03)	0.78 (0.04)
β_2		-2.0×10^{-3} (2.0×10^{-4})	-2.0×10^{-3} (1.0×10^{-4})				
β_6				227.30 (1.21)	229.59 (1.38)	230.60 (1.70)	229.89 (1.42)
Plateau nitrogen rate	129.60 (4.04)	210.39		150.00	200.00	250.00	210.20
Optimal nitrogen rate	129.60 (121.32, 136.68)	189.46 (174.42, 204.51)	207.98 (197.90, 218.07)	145.60 (144.19, 147.00)	157.18 (145.42, 168.93)	183.39 (168.40, 198.39)	179.51 (166.84, 192.17)
MSE	1028.90	1015.06	1022.67	1025.87	1016.22	1014.93	1015.60

Notes: Numbers in parentheses are standard errors. For the optimal nitrogen rate, the numbers in parentheses represent the 95% confidence interval.

Table 3. Estimation Results of Deterministic Models: Data from Experiment 502.

	Linear Plateau Model	Quadratic Plateau Model	Quadratic Model	Spline Plateau Model Knot 2 at 80	Spline Plateau Model Knot 2 at 90	Spline Plateau Model Knot 2 at 100	Spline Plateau Model Knot 2 at Quadratic Plateau N Rate
β_0	26.81 (0.78)	26.00 (0.85)	26.00 (0.86)	26.41 (0.91)	26.30 (0.90)	26.20 (0.89)	26.21 (0.89)
β_1	0.27 (0.02)	0.40 (0.04)	0.40 (0.04)	0.30 (0.05)	0.32 (0.04)	0.33 (0.04)	0.33 (0.04)
β_2		-2.0×10^{-3} (4.0×10^{-4})	-2.0×10^{-3} (4.0×10^{-4})				
β_6				44.98 (0.58)	45.38 (0.65)	45.66 (0.72)	45.63 (0.71)
Plateau nitrogen rate	67.49 (3.64)	98.66		80.00	90.00	100.00	98.73
Optimal nitrogen rate	67.49 (60.37, 74.62)	76.48 (65.51, 87.45)	76.45 (65.36, 87.53)	71.02 (66.83, 75.21)	75.16 (67.58, 82.74)	76.62 (65.03, 88.21)	76.59 (65.49, 87.70)
MSE	170.80	170.70	170.70	170.80	170.70	170.70	170.70

Notes: Numbers in parentheses are standard errors. For the optimal nitrogen rate, the numbers in parentheses represent the 95% confidence interval.

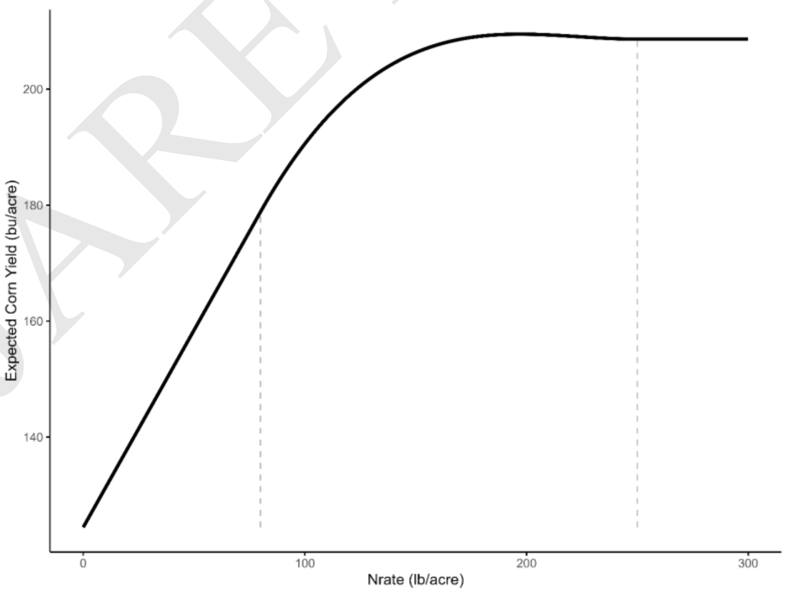

Figure 3. Example of a Fitted Spline Plateau Model That Failed the Concavity Check (Knots at 80 and 250 lb/acre, PRNT Corn Data).

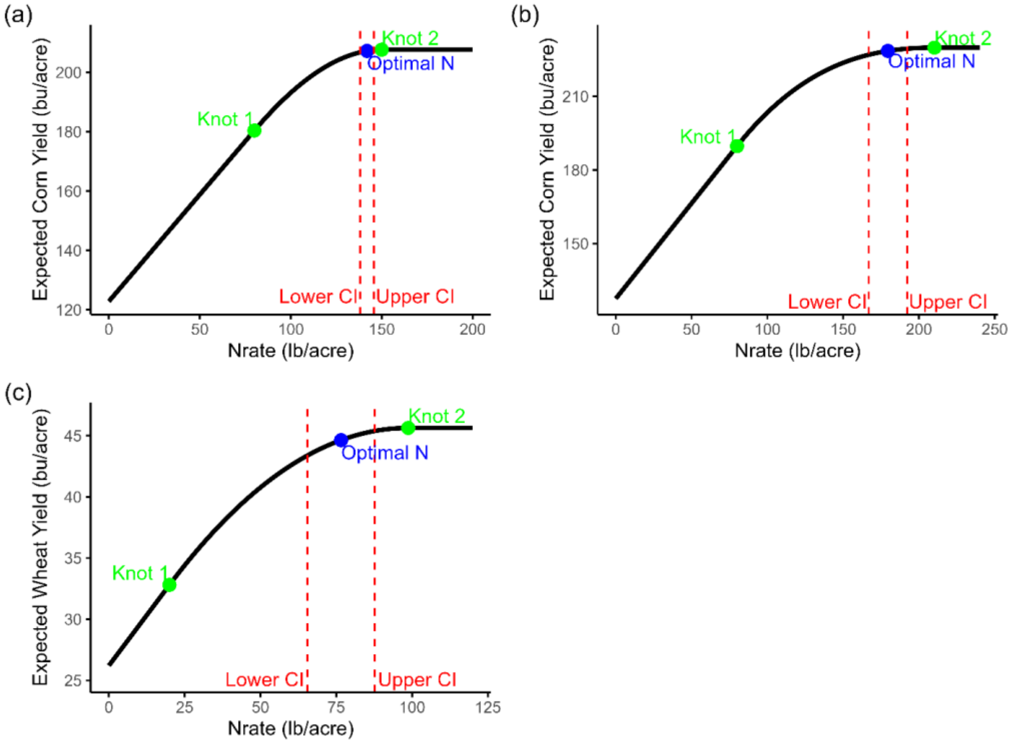

Figure 4. Fitted Spline Models of the Three Datasets with Knot 2 at the Quadratic Plateau Nitrogen Rates.

progressively higher plateau yields (β_6)—207.7 bu/acre, 208.4 bu/acre, 208.6 bu/acre, and 208.6 bu/acre for the knots at 150 lb/acre, 185 lb/acre, 200 lb/acre, and 250 lb/acre, respectively. Similarly, the optimal nitrogen rates also increased with higher second knot values, from 141.9 lb/acre to 163.2 lb/acre. Correspondingly, the optimal nitrogen rate was 111.2 lb/acre for the linear plateau model, 165.2 lb/acre for the quadratic plateau model, and 190.1 lb/acre for the quadratic model.

