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Summary

For two periods an expert E announces his forecast of the state to a decision-maker D who chooses action. They disagree about the precision of the probability assessments. At the end of period 1 the state is observed. In the last period E makes announcements more extreme than his forecasts. Despite countable equilibria, full revelation is never realised. When in period 1 E is interested in reputation only, the initial equilibrium partition is finite; E makes announcements of greater uncertainty with respect to his forecasts. When E is interested in action too, reputational concerns mitigate exaggerated reports.

Keywords: cheap-talk, expert, statistical bias

JEL Classification: D81, D84

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Forecasts as Repeated Cheap Talk from an Expert of Unknown Statistical Bias

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October 26, 2023

Abstract

For two periods an expert E announces his forecast of the state to a decision-maker D who chooses action. They disagree about the precision of the probability assessments. At the end of period 1 the state is observed. In the last period E makes announcements more extreme than his forecasts. Despite countable equilibria, full revelation is never realised. When in period 1 E is interested in reputation only, the initial equilibrium partition is finite; E makes announcements of greater uncertainty with respect to his forecasts. When E is interested in action too, reputational concerns mitigate exaggerated reports.

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1 Introduction

Quite often it happens that a decision-maker is uncertain about some event of interest and asks someone else for an opinion. However, for an advice to be of any use, the decision-maker must first know how to incorporate someone else's opinion into his own. That is, the decision-maker must make an assessment about how reliable the opinion of the advisor is, before processing the actual advice. Two potential sources of bias influence that preliminary assessment of reliability: a) the "statistical" bias that can affect the advisor's honest opinion, and b) the "strategic" bias that can make the expert's announcement different from his genuine opinion. When the advisor can observe a signal correlated to the event of interest, the statistical bias will depend on the properties that the decision-maker assigns to the stochastic process underlying the realizations of the signal observed by the expert. Instead, the strategic bias will depend on the

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verifiability of the announcements, jointly with the specific mode of interaction between the parties.

In most of the relevant literature both the receiver and the sender of an advice have common priors and agree upon the family of the likelihood functions relating the unknown event of interest to the signal observed by the expert. In that case only the strategic bias can matter. In particular, when the preferences of the advisor and the decision-maker are heterogenous, the resulting conflict of interest can induce the expert to misrepresent the signal by engaging into strategic communication. Consequently, an advice can be biased only because it is distorted at will by the advisor himself, and the decision-maker can have only strategic reasons to be skeptical (Sobel (2013)).

But consider a decision-maker who asks an expert to make a prediction. Notwithstanding common priors about the event of interest, the decision-maker can judge a forecast to be honest and statistically biased at the same time: the prediction can be considered true in the sense that the expert is reputed to report his genuine degree of confidence in the event of interest; the same prediction can be judged to be biased in the sense that the decision-maker believes that the forecasts of the expert are affected by some systematic error. For instance, the decision-maker can believe that the expert overestimates the quality of the signal he observes. In that case there will be no consensus upon the family of the likelihood functions relating the unknown event of interest to the signal perceived by the expert. And the decision-maker will judge the signal to be statistically biased, before the strategic interplay between the parties unfolds. As a consequence, both the statistical bias and the strategic bias will affect the reliability of the advice from the point of view of the decision-maker.

A decision-maker can take into account the possibility of honest and statistically biased forecasts for several reasons. For instance, the decision-maker can suppose the expert to be prey to some perception bias inducing overconfidence[1]. It is not uncommon to wonder how good an expert is, independently of how good an expert believes to be, while the advisor's genuine opinion can be insensitive to the consensus it meets. The statistical literature pays not scant attention to the use of predictions and to the merging of opinions. The issue itself of calibration, dealing with the agreement between the predictions of a forecaster and the actual relative frequency of the predicted event, would be of little interest if the forecasts were statistically unbiased perforce. Furthermore, since even evidence of the past performance of a particular forecaster may not be trivial to collect[2], the statistical bias of an expert may be evaluated by a decision-maker only subjectively. In addition, the confidence that people have in the genuine opinion of others tends to change over time, thereby indicating that reliability is subject to uncertainty.

[1] The idea that an agent can believe that the perceptions of another agent are faulty in some way is explored by Rabin and Schrag (1999). They consider the case in which the perception of the outcome of an information source may not always correspond to the actual informative outcome in a model of confirmatory bias.

[2] Often people need advice also at the preliminary stage at which they select the professionals to be consulted.

Moreover, not infrequently, agents are interested in inducing other agents to believe what they genuinely believe. There can be an issue of alignment of opinions, beyond the usual alignment of incentives. And in the realm of opinions reliability matters. Consider the issue of leadership from the economics of organizations. Hermalin (1998) distinguishes leadership from authority in that following a leader is a voluntary, rather than coerced, activity of the followers[3]. When subordinates receive instructions the rationale of which is beyond the scope of their direct observation or understanding, they will face the same situation of a decision-maker weighting the opinion of an expert. Since orders flow from the top to the bottom of hierarchies, the precision that subordinates expect to characterize the guidelines issued by their managers can reasonably have an impact on the achievable results, and a manager will exert real leadership only if his subordinates hold his competence in sufficiently high regard.

The repeated interaction between a decision-maker and an advisor of uncertain statistical bias is the concern of the present paper. In particular, for two periods, an expert makes predictions about the current state of the world, while the decision-maker chooses the current action to be implemented. In every period the current state of the world can be either -1 or 1, and the best action depends on the beliefs over the states. The expert and the decision-maker aim at maximising the same payoff function in period 2, while in period 1 two different cases are analysed. Either the expert is interested in reputational cheap talk only, or the initial forecast has an impact both on the current common payoff and the future reputation of the expert.

The expert is a probability assessor who believes in the validity of his own predictions: it is as if the expert looked at his own next forecast as a random variable having a probability density function conditional on the true current state only. In the paper the adjectives "honest" and "true" will denote a forecast that is unbiased from the expert's point of view. Instead, the decision-maker is uncertain about the statistical bias that affects the honest predictions of the advisor. In particular, the decision-maker believes the expert to be affected by a systematic error that is measured by the unknown value of a parameter denoted by η. The decision-maker believes that the true prediction of the expert is a random variable having a probability density function conditional on both the true current state and the unknown parameter η. In other words, the honest forecasts of the expert are of uncertain precision for the decision-maker. This is the distinctive assumption of the paper. The decision-maker believes the parameter η to be a drawing from some prior distribution function of η. The expected value of η in period t will be called the reputation of the expert in that period.

In every period the expert sends a message conditional on his true current forecast to the decision-maker. There are as many messages that the advisor can send as the different true predictions that the expert could make. In particular, a message will correspond to the announced probability that the current state

[3] In particular, in Hermalin leadership is the capacity of the leader to induce rational agents to exert effort when the leader can have incentives to mislead them.

is -1. Messages entail no direct cost so that talk is cheap and communication can be strategic. Having listened to the current message from the advisor, the decision-maker updates his own belief that the current state is -1; that posterior belief will be called the induced belief of the decision-maker hereafter. The decision-maker chooses the best current action from his point of view and the current payoff is realized.

At the end of the first period, the true current state becomes verifiable. The decision-maker will revise his assessment of the reliability of the expert, given the message he has received and the true state at the initial stage. In the second period the agents play a new round of interaction with a new state, prediction, message and action.

All aspects of the game are common knowledge, except the true forecasts of the expert in the two periods. Specifically, the decision-maker knows that the expert believes to make unbiased predictions, while the advisor knows that his honest predictions are considered to be statistically biased by the decision-maker. Consequently, the described cheap-talk game does not satisfy the consistency condition of Harsanyi (1967). The notion of equilibrium adopted in the paper is perfect Bayesian equilibria. Since the agents disagree about the informative value of the honest forecasts of the expert, the correspondence between true predictions and messages in equilibrium cannot be interpreted unequivocally in terms of value of the exchanged information.

The paper shows that, in the final period, for every history of the first stage, the advisor will not report his true forecasts. The reason is the following. Since the decision-maker cannot rule out the possibility of an honest but statistically biased expert, even if the advisor reported his genuine degree of confidence in the event of interest, the agents would disagree about the best current action. So, given that the decision-maker chooses the best current action coherently with his own beliefs, the final payoff expected by the expert will decrease in the distance between his true prediction and the final induced belief of the decision-maker. If the decision-maker expects honest messages, the advisor will have an incentive to send dishonest messages: when the expert believes that the final state is more likely to be -1 (1), his announced prediction will exceed (fall short of) his true forecast, if feasible, in order to neutralize the adjustments made by the decision-maker subsequently, because of the alleged statistical bias. In other words, the advisor will exaggerate his reports in order to lead the decision-maker to believe what he believes. The worse the expert's final reputation, the more distorted the announcements.

