The Stata Journal (2016)
**16**, Number 4, pp. 1039–1057

# Tests for normality based on the quantile-mean covariance

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**Abstract.** We present a new command, `qctest`, to implement tests for normality of a random variable based on the quantile-mean covariance. The test procedures are based on recent results by Bera et al. (2016, *Econometric Theory* 32: 1216–1252) and are an efficient alternative to existing normality tests in the literature.

**Keywords:** st0464, qctest, skewness, kurtosis, normality

## 1 Introduction

Testing for normality continues to be important in economic and financial data because it is central to many estimation, inference, and forecasting methods. The characterization of the normal distribution has been an object of interest and research to many scholars. Literature on the subject is vast and includes Lukacs (1942), Laha (1957), Rao (1958), Kagan, Linnik, and Rao (1973), Mathai and Pederzoli (1978), Riedel (1985), Chikuse (1990), and Xu (1998). Statisticians have long been interested in testing normality, and much literature includes tests based on the cumulative distribution function (see `ksmirnov`; Kolmogorov [1933]; Smirnof [1939]; Lilliefors [1967]; and Zheng [2000]); the normal probability plot (see `swilk`; Shapiro and Wilk [1965]; Shapiro and Francia [1972]; and Royston [1983]); the third and fourth moments (see `sktest`; Fisher [1930]; Geary [1947]; Bowman and Shenton [1975]; Jarque and Bera [1980]; White and MacDonald [1980]; D'Agostino, Belanger, and D'Agostino [1990]; and Bai and Ng [2005]); the characteristic function (Koutrouvelis 1980; Koutrouvelis and Kellermeier 1981; and

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Epps and Pulley 1983); the moment generating function (Epps, Singleton, and Pulley 1982); the Hermite polynomials (Kiefer and Salmon 1983; Hall 1990; van der Klaauw and Koning 2003; and Bontemps and Meddahi 2005); and the presence of serial correlation (Richardson and Smith 1993; Lobato and Velasco 2004). For an excellent historical review of tests for normality based on the above approaches, as well as others such as plotting methods, correlation-based tests, Edgeworth expansion, and outlier tests, see Thode (2002).

Bera et al. (2016) propose a novel way to test for normality of random variables using the quantile-mean covariance (QC) function, defined as the asymptotic covariance between the $\tau$th sample quantile and the sample mean. The QC function is given by the ratio of the expected quantile loss function over the density function evaluated at a particular quantile. Bera et al. (2016) show that this function is independent of the choice of the quantile $\tau$ and is equal to the variance if and only if the underlying distribution is normal. They thus propose a test for normality based on the constancy of the QC function across quantiles using Kolmogorov (supremum)- and the Cramér–von Mises (average)-type statistics that check over the entire distribution.

The new command `qctest` implements a battery of tests to identify nonnormality using Bera et al. (2016) QC test procedure. As is common in Cramér–von Mises-type statistics, the limiting distribution is nonstandard. We thus tabulate the asymptotic distribution for a sample size of 20,000 observations using 50,000 simulations. The results of the simulation are used to compute the $p$-value after the `qctest` run.

`qctest` also provides a QC plot, a graphical procedure for detecting nonnormality at particular quantiles, and a Q–Q plot comparing the given quantile distribution with that of a normal random variable. In the standard Q–Q plot, we look for a deviation from a 45-degree line, while in the QC plot, the reference line of comparison is a horizontal one; visually, the latter task is much easier. Also, a formal test statistic is available for making and using the QC function. The Q–Q plot does not possess such a counterpart. Large deviations from the horizontal line cast doubts on the normality of the data and also serve as a means to detect the "exact" locations of departures from normality. This is useful for statistical analysis based on particular locations of the distribution, such as tail behavior and extreme value analysis.

This article is organized as follows. Section 2 reviews the QC test of Bera et al. (2016) and presents the test statistics. Section 3 describes the `qctest` syntax and presents the code in which the asymptotic distribution of the Kolmogorov- and the Cramér–von Mises-type statistics was simulated. Section 4 presents simulations and a real data application. We conclude with practical suggestions on the proper use of the tests.

## 2 QC test for normality

Consider the QC function, simply denoted as $C(\tau)$, described in Ferguson (1999). Suppose $\{X_1, \dots, X_n\}$ is an independent and identically distributed sample with distribution function $F(x)$, density $f(x)$, quantile function $Q(\tau)$ $(0 < \tau < 1)$, and mean $\mu_X$.

Further suppose that the density $f(x)$ is continuous and positive at $Q(\tau)$. Denote the sample mean by $\overline{X}_n = 1/n\sum_{i=1}^n X_i$ and the sample $\tau$th quantile by $Y_{\tau,n} = X_{(n:\lceil n\tau\rceil)}$. The QC function, $C(\tau)$, is the asymptotic covariance between the sample quantiles with index $\tau \in (0,1)$ and the sample mean; that is,

$$C(\tau) = \lim_{n\to\infty} \text{Cov}\left(\sqrt{n}Y_{\tau,n}, \sqrt{n}\ \overline{X}_n\right)$$

Note that $Q(\tau) = \arg\min_a E\rho_\tau(X-a)$, where the quantile loss function is defined by $\rho_\tau(u) := \{\tau - \mathbf{1}(u \leq 0)\}u$; see Koenker and Bassett (1978). Thus the expected quantile loss function is given by $E\rho_\tau(X-a)$. Plugging in $Q(\tau)$, we obtain the minimized expected quantile loss function $\varpi(\tau) := E\rho_\tau\{X - Q(\tau)\}$. From Ferguson (1999),

$$C(\tau) = \frac{\varpi(\tau)}{f\{Q(\tau)\}}$$

where $1/[f\{Q(\tau)\}]$ (the inverse of the density evaluated at the $\tau$th quantile) is called the sparsity function (Tukey 1965).