For the MRTN corn data, the first knot of spline plateau model was 80 lb/acre and the second knots were 150 lb/acre, 200 lb/acre, 250 lb/acre, and the quadratic plateau nitrogen rate, 210.2 lb/acre. The quadratic plateau model emerged as the most accurate, with a MSE of 1015.1. In contrast, the multi-degree spline plateau models had slightly higher MSEs of 1025.9, 1016.2, 1014.9, and 1015.6 respectively, yet still performed better than both the linear plateau model (1028.9) and the quadratic model (1022.7). The spline plateau models and quadratic plateau models demonstrated a superior fit compared to the linear plateau model, effectively capturing the deceleration in corn yield growth as the nitrogen rate increases. The linear plateau model suggested the lowest plateau nitrogen rate of 129.6 lb/acre, whereas the quadratic plateau model indicated a higher rate of 210.4 lb/acre. The spline plateau models proposed optimal rates of 145.6 lb/acre, 157.2 lb/acre, 183.4 lb/acre, and 179.5 lb/acre, depending on the placement of the second knot. The quadratic plateau model suggested an optimal rate at 189.5 lb/acre, while the quadratic model had an optimal nitrogen rate at 208.0 lb/acre.

As for the wheat data from Experiment 502, the first knot for the spline plateau models was 20 lb/acre, and the second knots were at 80 lb/acre, 90 lb/acre, 100 lb/acre, and the quadratic plateau nitrogen rate, 98.7 lb/acre, respectively. All models obtained similar MSE, linear plateau model and spline plateau model with knot 2 at 80 lb/acre got slightly higher MSE (170.80) compared to other five models (170.70). It is worth noting that quadratic plateau model and quadratic model almost got the same results. The highest nitrogen application rate was 100 lb/acre in the dataset, and the plateau nitrogen rate was 98.7 lb/acre for the quadratic plateau model. Therefore, the quadratic plateau model and quadratic model got the same optimal nitrogen rate (76.5 lb/acre), which is higher than the linear plateau model (67.5 lb/acre) and spline plateau models (71.0 lb/acre, 75.2 lb/acre, 76.6 lb/acre, and 76.6 lb/acre). Figure 4 shows the fitted spline plateau models for all three datasets, using the quadratic plateau nitrogen rate as the second knot. In all three datasets, the optimal nitrogen rate and its 95% confidence interval lie entirely between knot 1 and knot 2.

Site-Year Models

The estimation results for the site-year specific models are reported in Tables 4, 5, and 6. The best-fitting model under the site-year specific estimation approach was also the quadratic plateau model. For the corn data from the PRNT project, quadratic plateau model had the lowest average SSE (18747). All four models converged for 49 site-years, and no model had a plateau nitrogen rate above the highest treatment level (280 lb/acre).

For the corn data from the MRTN project, quadratic plateau models were the best-fitting models with the average SSE at 111. For a few site-years linear plateau models and quadratic plateau models did not converge. Since nitrogen application rates in the MRTN data varied in each site-year, plateau nitrogen rates above 300 lb/acre were considered as above the highest treatment level. Although the quadratic plateau model had the best fit, it had two site-years that did not converge and nine site-years with plateau nitrogen rate higher than 300 lb/acre. The optimal nitrogen rate did not differ much from the deterministic models.

For the wheat data from Experiment 502, the quadratic plateau model fit the best, but it also had the highest number (6) of site-years that did not converge and highest number (16) of site-years with plateau nitrogen rate higher than the highest treatment level.

INLA Model

The spline plateau model was tested in R-INLA, which employs the inverse Laplacian approximation, using simulated data. To match how we envision the model being used in the precision agriculture case, the two knots were selected by first estimating a non-spatial model using all of the simulated data, treating the switch points as unknown parameters. These switch points were then used in the R-INLA model. Figure 5 presents the simulated and estimated plateau values (β_6) obtained by R-INLA, which closely align with the actual values. Few studies in precision agriculture reported computational time. Patterson (2023) required over a week to estimate a spatial varying linear plateau model with 1,056 observations using Bayesian Kriging. Similarly, Poursina and Brorsen (2025) reported runtimes ranging from 25 to 268 hours for estimating spatial varying linear plateau models using Bayesian Kriging, depending on the spatial covariance structure, using a grid with 486 observations for a single year. In contrast, the spline plateau model estimated with R-INLA presented in Figure 5 was estimated in just 18 seconds with a grid-size of 100 (1,000 observations, since ten years were simulated). The speed and accuracy of INLA make it a viable alternative to MCMC based and machine learning methods for estimating nitrogen response models in precision agriculture.

Discussion

The results of the deterministic models indicated that the spline plateau model has the potential to perform as effectively as linear plateau model and quadratic plateau model when the knot selection was appropriate. The spline plateau model had a poor fit in site-year specific estimation, one

Table 4. Estimation Results of Site-Year Specific Models: Data from the Performance and Refinement of Nitrogen Fertilization Tools (PRNT) Project.

	Linear Plateau	Quadratic Plateau	Quadratic	Spline Plateau Model Knot 2 at 150	Spline Plateau Model Knot 2 at 200	Spline Plateau Model Knot 2 at 250
β_0	121.56 (44.52)	118.90 (45.49)	124.84 (46.18)	122.51 (44.75)	123.21 (45.27)	124.14 (45.79)
β_1	0.76 (0.28)	1.07 (0.38)	0.80 (0.33)	0.72 (0.29)	0.70 (0.28)	0.68 (0.27)
β_2		-4.0×10^{-3} (2.0×10^{-3})	-2.0×10^{-3} (8.0×10^{-4})			
β_6				207.37 (34.34)	208.35 (34.40)	208.37 (34.56)
Plateau nitrogen rate	110.90 (35.48)	170.00 (69.35)		150.00	200.00	250.00
Optimal nitrogen rate	148.00	167.00	190.00	142.00	151.00	164.00
Average SSE	19236	18747	19594	19399	18937	19075
Converged models	49	49	49	49	49	49
Plateau not reached	0	0				

Notes: Number in parentheses are standard deviations.