The result of strategically biased announcements is not new in the cheap-talk literature. But, unlike what usually happens in cheap-talk games, the paper shows that every cardinality of the final partition of the true forecasts can be supported in equilibrium. There is no upper bound to the size of the equilibrium final partitions, that are indeed countable. The reason is that the parties have a conflict of opinion of variable intensity: not only the decision-maker's best final action can be higher as well as lower than the expert's best final action, but also the divergence about the informative power of the honest forecasts of E vanishes as the true prediction of E approaches 0.5. That "non-monotonic" and variable conflict of opinion between the parties is essential in yielding the result of no limit to the size of partitions in period 2 that can be supported by coherent beliefs. In particular, the paper derives the explicit solution for computing the endpoints of the intervals in the final partition. From that it follows that complete revelation could not be achieved even if the cardinality of the final partition tended to infinity.

The paper shows that, in correspondence to final partitions that can be supported by coherent beliefs, the final payoff expected by the expert is a function increasing in his final reputation: if the expected systematic error affecting the advisor's true predictions increases from period 1 to period 2, the expert's expected final payoff will decrease. For that reason the advisor can have instrumental reputational concerns in the initial period. The message that he chooses to send in period 1 will have a double impact: a direct effect on the decision-maker's induced beliefs in period 1, and an indirect effect on the expected final payoff through the updated bias. More precisely, the posterior distribution functions of η induced by different initial messages will be rankable according to the criterion of first order stochastic dominance.

When in period 1 the expert is concerned with reputational cheap talk only (i.e. he has no interest in the current action), the paper shows that again truthful revelation cannot occur in equilibrium. For the expert the choice of the initial message will be a choice over lotteries related to his future reputation. The higher the uncertainty of the true prediction of the expert, the higher the incentive to even more cautious announcement. The size of the initial equilibrium partition will have a finite upper bound related to the prior statistical bias.

When the expert is also interested in the initial action, instrumental reputational concerns will induce a trade-off between two conflicting purposes: a) the maximization of the initial payoff expected by the expert, and b) the minimization of the future loss expected by the advisor when his reputation falls. The initial payoff expected by the expert will increase if the initial messages are suitably inflated with respect to the true forecast in period 1. The future loss expected by the advisor will rise if the announced initial prediction turns out to be at odds with the actual state of the world observed at the end of period 1.

Instrumental reputational concerns will matter only if some level of communication is achieved in both the periods, namely if the size of the initial partition is greater than 2 and the final partitions are not the babbling partition. When reputation matters, in equilibrium both the parties will gain a higher initial expected payoff than they would in single-stage interactions. The reason is that reputational concerns attach opportunity costs to inflated announcements. By offsetting the incentive to misreport in period 1, reputational concerns will enhance the credibility of the initial messages and reduce the dispersion in the size of the subintervals that belong to the initial partition.

Moreover, the relevance of reputational concerns is related to the prior variance of the parameter η: the higher the prior variance of η, the higher will be the variation in the expert's reputation from period 1 to period 2. When the decision-maker is more prone to change his own opinion about the advisor's reliability, then the scope for reputational gains and losses will be greater.

The rest of the paper is organized as follows. I discuss the related literature in Section 2, and I describe the model in Section 3. Section 4 is concerned with the final period of play, while Section 5 focuses on perfect Bayesian equilibria. I conclude in Section 6. All proofs are in the Appendix.

2 Related Literature

Often professional advice is modelled like a case of asymmetric information. At least two parties are interested in some unknown state of the world ω. Only one party, called expert or advisor, can make an observation X. All the parties agree upon the state space, the space of the realizations of X and the distribution of X conditional on ω for every feasible ω (Bayarri-DeGroot (1991)). The consensus of all the parties upon the distribution of X conditional on every state makes expertise equivalent to private information.

Clearly professional advice refers to situations in which there can be communication between the parties. Along the path initiated by Crawford and Sobel (1982) and Green and Stokey (2007), in cheap-talk games, when no party can commit to a course of action, unverifiable information will be disclosed only limitedly, because strategic reasoning leads to strategically biased messages. In particular, before communication takes place, the sender observes the value of some random variable, that is modelled as the sender's type. The sender sends a message to the receiver who takes an action that, jointly with the sender's type, determines the welfare of both the parties. The sender is a "partisan expert" because he has preferences over actions, and his information is more precise than that of the receiver. Messages are not verifiable and are "cheap" in that they do not affect the payoffs directly. Equilibria are shown to involve noisy messages from which only subsets of the states of the world can be identified[4]. The present paper takes from Crawford and Sobel the idea of a partisan expert and the notion of partitional equilibria. However, while in Crawford and Sobel truthful messages can be sent in equilibrium only if the interests of the agents coincide, in the present paper noise will persist in equilibrium, despite the fact that the agents' preferences can be perfectly aligned. Moreover, no upper bound to communication is shown to prevail in equilibrium in period 2.

Ottaviani and Sorensen (2006a) consider a professional expert concerned about his reputation only. The advisor receives a private signal of the current state of the world. The amount of information encoded in the signal depends on the expert's ability, that is known to the advisor only ex-ante. After having

[4] While in cheap-talk games the focus is on the possibility of expertise being shared through communication, in organizational models the attention is on the allocation of decision prerogatives through the design of delegation in such a way that the most effective deployment and use of expertise can be fostered (for instance, Alonso-Matouschek (2008), Demski-Sappington (1987) and Li-Suen (2004)). The contributions concerned with credence goods take a different approach to the issue of professional advice. In that case, fraud and cheating are the major problems in the interaction between experts and consumers (see Dulleck and Kerschbamer (2006) for an extensive review).

observed the signal, the expert is free to send any public message. However, at the end of the period the realization of the state becomes public information, and an evaluator updates his belief regarding the expert's ability. That posterior belief is the advisor's reputation. The expert's payoff depends on his final reputation only. Truthful messages are shown to be incompatible with equilibrium. Specifically, if the evaluator's prior is already concentrated on a particular state, the advisor will always wish to bias his report in the same direction. In the present paper the analysis of the case of reputational cheap talk in period 1 confirms the idea of noisy initial announcements, strategically biased in order to protect future reputation. However, in the present paper reputational cheap talk is related to the fact that the expert is risk averse and cares about his view being shared by the decision-maker in the future[5].

In the literature the idea that agents can agree to disagree is analyzed by contributions that assume either different priors over the states of the world[6], or some disagreement over the advisor's competence[7].

Different priors over the states of the world are the starting point of Che and Kartik (2009). In a single-period game, an advisor and a decision-maker believe ex-ante that the unknown state is a variable normally distributed with given variance but different mean. The expert can exert costly effort in order to make an experiment. With a probability increasing in effort, the experiment is successful: the adviser observes a signal that is normally distributed with given variance and mean equal to the unknown state. If the experiment is not successful, the adviser will observe no informative signal[8]. Once he has performed the experiment, the advisor can choose between disclosing or not disclosing the signal, that cannot be falsified if divulged. Che and Kartik show that the difference of opinion entails a loss of information through strategic communication but creates incentives for information acquisition. First, since the signal is verifiable, the expert with a difference of opinion is motivated to persuade the decision-maker. Second, the advisor with a difference of opinion exerts effort

[5] Contributions concerned with the relative evaluation of many esperts (for instance, Ottaviani-Sorensen (2006b, 2006c) and Scharfstein-Stein (1990)) are less closely related to this paper.

[6] Banerjee and Somanathan (2001) consider the case in which agents share common goals but have different beliefs about which policies can achieve them most effectively. They assume that people have different priors about the state of nature, one agent at least observes an informative signal of the true state and is free to report it to the unique decision-maker. The only way in which agents can influence the decision-maker's action is by providing him with information that will change his assessment of the probabilities over the states, i.e. only "voice" can lead to persuasion. They show that, when agents can mislead, higher costs of communication will actually increase communication by lending greater credibility to reports.

[7] A different approach is developed by Harris and Raviv (1993). They consider a population of agents who receive public information but interpret that information differently and believe in the validity of their own judgements despite the lack of consensus.

[8] The assumption that effort can influence the informative content of the statistical experiment performed by the expert is also common to models in which the experts can be selected according to their preferences over outcomes (Dur-Swank (2005), Gerardi-Yariv (2008), Prendergast (2007)). Incentives for information acquisition are the issue in Szalay (2005), who shows that the delegation of only extreme options can be beneficial, despite the absence of any conflict of interest over actions ex-post.

in order to avoid rational prejudice. Although close in spirit, the set-up of the present paper is dissimilar from the above contribution under many respects. The game is not a single-period game, messages are not verifiable, there is no choice of effort and there is no problem of selection of the experts according to their prior beliefs. More relevantly, in the present paper the parties have the same priors about the current state: they disagree about the informative power of the genuine forecast of the expert. That disagreement can be lessened over time only by the expert improving his reputation as probability assessor.