Although desirable, except for some special cases (for example, normal distribution), closed-form expressions of $C(\tau)$ are not available. Following the standard notation, let $\phi$, $\Phi$, and $\Phi^{-1}$ denote, respectively, the density, distribution, and quantile functions of the standard normal random variable. Thus, if $X \sim N(\mu_X, \sigma_X^2)$, then it has density, distribution, and quantile functions $f(x) = (1/\sigma_X)\phi\{(x-\mu_X)/\sigma_X\}$, $F(x) = \Phi\{(x-\mu_X)/\sigma_X\}$, and $Q(\tau) = \sigma_X\Phi^{-1}(\tau) + \mu_X$, respectively.

Bera et al. (2016) key result is summarized in the following theorem.

**Theorem 1.** *Let $X$ be a random variable with finite mean $\mu_X \in \mathbb{R}$ and variance $\sigma_X^2 > 0$, and the density function $f(x)$ is positive and differentiable. Then, $C(\tau) = \sigma_X^2$ for all $\tau \in [0,1]$ if and only if $X \sim N(\mu_X, \sigma_X^2)$.*

The QC function, $C(\tau)$, of a random sample provides rich information about the underlying distribution, and the result in theorem 1 suggests a natural way to test normality based on the constancy of $C(\tau)$ across quantiles. Consider the null hypothesis ($H_0$) of normality based on this process. For technical reasons, we consider the QC process for $\tau \in \mathcal{T} = [\epsilon, 1-\epsilon]$, where $\epsilon$ is a small positive number (the uniform asymptotic representation holds on a subinterval of [0,1]). For similar treatment in other inference problems, see, for example, Andrews (1993) and Andrews and Ploberger (1994).

Given a normally distributed random sample $\{X_i\}_{i=1}^n$, define the QC process $C_n(\tau)$ as

$$C_n(\tau) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \left[\frac{X_i - \mu_X}{\sigma_X}\left(\tau - \mathbf{1}_{X_i<Q(\tau)}\right) - \phi\{\Phi^{-1}(\tau)\}\right]$$

According to theorem 1, for any $\tau$, as $n \to \infty$,

$$\frac{1}{f\{Q(\tau)\}\sigma_X^2}\frac{1}{n}\sum_{i=1}^n X_i\left(\tau - \mathbf{1}_{X_i<Q(\tau)}\right) \xrightarrow{p} 1$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{X_i\left(\tau-\mathbf{1}_{X_i<Q(\tau)}\right)}{f\left\{Q\left(\tau\right)\right\}\sigma_X^2}-1\right]\xrightarrow{d}N\left\{0,\sigma^2\left(\tau\right)\right\}$$

where

$$\sigma^2\left(\tau\right)=\frac{\tau^2}{f\left\{Q\left(\tau\right)\right\}^2\sigma_X^2}+(1-2\tau)\frac{EX_i^2\mathbf{1}_{X_i<Q(\tau)}}{f\{Q(\tau)\}^2\sigma_X^4}-1$$

The construction of $C_n\left(\tau\right)$ requires knowledge of unknown parameters $\mu_X$, $\sigma_X^2$, and $f\{Q\left(\tau\right)\}$; thus it is infeasible to use $C_n\left(\tau\right)$ directly in testing normality. To construct a feasible version of the QC process under $H_0$ in practice, we use

$$\begin{aligned}\widehat{\mu}_X &= \overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i,\quad \widehat{\sigma}_X^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\widehat{\mu}_X)^2\\ \widehat{Q}\left(\tau\right) &= \widehat{\mu}_X+\widehat{\sigma}_X\Phi^{-1}\left(\tau\right),\quad f\widehat{\{Q\left(\tau\right)\}}=\frac{1}{\widehat{\sigma}_X}\phi\left\{\Phi^{-1}\left(\tau\right)\right\}\end{aligned}$$

and consider the following feasible QC process:

$$\widehat{C}_n\left(\tau\right)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{X_i-\widehat{\mu}_X}{\widehat{\sigma}_X}\left(\tau-\mathbf{1}_{X_i<\widehat{Q}(\tau)}\right)-\widehat{f}\left\{Q\left(\tau\right)\right\}\widehat{\sigma}_X\right]$$

To test normality, Bera et al. (2016) consider functionals of $\widehat{C}_n(\tau)$ over $\tau\in\mathcal{T}$. In particular, they propose the following test statistics:

$$\begin{aligned}T_{1n} &:= \sup_{\tau\in\mathcal{T}}\left|\widehat{C}_n(\tau)\right|\\ T_{2n} &:= \sup_{\tau\in\mathcal{T}}\widehat{C}_n(\tau)^2\\ T_{3n} &:= \int_{\tau\in\mathcal{T}}\widehat{C}_n(\tau)^2d\tau\end{aligned}$$

The first two are Kolmogorov suptype statistics, and the third one is the Cramér–von Mises (average)-type statistic. Because the distributions of the test statistics are not standard under the null hypothesis of normality $H_0$, we need to construct their critical values. These are calculated by simulation using a sample size of 20,000, over 50,000 replications. Table 1 provides 1%, 5%, and 10% critical values for the test statistics $T_{1n}$, $T_{2n}$, and $T_{3n}$. The table covers $\mathcal{T}=[\epsilon,1-\epsilon]$ with $\epsilon\in\{0.001,0.01,0.05,0.10,0.15,0.20\}$, and the approximating grid is 0.001. These simulations are also used to construct $p$-values for the QC tests.