Table 5. Estimation Results of Site-Year Specific Models: Data from the Maximum Return to N (MRTN) Project.

	Linear Plateau	Quadratic Plateau	Quadratic	Spline Plateau Model Knot 2 at 150	Spline Plateau Model Knot 2 at 200	Spline Plateau Model Knot 2 at 250
β_0	120.96 (39.36)	112.00 (48.03)	121.51 (40.46)	124.09 (42.91)	121.89 (40.57)	122.73 (40.51)
β_1	0.88 (0.34)	1.32 (0.73)	1.03 (0.35)	0.83 (0.41)	0.85 (0.31)	0.83 (0.29)
β_2		-4.0×10^{-3} (4.0×10^{-3})	-2.0×10^{-3} (9.0×10^{-4})			
β_6				227.15 (28.99)	229.12 (29.06)	229.64 (29.15)
Plateau nitrogen rate	124.70 (28.63)	192.40 (56.39)		150	200	250
Optimal nitrogen rate	160	180	193	145	171	177
Average SSE	192	111	204	224	163	169
Converged models	216	217	219	219	219	219
Plateau not reached	0	9				

Notes: Number in parentheses are standard deviations.

Table 6. Estimation Results of Site-Year Specific Models: Data from Experiment 502.

	Linear Plateau	Quadratic Plateau	Quadratic	Spline Plateau Model Knot 2 at 80	Spline Plateau Model Knot 2 at 90	Spline Plateau Model Knot 2 at 100
β_0	25.81 (7.53)	26.38 (7.80)	25.97 (7.92)	26.40 (7.66)	26.28 (7.71)	26.17 (7.78)
β_1	0.54 (0.85)	0.51 (0.32)	0.40 (0.26)	0.30 (0.20)	0.32 (0.21)	0.33 (0.22)
β_2		-5.0×10^{-3} (7.0×10^{-3})	-2.0×10^{-3} (2.0×10^{-3})			
β_6				45.01 (14.91)	45.39 (15.48)	45.66 (16.00)
Plateau nitrogen rate	65.10 (37.62)	256.80 (1121.75)		80	90	100
Optimal nitrogen rate	82	78	77	71	75	76
Average SSE	762.50	708.90	732.00	759.70	740.60	732.00
Converged models	46	41	47	47	47	47
Plateau not reached	10	16				

Notes: Number in parentheses are standard deviations.

reason could be that the data are different for each site-year, and the spline plateau model makes some site-years hit the plateau too early or too late. If different knots had been chosen for each site-year, the spline plateau model may have a better fit. The reason that some of the models fail to converge may be due to poor starting values in model estimation.

For the corn data from the PRNT project, the derived optimal nitrogen rates are slightly higher than the findings of Ransom et al. (2020) using the same prices, who reported average economically optimal nitrogen rates (EONR) of 150.8 lb/acre (169 kg/ha) for at-planting applications and 141.9 lb/acre (159 kg/ha) for split nitrogen applications. One possible reason could be they estimated the optimal nitrogen rates for the two nitrogen application methods separately, whereas in this paper the data were pooled. In contrast, the optimal nitrogen rates from the MRTN project closely matched the rates generated by the N rate calculator (http://cnrc.agron.iastate.edu/), which suggested 180.2 lb/acre for corn following soybean. Similarly, the optimal nitrogen rate for wheat in Experiment 502 was consistent with the findings of Boyer et al. (2012). Throughout the analysis, the quadratic plateau model consistently demonstrated a strong fit across all three datasets, reinforcing its widespread adoption in nitrogen response estimation (Alotaibi et al., 2018; Nafziger and Rapp, 2020; Baum et al., 2023). Note that the plot of a linear stochastic plateau model will look like a quadratic plateau model. The support for the quadratic plateau model may be due to not letting the plateau be stochastic over space (for site-year models) or time (for deterministic models). The INLA estimated plateau values closely aligned with actual values, showing that it can be a viable alternative to MCMC-based and machine learning methods for estimating nitrogen response models in precision agriculture. As Pautsch et al. (1998) argued regarding soil sampling, information uncertainty creates a Bayesian problem in determining variable rate fertilizer rates.