In one-shot cheap talk games disagreement about the expert's competence is directly addressed by Admati and Pfleinderer (2004). An unknown parameter of interest is modelled as a random variable with continuous distribution on a finite interval of the real line. The sender observes a signal, makes his own assessment of the unknown parameter, and chooses a message to be sent to a set of homogenous receivers. The receivers believe that the signal observed by the sender is perfectly informative of the true parameter with probability λ, while it is pure noise with probability $(1-\lambda)$. Admati and Pfleinderer consider two possible scenarios. In the first scenario, the sender is "rational" in that also he believes to observe a perfect signal with probability λ. Instead, in the second scenario, the sender is "overconfident" because he trusts that his observation is precise with some probability higher than λ. In both the cases the sender always understands how the homogeneous receivers will respond to his messages, and he always wants the receivers' updated value of the unknown parameter to be as close as possible to his own assessment out of altruism. Admati and Pfleinderer consider only equilibria that they define to be "expressive", i.e. such that the partition of the states of the world has as many elements as the given and finite message set. They show that an expressive equilibrium always exists[9]. If the underlying distribution of the unknown parameter is uniform, overconfidence on the sender's side will always lead to a decrease in the amount of information transmitted in equilibrium, because the sender has a tendency to exaggerate. However, for some general distributions, equilibria with a rational sender can be multiple and involve asymmetric partitions of the state set. In such cases, overconfidence can yield equilibria that are more informatively efficient. Also in the present paper the expert can be represented like an overconfident agent, though in the context of statistically biased predictions. The specific contribution of the present paper is that the systematic error that affects the advisor's forecasts is not a dichotomous variable, and the interaction between the parties is repeated so that reputation is a stake for the expert. Moreover, the paper provides an explicit solution for the equilibrium partitions in the last period, and the adverse impact of the conflict of opinion on the information disclosed

[9] Dimitrakas and Sarafidis (2005) consider the case in which, given quadratic loss functions, the value of the constant bias in the sender's preferences is private information. They show that, when the receiver presumes that the sender may have zero bias, there will be a partitional equilibrium with every integer size. In Gordon (2010) the existence of an equilibrium with infinitely (either countably or uncountably) many actions is analysed for the case in which not only the magnitude, but also the direction of the bias that affects the sender's preferences depend on his type.

in period 1 is shown to be mitigated by reputational concerns to some extent.

In Kawamura (2012) the focus is on the interplay between overconfidence and conflict of interest. For the uniform quadratic case, when preferences are perfectly aligned, the sender's overconfidence is proved to reduce the quality of communication, while, for suitably biased preferences, information transmission can even be enhanced by overconfidence, if biased preferences and overconfidence work into opposite directions.

The dynamics of credibility in communication is analyzed by relatively few papers[10]. In Sobel (1985), at every stage of interaction, only the sender receives a signal, that is a binary random variable, independently and identically distributed across time. In every period the sender sends a message to the receiver, who takes an action affecting the current welfare of both the agents. In particular, the per-period payoff of the parties depends on the distance between current action and current signal. The problem is that two different types of sender exist. If the sender is an "enemy", his best current action will be the opposite of the receiver's best current action. If the sender is a "friend", his best current action will be equal to the receiver's best current action. At the end of each period the decision-maker can verify whether the sender sent a truthful message or not. Sender and receiver interact for a finite number of times. The players maximize the undiscounted sum of their single-period payoffs. Repeated interaction between the agents, coupled with verifiable information at the end of each period, is shown to make it worthwhile for the receiver to build a reputation for truthfulness. In equilibrium the sender typically conveys accurate information for the first several periods. An enemy will eventually take advantage of the receiver by misleading him and losing all opportunities for deception in the future. The present paper shares with Sobel the assumption of repeated interaction between expert and decision-maker, both interested in the maximization of the undiscounted sum of the single-period payoffs. However, while in Sobel the decision-maker is uncertain about the preferences of the expert and information is verifiable, in the present paper the decision-maker is uncertain about the reliability of the expert as probability assessor and forecasts can never be verified. Reputational concerns in period 1 are shown to affect the initial equilibrium partition as long as that partition has cardinality higher than 2 and some communication other than babbling occurs in period 2.

In Morris (2001) an informed advisor wishes to convey his valuable information to an uninformed decision-maker in a cheap-talk game that lasts two periods. The decision-maker believes that the expert can have preferences different from his own, biased in favor of particular decisions. In every period, the advisor observes a noisy signal of the binary state of the world, and sends a message to the decision-maker. The decision-maker takes an action in every period, affecting the welfare of both the agents. At the end of period 1 the

[10] Repeated rounds of information transmission within a single period of interaction between expert and decision-maker are analyzed by Li (2007) and Prendergast-Stole (1996) for the case in which the expert receives multiple signals about the same state of the world, and by Krishna-Morgan (2004)) for the case of sequential reports designed as jointly controlled lotteries.

state of the world in period 1 is verified and the decision-maker can update his beliefs about the advisor's preferences. Then, the game is played again, with the same advisor but a new state, signal, message and action. The advisor has no intrinsic reputational concerns like he has in Ottaviani and Sorensen (2006a), but he has instrumental reputational concerns. They arise exclusively from the desire of the advisor to have his advice listened to in the future. Morris shows that, when the advisor and the decision-maker have the same preferences, the advisor can have an incentive to lie in the first period, just in order not to be mistaken for a biased expert. So, when reputational concerns are sufficiently important relatively to the current payoff, no information will be conveyed in equilibrium in the first period. The present paper shares with Morris the idea that reputational concerns are instrumental and it adopts the approach of repeated cheap talk. However, while in Morris an expert is biased because he has biased preferences, here it is the decision-maker who believes that the signal observed by the expert can be affected by an unknown statistical bias. Moreover, in the present paper reputational concerns are shown to be welfare-improving when they mitigate the drive towards exaggerated reports.

While the economic contributions are mostly concerned with strategic information transmission, the statistical literature pays particular attention to the use of predictions. French (1986) summarizes the expert problem in the following way: a decision-maker needs to assess his subjective probability for an event ω of interest; having little substantive knowledge of the factors affecting ω, the decision-maker asks another person for advice. An expert is anyone who can give predictions, i.e. anyone who can make probability statements, called judgments or opinions, concerning the event of interest. The problem is: how should the decision-maker incorporate the honest opinion of an expert into his own? Morris (1974, 1977), Lindley et al. (1979), and French (1980) suggest a Bayesian modeling approach to the use of experts[11]. The decision-maker should look upon the true opinion of the expert as a piece of data: consulting an expert is like performing an experiment, and just as the results of an experiment are a priori unknown to the experimenter, so the advice of the expert is uncertain to the decision-maker before he receives it. According to Morris (1974), the model of the expert in the decision-maker's mind is a likelihood function $l\left(p\left(\omega\right) \mid \omega\right)$, that represents the probability of the event that the expert's prior is $p\left(\omega\right)$, given the event of interest ω (not the probability of a probability in the classical sense). The likelihood function $l\left(p\left(\omega\right) \mid \omega\right)$ is meant to summarize the decision-maker's subjective measure of the expert's reliability[12]. Consequently, a distinction is required between the meaning of an honest probability assessment to the decision maker and the expert himself: the expert looks at his probability assessment as the reflection of his own information, while the decision-maker takes the expert's true opinion as information itself[13]. The present paper borrows from the

[11] Related works are Morris (1983) and Genest-Schervish (1985).

[12] According to Lindley (1982), an expert will be probability calibrated if the decision maker adopts the expert's opinion for his own. Other concepts of calibration are discussed by DeGroot-Fienberg (1983).

[13] Strategic communication from the experts to the decision-maker is not taken into account

statistical literature the idea that the decision-maker has his own model of the true opinion of the expert. Further, it tries to combine it with strategic behavior on the expert's side, as it is typical of the economic literature.

3 Set-up

Players and payoffs. For two periods a decision-maker and an expert, denoted by D and E respectively, interact as follows. In every period the current state of the world is unknown to both the agents. In every period, given his expertise, E is called to assess the probability distribution function of the current state. Once E has announced his forecast, D chooses the current action, and the current payoffs are realised.

The payoff of D in period t depends on the current state of the world, denoted by ω_t, and the current action, denoted by a_t. For simplicity, the payoff of D in period t, denoted by π_t^D, is given by the quadratic loss function[14], i.e.:

$$\pi_t^D(\omega_t, a_t) = -(\omega_t - a_t)^2 \tag{1}$$

The state space is binary with $\omega_t \in \{-1, 1\}$, while the action set, denoted by A, is the closed interval $[-1, 1]$[15].

D puts equal weights on period 1 and period 2 decisions so that his total payoff is given by $\sum_{t=1}^{2} \pi_t^D(\omega_t, a_t)$. Instead, E may put different weights on the decisions in different periods. Two cases will be considered:

- either E is concerned with period 2 decision only, and his payoff in period 2 is given by $-(\omega_2 - a_2)^2$,

- or his preferences are identical to those of D in both periods.