Table 1. Critical values for $T$ statistics

| $\epsilon$ | $T$ statistic | 10% | 5% | 1% |
|---|---|---|---|---|
| 0.001 | $T_{1n}$ | 0.8144674 | 0.8868363 | 1.0426934 |
| | $T_{2n}$ | 0.6633571 | 0.7864787 | 1.0872095 |
| | $T_{3n}$ | 0.0785807 | 0.0964926 | 0.1397021 |
| 0.01 | $T_{1n}$ | 0.8142603 | 0.8867764 | 1.0426934 |
| | $T_{2n}$ | 0.6630199 | 0.7863724 | 1.0872095 |
| | $T_{3n}$ | 0.0777682 | 0.0955875 | 0.138823 |
| 0.05 | $T_{1n}$ | 0.7814074 | 0.8563241 | 1.0122919 |
| | $T_{2n}$ | 0.6105975 | 0.7332909 | 1.0247349 |
| | $T_{3n}$ | 0.0681487 | 0.0845204 | 0.122791 |
| 0.10 | $T_{1n}$ | 0.7037061 | 0.775032 | 0.9195223 |
| | $T_{2n}$ | 0.4952022 | 0.6006745 | 0.8455212 |
| | $T_{3n}$ | 0.0545155 | 0.0684603 | 0.1003814 |
| 0.15 | $T_{1n}$ | 0.6192168 | 0.6864218 | 0.8223042 |
| | $T_{2n}$ | 0.3834294 | 0.4711749 | 0.6761842 |
| | $T_{3n}$ | 0.0428251 | 0.054668 | 0.0839317 |
| 0.20 | $T_{1n}$ | 0.5396766 | 0.5985859 | 0.722712 |
| | $T_{2n}$ | 0.2912508 | 0.3583051 | 0.5223127 |
| | $T_{3n}$ | 0.0325979 | 0.0431813 | 0.0695557 |

# 3 The qctest command

## 3.1 Syntax

The syntax is

`qctest` *depvar* [ *indepvars* ] [ *if* ] [ , `nograph` `addqqplot` `allstats` `level(#)` *twoway_options* ]

## 3.2 Options

`nograph` suppresses QC plot display. The default is to show the QC plot.

`addqqplot` adds a Q–Q plot to the QC plot. This option works only if `nograph` is not specified.

`allstats` reports all the $T$ statistics. The default is $T_3$.

`level(#)` sets the confidence level. The default is `level(95)`.

*twoway_options*; see [G-3] ***twoway_options***.

### 3.3 Stored results

`qctest` stores the following in `r()`:

Matrices

| | |
|---|---|
| `r(QCplot)` | a matrix with numerical coordinates of the QC plot and their confidence intervals |
| `r(Ts)` | table of results; first column: epsilon values; second: $T$ statistics; third: $p$-values |

The matrix `r(QCplot)` is useful to replicate the QC plot with other graph formats provided by Stata.

Bera et al. (2016) show that the $T$ statistics have nonnormal asymptotic distributions, and then $p$-values stored in `r(Ts)` are computed using a simulation of 50,000 random samples with 20,000 observations each. Results of the simulated database are provided in the companion databases `qctest-Tdist.dta` for obtaining the $p$-values of the $T$ statistics and `qctest-QCdist.dta` for the QC plot.

## 4 Simulation and empirical application

This section illustrates the `qctest` command with two examples. First, we use different random samples to evaluate the performance of the tests. Second, we use real data stored in Stata's `auto.dta`.

### 4.1 Simulations

For comparison, we develop the examples in Bera et al. (2016) by drawing random samples from $N(0,1)$, Student's $t_3$, and $\chi^2_1$.

Consider a sample size of $n = 50$. First, we evaluate the tests' performance for normal distributions. As expected, we cannot reject the null hypothesis of normality.

```
. version 13
. set seed 123
. set obs 50
number of observations (_N) was 0, now 50
. generate x = rnormal()
. qctest x, addqqplot allstats
Computing plots and T statistics...
Test for normality based on the quantile-mean covariance process
    Ho: x has a normal distribution
    Ha: x hasn´t a normal distribution
```

| epsilon and T | Stat. | p-value |
|---|---|---|
| 0.001 | | |
| T1 | 0.4764 | 0.8254 |
| T2 | 0.2270 | 0.8254 |
| T3 | 0.0206 | 0.8608 |
| 0.01 | | |
| T1 | 0.4764 | 0.8210 |
| T2 | 0.2270 | 0.8210 |
| T3 | 0.0203 | 0.8560 |
| 0.05 | | |
| T1 | 0.4764 | 0.7222 |
| T2 | 0.2270 | 0.7222 |
| T3 | 0.0186 | 0.8032 |
| 0.10 | | |
| T1 | 0.3857 | 0.8052 |
| T2 | 0.1488 | 0.8052 |
| T3 | 0.0130 | 0.8106 |
| 0.15 | | |
| T1 | 0.3857 | 0.6213 |
| T2 | 0.1488 | 0.6213 |
| T3 | 0.0117 | 0.6729 |
| 0.20 | | |
| T1 | 0.3857 | 0.4162 |
| T2 | 0.1488 | 0.4162 |
| T3 | 0.0097 | 0.5524 |