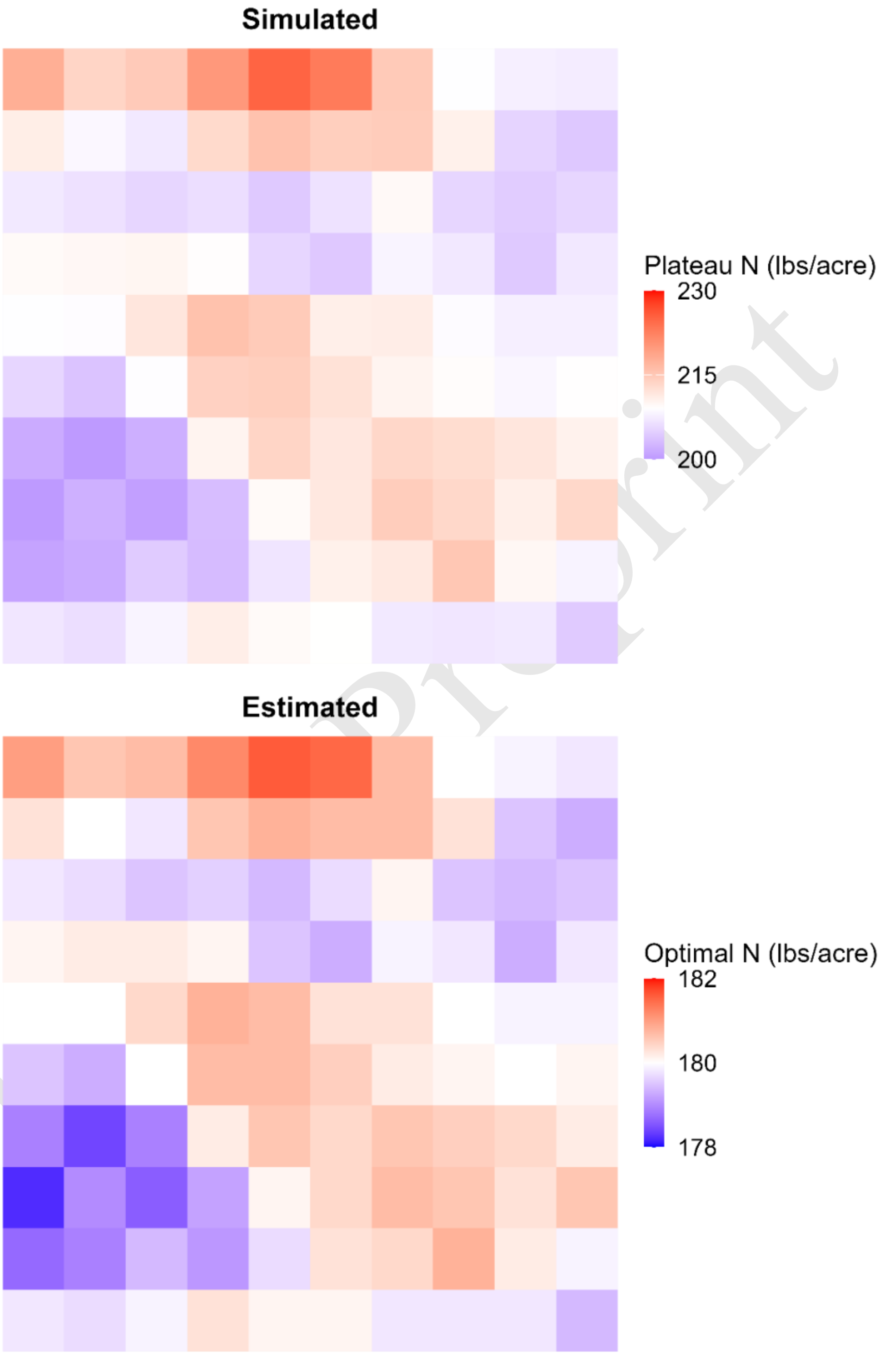

Figure 5. Simulated and Estimated Plateau Value (bu/acre) from the Spline Plateau Model.

Conclusion

The multi-degree spline plateau model achieved similar accuracy to the widely used linear and quadratic plateau models, especially if the knot selection is optimized. The flexibility and interpretability of the spline model make it a compelling alternative, especially given its linearity in parameters, which simplifies estimation and enhances computational efficiency. The concavity evaluation further highlighted the importance of economic plausibility in model selections. Several spline models that offered strong in-sample fit failed to meet concavity requirements and thus lacked interpretability. However, the analysis also reveals limitations in the datasets used—specifically, the E502 dataset—where the highest nitrogen levels may be insufficient to fully reach the plateau.

The INLA results confirmed that it can efficiently estimate plateau values while maintaining strong accuracy with simulated data. The fast computational speed of INLA makes it a promising tool for spatially varying nitrogen response estimation. Future research will focus on incorporating spatial random effects into the spline plateau model to better capture field-level variability and further assess its computational efficiency and model fit. A critical component of model refinement will be optimizing knot selection, as knot placement significantly influences predictive accuracy. Since the primary goal is to predict future outcomes and optimize input recommendations, pretesting is not only acceptable but also essential to ensure robust and reliable model performance in real-world applications.

[First submitted March 2025; accepted for publication September 2025.]

References

Ahmed, M., M. Rauf, Z. Mukhtar, and N.A. Saeed. 2017. “Excessive Use of Nitrogenous Fertilizers: An Unawareness Causing Serious Threats to Environment and Human Health.” *Environmental Science and Pollution Research* 24(35):26983–26987. doi: 10.1007/s11356-017-0589-7

Alotaibi, K.D., A.N. Cambouris, M. St. Luce, N. Ziadi, and N. Tremblay. 2018. “Economic Optimum Nitrogen Fertilizer Rate and Residual Soil Nitrate as Influenced by Soil Texture in Corn Production.” *Agronomy Journal* 110(6): 2233–2242. doi: 10.2134/agronj2017.10.0583

Bäckman, S.T., S. Vermeulen, and V.-M. Taavitsainen. 1997. “Long-Term Fertilizer Field Trials: Comparison of Three Mathematical Response Models.” *Agricultural and Food Science* 6(2):151–160. doi: 10.23986/afsci.72778

Barbosa, A., R. Trevisan, N. Hovakimyan, and N.F. Martin. 2020. “Modeling Yield Response to Crop Management Using Convolutional Neural Networks.” *Computers and Electronics in Agriculture* 170:105197. doi: 10.1016/j.compag.2019.105197

Bashir, M.T., S. Ali, M. Ghauri, A. Adris, and R. Harun. 2013. “Impact of Excessive Nitrogen Fertilizers on the Environment and Associated Mitigation Strategies.” *Asian Journal of Microbiology Biotechnology and Environmental Sciences* 15(2): 213-221. doi: 10.13140/RG.2.1.2754.7606

Baum, M.E., J.E. Sawyer, E.D. Nafziger, I. Huber, P.J. Thorburn, M.J. Castellano, and S.V. Archontoulis. 2023. “Evaluating and Improving APSIM’s Capacity in Simulating Long-Term Corn Yield Response to Nitrogen in Continuous- and Rotated-Corn Systems.” *Agricultural Systems* 207:103629. doi: 10.1016/j.agsy.2023.103629

Beccari, C.V., G. Casciola, and S. Morigi. 2017. “On Multi-Degree Splines.” *Computer Aided Geometric Design* 58: 8–23. doi: 10.1016/j.cagd.2017.10.003.

Berck, P., and G. Helfand. 1990. “Reconciling the von Liebig and Differentiable Crop Production Functions.” *American Journal of Agricultural Economics* 72(4): 985–996. doi: 10.2307/1242630.

Biermacher, J.T., B.W. Brorsen, F.M. Epplin, J.B. Solie, and W.R. Raun. 2009. "The Economic Potential of Precision Nitrogen Application with Wheat Based on Plant Sensing." *Agricultural Economics* 40(4):397–407. doi: 10.1111/j.1574-0862.2009.00387.x

Boyer, C.N., B.W. Brorsen, W.R. Raun, D.B. Arnall, and J.B. Solie. 2012. "Efficiency of Pre-Plant, Topdress, and Variable Rate Application of Nitrogen in Winter Wheat." *Journal of Plant Nutrition* 35(12): 1776–1790. doi: 10.1080/01904167.2012.706675.

