Shadow Price Implications of Second Degree Stochastic Dominance Efficiency

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Abstract

Second degree stochastic dominance (SSD) can be, but seldom is explicitly, applied to problems having continuous variables. A model is presented which, for any SSD efficient solution, facilitates exploration of the set of SSD consistent shadow prices. The model is tested by applying it to a problem described by Hazell.

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Agricultural economists use several approaches to evaluate risky alternatives. We tend to use mathematical programming with specific criteria or utility functions when the decision variables are continuous. Stochastic dominance criteria are often applied when several discrete alternatives are feasible. It is possible to apply stochastic dominance criteria to problems having continuous decision variables. Although this is seldom explicitly done, criteria consistent with expected utility maximization, and thus with stochastic dominance, are often used.

Even though we are typically more interested in optimal or efficient primal solutions, shadow prices are sometimes of interest. Primal solutions and shadow prices are influenced both by the specific criteria which are maximized and by the problems considered.

Several years ago, one of us read an paper (Anonymous) which asserted that although both primal and shadow prices are influenced by the choice of criterion, the effect of criterion on shadow prices tends to be greater than its effect on the primal solution.

Our paper is both more limited and more general than the earlier paper. It is more limited because we consider the shadow prices associated with specific enterprise activity level vectors (primal solutions) rather than allow both the primal solutions and the shadow prices to vary as the criterion or utility function changes. Our paper is more general because we explore the set of shadow price vectors consistent with second degree stochastic dominance (SSD) efficiency rather than simply compare the shadow prices implied by a few specific criteria.

Nature of Our Approach

Our approach to examining the shadow price implications of SSD efficiency for shadow prices is related to the work of Dybvig and Ross. We exploit selected characteristics of mathematical programming problems and of the class of problems which we consider.

Consider the maximization of a (an at least weakly) concave function, f(y), on a convex set, S. If y* is an optimal solution vector then there is a (at least one) linear approximation, g(y), of f(y) at y* which has useful properties. Specifically, there exists at least one vector, w, such that

(1) $g(y) = (y - y^*)'w + f(y^*) = f(y)$ when y equals y*

Moreover,

(2) $(y - y^*)'w \geq f(y) - f(y^*)$ and

(3) $(y - y^*)'w \leq 0$ for all y vectors which belong to S.

That is, the linear approximation equals f(y*) at y*, is never smaller than f(y) for any feasible y and attains a (not necessarily unique) restricted maximum at y*. A desirable feature of the linear approximation is that, for problems with linear constraints, it can simplify the computation of shadow prices.[1] When the partial derivatives of f(y) exist the elements of w are the partial derivatives of f at y*. When the partial derivatives do not exist, w need not be unique. The possible lack of uniqueness is considered and indeed is a relevant feature of our approach.

Class of Problems Considered

We are interested in problems for which the number of mutually exclusive states of nature is finite and for which the elements of a vector, p, are the probabilities of occurrence of the various states of nature.

Constraints

Another feature of our class of problems is that the vector, y, of net returns associated with the various states is a linear function of the vector, x.

(4) $Cx - y = 0$

The x vector is constrained by a set of linear resource constraints,

(5) $Ax \leq b$, and

by nonnegativity constraints,

(6) $x \geq 0$.

Utility Functions

We assume what seems to be about the broadest possible class of utility functions which can be associated with second degree stochastic dominance. Its members are continuous, increasing and at least weakly concave functions of net return. The most interesting members are those which have at least some strictly concave portions. Our class of utility functions include members of the families of utility functions associated with the Target MOTAD models of Tauer and Watts, Held and Helmers.

Our Model

For a specific utility function, it is possible to determine the (a) primal solution, y*, which maximizes expected utility. The expected utility function associated with this utility function can be replaced by a linear approximation and the dual of the resulting problem can be solved. The solution to the dual can then be converted to monetary units to determine shadow prices implied by the specific utility function. In principle, this process can be repeated for each utility function in the relevant class for which expected utility is maximized by y*. Unfortunately, this can be rather tedious. Our approach essentially reverses this process and, in effect, asks whether a shadow price vector, v, is appropriate. That is, is there a vector w which is consistent with both v and at least one utility function in the relevant class.