Second, we consider a random sample from a Student's $t$ distribution with three degrees of freedom. This example is of interest because the difference with the normal distribution corresponds only to excess kurtosis. The results are sensitive to the $T$ statistic and the epsilon chosen. Note that `T3` detects nonnormality of the data-generating process while the others do not. As shown below, the lack of power is due to the small sample chosen for this example.

```
. generate y = rt(3)

. qctest y, addqqplot allstats
Computing plots and T statistics...

Test for normality based on the quantile-mean covariance process

    Ho: y has a normal distribution
    Ha: y hasn´t a normal distribution
```

| epsilon and T | Stat. | p-value |
|---|---|---|
| 0.001 | | |
| T1 | 0.6948 | 0.2655 |
| T2 | 0.4827 | 0.2655 |
| T3 | 0.1674 | 0.0037 |
| 0.01 | | |
| T1 | 0.6948 | 0.2645 |
| T2 | 0.4827 | 0.2645 |
| T3 | 0.1655 | 0.0037 |
| 0.05 | | |
| T1 | 0.6948 | 0.2030 |
| T2 | 0.4827 | 0.2030 |
| T3 | 0.1625 | 0.0022 |
| 0.10 | | |
| T1 | 0.6948 | 0.1086 |
| T2 | 0.4827 | 0.1086 |
| T3 | 0.1469 | 0.0013 |
| 0.15 | | |
| T1 | 0.6948 | 0.0453 |
| T2 | 0.4827 | 0.0453 |
| T3 | 0.1316 | 0.0009 |
| 0.20 | | |
| T1 | 0.6948 | 0.0144 |
| T2 | 0.4827 | 0.0144 |
| T3 | 0.1242 | 0.0005 |

Third, we consider the $\chi_1^2$ case characterized by the asymmetry due to a large mass of probability on the right tail of the distribution. Clearly, all results show a rejection of the null hypothesis of normality.

```
. generate z = rnormal()^2
. qctest z, addqqplot allstats
Computing plots and T statistics...
Test for normality based on the quantile-mean covariance process
    Ho: z has a normal distribution
    Ha: z hasn´t a normal distribution
```

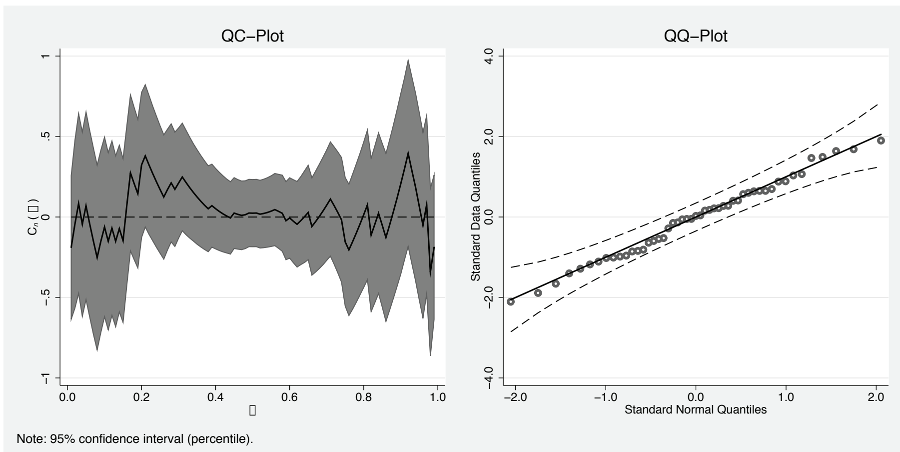
| epsilon and T | Stat. | p-value |
|---|---|---|
| 0.001 | | |
| T1 | 2.2374 | 0.0000 |
| T2 | 5.0061 | 0.0000 |
| T3 | 0.8009 | 0.0000 |
| 0.01 | | |
| T1 | 2.2374 | 0.0000 |
| T2 | 5.0061 | 0.0000 |
| T3 | 0.7927 | 0.0000 |
| 0.05 | | |
| T1 | 2.2374 | 0.0000 |
| T2 | 5.0061 | 0.0000 |
| T3 | 0.7485 | 0.0000 |
| 0.10 | | |
| T1 | 2.2374 | 0.0000 |
| T2 | 5.0061 | 0.0000 |
| T3 | 0.6892 | 0.0000 |
| 0.15 | | |
| T1 | 2.2374 | 0.0000 |
| T2 | 5.0061 | 0.0000 |
| T3 | 0.5824 | 0.0000 |
| 0.20 | | |
| T1 | 2.2374 | 0.0000 |
| T2 | 5.0061 | 0.0000 |
| T3 | 0.4161 | 0.0000 |

We also evaluate the tests' performance when we increase the sample size. This aspect is analyzed using the plots generated by `qctest` (figures 1 to 3). In each figure, the graph on the left contains the QC plots (black solid lines) with the 95% uniform confidence bands (in shaded region), which are calculated using the simulation described in section 3.3. The graph on the right displays the Q–Q plots (scatterplots). For each of the Q–Q plots, we added a solid straight line that connects the quantile pairs and a 95% confidence band (in dashed lines).

The results clearly show that the tests work well when the data are generated using normal distributions. For nonnormal distributions, the QC plots show departures from the horizontal line at the extreme quantiles. In the Student's $t$ case, rejections are clearly observed for extreme quantiles even for $n = 50$.

The figures also show the disadvantages of the Q–Q plots with respect to the QC plots. For example, on the Q–Q plots, the scales on the graphs increase considerably as we increase the sample size (see, for instance, figure 2). The QC plots, however, do not suffer from these scale distortions. Moreover, the 45-degree line in the Q–Q plots becomes more horizontal as we increase the sample size. Thus we find that the QC plots provide a useful graphical alternative analysis to the popular Q–Q plots.

