Shadow Price Implications of Several Stochastic Dominance Criteria

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Abstract: Stochastic dominance criteria can be, but seldom are explicitly, applied to problems having continuous variables. A previously developed model is modified to facilitate exploration of sets of shadow price vectors for decreasing (non-increasing) absolute risk aversion stochastic dominance (DSD), a combination, TGSD, of third degree stochastic dominance (TSD) and generalized stochastic dominance (GSD) and a combination, DGSD, of DSD and GSD. The model is illustrated by applying it to two risk efficient (primal) solutions of a problem by Anderson, Dillon and Hardaker. For each of the two primal solutions and, where relevant, three risk aversion coefficient intervals, selected aspects of the sets of shadow price vectors consistent with TSD, DSD, TGSD and DGSD are compared with each other and with sets of shadow price vectors consistent with GSD and second degree stochastic dominance (SSD).

Keywords: shadow prices, stochastic dominance

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Introduction

Agricultural economists use several approaches to evaluate risky alternatives. We tend to use mathematical programming with specific criteria or utility functions when the decision variables are continuous. Stochastic dominance criteria are often applied when several discrete alternatives are feasible. Although stochastic dominance criteria are seldom explicitly applied in the first case, criteria consistent with expected utility maximization, and thus with stochastic dominance, are often used.

Even though we are typically more interested in optimal or efficient primal solutions, shadow prices are sometimes of interest. Both primal solutions and shadow prices are influenced by the specific objective function which is maximized and by the problems considered. It is relatively common to consider the effect of changes in objective function coefficients on primal solutions. It is less common to consider the effect of changes in the objective function on shadow prices. Stochastic dominance criteria can be useful for both types of sensitivity analyses.

In an earlier paper (McCamley and Rudel 2000), we presented a model which facilitates the exploration of sets of shadow price vectors consistent with several stochastic dominance criteria. A specific version of the model was used to explore the set of shadow price vectors consistent with second degree stochastic dominance (SSD) for a simple problem. The SSD criterion has the advantage of being consistent with many, but not all, risk efficient solutions. An important limitation is that, even for a single primal solution, the set of shadow price vectors consistent with SSD tends to be very large. Partly for that reason, we also examined the shadow price implications of third degree stochastic dominance (TSD) and (a risk averse version) of generalized stochastic dominance (GSD) (McCamley and Rudel 2001a, 2001b).

Rabin suggests that increasing and constant absolute risk aversion (CARA) are inconsistent with observed behavior. He uses that inconsistency as a basis for criticizing the expected utility hypothesis. His argument motivates us to consider the decreasing (non-increasing) absolute risk aversion stochastic dominance (DSD) criterion. It is considered by itself and also in combination with the GSD criterion. Inasmuch as TSD is often used as a substitute for DSD because it easier to implement, we also consider a combination of TSD and GSD.

This is a shorter version of a paper which was presented at the 2001 WAEA meeting. The longer paper can be obtained from the first author.

Nature of Our Approach

Our approach to examining the shadow price implications of several stochastic dominance criteria is based on some of the ideas in Dybvig and Ross. We exploit selected characteristics of solutions of mathematical programming problems, the class of problems which we consider and selected classes of utility functions.

Class of Problems Considered

We are interested in problems for which s, the number of mutually exclusive states of nature, is finite and for which the elements of a vector, p, are the probabilities of occurrence of the various states of nature. A vector, y, of net returns associated with the various states is a linear function of the vector, x.

(1) $Cx - y = 0$

The x vector is constrained by a set of linear resource constraints,

(2) Ax # b,

and by nonnegativity constraints,

(3) x $ 0.

Classes/Families of Utility Functions

Our model is consistent with some, but not all, risk analysis criteria. The fact that it assumes expected utility maximization suggests that it may not be applicable to criteria which do not assume expected utility maximization. Partly for that reason, all of the criteria considered in this paper are consistent with expected utility maximization. The model is easier to use when the feasible set of shadow price vectors is convex. This is the case for SSD, TSD, risk averse/neutral versions of GSD, DSD and combinations of GSD with TSD and DSD. The set of shadow price vectors for CARA is not always convex. Fortunately, it is relatively easy to explore the set of shadow price vectors for the CARA criterion in other ways.

Levy describes the classes of utility functions commonly believed to be associated with SSD, TSD and DSD. In practice, these criteria are usually associated with somewhat larger classes of utility functions those described by Levy. For example, the class of functions actually associated with SSD usually includes continuous, weakly concave functions of net return or wealth. This class includes the family of utility functions associated with Target MOTAD.