Brorsen, B.W., and F.G.-C. Richter. 2012. "Experimental Designs for Estimating Plateau-Type Production Functions and Economically Optimal Input Levels." *Journal of Productivity Analysis* 38(1):45–52. doi: 10.1007/s11123-010-0204-0

Bullock, D.G., and D.S. Bullock. 1994. "Quadratic and Quadratic-Plus-Plateau Models for Predicting Optimal Nitrogen Rate of Corn: A Comparison." *Agronomy Journal* 86(1): 191–195. doi: 10.2134/agronj1994.00021962008600010033x.

Bullock, D.S., M. Boerngen, H. Tao, B. Maxwell, J.D. Luck, L. Shiratsuchi, L. Puntel, and N.F. Martin. 2019. "The Data-Intensive Farm Management Project: Changing Agronomic Research Through On-Farm Precision Experimentation." *Agronomy Journal* 111(6): 2736–2746. doi: 10.2134/agronj2019.03.0165.

Bullock, D.S., T. Mieno, and J. Hwang. 2020. "The Value of Conducting On-Farm Field Trials Using Precision Agriculture Technology: A Theory and Simulations." *Precision Agriculture* 21(5):1027–1044. doi: 10.1007/s11119-019-09706-1

Cate, R.B., and L.A. Nelson. 1971. "A Simple Statistical Procedure for Partitioning Soil Test Correlation Data Into Two Classes." *Soil Science Society of America Journal* 35(4): 658–660. doi: 10.2136/sssaj1971.03615995003500040048x.

Cho, W., D.M. Lambert, A. Fornah, and W.R. Raun. 2020. "Bayesian Estimation and Economic Analysis of Under-Replicated Field Trials With a Linear Response Plateau Function." *Journal of Agricultural Science* 12(10): 1. doi: 10.5539/jas.v12n10p1.

Coleman, E., B.W. Brorsen, and T. Mieno. 2023. "Optimal Nitrogen Rates: Generalized Additive Model with Spatially Varying Coefficients Using INLA. " Working Paper. Oklahoma State University.

Gauthier, J., Q.V. Wu, and T.A. Gooley. 2020. "Cubic Splines to Model Relationships Between Continuous Variables and Outcomes: A Guide for Clinicians." *Bone Marrow Transplantation* 55(4):675–680. doi: 10.1038/s41409-019-0679-x

Harmon, X., C.N. Boyer, D.M. Lambert, J.A. Larson, and C.O. Gwathmey. 2016. "Comparing the Value of Soil Test Information Using Deterministic and Stochastic Yield Response Plateau Functions." *Journal of Agricultural and Resource Economics* 41(2):307-323. doi: 10.22004/ag.econ.235192

Hegedus, P.B., B.D. Maxwell, and T. Mieno. 2023. "Assessing Performance of Empirical Models for Forecasting Crop Responses to Variable Fertilizer Rates Using On-Farm Precision Experimentation." *Precision Agriculture* 24(2):677–704. doi: 10.1007/s11119-022-09968-2

Hegedus, P.B., S.A. Ewing, C. Jones, and B.D. Maxwell. 2023. "Using Spatially Variable Nitrogen Application and Crop Responses to Evaluate Crop Nitrogen Use Efficiency." *Nutrient Cycling in Agroecosystems* 126(1):1–20. doi: 10.1007/s10705-023-10263-3

Lambert, D.M., and W. Cho. 2022. "Geographically Weighted Regression Estimation of the Linear Response and Plateau Function." *Precision Agriculture* 23(2):377–399. doi: 10.1007/s11119-021-09841-8

Largo, R.H., Th. Gasser, A. Prader, W. Stuetzle, and P.J. Huber. 1978. "Analysis of the Adolescent Growth Spurt Using Smoothing Spline Functions." *Annals of Human Biology* 5(5):421–434. doi: 10.1080/03014467800003071

Lindgren, F., and H. Rue. 2015. "Bayesian Spatial Modelling with *R* - **INLA**." *Journal of Statistical Software* 63(19). doi: 10.18637/jss.v063.i19.

Makowski, D., D. Wallach, and J. Meynard. 1999. “Models of Yield, Grain Protein, and Residual Mineral Nitrogen Responses to Applied Nitrogen for Winter Wheat.” *Agronomy Journal* 91(3): 377–385. doi: 10.2134/agronj1999.00021962009100030005x.

Mieno, T., X. Li, and D.S. Bullock. 2024. “Bias in Economic Evaluation of Variable Rate Application Based on Geographically Weighted Regression Models with Misspecified Functional Form.” *Journal of the Agricultural and Applied Economics Association* 3(1): 135–151. doi: 10.1002/jaa2.102.

Monod, H., D. Makowski, M. Sahmoudi, and D. Wallach. 2002. “Optimal Experimental Designs for Estimating Model Parameters, Applied to Yield Response to Nitrogen Models.” *Agronomie* 22(2):229–238. doi: 10.1051/agro:2002005

Morales, G., J. Sheppard, A. Peerlinck, and P. Hegedus. 2022. “Generation of Site-Specific Nitrogen Response Curves for Winter Wheat Using Deep Learning.” In Proceedings of the 15th International Conference on Precision Agriculture (unpaginated, online). Monticello, IL: International Society of Precision Agriculture.

Nafziger, E. and D. Rapp. 2021. “Corn Yield Response to Late-Split Nitrogen Fertilizer.” *Agronomy Journal* 113:527–536. doi: 10.1002/agj2.20472

Nafziger, E., J. Sawyer, C. Laboski, and D. Franzen. 2022. “The MRTN Approach to Making Nitrogen Rate Recommendations: Background and Implementation.” *Crops & Soils* 55(2): 4–11. doi: 10.1002/crso.20180.

Ouedraogo, F., and B.W. Brorsen. 2018. “Hierarchical Bayesian Estimation of a Stochastic Plateau Response Function: Determining Optimal Levels of Nitrogen Fertilization.” *Canadian Journal of Agricultural Economics* 66(1):87–102. doi: 10.1111/cjag.12139

Paccioretti, P, C. Bruno, F. G. Kurina, M. Córdoba, M A, D. S. Bullock, and M Balzarini. 2021. “Statistical Models of Yield in on-Farm Precision Experimentation.” *Agronomy Journal* 1134916–4929. doi: 10.1002/agj2.20833

Panyi, A.F., and B.W. Brorsen. 2024. “Trends in the Yield Response to Nitrogen of Winter Wheat in Oklahoma.” *Agricultural & Environmental Letters* 9(2):e20143. doi: 10.1002/ael2.20143

Park, E., B.W. Brorsen, and X. Li. 2024. “Using Data from Uniform Rate Applications for Site-Specific Nitrogen Recommendations.” *Journal of Agricultural and Applied Economics* 56(1): 138–154. doi: 10.1017/aae.2023.44.