Consequently, the total payoff of E can be expressed as:

$$\sum_{t=1}^{2} \pi_t^E(\omega_t, a_t) = -\upsilon(\omega_1 - a_1)^2 - (\omega_2 - a_2)^2 \qquad \text{with } \upsilon \in \{0, 1\} \tag{2}$$

Beliefs. Ex-ante both the agents believe each state to occur with equal probability. The states in different periods are stochastically independent. Let p_t denote the probability, assessed by E, that ω_t is equal to -1 during period t. Probability p_t is the true/honest forecast or prediction of E. The sample space of p_t is the unit interval of the real line.

Both the agents agree about the fact that the true prediction of E is informative. However, E and D disagree about how reliable p_t can be. E cannot but trust his own true assessment, and, since p_t is his forecast, it is like if E

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also by Lehrer (1998). He considers the case in which different experts propose their own assessments regarding the real distribution over the states of the world. The focus is on the conditions that the proposed distributions must satisfy in order to rank an expert more knowledgeable than another.

[14] The quadratic loss function is a common specification in the literature, with the implication that the best action is the expected state.

[15] An optimal action is always feasible for every belief over the states.

believed that the probability density function (p.d.f.) of p_t conditional on ω_t, denoted by $l^E(p_t \mid \omega_t)$, were as follows:

$$l^E(p_t \mid -1) = 2p_t \qquad l^E(p_t \mid 1) = 2(1 - p_t) \tag{3}$$

(3) says that E looks at his own forecast as an unbiased estimate. If E were announced his own prediction, he would adopt it.

On the contrary, D looks at the true opinion of E as a piece of data that can be generated by different types of statistical experiments. D believes that consulting an expert is like running an experiment characterized by a specific and unknown statistical bias η. The parameter η takes value in the closed unit interval of the real line. When η is equal to 0, the expert is perfectly calibrated and makes honest forecasts that should be adopted by D as they are. When η is equal to 1, the opinion of the expert should be totally disregarded by D. For intermediate values of η, some adjustment should be in place.

In particular, D believes that the p.d.f. of p conditional on (ω_t, η), denoted by $l^D(p_t \mid \omega_t, \eta)$, is as follows:

$$\begin{aligned} l^D(p_t \mid -1, \eta) &= 2(1-\eta)\, p_t + \eta \\ l^D(p_t \mid 1, \eta) &= 2(1-\eta)(1 - p_t) + \eta \end{aligned} \tag{4}$$

(4) says that the likelihood of p_t conditional on η is a mixture of the likelihoods of p_t for a perfectly calibrated and a totally uninformative expert, with weights $(1-\eta)$ and η respectively. Looking at the distance between the true beliefs of an expert of bias η and the posterior beliefs of D conditional on (p_t, η), (4) implies that the conflict of opinion increases as p_t moves away from 0.5. The more extreme is the true prediction of E, the heavier the adjustment that D would consider appropriate[16].

[16] Both (3) and (4) can be obtained as results of preliminary assumptions that deal directly with the information that E gathers in order to formulate his forecasts. Consider the following assumptions. There is an information source, denoted by K_t. The sample space of K_t is the unit interval of the real line. The p.d.f. of K_t conditional on ω_t, denoted by $f(k_t \mid \omega_t)$, is as follows:

$$f(k_t \mid -1) = 2k_t; \quad f(k_t \mid 1) = 2(1 - k_t)$$

Only E can observe a signal, denoted by S_t, related to the outcome k_t. Every signal is non-verifiable and private information of E. The sample space of S_t is the unit interval.

E and D disagree about the quality of the signal S_t. E believes that the p.d.f. of S_t conditional on k_t, denoted by $f^E(s_t \mid k_t)$, is as follows:

$$f^E(s_t \mid k_t) = \begin{cases} 1 \text{ if } & s_t = k_t \\ 0 \text{ if } & s_t \neq k_t \end{cases}$$

From Bayes' rule, p_t is always equal to s_t so that p_t always belongs to the unit interval.

Instead, according to D the p.d.f. of S_t conditional on (k_t, η), denoted by $f^D(s_t \mid k_t, \eta)$, is as follows:

$$f^D(s_t \mid k_t, \eta) = \begin{cases} 1-\eta & \text{if } s_t = k_t \\ \eta & \text{if } s_t \neq k_t \end{cases}$$

That is, according to D, E can misread the outcome k, as if E were a noisy channel of the information source K with equivocation η.

An alternative specification is the following. Both the agents believe that the p.d.f. of S_t conditional on (k_t, η) is $f^D(s_t \mid k_t, \eta)$, but the agents have different priors on η. In particular, E assigns prior probability 1 to the event "η is equal to 0".

True predictions at different times are supposed to be stochastically independent by both E and D.

D is uncertain about the value of η affecting the predictions of E. Initially D believes that the unknown parameter η has a p.d.f., denoted by $g_1(\eta)$, supported on $[0,1]$, with variance σ^2. Let $\overline{\eta}_t$ denote the value of η expected by D in period t. Since $\overline{\eta}_t$ is a measure of the systematic error built into the true opinion of E, $\overline{\eta}_t$ will be called the reputation of E in period t.

Observables. Only E can know his own mind, included his true prediction p_t. What D can observe is the forecast that E decides to announce conditional on p_t. E chooses a message m_t from the unit interval of the real line: the range of the messages that E can send corresponds to the entire spectrum of the honest opinion that E can entertain. Having received m_t, D chooses a_t. Neither E can commit to a message rule, nor D can commit to a decision rule at any time.

At the end of period 1, ω_1 is publicly observed. Given history $h_1 = (\omega_1, m_1)$, D will update his belief about the bias η of E, and the p.d.f. $g_{h_1}(\eta)$ will be relevant in the second period.

Since D revises his opinion about the professional ability of E at the end of period 1, I will refer to the case in which E is interested only in his payoff in period 2 as the case of reputational cheap talk, while I will call the case in which E has the same preferences of D in both periods the case of conflict between inflated reports and reputation.

Timing. In every period the sequence of the events is as follows:
i) nature selects the current state ω_t;
ii) E makes his forecast p_t and sends a message m_t to D;
iii) D updates his beliefs about ω_t and chooses a_t.

Common knowledge. All aspects of the interaction between E and D are common knowledge except p_t in every period. In particular, the conflict of opinion is common knowledge: E knows that his own true predictions are believed to be statistically biased by D. So the described set-up does not conform to the Harsanyi's doctrine[17] because, in Feinberg's (2000) words, even if p_t were public information, E and D would agree to disagree about the posterior probability that ω_t is equal to -1, and that disagreement would be commonly known between them[18].

[17] According to Harsanyi (1967) consistent Bayesian games are games in which the players are uncertain about some parameters, and their subjective probability distributions over the unknown parameters correspond to conditional probability distributions from a unique prior joint p.d.f. known to all the players. That consistency condition, or common prior assumption, is not satisfied by the game played by E and D in the present paper.

[18] The common prior assumption is a subject of debate. On one side, reconcilable priors are pervasive in the economic literature. Dekel, Fudenberg and Levine (2004) argue that the absence of common priors is difficult to justify as the long-run result of a learning process. On the other side, Aumann (1974, 1987) underlines that distinct priors are mathematically perfectly consistent. Acemoglu et al. (2006) consider the case in which two individuals observe the same infinite sequence of signals about some underlying parameter, have different priors and have non degenerate probability distributions over the conditional distribution of signals given the unknown parameter. They show that the individuals can agree to eventually disagree, because the common observation of the same sequence of signals can lead to a

The solution concept is perfect Bayesian equilibrium.

4 Strategies and Beliefs for One Period Ahead

Consider the last stage of the game after history h_1 has occurred.

E will choose a message rule μ_{h_1}, i.e. a family of conditional probability density functions of m_2, each one denoted by $\mu_{h_1}(m_2 \mid p_2)$, for every possible p_2. Having received message m_2, D will assess his posterior probability $w_{h_1}(m_2)$ that ω_2 is equal to -1. Let the set of posterior beliefs of D be denoted by w_{h_1}. That set will be called coherent with message rule μ_{h_1} when, from (4), every $w_{h_1}(m_2)$ with m_2 in the support of μ_{h_1} is the outcome of Bayes rule as follows:

$$
\begin{aligned}
w_{h_1}(m_2) &= \left(1-\overline{\eta}_{h_1}\right)\frac{\int_0^1 p_2\mu_{h_1}(m_2 \mid p_2)dp_2}{\int_0^1 \mu_{h_1}(m_2 \mid p_2)dp_2}+\frac{1}{2}\overline{\eta}_{h_1} \qquad (5)\\
&= \left(1-\overline{\eta}_{h_1}\right)E_{\mu_{h_1}}\left[p_2 \mid m_2\right]+\frac{1}{2}\overline{\eta}_{h_1}
\end{aligned}
$$

I will say that a belief is an induced belief if it is related to a final message sent with positive probability. From (5) it follows that every induced belief belongs to the real interval $\left[\frac{1}{2}\overline{\eta}_{h_1}, 1-\frac{1}{2}\overline{\eta}_{h_1}\right]$.