$n = 50$




$n = 500$

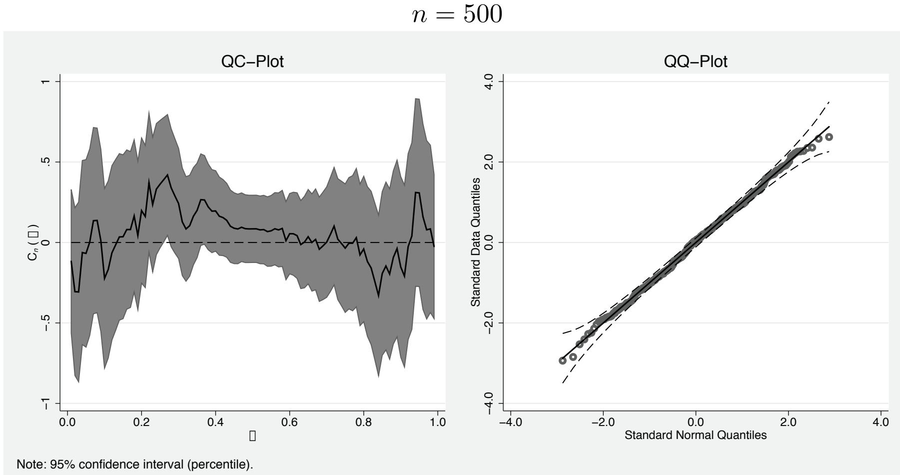
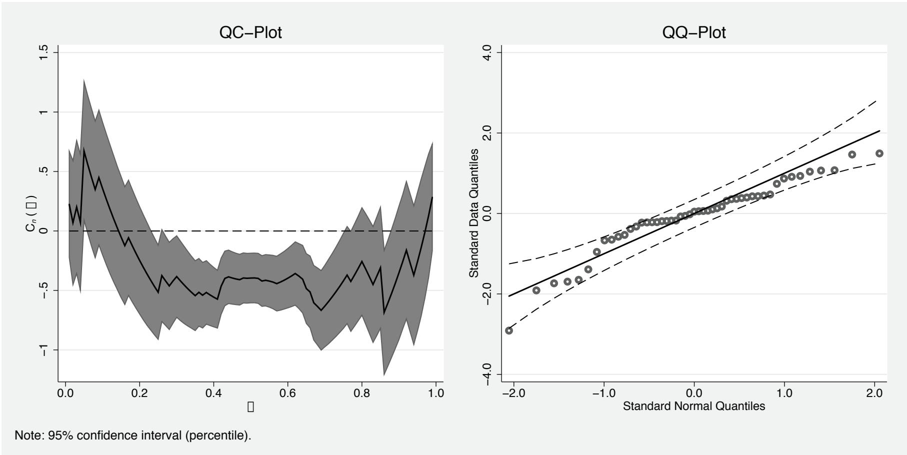



Figure 1. Normal distribution, $n = 50500$

$n = 50$




$n = 500$

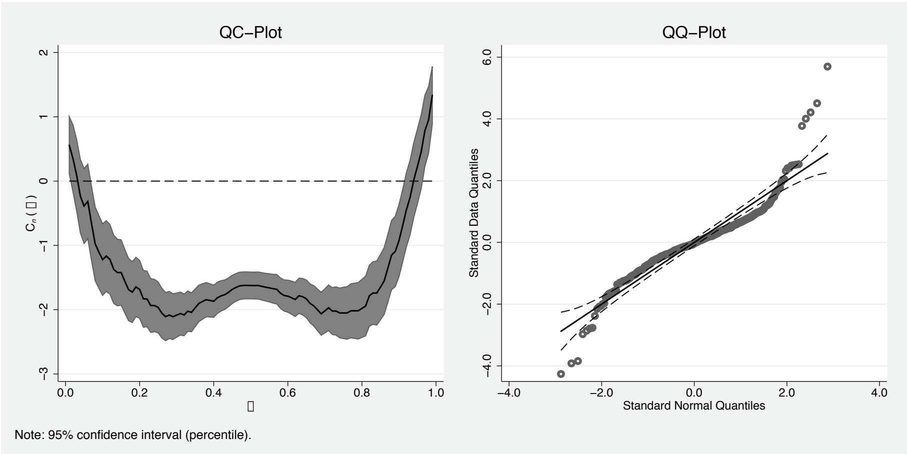
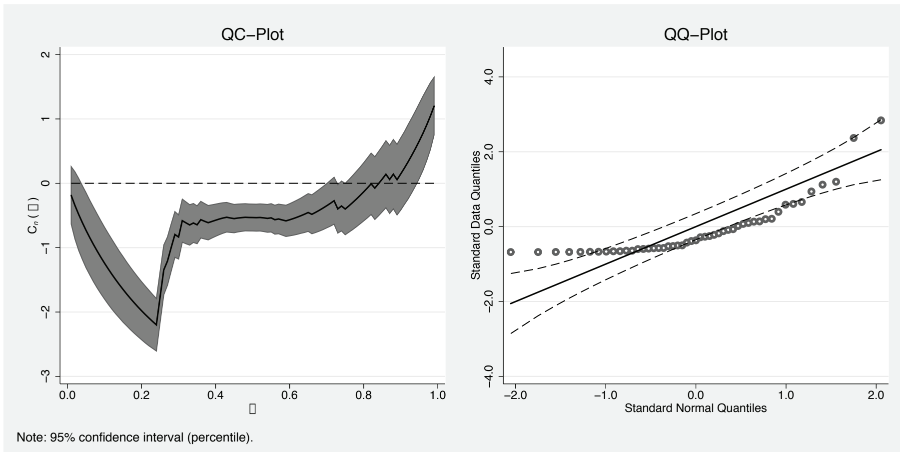