Several versions of GSD are used by agricultural economists. The version which we call GSD here is less general than the version described by Meyer. We allow the Pratt-Arrow risk aversion coefficient (RAC) to vary with net return as long as it is never smaller than a nonnegative constant, r_1, and never larger than a second constant, r_2. Another version, CARA, which seems to be implicitly accepted by many agricultural economists, requires the RAC to be constant for all levels of net return. It is associated with a subset of the family of negative exponential utility functions.

The class of utility functions associated with TGSD is the intersection of the classes of functions associated with TSD and GSD. Likewise, the class of utility functions for DGSD is the intersection of the classes of functions associated with DSD and GSD.

Our Model

Our model is very similar to the model which we presented in earlier papers. The major difference is that (11) is revised to be

(11) $R(w) \# 0$.

This version of (11) allows linear and/or nonlinear constraints to be imposed on the elements of w. w is a vector which is related to the vector of marginal utilities for the various states of nature; the i^{th} element of the vector of marginal utilities is w_i/p_i. This marginal utility is associated with y_i^* where y^* is the y vector associated with the primal solution for which the shadow price vector set is being explored.

Our model is tailored to a specific stochastic dominance criterion by appropriate specification of (11). To simplify the discussion, we assume that the states of nature have been reordered so that the elements of y^* are in descending order. We also assume that each element of y^* is unique to eliminate the necessity of discussing the complications associated with "ties".

Alternative Specifications of (11)

For SSD, marginal utility must be a non-increasing function of net return. This means that $w_i/p_i \# w_j/p_j$ for all i and j for which $s \$ j > i \$ 1$. Many of these (s - 1)(s - 2) constraints are redundant; s - 1 non-redundant constraints are

(11.SSD) $p_j w_i - p_i w_j \# 0$ for $j = i + 1$ and $i = 1$ to $s - 1$.

For the GSD criterion, the marginal utility values must be consistent with the selected RAC interval. McCamley and Kliebenstein showed that this implies

$(w_j/p_j)\exp[-r_1(y_i^* - y_j^*)]$ \$ w_i/p_i \$ $(w_j/p_j)\exp[-r_2(y_i^* - y_j^*)]$ for all i and j for which s \$ j > i \$ 1. s - 1 pairs of these constraints which are not redundant are

(11a.GSD) $p_j w_i - p_i w_j\{\exp[-r_1(y_i^* - y_j^*)]\}$ # 0 and

(11b.GSD) $-p_j w_i + p_i w_j\{\exp[-r_2(y_i^* - y_j^*)]\}$ # 0

for j = i + 1 and i = 1 to s - 1.

For TSD, marginal utility is a non-increasing function of net return and the rate of change in marginal utility must be a non-decreasing function of net return. For our class of problems, this means that 0 \$ $(w_i/p_i - w_j/p_j)/(y_i^* - y_j^*)$ \$ $(w_j/p_j - w_k/p_k)/(y_j^* - y_k^*)$ for all i, j and k for which s \$ k > j > i \$ 1. The left inequalities are equivalent to the constraints in (11.SSD). The right inequalities make all but one of the left inequalities redundant. All but s - 2 of the right inequalities are also redundant. For TSD, (11) includes the constraint in (11.SSD) for which i = 1 and the constraints

(11.TSD) $- p_j p_k(y_j^* - y_k^*)w_i + p_i p_k(y_i^* - y_k^*)w_j - p_i p_j(y_i^* - y_j^*)w_k$ # 0

for k = i + 2, j = i + 1 and i = 1 to s - 2.

DSD requires marginal utility values to be consistent with non-increasing absolute risk aversion. There are several ways of formulating this requirement. One set of non-redundant constraints for DSD includes the constraint from (11.SSD) for which i = 1 and

(11.DSD) $((p_i w_j)/(p_j w_i))^\alpha - ((p_j w_k)/(p_k w_j))^\beta$ # 0 for k = i + 2, j = i + 1 and i = 1 to s - 2

where $\alpha = 1/(y_i^* - y_j^*)$ and $\beta = 1/(y_j^* - y_k^*)$. The terms in (11.DSD) are the natural logarithms of the RAC's for the net return intervals from y_j^* to y_i^* and y_k^* to y_j^*, respectively.

TGSD requires the marginal utility values to be consistent with both GSD and TSD. The set of non-redundant constraints for TGSD includes (11.TSD), (11a.GSD) and (11b.GSD). The

(11b.GSD) constraints eliminate the need to include a constraint from (11.SSD).

DGSD requires the marginal utility values to be consistent with both GSD and DSD. The set of non-redundant constraints for DGSD includes (11.DSD), the constraint from (11a.GSD) for which $i = 1$ and the constraint from (11b.GSD) for which $i = s - 1$. The other constraints in (11a.GSD) and (11b.GSD) are redundant.