Patterson, G.C. 2023. “Using Informative Bayesian Priors and on-Farm Experimentation to Predict Optimal Site-Specific Nitrogen Rates.” Master's thesis. Stillwater, OK: Oklahoma State University.

Pautsch, G.R., B.A. Babcock, and F.J. Breidt. 1999. “Optimal Information Acquisition Under a Geostatistical Model.” *Journal of Agricultural and Resource Economics* 24(2):342-366. doi: 10.22004/ag.econ.30797

Peerlinck, A., G. Morales, J. Sheppard, P. Hegedus, and B. Maxwell. 2022. “Optimizing Nitrogen Application to Maximize Yield and Reduce Environmental Impact in Winter Wheat Production.” In Proceedings of the 15th International Conference on Precision Agriculture (unpaginated, online). Monticello, IL: International Society of Precision Agriculture.

Perperoglou, A., W. Sauerbrei, M. Abrahamowicz, and M. Schmid. 2019. “A review of Spline Function Procedures in R.” *BMC Medical Research Methodology* 19(1):46. doi: 10.1186/s12874-019-0666-3

Poursina, D., and B.W. Brorsen. 2025. “Site-Specific Nitrogen Recommendation: Fast, Accurate, and Feasible Bayesian Kriging.” *Computational Statistics* 40(2): 1053–1069. doi: 10.1007/s00180-024-01527-9.

Ransom, C.J., N.R. Kitchen, J.J. Camberato, P.R. Carter, R.B. Ferguson, F.G. Fernández, D.W. Franzen, C.A.M. Laboski, E.D. Nafziger, J.E. Sawyer, P.C. Scharf, and J.F. Shanahan. 2020. “Corn Nitrogen Rate Recommendation Tools’ Performance Across Eight US Midwest Corn Belt States.” *Agronomy Journal* 112(1):470–492. doi: 10.1002/agj2.20035

Ransom, C.J., J. Clark, G.M. Bean, C. Bandura, M.E. Shafer, N.R. Kitchen, J.J. Camberato, P.R. Carter, R.B. Ferguson, F. Fernández, D.W. Franzen, C.A.M. Laboski, D.B. Myers, E.D. Nafziger, J.E. Sawyer, and J. Shanahan. 2021. "Data from a Public–Industry Partnership for Enhancing Corn Nitrogen Research." *Agronomy Journal* 113(5):4429–4436. doi: 10.1002/agj2.20812

Rue, H., S. Martino, and N. Chopin. 2009. "Approximate Bayesian Inference for Latent Gaussian Models by Using Integrated Nested Laplace Approximations." *Journal of the Royal Statistical Society Series B: Statistical Methodology* 71(2): 319–392. doi: 10.1111/j.1467-9868.2008.00700.x.

Schuster, N.A., J.J.M. Rijnhart, J.W.R. Twisk, and M.W. Heymans. 2022. "Modeling Non-Linear Relationships in Epidemiological Data: The Application and Interpretation of Spline Models." *Frontiers in Epidemiology* 2:975380. doi: 10.3389/fepid.2022.975380

Sellars, S.C., G.D. Schnitkey, and L.F. Gentry. 2020. "Do Illinois Farmers Follow University-Based Nitrogen Recommendations?" Selected paper. In: *Agricultural and Applied Economics Association Annual Meeting*, Kansas City, Mo.

Shen, W., and G. Wang. 2010. "Changeable Degree Spline Basis Functions." *Journal of Computational and Applied Mathematics* 234(8):2516–2529. doi: 10.1016/j.cam.2010.03.015

Tanaka, T.S.T., G.B.M. Heuvelink, T. Mieno, and D.S. Bullock. 2024. "Can Machine Learning Models Provide Accurate Fertilizer Recommendations?" *Precision Agriculture* 25(4): 1839–1856. doi: 10.1007/s11119-024-10136-x.

Tembo, G., B.W. Brorsen, F.M. Epplin, and E. Tostão. 2008. "Crop Input Response Functions with Stochastic Plateaus." *American Journal of Agricultural Economics* 90(2): 424–434. doi: 10.1111/j.1467-8276.2007.01123.x.

Toshniwal, D., H. Speleers, R.R. Hiemstra, C. Manni, and T.J.R. Hughes. 2020. "Multi-Degree B-Splines: Algorithmic Computation and Properties." *Computer Aided Geometric Design* 76:101792. doi: 10.1016/j.cagd.2019.101792

Trevisan, R.G., D.S. Bullock, and N.F. Martin. 2021. "Spatial Variability of Crop Responses to Agronomic Inputs in On-Farm Precision Experimentation." *Precision Agriculture* 22(2):342–363. doi: 10.1007/s11119-020-09720-8

Tumusiime, E., B. W. Brorsen, J. Mosali, J. Johnson, J. Locke, and J.T. Biermacher. 2011. "Determining Optimal Levels of Nitrogen Fertilizer Using Random Parameter Models." *Journal of Agricultural and Applied Economics* 43(4):541–552. doi: 10.1017/S1074070800000067

Zhang, C., X. Li, T. Mieno, C. Diao, and D. S. Bullock. "Quadratic-Plateau Geographically Weighted Regression Model for Estimating Site-Specific Economically Optimal Input Rates." *Computers and Electronics in Agriculture* 237 (2025): 110655 doi: 10.1016/j.compag.2025.110655

Appendix 1.

The three-degree spline plateau model can be represented in dummy variable form as

(A1) $$y_{it} = D_{1it}(\beta_0 + \beta_1 N_{it}) + D_{2it}(\beta_2 + \beta_3 N_{it} + \beta_4 N_{it}^2 + \beta_5 N_{it}^3) + D_{3it}\beta_6 + \varepsilon_{it}$$

where y_{it} is the corn yield of observation i in year t, N_{it} is the nitrogen application rate of observation i in year t, D_{1it}, D_{2it}, and D_{3it} are dummy variables, D_{1it} equals 1 when the nitrogen rate N_{it} is less than or equal to N_1^*, D_{2it} equals 1 when the nitrogen rate N_{it} is greater than N_1^* and less than or equal to N_2^*, D_{3it} equals 1 when the nitrogen rate N_{it} is greater than N_2^*, and ε_{it} is a random error term that is normally distributed. The first part of the model (when $N_{it} \le N_1^*$) is linear