Figure 2. Student's $t$ distribution with 3 degrees of freedom, $n = 50500$

$n = 50$




$n = 500$

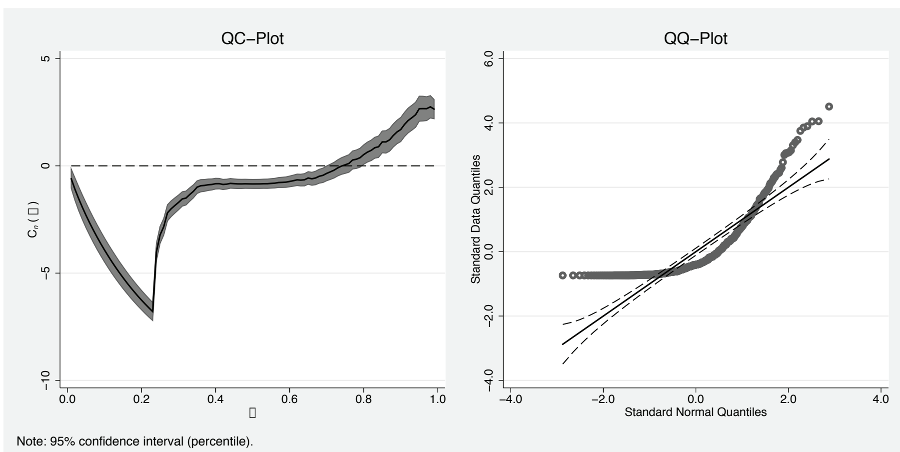



Figure 3. Chi-squared distribution with 1 degree of freedom, $n = 50500$

## 4.2 Example with real data

We use real data from the 1978 automobile dataset (`auto.dta`) provided by Stata. We test the normality of the `price` and `gear_ratio` variables using `qctest`.

```
. sysuse auto, clear
(1978 Automobile Data)

. qctest price, addqq allstats
Computing plots and T statistics...

Test for normality based on the quantile-mean covariance process

    Ho: price has a normal distribution
    Ha: price hasn´t a normal distribution
```

| epsilon and T | Stat. | p-value |
|---|---|---|
| 0.001 | | |
| T1 | 2.1675 | 0.0000 |
| T2 | 4.6982 | 0.0000 |
| T3 | 0.8984 | 0.0000 |
| 0.01 | | |
| T1 | 2.1675 | 0.0000 |
| T2 | 4.6982 | 0.0000 |
| T3 | 0.8921 | 0.0000 |
| 0.05 | | |
| T1 | 2.1675 | 0.0000 |
| T2 | 4.6982 | 0.0000 |
| T3 | 0.8129 | 0.0000 |
| 0.10 | | |
| T1 | 2.1675 | 0.0000 |
| T2 | 4.6982 | 0.0000 |
| T3 | 0.6711 | 0.0000 |
| 0.15 | | |
| T1 | 2.1675 | 0.0000 |
| T2 | 4.6982 | 0.0000 |
| T3 | 0.4669 | 0.0000 |
| 0.20 | | |
| T1 | 2.1265 | 0.0000 |
| T2 | 4.5220 | 0.0000 |
| T3 | 0.2353 | 0.0000 |

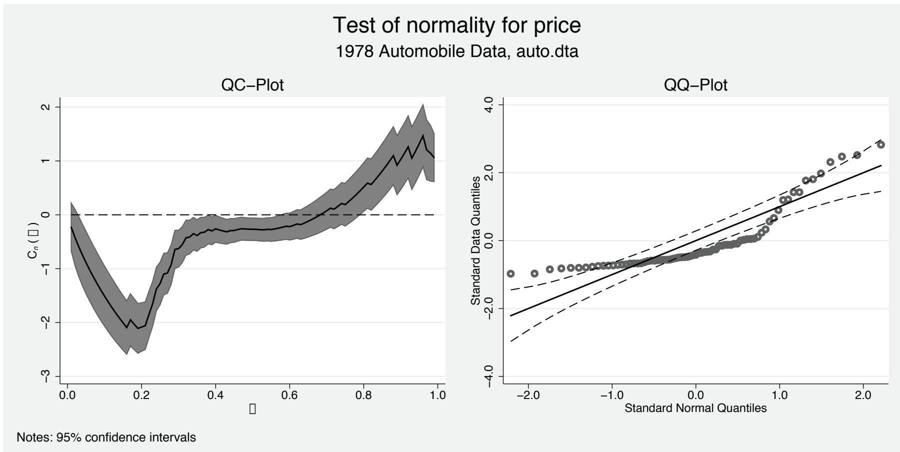
```
. qctest gear_ratio, addqq allstats
Computing plots and T statistics...
Test for normality based on the quantile-mean covariance process

    Ho: gear_ratio has a normal distribution
    Ha: gear_ratio hasn´t a normal distribution
```

| epsilon and T | Stat. | p-value |
|---|---|---|
| 0.001 | | |
| T1 | 1.1201 | 0.0039 |
| T2 | 1.2547 | 0.0039 |
| T3 | 0.1260 | 0.0167 |
| 0.01 | | |
| T1 | 1.1201 | 0.0039 |
| T2 | 1.2547 | 0.0039 |
| T3 | 0.1256 | 0.0162 |
| 0.05 | | |
| T1 | 1.1201 | 0.0028 |
| T2 | 1.2547 | 0.0028 |
| T3 | 0.1102 | 0.0175 |
| 0.10 | | |
| T1 | 1.1201 | 0.0008 |
| T2 | 1.2547 | 0.0008 |
| T3 | 0.0861 | 0.0206 |
| 0.15 | | |
| T1 | 0.8852 | 0.0043 |
| T2 | 0.7836 | 0.0043 |
| T3 | 0.0465 | 0.0798 |
| 0.20 | | |
| T1 | 0.5315 | 0.1089 |
| T2 | 0.2824 | 0.1089 |
| T3 | 0.0208 | 0.2322 |