Finally, D will choose an action rule α_{h_1}, i.e. a function selecting action a_2 for every possible message m_2[19].

Strategies and beliefs in period 2 will constitute a Bayesian Nash equilibrium when the following property is satisfied.

Property $\hat{H}$. Given the p.d.f. $g_{h_1}(\eta)$:

1) action rule $\widehat{a}_{h_1}$ maximizes the final payoff expected by D for every belief $\hat{w}_{h_1}(m_2)$;

2) message rule $\widehat{\mu}_{h_1}$ is such that every p.d.f. $\widehat{\mu}_{h_1}(m_2 \mid p_2)$ maximizes the final payoff expected by E, given $\widehat{a}_{h_1}$;

3) every induced belief $\hat{w}_{h_1}(m_2)$ is coherent with $\widehat{\mu}_{h_1}$[20].

Property $\hat{H}$ guarantees that each party responds optimally to the opponent's strategy choice, taking into account its implications in the light of his probabilistic beliefs and maximising the expected payoff over his possible strategy

divergence of opinion. Finally, Maschler et al (2013) underline that Bayesian Nash equilibria are both computable and applicable in inconsistent situations, being the set of consistent belief spaces a set of measure zero within the set of belief spaces.

[19] Since from (1) the expected payoff of D in period 2 is strictly concave in a_2, D will never use mixed strategies in equilibrium. So, randomized actions can never be optimal for D, once he has come up with his final forecast of the current state of the world. Hence, attention will be restricted to deterministic action rules.

[20] Without loss of generality, I assume that D has posterior beliefs in the set of induced beliefs for messages not in the support of $\widehat{\mu}_{h_1}$.

choices. Under property $\hat{H}$ the parties' conditional probabilistic beliefs about each other's moves and characteristics are self-confirming.

Analogously to one-shot cheap-talk games, truthtelling in the final period cannot be an equilibrium outcome. The problem is that truthtelling implies continuity and monotonicity of the induced beliefs of D in correspondence to some suitably arranged subsets of messages sent with positive probability. But, when the induced beliefs of D are continuous and monotonic, then either they are incoherent with the message rule, or E has not adopted a best strategy. Indeed, on one side, D cannot adopt the honest forecasts of E for his own, because he expects them to be statistically biased; on the other side, the final payoff expected by E is decreasing in the distance between his true prediction and the induced belief of D in period 2. So E has an interest in announcing an inflated forecast in favour of the state of nature that he believes to be most likely, in order to counterbalance the adjustment that D will make subsequently. As a consequence, property $\hat{H}$ is always violated under truthtelling as Lemma 1 implies.

Lemma 1 *: under property $\hat{H}$ no real interval can belong to the set $\widehat{w}_{h_1}$ of induced beliefs of D.*

Since Lemma 1 rules out the possibility of an uncountable set, consider a countable set for the induced beliefs of D. In that case E can find an optimal partition of his predictions for every set of induced beliefs.

Let $Y_{n_2|h_1} = \left(y_{0,n_2|h_1}, ..., y_{n_2,n_2|h_1}\right)$ denote a partition of the unit interval of the real line, where $0 = y_{0,n_2|h_1} < y_{i,n_2|h_1} < y_{n_2,n_2|h_1} = 1$ for every i from 1 to $(n_2 - 1)$. Let $Y_{i,n_2|h_1}$ denote the interval $\left[y_{i-1,n_2|h_1}, y_{i,n_2|h_1}\right]$.

Consider an ordered set of n_2 beliefs, with representative element denoted by $w_{i,n_2|h_1}$, so that $w_{i-1,n_2|h_1} < w_{i,n_2|h_1} < w_{i+1,n_2|h_1}$, with $i = 2, ...(n_2 - 1)$. Finally, consider a family of subsets of messages, with representative element denoted by $M_{i,n_2|h_1}$, such that:

$$M_{i,n_2|h_1} = \left\{m_2 \mid w_{h_1}(m_2) = w_{i,n_2|h_1}\right\} \tag{6}$$

In cheap-talk games à la Crawford and Sobel (1982), since the role played by messages is independent from their meaning in any natural language, an equilibrium action profile can be supported by different equilibrium message strategies as long as they induce the receiver to hold identical posterior beliefs about the states of the world. Green and Stokey (2007) use the expression "essential equilibria": the strategies of each player are essentially unique when choosing a different strategy from the optimal set does not alter the statistical relationship between the observations and the action taken.

The following Proposition shows that for every cardinality n_2 there always exist a final partition $\hat{Y}_{n_2|h_1}$ of the true predictions of E and a unique set $\hat{w}_{n_2|h_1}$ of induced beliefs of D consistent with property $\hat{H}$.

Proposition 1 *: given the p.d.f.* $g_{h_1}(\eta)$*, under a message rule where* $\hat{\mu}(m_2 \mid p_2)$ *is uniform, supported on* $\hat{M}_{i,n_2|h_1}$ *if* $p_2 \in \left(\hat{y}_{i-1,n_2|h_1}, \hat{y}_{i,n_2|h_1}\right)$*, for every natural number* n_2 *there is an essential equilibrium such that:*

$$\begin{aligned}
\widehat{y}_{i,n_2|h_1} &= \frac{1}{2} - \frac{1}{2}\frac{\gamma_{h_1}^{i} - \gamma_{h_1}^{n_2-i}}{1-\gamma_{h_1}^{n_2}} \qquad i = 0, ..., n_2 \qquad (7)\\
\widehat{w}_{i,n_2|h_1} &= \frac{1}{2} - \frac{1}{2}\frac{2\gamma_{h_1}}{1+\gamma_{h_1}}\frac{\gamma_{h_1}^{i-1} - \gamma_{h_1}^{n_2-i}}{1-\gamma_{h_1}^{n_2}} \qquad i = 1, ..., n_2\\
\textit{where} \quad \gamma_{h_1} &= \frac{1-\sqrt{\eta_{h_1}}}{1+\sqrt{\eta_{h_1}}}
\end{aligned}$$

Proposition 1 is an example of the result of Crawford and Sobel (1982) concerning partitional equilibria. In line with Lemma 3 of Crawford and Sobel (1982), the equilibrium partitions for n_2 and $n_2 + 1$ are such that $\hat{y}_{i,n_2+1|h_1}$ belongs to the interval $\left(\hat{y}_{i-1,n_2|h_1}, \hat{y}_{i,n_2|h_1}\right)$ for every i, since there is one solution for a fixed cardinality of the equilibrium partition. However, like in Admati and Pfleiderer (2004), Proposition 1 departs from the usual outcomes of cheap-talk games in that there is no upper bound to the size of the partitions compatible with property $\hat{H}$ for every p.d.f. $g(\eta)$. The reason is the following. Call the action $a_u^E(p_2) \in \arg\max \pi_2^E(a_2 \mid p_2)$ the unconstrained best action of E. Cheap-talk games are often marked by some "monotonic" conflict of interest of given intensity between the parties: for every state of the world the absolute distance between the unconstrained best action of E and the best action of D is never below some threshold. Instead, in the present paper the problem lies with a variable conflict of opinion between the parties: the best action of D is higher than the best action of E when the true prediction of E is above 0.5, while it is lower otherwise, and the absolute distance between the actions decreases as the true prediction of E approaches 0.5. That property of preference reversal over actions, described by Melumad and Shibano (1991), is essential in yielding the result of no finite upper bound to the size of the partitions satisfying property $\hat{H}$. For the same reason every partition that satisfies property $\hat{H}$ is symmetric around 0.5.

Unfortunately, the results that the set of partitional equilibria is countable remains incompatible with complete communication, despite the fact that communication improves as n_2 increases. Indeed:

Corollary 1 *: the equilibrium will never converge to full revelation.*

In correspondence to every final partition that satisfy property $\hat{H}$, the final payoff expected by each party at the end of the first period can be computed from (7). In particular, it can be shown that:

Corollary 2 *: under property* $\hat{H}$ *the final payoff expected by party* r*, with* $r = E, D$*, is equal to:*

$$\widehat{\pi}^r(n_2 \mid h_1) = -1 \qquad \textit{if} \;\; n_2 = 1$$

$$\begin{aligned}\widehat{\pi}^{E}\left(n_2 \mid h_1\right) &= -1+\frac{1}{4}\rho^{E}_{n_2}\left(\overline{\eta}_{h_1}\right) \qquad \textit{if } \; n_2>1 \\ \widehat{\pi}^{D}_{2}\left(n_2 \mid h_1\right) &= -1+\frac{1}{4}\rho^{D}_{n_2}\left(\overline{\eta}_{h_1}\right)\end{aligned} \tag{8}$$

where:

$$\begin{aligned}\rho^{E}_{n_2}\left(\overline{\eta}_{h_1}\right) &= \left(1-\overline{\eta}^{2}_{h_1}\right)\left[1+\frac{1-\overline{\eta}_{h_1}}{3+\overline{\eta}_{h_1}}\varepsilon_{n_2}\left(\overline{\eta}_{h_1}\right)\right] \\ \rho^{D}_{n_2}\left(\overline{\eta}_{h_1}\right) &= \left(1-\overline{\eta}_{h_1}\right)^{2}\left[1+\frac{1-\overline{\eta}_{h_1}}{3+\overline{\eta}_{h_1}}\varepsilon_{n_2}\left(\overline{\eta}\right)\right] \\ \varepsilon_{n_2}\left(\overline{\eta}_{h_1}\right) &= \frac{1-\gamma^{n_2-2}_{h_1}-\gamma^{n_2+2}_{h_1}+\gamma^{2n_2}_{h_1}}{1-2\gamma^{n_2}_{h_1}+\gamma^{2n_2}_{h_1}}\end{aligned} \tag{9}$$

Provided $n_2>1$, $\widehat{\pi}^{r}_{2}\left(n_2 \mid h_1\right)$ is decreasing and concave in $\overline{\eta}_{h_1}$, and increasing in n_2. Under coherent beliefs of D but no interim incentive constraint for E, the partition of the true forecasts that would maximize the final payoff expected by each party has intervals of equal size.