(A2) $$y_{it} = \beta_0 + \beta_1 N_{it} + \varepsilon_{it}$$

The second part of the model (when $N_1^* < N_{it} \le N_2^*$) is cubic

(A3) $$y_{it} = \beta_2 + \beta_3 N_{it} + \beta_4 N_{it}^2 + \beta_5 N_{it}^3 + \varepsilon_{it}$$

And the third part (when $N_{it} > N_2^*$) is the plateau:

(A4) $$y_{it} = \beta_6 + \varepsilon_{it}$$

This model has five constraints, two constraints to make sure the model is continuous at the two knots N_1^* and N_2^*:

(A5) $$\beta_0 + \beta_1 N_1^* = \beta_2 + \beta_3 N_1^* + \beta_4 {N_1^*}^2 + \beta_5 {N_1^*}^3$$

(A6) $$\beta_2 + \beta_3 N_2^* + \beta_4 {N_2^*}^2 + \beta_5 {N_2^*}^3 = \beta_6$$

two constraints to make sure the model is differentiable at the two knots:

(A7) $$\beta_1 = \beta_3 + 2\beta_4 N_1^* + 3\beta_5 {N_1^*}^2$$

(A8) $$\beta_3 + 2\beta_4 N_2^* + 3\beta_5 {N_2^*}^2 = 0$$

Given the four constraints, β_2, β_3, β_4, and β_5 can be represented in terms of β_0, β_1 and β_6. From equation (A6), β_2 can be represented byβ_3, β_4, β_5, and β_6 as follows:

(A9) $$\beta_2 = \beta_6 - \beta_3 N_2^* - \beta_4 {N_2^*}^2 - \beta_5 {N_2^*}^3$$

Substituting equation (A9) into equation (A5):

(A10) $$\beta_0 + \beta_1 N_1^* = \beta_6 - \beta_3 N_2^* - \beta_4 {N_2^*}^2 - \beta_5 {N_2^*}^3 + \beta_3 N_1^* + \beta_4 {N_1^*}^2 + \beta_5 {N_1^*}^3$$

From equation (A8), β_3 can be represented by β_4 and β_5 as follows:

(A11) $$\beta_3 = -2\beta_4 N_2^* - 3\beta_5 {N_2^*}^2$$

From equation (A7), β_4 can be represented by β_1, β_3, and β_5 as follows:

(A12) $$\beta_4 = \frac{\beta_1 - \beta_3 - 3\beta_5 {N_1^*}^2}{2N_1^*}$$

Substituting equation (A12) into equation (A11):

$$\beta_4 = \frac{\beta_1 + 2\beta_4 N_2^* + 3\beta_5 {N_2^*}^2 - 3\beta_5 {N_1^*}^2}{2N_1^*}$$

(A13)
$$\beta_4 = \frac{\beta_1 + 3\beta_5 {N_2^*}^2 - 3\beta_5 {N_1^*}^2}{2N_1^* - 2N_2^*}$$

Substituting equation (A11) into equation (A6):

$$\beta_0 + \beta_1 N_1^* = \beta_6 - (-2\beta_4 N_2^* - 3\beta_5 {N_2^*}^2)N_2^* - \beta_4 {N_2^*}^2 - \beta_5 {N_2^*}^3 + (-2\beta_4 N_2^* - 3\beta_5 {N_2^*}^2)N_1^* + \beta_4 {N_1^*}^2 + \beta_5 {N_1^*}^3$$

$$\beta_6 = \beta_0 + \beta_1 N_1^* + \beta_4\left(2N_2^* N_1^* - {N_1^*}^2 - {N_2^*}^2\right) + \beta_5(3{N_2^*}^2 N_1^* - {N_1^*}^3 - 2{N_2^*}^3)$$

(A14)
$$\beta_6 = \beta_0 + \beta_1 N_1^* + \beta_4\left(2N_2^* N_1^* - {N_1^*}^2 - {N_2^*}^2\right) + \beta_5(3{N_2^*}^2 N_1^* - {N_1^*}^3 - 2{N_2^*}^3)$$

Substituting equation (A13) into equation (A14):

$$\beta_6 = \beta_0 + \beta_1 N_1^* + \frac{\beta_1 + 3\beta_5 {N_2^*}^2 - 3\beta_5 {N_1^*}^2}{2N_1^* - 2N_2^*}\left(2N_2^* N_1^* - {N_1^*}^2 - {N_2^*}^2\right) + \beta_5(3{N_2^*}^2 N_1^* - {N_1^*}^3 - 2{N_2^*}^3)$$

$$\beta_6 = \beta_0 + \beta_1 N_1^* - \frac{1}{2}\beta_1(N_1^* - N_2^*) + \beta_5(\frac{1}{2}{N_1^*}^3 - \frac{1}{2}{N_2^*}^3 + \frac{3}{2}{N_2^*}^2 N_1^* - \frac{3}{2}{N_1^*}^2 N_2^*)$$

$$\beta_5 = \frac{2\beta_6 - 2\beta_0 - \beta_1 N_1^* - \beta_1 N_2^*}{{N_1^*}^3 - {N_2^*}^3 + 3{N_2^*}^2 N_1^* - 3{N_1^*}^2 N_2^*}$$

(A15)
$$\beta_5 = \frac{2\beta_6 - 2\beta_0 - \beta_1(N_1^* + N_2^*)}{(N_1^* - N_2^*)^3}$$

Substituting equation (A15) to equation (A13):

$$\beta_4 = \frac{\beta_1 + 3({N_2^*}^2 - {N_1^*}^2)\beta_5}{2N_1^* - 2N_2^*}$$

$$\beta_4 = \frac{\beta_1}{2N_1^* - 2N_2^*} + \frac{3({N_2^*}^2 - {N_1^*}^2)\beta_5}{2N_1^* - 2N_2^*}$$

$$\beta_4 = \frac{\beta_1}{2N_1^* - 2N_2^*} + \frac{-3(N_1^* - N_2^*)(N_1^* + N_2^*)\beta_5}{2N_1^* - 2N_2^*}$$

$$\beta_4 = \frac{\beta_1}{2N_1^* - 2N_2^*} + \frac{-3(N_1^* + N_2^*)}{2}\frac{2\beta_6 - 2\beta_0 - \beta_1 N_1^* - \beta_1 N_2^*}{(N_1^* - N_2^*)^3}$$

$$\beta_4 = \frac{\frac{1}{2}\beta_1(N_1^* - N_2^*)^2 - \frac{3}{2}(N_1^* + N_2^*)(2\beta_6 - 2\beta_0 - \beta_1 N_1^* - \beta_1 N_2^*)}{(N_1^* - N_2^*)^3}$$