CARA requires the marginal utility values to be consistent with DGSD and also consistent with non-decreasing absolute risk aversion. The second requirement implies constraints like (11.DSD) except that they are greater-than-or-equal constraints rather than less-than-or-equal constraints. Unfortunately, the greater-than-or-equal constraints do not define a convex set.

Expected Characteristics of Sets of Shadow Price Vectors

Our model is simply a tool which can be used to explore the set of shadow price vectors implied by SSD, TSD, GSD, DSD, TGSD or DGSD efficiency of a given primal solution. The nature and extent of any exploration depend on the user's interest. Features which seem interesting to us are the ranges of the shadow prices associated with individual resources, the range of the total implied value of the resources, the dimensionality of the set of shadow price vectors and the "shape" of the set of shadow price vectors.

With respect to "shape", it is known that the SSD, TSD, GSD and TGSD shadow price vector sets are convex with linear boundaries. DGSD shadow price vector sets are also convex. They may have linear and/or nonlinear boundaries.

The dimensionality of a set of shadow price vectors can be anticipated to some degree. Ordinarily, the dimensionality can be no greater than the smallest of the number of positive elements of x^*, the number of limiting resources and one less than the number of states of nature. This limit may be exceeded for problems for which the dual solution is not unique for a specific combination of x^* and utility function.

Example

A problem based on data from Anderson, Dillon and Hardaker is used to illustrate our ideas. Tauer used the same problem and may be the best source for the constraints and returns data. The problem has three activities (wheat, oats and new wheat), four resource constraints (crop land, total wheat area, operating capital and labor) and five equally likely states of nature. We consider two primal solutions. The first, solution A, maximizes expected net return and also belongs to all GSD efficient sets for which the RAC interval has a nonnegative upper limit no larger than (approximately) .005346. A is a corner solution which is constrained by three resource limits. Thus, any set of SSD, TSD, GSD, DSD, TGSD or DGSD consistent shadow price vectors for A might have dimensionality as large as three. The second primal solution, solution a, is an edge solution which is constrained by two resource limits. Primal solution a belongs to every efficient set for which the RAC interval includes .00831955. The activity levels associated with primal solutions A and a are presented in table 1.

Table 1. Selected Primal Solutions

Name	Wheat x_1	Oats x_2	New Wheat x_3	Expected Return	Limiting Resource(s)
	-----------(hectares)--------			---($)----	
A	4/3	4.0	20/3	1260.37	Crop Land Wheat Area Labor
a	1.0	3.8	7.0	1253.53	Wheat Area Labor

For primal solutions A and a, the sets of shadow price vectors consistent with SSD, TSD, DSD and, for selected RAC intervals, GSD, TGSD, DGSD and CARA are explored. For each criterion (and each selected RAC interval, where relevant), we determine the maximum and minimum shadow prices for each limiting resource. We also determine the maximum and minimum total implied value of the (constraining) resources.

Results

Shadow Prices for Primal Solution A

The SSD, TSD, GSD, DSD, TGSD and DGSD shadow price vector sets are three dimensional and convex. For any RAC interval, the set of shadow price vectors associated with the CARA criterion is neither three dimensional nor convex. It is a curve in the three dimensional set associated with the GSD criterion for the same RAC interval.

Three RAC intervals are explicitly considered for GSD, TGSD and DGSD. They are 0 to .002, .002 to .004 and 0 to .004. Inasmuch as primal solution A is the optimal CARA solution for any RAC in the interval from 0 to approximately .005346, this longer interval is also implicitly considered. For any (proper or improper) subinterval of 0 to .005346, it appears that the shadow price vectors which maximize and minimize total implied resource value are the same for the GSD, TGSD, DGSD and CARA criteria. Total implied resource value is maximized by setting the RAC at the lower limit of the subinterval and minimized by setting the RAC at the upper limit of the subinterval.

For RAC intervals with upper limits larger than .005346, the pattern is somewhat different for shadow price vectors which minimize total implied resource value. CARA and DSD share the same shadow price vector for RAC intervals with upper limits larger than .005346 because, in order to maintain stochastic dominance efficiency of point A, these two criteria effectively limit the RAC to (the interval from 0 to) approximately .005346. For the more flexible TSD and SSD criteria, the minimum total implied resource values are smaller.

The total implied resource value results are largely dependent on either the lower or upper limit of the RAC interval but usually not on both limits or on the criterion. The criterion and both limits of the RAC interval are more important for individual shadow prices than they are for total implied resource value. This is related to the fact that the ranges in the shadow prices for individual resources are usually longer for the 0 to .004 RAC interval than the sum of the corresponding ranges for the 0 to .002 and .002 to .004 RAC intervals. For example, consider the GSD consistent shadow price of crop land. For the RAC interval, 0 to .004, this shadow price can be as small as $7.49 or as large as $51.97. The union of the GSD consistent crop land shadow prices for the 0 to .002 and .002 to .004 RAC intervals does not include either of these values.