For partitions satisfying property $\hat{H}$ the expected distance between the true forecasts of E and the induced beliefs of D can be taken to measure the conflict of opinion between the parties, which depends on the reputation of E in period 2. With the exception of the babbling equilibrium, the final payoff expected by each party will decrease as that conflict of opinion rises. If the parties could commit to the distribution of the intervals within a partition of fixed cardinality n_2 under coherent beliefs of D, they would prefer intervals of equal size, that are the ones that would prevail if $\overline{\eta}_{h_1}$ tended to 0. But, intervals of equal size are incompatible with property $\hat{H}$ when n_2 is greater than 2.

5 Strategies and Beliefs for Two Periods Ahead

Having analysed the Bayesian Nash equilibria that can prevail in period 2, let me go back and consider period 1.

In the first period E will choose an initial message rule μ_1, i.e. a p.d.f. of m_1 conditional on his true initial forecast p_1, denoted by $\mu(m_1 \mid p_1)$, for every possible p_1. D will associate every message m_1 to his own posterior probability $w\left(m_1\right)$ that ω_1 is equal to -1. The set w_1 of induced beliefs will be coherent with message rule μ_1 when the belief $w\left(m_1\right)$ is the outcome of Bayes rule for every message m_1 in the support of μ_1, so that $w\left(m_1\right) = \left\{\left(1-\bar{\eta}_1\right)E\left[p_1 \mid \mu_1\left(m_1\right)\right]+\frac{1}{2}\bar{\eta}_1\right\}$. Finally, D will choose an action rule α_1, i.e. a function selecting an action a_1 for every possible message m_1.

Message m_1 will have a straightforward direct effect on the beliefs of D in period 1. But, at the end of period 1, D will take message m_1 into account again in order to revise his assessment of the unknown statistical bias of E, and, from Corollary 2, the equilibrium expected payoffs in period 2 will depend on the estimated bias, provided $n_2 > 1$[21]. Consequently, when no babbling equilibrium is expected in the final period, message m_1 will have an indirect effect as well. In other words, reputational concerns will arise when the prospective reputation of E matters and changes with different initial messages.

Since it is D who updates both his beliefs about ω_1, given m_1, and the final p.d.f. of η, given h_1, and since the same conjecture about the initial message rule is needed for both the processes of updating, the present and the future consequences of the choice over the initial message m_1 are functionally related. In particular, the updated p.d.f. of η will be:

$$g(\eta \mid m_1, \omega_1) = \frac{\displaystyle\int_0^1 \mu(m_1 \mid p_1) l^D (p_1 \mid \omega_1, \eta)\, dp_1}{\displaystyle\int_0^1 \int_0^1 \mu(m_1 \mid p_1) l^D (p_1 \mid \omega_1, \eta)\, g_1(\eta)\, dp_1 d\eta} g_1(\eta) \tag{10}$$

In particular:

$$\begin{aligned} g(\eta \mid m_1, -1) &= \frac{(1-\eta)\, E_{\mu_1}[p_1 \mid m_1] + \frac{1}{2}\eta}{(1-\overline{\eta}_1)\, E_{\mu_1}[p_1 \mid m_1] + \frac{1}{2}\overline{\eta}_1} g_1(\eta) \\ g(\eta \mid m_1, 1) &= \frac{(1-\eta)\, E_{\mu_1}[1-p_1 \mid m_1] + \frac{1}{2}\eta}{(1-\overline{\eta}_1)\, E_{\mu_1}[1-p_1 \mid m_1] + \frac{1}{2}\overline{\eta}_1} g_1(\eta) \end{aligned}$$

Provided $E_{\mu_1}[p_1 \mid m_1]$ is different from $\frac{1}{2}$, the posterior distribution functions $G(\eta \mid m_1, -1)$ and $G(\eta \mid m_1, 1)$ will be orderable for first-order stochastic dominance. In particular, for $\eta > \eta'$:

$$\frac{g(\eta \mid m_1, -1)}{g(\eta' \mid m_1, -1)} \gtrless \frac{g(\eta \mid m_1, 1)}{g(\eta' \mid m_1, 1)} \text{ if } E_{\mu_1}[p_1 \mid m_1] \lessgtr \frac{1}{2}$$

Indeed, when D expects the initial true forecast to be lower than 0.5, then the reputation of E will be better in case of history $(m_1, 1)$ than in case of history $(m_1, -1)$.

Given message rule $\mu_1(m_1)$, at the end of the first period the expected bias will be:

$$\begin{aligned} E[\eta \mid m_1, -1] &= \overline{\eta}_1 - \sigma^2 \frac{E_{\mu_1}[p_1 \mid m_1] - \frac{1}{2}}{(1-\overline{\eta}_1) E_{\mu_1}[p_1 \mid m_1] + \frac{1}{2}\overline{\eta}_1} \\ E[\eta \mid m_1, 1] &= \overline{\eta}_1 + \sigma^2 \frac{E_{\mu_1}[p_1 \mid m_1] - \frac{1}{2}}{(1-\overline{\eta}_1) E_{\mu_1}[1-p_1 \mid m_1] + \frac{1}{2}\overline{\eta}_1} \end{aligned} \tag{11}$$

Recalling that m_1 represents the reported probability that the current state ω_1 is equal to -1, let me say that an announced forecast m_1 is "at odds with evidence" in the cases in which:

[21] If $n_2 = 1$, the expected future payoff will be independent of $\overline{\eta}_{h_1}$.

- $E_{\mu_1}[p_1 \mid m_1] < \frac{1}{2}$ and $\omega_1 = -1$ or
- $E_{\mu_1}[p_1 \mid m_1] > \frac{1}{2}$ and $\omega_1 = 1$.

Let me say that an announced prediction is "in line with evidence" in the cases in which:

- $E_{\mu_1}[p_1 \mid m_1] > \frac{1}{2}$ and $\omega_1 = -1$ or
- $E_{\mu_1}[p_1 \mid m_1] < \frac{1}{2}$ and $\omega_1 = 1$.

Hence, if the announcement in period 1 turns out to be at odds with evidence, from (11) the reputation of E at the start of the second period will be worse than before. On the contrary, if the announcement in period 1 is in line with evidence, the reputation of E at the start of the second period will improve.

Now let me consider different initial messages. When m_1' and m_1'' induce different beliefs, the ratio $g(\eta \mid m_1', \omega_1) / g(\eta \mid m_1'', \omega_1)$ of the posterior probability density functions will satisfy the property of monotone likelihood, making also the distribution functions $G(\eta \mid m_1', \omega_1)$ and $G(\eta \mid m_1'', \omega_1)$ orderable for first-order stochastic dominance. Indeed, $w(m_1')$ will be greater than $w(m_1'')$ only if $E_{\mu_1}[p_1 \mid m_1']$ is higher than $E_{\mu_1}[p_1 \mid m_1'']$. It follows that:

$$\frac{g(\eta \mid m_1'', -1)}{g(\eta' \mid m'', -1)} > \frac{g(\eta \mid m_1', -1)}{g(\eta' \mid m_1', -1)} \quad \text{and} \quad \frac{g(\eta \mid m_1', 1)}{g(\eta' \mid m_1', 1)} > \frac{g(\eta \mid m_1'', 1)}{g(\eta' \mid m_1'', 1)}$$
$$\text{for } \eta > \eta'$$

The interpretation is the following: when m_1' induces D to put a higher probability on the event "ω_1 is equal to -1" than m_1'' does, then D will expect E to be more reliable in case of history $(m_1', -1)$ than in case of history $(m_1'', -1)$, while the opposite will hold if the initial state turns out to be 1. The correspondence between the difference in the induced beliefs, $(w(m_1') - w(m_1''))$, and the difference in the reputation of E, $\left(\overline{\eta}_{m_1', \omega_1} - \overline{\eta}_{m_1'', \omega_1}\right)$, is the connecting link between the direct and the indirect effects generated by the choice of the initial message.