The results suggest that price is clearly nonnormal, while there is mixed evidence for gear ratio. The following code and figure 4 show the plot analysis where the price distribution has large differences with the zero line in the QC plot, while the gear ratio shows discrepancies only for the 0.80 to 0.90 percentiles. `qctest` supports some customizations to display graphics. The following code shows two examples of customization with a title, a subtitle, and a note.

```
. qctest price, addqq
> title("Test of normality for price")
> subtitle("1978 automobile data, auto.dta")
> note("Notes: 95% confidence intervals")
. qctest gear_ratio, addqq
> title("Test of normality for grear_ratio")
> subtitle("1978 automobile data, auto.dta")
> note("Notes: 95% confidence intervals")
```

Price




Gear ratio

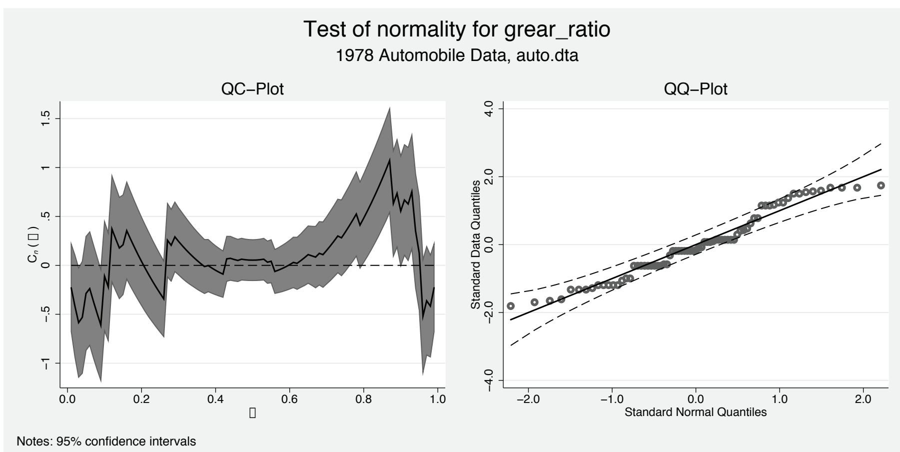



Figure 4. Automobile price and gear ratio from 1978 automobile data

## 4.3 Monte Carlo study comparing QC test with other tests for normality

We implement the proposed test in a Monte Carlo experiment and compare it with three popular tests available in Stata. In particular, we compare it with `sktest` (Fisher 1930; Geary 1947; Bowman and Shenton 1975; Jarque and Bera 1980; White and MacDonald 1980; D'Agostino, Belanger, and D'Agostino 1990; Bai and Ng 2005), `swilk` (Shapiro and Wilk 1965; Shapiro and Francia 1972; Royston 1983), `mvtest` (Mardia, Kent, and Bibby 1979), and `ksmirnov` (Kolmogorov–Smirnov-type test).

We consider 1,000 replications of a different data-generating process [$N(0,1)$, Student's $t$ with 3 degrees of freedom, and $\chi^2_1$] and sample sizes $(50, 100, 500)$. We evaluate `qctest` for the $T_3$ statistic only (that is, we do not use the `allstats` option) and construct the simulation using the `simulate` command in Stata. The results appear in table 2.