(A16)
$$\beta_4 = \frac{3(\beta_0 - \beta_6)(N_1^* + N_2^*) + 2\beta_1({N_1^*}^2 + {N_2^*}^2 + N_1^* N_2^*)}{(N_1^* - N_2^*)^3}$$

Substituting equation (A15) and equation (A16) into equation (A11):

$$\beta_3 = -2\beta_4 N_2^* - 3\beta_5 {N_2^*}^2$$

$$\beta_3 = -2N_2^* \frac{3(\beta_0 - \beta_6)(N_1^* + N_2^*) + 2\beta_1\left({N_1^*}^2 + {N_2^*}^2 + N_1^* N_2^*\right)}{(N_1^* - N_2^*)^3} - 3{N_2^*}^2 \frac{2\beta_6 - 2\beta_0 - \beta_1(N_1^* + N_2^*)}{(N_1^* - N_2^*)^3}$$

$$\beta_3 = \frac{-6\beta_0 N_1^* N_2^* - 6\beta_0 {N_2^*}^2 + 6\beta_6 N_1^* N_2^* + 6\beta_6 {N_2^*}^2 - 4\beta_1 {N_1^*}^2 N_2^* - 4\beta_1 {N_2^*}^3}{(N_1^* - N_2^*)^3} + \frac{-4\beta_1 {N_2^*}^2 N_1^* - 6\beta_6 {N_2^*}^2 + 6\beta_0 {N_2^*}^2 + 3\beta_1 {N_2^*}^2 N_1^* + 3\beta_1 {N_2^*}^3}{(N_1^* - N_2^*)^3}$$

(A17) $$\beta_3 = \frac{6N_1^* N_2^*(\beta_6 - \beta_0) + (-4{N_1^*}^2 N_2^* - {N_2^*}^3 - {N_2^*}^2 N_1^*)\beta_1}{(N_1^* - N_2^*)^3}$$

Substituting equation (A15), equation (A16), and equation (A17) into equation (A9):

$$\beta_2 = \beta_6 - \beta_3 N_2^* - \beta_4 {N_2^*}^2 - \beta_5 {N_2^*}^3$$

$$\beta_2 = \beta_6 - N_2^* \frac{6N_1^* N_2^*(\beta_6 - \beta_0) + \left(-4{N_1^*}^2 N_2^* - {N_2^*}^3 - {N_2^*}^2 N_1^*\right)\beta_1}{(N_1^* - N_2^*)^3} - {N_2^*}^2 \frac{3(\beta_0 - \beta_6)(N_1^* + N_2^*) + 2\beta_1\left({N_1^*}^2 + {N_2^*}^2 + N_1^* N_2^*\right)}{(N_1^* - N_2^*)^3} - {N_2^*}^3 \frac{2\beta_6 - 2\beta_0 - \beta_1(N_1^* + N_2^*)}{(N_1^* - N_2^*)^3}$$

$$\beta_2 = \beta_6 \frac{{N_1^*}^3 - 3{N_1^*}^2 N_2^* + 3N_1^* {N_2^*}^2 - {N_2^*}^3 - 6N_1^* {N_2^*}^2 + 3N_1^* {N_2^*}^2 + 3{N_2^*}^3 - 2{N_2^*}^3}{(N_1^* - N_2^*)^3} + \beta_0 \frac{6N_1^* {N_2^*}^2 - 3N_1^* {N_2^*}^2 - 3{N_2^*}^3 + 2{N_2^*}^3}{(N_1^* - N_2^*)^3} + \beta_1 \frac{\left(4{N_1^*}^2 N_2^* + {N_2^*}^3 + {N_2^*}^2 N_1^*\right)N_2^* - 2\left({N_1^*}^2 + {N_2^*}^2 + N_1^* N_2^*\right){N_2^*}^2 + (N_1^* + N_2^*){N_2^*}^3}{(N_1^* - N_2^*)^3}$$

(A18) $$\beta_2 = \frac{({N_1^*}^3 - 3{N_1^*}^2 N_2^*)\beta_6 + (3N_1^* {N_2^*}^2 - {N_2^*}^3)\beta_0 + 2{N_1^*}^2 {N_2^*}^2 \beta_1}{(N_1^* - N_2^*)^3}$$

Therefore, the model can be written as:

(A19) $$y_{it} = D_{1it}(\beta_0 + \beta_1 N_{it}) + \frac{D_{2it}}{(N_1^* - N_2^*)^3}\Big(({N_1^*}^3 - 3{N_1^*}^2 N_2^*)\beta_6 + (3N_1^* {N_2^*}^2 - {N_2^*}^3)\beta_0 + 2{N_1^*}^2 {N_2^*}^2 \beta_1 + (6N_1^* N_2^*(\beta_6 - \beta_0) + (-4{N_1^*}^2 N_2^* - {N_2^*}^3 - {N_2^*}^2 N_1^*)\beta_1)N_{it} + (3(\beta_0 - \beta_6)(N_1^* + N_2^*) + 2\beta_1({N_1^*}^2 + {N_2^*}^2 + N_1^* N_2^*))N_{it}^2 + (2\beta_6 - 2\beta_0 - \beta_1(N_1^* + N_2^*))N_{it}^3\Big) + D_{3it}\beta_6 + \varepsilon_{it}$$

For estimation, the required function here is:

(A20) $$y_{it} = \delta_0 \beta_0 + \delta_1 \beta_1 + \delta_6 \beta_6 + \varepsilon_{it}$$

From equation (A19),

(A21) $$\delta_0 = D_{1it} + \frac{D_{2it}}{(N_1^* - N_2^*)^3}\left(\left(3N_1^* N_2^{*2} - N_2^{*3}\right) - 6N_1^* N_2^* N_{it} + 3(N_1^* + N_2^*)N_{it}^2 - 2N_{it}^3\right)$$

(A22) $$\delta_1 = D_{1it}N_{it} + \frac{D_{2it}}{(N_1^* - N_2^*)^3}(2N_1^{*2}N_2^{*2} + (-4N_1^{*2}N_2^* - N_2^{*3} - N_2^{*2}N_1^*)N_{it} + 2(N_1^{*2} + N_2^{*2} + N_1^* N_2^*)N_{it}^2 - (N_1^* + N_2^*)N_{it}^3)$$

(A23) $$\delta_6 = (\frac{D_{2it}}{(N_1^* - N_2^*)^3}\left(N_1^{*3} - 3N_1^{*2}N_2^* + 6N_1^* N_2^* N_{it} - 3(N_1^* + N_2^*)N_{it}^2 + 2N_{it}^3\right) + D_{3it}$$