The Basic Version

The basic version of our model is

(7) $y^{*\prime}w - z = 0$,

(8) $hz - Y'w \geq 0$,

(9) $q'w = 1$,

(10) $R'w \leq 0$,

(11) $C'w - A'v = 0$,

(12) $w \geq 0$ and

(13) $v \geq 0$.

y* is the y vector for which the associated shadow prices are being examined. For the basic version of our model, y* is assumed to *not* be an extreme (corner) vector of the set of feasible y vectors defined by (3) through (5).[2] One way of defining Y is for the rows of Y to be transposes of the extreme vectors (corners) of the feasible set of y vectors. The i^{th} element of the vector, w, can be interpreted as the product of p_i, the probability that the i^{th} state of nature will occur, and the marginal utility of net return for the i^{th} state of nature. z is the total implied value of the limiting resources. h is a column vector of ones.

The role of (7) and (8) is to ensure that we consider only marginal utility vectors which are consistent with utility functions whose expected values attain constrained maxima at y*. Demonstrating that (7) and (8) accomplish this depends in part on (1) through (3). For a convex set defined by linear constraints it is not necessary to explicitly consider each feasible y vector. It is usually sufficient to consider only corner solutions which are "adjacent" to y.[3]

q is a column vector of ones. (9) limits consideration to utility functions for which the expected marginal utility of net return equals 1.0. This is done for convenience and does not affect shadow prices for the class of utility functions which we consider here. If a utility function, u(y), is a member of our class of utility functions, then each positive linear transformation, a + bu(y) (where b is positive), of u(y) is also a member. All positive transformations of u(y) have the same shadow price implications as u(y). (9) merely eliminates the necessity of dividing v by expected marginal utility to convert the dual solution to monetary units.

(10) constrains marginal utilities of net return for the various states of nature in ways that are appropriate for our class of utility functions. For example, it ensures that the marginal utility of net return for state i cannot be greater than the marginal utility of net return for state j if y_i is larger than y_j. One way of formulating (10) is with constraints of the form, $p_j w_i - p_i w_j \leq 0$. If this approach is taken, each element of R is a zero or a probability.

(11) assumes that all elements of x*, the enterprise activity level vector associated with y*, have positive values and all resources are constraining. v is a vector of shadow prices. The relation of (11) to the dual of a linear program is fairly obvious when is recognized that, for our model, c' = C'w.

(12) and (13) merely require the elements of w and v to be nonnegative.

If y* is SSD efficient, then there is a solution to (7) through (13). However, if, as assumed in this section, y* is not a corner solution, the software which is used to solve the system may fail to find a solution to (7) through (13) because of rounding errors. Minor perturbation of (7) may be required.

Variations

This section considers some of the modifications which may be required if one or more of the assumptions associated with the basic version of the model are not valid.

If y* is a corner, then the row of Y which equals y* can, and probably should be, deleted. If y* is a corner, perturbation of (7) should not be necessary.

The basic version of the model assumes that each element of x* is positive. If an element of x* is zero, the corresponding equality in (11) is either deleted or converted to a "$\leq$" inequality.[4]

The basic version of the model also assumes that every resource is limiting. If a resource is not limiting, the corresponding column (and element of v) is either dropped or the constraint on the corresponding element of v is changed to "≤ 0".[5]

The two adjustments which have just been described are required to ensure that the complementary slackness portion of the Kuhn-Tucker conditions are enforced. These adjustments are necessary because the model does not always automatically make the appropriate adjustments.

The model also does not always deal well with situations where the dual solution is not unique for a given y* and utility function. This limitation is not unique to our model. As is the case for other situations when the dual solution is not unique, active intervention may be needed to determine the complete set of shadow prices.

Other Considerations

If there is no feasible solution to (7) through (13), then y* and the corresponding x* vector are not SSD efficient. Unfortunately, the existence of a feasible solution does not necessarily mean that y* is SSD efficient. Our model is based on a set of conditions which are necessary, but not always sufficient, for SSD efficiency. As a test for SSD efficiency, (7) through (13) is weaker than the test presented by McCamley and Kliebenstein (1987b).