Table 2. Monte Carlo simulations

| | | $n=50$ | | | $n=100$ | | | $n=500$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $N(0,1)$ | $t_3$ | $\chi^2_1$ | $N(0,1)$ | $t_3$ | $\chi^2_1$ | $N(0,1)$ | $t_3$ | $\chi^2_1$ |
| | | | | | QC test (`T3`) | | | | | |
| $\epsilon=0.001$ | $\alpha=0.01$ | 0.005 | 0.515 | 1.000 | 0.005 | 0.814 | 1.000 | 0.010 | 1.000 | 1.000 |
| | $\alpha=0.05$ | 0.038 | 0.671 | 1.000 | 0.040 | 0.901 | 1.000 | 0.046 | 1.000 | 1.000 |
| | $\alpha=0.10$ | 0.090 | 0.742 | 1.000 | 0.102 | 0.925 | 1.000 | 0.099 | 1.000 | 1.000 |
| $\epsilon=0.01$ | $\alpha=0.01$ | 0.005 | 0.513 | 1.000 | 0.005 | 0.814 | 1.000 | 0.010 | 1.000 | 1.000 |
| | $\alpha=0.05$ | 0.038 | 0.666 | 1.000 | 0.041 | 0.901 | 1.000 | 0.045 | 1.000 | 1.000 |
| | $\alpha=0.10$ | 0.091 | 0.742 | 1.000 | 0.102 | 0.923 | 1.000 | 0.096 | 1.000 | 1.000 |
| $\epsilon=0.05$ | $\alpha=0.01$ | 0.006 | 0.531 | 1.000 | 0.005 | 0.833 | 1.000 | 0.008 | 1.000 | 1.000 |
| | $\alpha=0.05$ | 0.041 | 0.671 | 1.000 | 0.047 | 0.902 | 1.000 | 0.042 | 1.000 | 1.000 |
| | $\alpha=0.10$ | 0.094 | 0.744 | 1.000 | 0.109 | 0.924 | 1.000 | 0.099 | 1.000 | 1.000 |
| $\epsilon=0.10$ | $\alpha=0.01$ | 0.007 | 0.565 | 1.000 | 0.009 | 0.855 | 1.000 | 0.008 | 1.000 | 1.000 |
| | $\alpha=0.05$ | 0.040 | 0.697 | 1.000 | 0.050 | 0.911 | 1.000 | 0.047 | 1.000 | 1.000 |
| | $\alpha=0.10$ | 0.096 | 0.761 | 1.000 | 0.107 | 0.936 | 1.000 | 0.095 | 1.000 | 1.000 |
| $\epsilon=0.15$ | $\alpha=0.01$ | 0.006 | 0.577 | 0.997 | 0.008 | 0.861 | 1.000 | 0.009 | 1.000 | 1.000 |
| | $\alpha=0.05$ | 0.038 | 0.707 | 0.999 | 0.053 | 0.918 | 1.000 | 0.046 | 1.000 | 1.000 |
| | $\alpha=0.10$ | 0.090 | 0.767 | 0.999 | 0.103 | 0.941 | 1.000 | 0.095 | 1.000 | 1.000 |
| $\epsilon=0.20$ | $\alpha=0.01$ | 0.009 | 0.582 | 0.840 | 0.009 | 0.868 | 0.977 | 0.009 | 1.000 | 1.000 |
| | $\alpha=0.05$ | 0.042 | 0.712 | 0.906 | 0.057 | 0.916 | 0.988 | 0.048 | 1.000 | 1.000 |
| | $\alpha=0.10$ | 0.087 | 0.772 | 0.938 | 0.099 | 0.942 | 0.996 | 0.091 | 1.000 | 1.000 |
| | | | | | Skewness and kurtosis (SK) test | | | | | |
| | $\alpha=0.01$ | 0.009 | 0.502 | 0.976 | 0.007 | 0.747 | 1.000 | 0.009 | 1.000 | 1.000 |
| | $\alpha=0.05$ | 0.057 | 0.661 | 0.999 | 0.044 | 0.865 | 1.000 | 0.063 | 1.000 | 1.000 |
| | $\alpha=0.10$ | 0.097 | 0.741 | 1.000 | 0.094 | 0.907 | 1.000 | 0.111 | 1.000 | 1.000 |
| | | | | | Multivariate (MV) tests | | | | | |
| | $\alpha=0.01$ | 0.012 | 0.573 | 1.000 | 0.005 | 0.833 | 1.000 | 0.016 | 1.000 | 1.000 |
| | $\alpha=0.05$ | 0.045 | 0.714 | 1.000 | 0.046 | 0.896 | 1.000 | 0.065 | 1.000 | 1.000 |
| | $\alpha=0.10$ | 0.088 | 0.776 | 1.000 | 0.076 | 0.927 | 1.000 | 0.100 | 1.000 | 1.000 |
| | | | | | Shapiro–Wilk (SW) test | | | | | |
| | $\alpha=0.01$ | 0.009 | 0.520 | 1.000 | 0.003 | 0.803 | 1.000 | 0.014 | 1.000 | 1.000 |
| | $\alpha=0.05$ | 0.051 | 0.640 | 1.000 | 0.039 | 0.881 | 1.000 | 0.059 | 1.000 | 1.000 |
| | $\alpha=0.10$ | 0.096 | 0.718 | 1.000 | 0.093 | 0.910 | 1.000 | 0.111 | 1.000 | 1.000 |
| | | | | | Kolmogorov–Smirnov test | | | | | |
| | $\alpha=0.01$ | 0.000 | 0.053 | 0.596 | 0.000 | 0.114 | 1.000 | 0.000 | 0.870 | 1.000 |
| | $\alpha=0.05$ | 0.001 | 0.116 | 0.926 | 0.000 | 0.248 | 1.000 | 0.000 | 0.967 | 1.000 |
| | $\alpha=0.10$ | 0.001 | 0.177 | 0.982 | 0.000 | 0.340 | 1.000 | 0.000 | 0.987 | 1.000 |

Note: Monte Carlo simulations based on 1,000 replications.

The Monte Carlo experiment shows an excellent performance of the Bera et al. (2016) tests for normality. The QC tests have correct empirical size for different $\alpha$ theoretical values and for all $\epsilon$. Note also that the SK, MV, and SW tests have a similar size performance but that the Kolmogorov–Smirnov tests are largely undersized. The QC tests also have good power performance. For $t_3$ random variables, the QC tests have more power than the SK and SW tests, but they are outperformed by the MV tests. Note that the power increases as $\epsilon$ increases. For $\chi^2_1$ random variables, all tests have power close to 1, which makes it difficult to compare them. Note that the QC test reduces power as $\epsilon$ increases, probably because the non-Gaussian features are better observed in the tails for asymmetric distributions.

## 5 Conclusion

The new `qctest` command develops Bera et al. (2016) QC tests for normality. These provide a novel way to implement tests for normality of a random variable based on the covariance between the sample mean and sample quantiles. The new procedure also comes with a graphical alternative to visualize departures from Gaussianity.

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