Our results for primal solution A show that the sets of shadow price vectors for TGSD and DGSD are sometimes smaller that the corresponding sets for GSD. The TGSD set for the RAC interval, 0 to .004, is smaller than either the TSD set or the GSD set for the same RAC interval. A similar statement is true for the RAC interval, 0 to .002. A similar statement may be true for the RAC interval, .002 to .004. However, our explorations do not confirm that conclusion.

Shadow Prices for Primal Solution a

For any (proper or improper) subinterval of the interval of RAC values for which primal solution A is CARA efficient, the shadow price vectors which maximize and minimize total implied resource values are the same for the GSD, TGSD, DGSD and CARA criteria. A similar statement is trivially true but less significant for primal solution a since it is CARA efficient for only one RAC value, approximately .00831955. When the RAC interval is a single value, GSD, TGSD and DGSD are the same as the CARA criterion.

Inasmuch as solution a is efficient only for RAC intervals which include (approximately) .00831955, we choose RAC intervals which include that value. The RAC intervals which we consider are 0 to .01, .006 to .01 and .008 to .009. As would be expected, the "size" of the set of shadow price vectors increases as the RAC interval changes by lowering the lower limit and/or increasing the upper limit.

For each RAC interval considered, the shadow price vector which maximizes total implied resource value is shared by GSD, TGSD and DGSD. The fact that the shadow price vector which maximizes total implied resource value for SSD, TSD and DSD is different from the shadow price vector which maximizes total implied resource value for GSD, TGSD and DGSD for the 0 to .01 RAC interval suggests that, at least when the lower limit of the RAC interval is zero, the upper limit of RAC interval as well as the lower limit affect total implied resource value at solution a.

When total implied resource value is minimized, neither the lower nor upper limit of the RAC interval matters for DGSD as long as the RAC interval includes .00831955. For TGSD and GSD, the lower limit of the RAC interval doesn't seem to have any effect. The upper limit seems to affect the sets of shadow price vectors for these two criteria in the same way. However, the fact that the shadow price vectors which minimize total implied resource value are not the same for TSD and SSD implies that TGSD and GSD are not always affected in the same way by the upper limit of the RAC interval.

For each combination of criterion and RAC interval, the same shadow price vector maximizes total implied value of the resources, minimizes the shadow price of wheat area and maximizes the shadow price of labor. At the other extremes, for each combination of criterion and RAC interval, a common shadow price vector minimizes total implied value of the resources, maximizes the shadow price of wheat area and minimizes the shadow price of labor. Thus, the maxima and minima results are consistent with one dimensional shadow price vector sets.

Additional exploration confirms that, with the exception of the CARA shadow price set which has zero dimension, the sets of shadow price vectors are two dimensional. The SSD, TSD, GSD and CARA shadow price vector sets are "nested". This is, of course, due to our RAC intervals.

The fact that the GSD shadow price vector sets are subsets of the TSD shadow price vector set might seem to suggest that each TGSD shadow price vector set is the same as the GSD shadow price vector set for the same RAC interval. Initially, we also thought that the DGSD shadow price vector set for any RAC interval is the intersection of the DSD shadow price vector set with the GSD shadow price vector set for the same RAC interval. Neither conclusion is valid. For the RAC interval, 0 to .01, the TGSD shadow price vector set is slightly smaller than the GSD shadow price vector set. The DGSD shadow price vector set is somewhat smaller than the intersection of the DSD and GSD shadow price vector sets.

Concluding Remarks

This paper explores the set of shadow price vectors associated with DSD and examines the effect of adding the conditions for TSD or DSD efficiency to the conditions for GSD efficiency. Two conclusions are consistent with economists' folklore. TSD is easier to apply than DSD because, for our class of problems, its restrictions are linear rather than nonlinear. DSD is more effective, both alone and in combination with GSD, in reducing the size of the set of shadow price vectors. The reduction is not always large but is obtained by paying a small "price" in the form of a reasonable assumption.

It is appropriate to emphasize that each set of shadow price vectors which we explore is associated with a single primal solution. For example, the set of GSD (.008 to .009) consistent shadow price vectors includes only shadow price vectors which are consistent both with primal solution a being the optimal solution and with the set of utility functions consistent with the GSD criterion for a RAC interval of .008 to .009. The union of the sets of shadow price vectors associated with all primal solutions which are GSD efficient for the same RAC interval is larger than the set which we explore here. Although this may seem too limiting, it does remind us that obtaining a primal solution for a specific objective function does not mean that the associated shadow price vector is the appropriate one unless we are certain about the specification and coefficients of the objective function.

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