Even when y* is SSD efficient, a shadow price vector may satisfy (7) through (13) without being consistent with SSD efficiency. One way that this can happen is if the solution to (7) through (13) includes an element of w which is zero rather than greater than zero as implied by our class of utility functions. This and the other "failures" which we have identified involve cases where limit points of the set of shadow prices are erroneously included in the set. Although these failures are of theoretical importance, their practical importance is limited.

An example which illustrates the two limitations discussed above has been constructed by the authors. It and a partial analysis of it are available from the authors.

We regard the model which we have presented as simply a tool which can be used to explore the set of shadow prices implied by SSD efficiency for a given primal solution. The nature and extent of any exploration will depend on the researcher's interest. Features which seem interesting to us are the ranges for the shadow prices associated with individual resources, the range in the total implied value of the resources, the dimensionality of the set of shadow prices and the "shape" of the set of shadow prices.

With respect to the "shape", it is known that the set is convex with linear boundaries. As suggested earlier, portions of the boundaries may not be part of the set.

The dimensionality of a set of shadow prices can be anticipated to some degree. Ordinarily, the dimensionality can be no greater than the smallest of the number of states of nature, the number of positive elements of x^* and the number of limiting resources. This limit may be exceeded for problems where the dual solution is not unique for a specific combination of x^* and utility function.

Example

A problem based on Hazell's data is used to illustrate our ideas. A practical advantage of this example is that many SSD efficient solutions are presented by McCamley and Kliebenstein (1987a). We consider seven of these SSD efficient primal solutions. The solutions vary with respect to several characteristics which might affect, or be related to, the dimensionality, size and other attributes of the SSD consistent set of shadow price vectors.

We selected two primal vectors (TM04 and NEXP) which are constrained by one resource limitation, two (TM05 and TM09) which are constrained by two resource limitations and three (TM01, TM11 and NEUT) which are constrained by three resource limitations.

Six of the primal solutions are Target MOTAD solutions. The one (NEXP) which is not a Target MOTAD solution maximizes expected utility for a specific negative exponential function. Four of the Target MOTAD solutions (TM01, TM04, TM05 and TM09) appear to have unique Target MOTAD (resource) shadow price vectors. Because TM11 has at least five different Target MOTAD shadow price vectors and NEUT has at least seven, their SSD consistent sets of shadow price vectors are expected to be of maximum (three) dimensionality.

Table 1 presents selected information about the primal solutions which we consider. For each Target MOTAD primal solution the second and third columns give (one) combination of target level and upper limit on expected deviations for which the primal solution is optimal. The name of each Target MOTAD primal solution is related to the (a) subset to which it belongs. The nature of the subsets is discussed by McCamley and Kliebenstein (1987a). An important thing to know is that all members of a subset share a (at least one) shadow price vector. The same enterprise activity level vector may be optimal for more than one combination of target level and upper limit on expected deviations and, therefore, may be associated with more than one Target MOTAD subset as suggested by footnotes b and d following table 1.

The sets of SSD consistent shadow price vectors associated with these primal solutions are explored. We do not attempt to completely identify each set. We do determine the maximum and minimum shadow prices for each limiting resource. We also determine the maximum and minimum total (implied) value of the (constraining) resources. Knowing this allows us to determine the largest and smallest values of the (implied) return to risk bearing which are consistent with SSD efficiency.

We also verify that the SSD consistent set of shadow price vectors associated with each primal solution includes the shadow price vector(s) implied by the criterion which was originally used to identify that primal solution. For the Target MOTAD solutions, the shadow price vectors were presented in McCamley and Kliebenstein (1991).[6]

Results

In the interest of brevity, the maxima and minima findings as well as the shadow prices known from other sources are not presented here. A long table which includes those results is available from the authors. All previously known shadow price vectors were verified as being feasible solutions to our model.

The dimensionality of the set of SSD consistent shadow prices associated with each primal solutions is equal to the corresponding number of limiting resources. For several of the primal solutions, this can be inferred from previous shadow price evidence and the maxima and minima computed for this paper. For others, addition exploration was necessary.

One measure of the size of an SSD consistent set of shadow prices is the range of its implied set of returns to risk bearing. This measure varies from $817.71 for the SSD consistent set associated with TM04 to $14761.76 for the set associated with NEUT.