Since what is expected to happen in period 2 can matter for strategies in period 1, I will focus on perfect Bayesian equilibria. A perfect Bayesian equilibrium meets the following conditions:

1) the initial message rule $\widetilde{\mu}_1$ maximizes the total payoff expected by E, given the initial forecast p_1, the initial action rule $\widetilde{\alpha}_1$ and the equilibrium sets of final strategies and beliefs conditional on the posterior distributions of η for every history (m_1, ω_1).

2) The induced beliefs of D in period 1 are coherent with $\widetilde{\mu}_1$ for every message m_1 sent with positive probability.

3) The initial action rule $\widetilde{\alpha}_1$ maximizes the initial payoff expected by D[22], given the set of his induced belief $\tilde{w}_1$.

4) For every message m_1 sent with positive probability, the posterior p.d.f $\widetilde{g}(\eta \mid m_1, \omega_1)$ is as in (10).

5) Property $\hat{H}$ holds for every posterior p.d.f. $\widetilde{g}(\eta \mid m_1, \omega_1)$ with m_1 in the support of $\widetilde{\mu}_1$.

[22] The final payoff expected by D is not affected by his initial action rule, because D cannot commit to any specific plan of action in period 2.

In period 2 the Bayesian Nash equilibria are partitional and countable,hence the final payoff expected at the beginning of period 1 will depend also on the cardinality n_2 of the future equilibrium partitions. For simplicity, in what follows I will consider only perfect Bayesian equilibria where the size n_2 is constant, whether the true initial state ω_1 turns out to be -1 or 1[23].

5.1 Reputational cheap talk

Consider the case in which the initial period is a sort of trial period: E is not interested in the current action, however the current forecast is announced by E to D, while in period 2 E will have an interest in influencing the beliefs of D. Which reporting strategy would E choose in period 1? By the end of the period, the state of the world becomes publicly observable. Consequently, past announcements and evidence will be associated.

Once again, there cannot be truthful revelation in period 1.

Lemma 2 *: truthtelling cannot be supported in equilibrium.*

The interpretation is as follows. When D believes E to be honest and to report his true prediction exactly, the future expected payoff of E will be a continuous and monotonic function of m_1. If E has a true prediction relatively extreme (i.e. close to 0 or 1), he may exaggerate his report, in order to enhance his future reputation. Instead, if E is rather uncertain and has a true prediction relatively close to $\frac{1}{2}$, he can smooth his announcement by delivering a message even closer to $\frac{1}{2}$, in order to protect his reputation in case of reports at odds with evidence. In terms of final expected bias, a report at odds with evidence will hurt the reputation of E more than a report in line with evidence will improve it.

Since I want to focus on reputational concerns, in what follows the size n_2 of the equilibrium final partitions is supposed to be greater than 1, and it will not be added to the notation.

Let $X_{n_1} = (x_{i,n_1}, ..., x_{n_1,n_1})$ denote a partition of the unit interval of the real line, where $0 = x_{0,n_1} < x_{i,n_1} < x_{n_1,n_1} = 1$ for every i from 1 to $(n_1 - 1)$. Let X_{i,n_1} denote the interval $[x_{i-1,n_1}, x_{i,n_1}]$.

Consider an ordered set of n_1 beliefs, with representative element denoted by w_{i,n_1}, so that $w_{i-1,n_1} < w_{i,n_1} < w_{i+1,n_1}$, with $i = 2, ...(n_1 - 1)$. Finally, consider a family of subsets of messages, with representative element denoted by M_{i,n_1}, such that:

$$M_{i,n_1} = \{m_1 \mid w_1(m_1) = w_{i,n_1}\}$$

Let m_{i,n_1} denote an initial message in M_{i,n_1}, and let $\overline{\eta}_{i,n_1|\omega_1}$ denote the reputation of E at the final period after history (m_{i,n_1}, ω_1). Define the following

[23] More generally, the size of the final partitions could depend on the particular history (m_1, ω_1) of the game in period 1.

function:

$$
\begin{aligned}
\psi_i(p_1) &= \hat{\pi}_2^E(\overline{m}_{i+1,n_1} \mid p_1) - \hat{\pi}_2^E(\overline{m}_{i,n_1} \mid p_1) \\
= \quad & p_1 \frac{\rho^E\left(\overline{\eta}_{i+1,n_1|-1}\right) - \rho^E\left(\overline{\eta}_{i,n_1|-1}\right)}{4} \\
& - (1-p_1) \frac{\rho^E\left(\overline{\eta}_{i,n_1|+1}\right) - \rho^E\left(\overline{\eta}_{i+1,n_1|+1}\right)}{4}
\end{aligned}
$$

$\psi_i(\cdot)$ represents the difference between the future payoffs expected by E when he sends messages m_{i+1,n_1} and m_{i,n_1}, conditional on E having true prediction p_1. Hence, the function $\psi_i(\cdot)$ depends on the expected statistical biases induced by m_{i,n_1} and m_{i+1,n_1}. I will refer to the function $\psi_i(\cdot)$ as the relative indirect effect of sending message m_{i+1,n_1} instead of message m_{i,n_1}.

Proposition 2 *: under a message rule where $\tilde{\mu}(m_1 \mid p_1)$ is uniform, supported on $\tilde{M}_{i,n_1}$ if $p_1 \in (\tilde{x}_{i-1,n_1}, \tilde{x}_{i,n_1})$, the equilibrium initial partition is symmetric at* 0.5 *and every element satisfies the following condition:*

$$\psi_i(\tilde{x}_{i,n_1}) = 0 \qquad i = 1, ..., n_1$$

Moreover:

$$
\begin{aligned}
\overline{\eta}_{i,n_1|-1} &= \overline{\eta}_1 + \sigma_1^2 \frac{1 - \tilde{x}_{i-1,n_1} - \tilde{x}_{i,n_1}}{(1-\overline{\eta}_1)(\tilde{x}_{i-1,n_1} + \tilde{x}_{i,n_1}) + \overline{\eta}_1} \\
\overline{\eta}_{i,n_1|1} &= \overline{\eta}_1 - \sigma_1^2 \frac{1 - \tilde{x}_{i-1,n_1} - \tilde{x}_{i,n_1}}{(1-\overline{\eta}_1)(2 - \tilde{x}_{i-1,n_1} - \tilde{x}_{i,n_1}) + \overline{\eta}_1}
\end{aligned}
\tag{12}
$$

Further, the set of induced beliefs in equilibrium is finite. In particular, $\left\langle \frac{n_1}{2} \right\rangle < 1 + \frac{1+\overline{\eta}_1}{2\overline{\eta}_1}$, where $\left\langle \frac{n_1}{2} \right\rangle$ denotes the highest natural number not greater than $\frac{n_1}{2}$

The interpretation is the following. In announcing his forecast, E chooses between lotteries where each lottery has two different prizes, that are the two expected statistical biases induced by a particular message according to the evidence in period 1. In a partitional equilibrium D will expect E to have lower true forecasts when he receives message m_{i,n_1} than when he receives message m_{i+1,n_1}, i.e. $E_{\tilde{\mu}_1}[p_1 \mid \widetilde{m}_{i,n_1}] < E_{\tilde{\mu}_1}[p_1 \mid \widetilde{m}_{i+1,n_1}]$. It follows that the differences $\left[\rho^E\left(\overline{\eta}_{i+1,-1}\right) - \rho^E\left(\overline{\eta}_{i,-1}\right)\right]$ and $\left[\rho^E\left(\overline{\eta}_{i,1}\right) - \rho^E\left(\overline{\eta}_{i+1,1}\right)\right]$ have always positive sign, because of first order stochastic dominance across the posterior distribution functions of η[24]. The difference $\left[\rho^E\left(\overline{\eta}_{i+1,-1}\right) - \rho^E\left(\overline{\eta}_{i,-1}\right)\right]$ measures the opportunity cost of $\widetilde{m}_{i,n_1}$ instead of $\widetilde{m}_{i+1,n_1}$ in terms of future reputation when the true state ω_1 is -1: in sending $\widetilde{m}_{i,n_1}$, E reveals that his true prediction is poorer than in case of $\widetilde{m}_{i+1,n_1}$ (and $\overline{\eta}_{i,-1} > \overline{\eta}_{i+1,-1}$). On the contrary, the

[24]In the terminology of Quah and Strulovici (2009), the collection $\{G_{i,-1}(\eta)\}$ will be MLR ordered in the sense that $G_{i+1,-1}(\eta)$ is a MLR shift of $G_{i,-1}(\eta)$. The opposite occurs with $\{G_{i,1}(\eta)\}$.

difference $\left[\rho^E\left(\overline{\eta}_{i,1}\right)-\rho^E\left(\overline{\eta}_{i+1,1}\right)\right]$ measures the expected benefit of $\widetilde{m}_{i,n_1}$ instead of $\widetilde{m}_{i+1,n_1}$ in terms of future reputation when the true state ω_1 is 1: in signaling $\widetilde{m}_{i,n_1}$, E reveals that his true prediction is relatively more adequate (and $\overline{\eta}_{i+1,1}>\overline{\eta}_{i,1}$).