One aspect of the SSD consistent set for NEXP is less easily anticipated than other results. In spite of the fact that the NEXP primal is not compatible with Target MOTAD, its SSD consistent set of shadow price vectors shares at least one vector with Target MOTAD subset 4. TM SS04L is a shadow price vector associated with the lower boundary (in target and expected deviations space) of subset 4. The lower boundary of subset 4 is also a lower boundary of the set of feasible combinations of target level and expected deviations.

Concluding Remarks

When formulating mathematical programming models for risk averse decision makers, we tend to select criteria which are well known and/or make the models easy to solve. For criteria which are consistent with expected utility maximization, the approach described in our paper provides one way of examining the potential sensitivity of the shadow price vector to the criterion selected.

Our results seem to validate the assertions of Anonymous that the criterion and/or utility function which is chosen can influence shadow prices.

SSD inconsistent "limit" vectors of a SSD consistent set of shadow prices may be feasible solutions to our model. Other than that, our model seems to give valid results. Its results for the Hazell problem are consistent with all of the shadow price information which is known by us. It is important to note that our model has been tested on only one problem.

Just as the SSD efficient set can be relatively large, sets of SSD consistent shadow price vectors can also be large. Moreover, some SSD consistent shadow price vectors may be consistent only with utility functions which are not very "nice". Fortunately, our approach can be (further) modified to examine the implications of more restrictive criteria. It is relatively easy to replace the SSD criterion with an appropriate (for risk averse persons) version of the generalized stochastic dominance criterion.

Footnotes

1. Inasmuch as shadow prices are computed by many optimization packages, users may not need to know how shadow prices are computed. The primary purpose of this discussion is to help describe our model.

2. There is not always a SSD efficient solution which is not a corner solution. We start with this case because it seems the simplest way to begin.

3. Our approach implicitly assumes that the feasible set can be defined by its corner solutions. Alternative formulations of Y or portions of it can be employed when the feasible set can not be completely defined by corner solutions.

4. Converting to an inequality may seem to be better. However, the inequality should already be implied by constraints (7) and (8).

5. Again, converting to an inequality may be seem to be better. However, this inequality should also be implied by constraints (7) and (8).

6. They noted that the resource shadow prices implied by Target MOTAD depend on the assumptions used to compute them. The Target MOTAD shadow prices which we use for comparative purpose are their adjusted shadow prices. Their adjusted shadow prices seem most consistent with expected utility maximization.

Table 1. Primal Solutions for Which Shadow Prices are Computed

Name[a]	T	λ	x_1	x_2	x_3	x_4	Expected Return	Limiting Resource(s)
	---(Dollars)---		------------------(Acres)------------------				-(Dollars)-	
TM01	50000	1000	66.36	24.85	33.64	75.15	76091.31	All
TM04	65000	2500	72.82	33.48	39.16	54.53	72478.30	Land
TM05	60000	2000	92.07	32.35	7.93	67.66	74746.02	Land, Rotate
TM09	80520	7160	2.84	26.95	97.49	72.72	77829.59	Land, Labor
TM11[b]	80440	7145	.21	27.44	99.79	72.56	77952.12	All
NEXP		NA[c]	70.43	29.53	70.70	29.34	66080.57	Land
NEUT	35000	0[d]	0	27.45	100.00	72.55	77958.17	All

[a]The names are related to the methods used to obtain the primal solutions. The "TM" names identify Target MOTAD primal solutions. The number following TM in each of "TM" names indicates a (one of the) subset(s) to which the primal solution belongs. The NEXP solution was obtained by maximizing expected utility for a negative exponential utility function. NEUT refers to a risk neutral solution which was obtained by maximizing expected return without target or expected deviations constraints. NEUT may be regarded as a Target MOTAD solution.

[b]This primal solution is also a member of Target MOTAD subsets 1, 7, 8 and 9.

[c]An absolute risk aversion coefficient of .000229 was used to obtain this primal solution.

[d]Although Target MOTAD was not used to obtain this solution, it does solve the Target MOTAD model for this (and many other) combinations of target level and upper limit on expected deviations. This solution is also a member of Target MOTAD subsets 1, 7, 8, 9, 10 and 11.

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