Consider $\widetilde{x}_{i,n_1}$ lower than $\frac{1}{2}$. Since the future expected payoff is strictly concave in $\overline{\eta}_{h_1}$, because of first order stochastic dominance, in equilibrium, $E\left[\eta_2 \mid \widetilde{m}_{i,n_1}\right]$ must be lower than $E\left[\eta_2 \mid \widetilde{m}_{i+1,n_1}\right]$, that is not greater than $\overline{\eta}_1$. In other words, the lottery induced by $\widetilde{m}_{i,n_1}$ must be characterized by a low enough expected final bias with respect to the lottery induced by $\widetilde{m}_{i+1,n_1}$, in order to be selected by E when p_1 belongs to $\tilde{X}_{i,n_1}$. That condition implies that $(\widetilde{x}_{i+1,n_1}-\widetilde{x}_{i,n_1})$ must always be greater than some function of $\overline{\eta}_1$. Hence, the cardinality of the equilibrium partition cannot be countable. In other words, reputational cheap talk under coherent beliefs will lead to a finite partition since E is risk averse. Further, the upper bound to the size of the initial equilibrium partition will be lower the lower is the initial reputation of E.

In a sense, the underlying forces in a partitional equilibrium with reputational cheap talk work exactly in the opposite direction of what happens when E is concerned with the current action only, as it can be induced from Proposition 1. In the first case information transmission is limited by the risk aversion of E, that leads to more ambiguous announcements. In the second case it is limited by exaggerated reports towards extreme forecasts in order to countervail the discounting of the reports by D[25].

In addition to the babbling equilibrium, there will always be an equilibrium of cardinality 2 because the function $\psi_i(\cdot)$ is zero, given $\tilde{X}_{1,2}=\left[0,\frac{1}{2}\right]$ and $\tilde{X}_{2,2}=\left[\frac{1}{2},1\right]$.

5.2 Conflict between inflated reports and reputation

Now let me consider the case in which also E is interested in period 1 action, and his total payoff is $\left[-\left(\omega_1-a_1\right)^2-\left(\omega_2-a_2\right)^2\right]$. In that case E will be subject to conflicting aim: on one side, he wants to induce D to entertain beliefs very close to his own forecasts through inflated reports, on the other side E is concerned about his future reputation as probability assessor, because it will have an impact on his payoff in period 2.

Once again, consider a partition $X_{n_1}=(x_{i,n_1},...,x_{n_1,n_1})$ of the unit interval of the real line. Consider an ordered set of n_1 beliefs, with representative element denoted by w_{i,n_1}, so that $w_{i-1,n_1}<w_{i,n_1}<w_{i+1,n_1}$, with $i=2,...(n_1-1)$. Define the following function:

$$\chi_i\left(p_1\right)=\pi_1^E\left(w_{i,n_1} \mid p_1\right)-\pi_1^E\left(w_{i+1,n_1} \mid p_1\right)$$

[25] Consider the case in which the p.d.f. $g_1(\eta)$ is unform in $[0,1]$, and $n_2=2$. Under reputational cheap talk the initial equilibrium partition of size 4 would be: $X_{1,4}=[0,0.295\,98]$, $X_{2,4}=[0.295\,98,0.5]$, $X_{3,4}=[0.5,0.704\,02]$, $X_{4,4}=[0.704\,02,1]$. Instead, if E were interested in the current action only, the equilibrium partition of size 4 would be: $X_{1,4}=[0,0.428\,57]$, $X_{2,4}=[0.428\,57,0.5]$, $X_{3,4}=[0.5,0.571\,43]$, $X_{4,4}=[0.571\,43,1]$.

$\chi_i(p_1)$ represents the difference between the initial payoff expected by E when he induces beliefs w_{i,n_1} and w_{i+1,n_1}, conditional of E having true prediction p_1. I will refer to the function $\chi_i(p_1)$ as the relative direct effect of inducing belief w_{i,n_1} instead of belief w_{i+1,n_1}.

The following Proposition characterizes the properties that an initial partition X_{n_1} and the corresponding set w_{n_1} of induced beliefs need to satisfy in equilibrium.

Proposition 3 *: provided n_1 and n_2 are greater than 1, under a message rule where $\tilde{\mu}(m_1 \mid p_1)$ is uniform, supported on $\tilde{M}_{i,n_1}$ if $p_1 \in (\tilde{x}_{i-1,n_1}, \tilde{x}_{i,n_1})$, a pair $\left(\widetilde{X}_{n_1}, \widetilde{w}_{n_1}\right)$ can be supported in equilibrium if and only if:*

$$\chi_i(\widetilde{x}_{i,n_1}) = \psi_i(\widetilde{x}_{i,n_1}) \tag{13}$$

for every i, with $i = 1, ...(n_1 - 1)$, and:

$$\widetilde{w}_{i,n_1} = (1 - \overline{\eta}_1)\frac{\widetilde{x}_{i-1,n_1} + \widetilde{x}_{i,n_1}}{2} + \frac{1}{2}\overline{\eta}_1 \tag{14}$$

for every i. If $n_1 \leq 2$ or $n_2 = 1$, an equilibrium initial partition satisfies the conditions of Proposition 1. If $n_1 > 2$ and $n_2 > 1$, the dispersion in the size of the elements of an equilibrium initial partition is lower than it would be in single-stage interactions.

In equilibrium both the direct effect and the indirect effect of the initial message m_1 need be evaluated under coherent posterior beliefs of D in the first period, that imply condition (14). An initial partition cannot be influenced by what happens in the next stage of the game when there is no scope for reputational concerns. Hence, when either n_1 or n_2 are equal to 1, Proposition 1 holds with respect to the initial partitions in equilibrium, and both stages of the game can be analysed as if they were a single-stage game. Moreover, an initial partition is always symmetric around 0.5 in equilibrium. So the only initial partition of size 2 that can be supported in equilibrium is such that $\widetilde{x}_{1,2}$ is equal to 0.5. In that case both the functions $\chi_1(0.5)$ and $\psi_1(0.5)$ are equal to 0: despite the fact the future payoff expected by E depends on $\overline{\eta}_{m_1,\omega_1}$, the expected future benefit from sending $\widetilde{m}_{1,2}$ exactly offsets its opportunity cost. To sum up, in equilibrium an initial partition of cardinality n_1 is identical to the partition of equal size that would prevail in one-shot games when either reputational concerns do not exist (i.e. there is a babbling partition in period 1 and/or in period 2), or when reputational concerns exist but n_1 is equal to 2. An equilibrium initial partition of size 2 always exists.

When both n_1 is greater than 2 and n_2 is greater than 1, reputational concerns not only exist but also matter, because they make the equilibrium initial partition satisfy different conditions from the ones relevant for single-shot games. Consider a symmetric partition X_{n_1} and an interval X_{i,n_1} with $i \leq \frac{n_1}{2}$ so that x_{i,n_1} is not higher than 0.5. When the true prediction of E is equal to x_{i,n_1} and the function $\chi_i(x_{i,n_1})$ has value equal to 0, the function $\psi_i(x_{i,n_1})$ can be proved to be strictly positive. However, in equilibrium E must expect a relative current advantage from $\widetilde{m}_{i,n_1}$ that is exactly equal to the relative future advantage from $\widetilde{m}_{i+1,n_1}$. Hence, reputational concerns reduce the incentive of E to announcements that claim a relatively low probability that ω_1 is equal to -1. When x_{i,n_1} is not lower than 0.5, reputational concerns will work in the opposite direction, increasing the opportunity cost of announcing a relatively high probability that ω_1 is equal to -1 in terms of future reputation.

More generally, when reputational concerns matter, the direct and the indirect effects generated by the initial message makes strategic communication in the first period sensitive to conflicting purposes. In this way, the benefits in terms of current payoff that are associated to some messages different from the honest prediction are partially offset by their potentially adverse consequences on the expected future payoff. In a sense, reputational concerns enhance the credibility of the initial messages, despite the fact that communication is not verifiable.

Looking at the welfare achieved in equilibrium, instrumental reputational concerns cannot make the equilibrium initial payoff expected by the agents decrease. In particular, when reputation matters, in the two-stage game the equilibrium initial payoff expected by both the agents is higher than the equilibrium payoff that the agents would expect in single-stage games.

The reason is that payoff improvements can occur not only when the number of the elements in the relevant partition increases, but also when the distribution of the intervals within the partition changes. From Corollary 2, the initial payoff expected by both the agents will be maximized if the intervals in the initial partition have equal size. When reputational concerns matter in the first period, in equilibrium the dispersion in the size of the different intervals $\widetilde{X}_{i,n_1}$ is lower than it would be in their absence, because reputational concerns contribute to the reliability of the initial messages. In this sense reputational concerns reduce noise in communication under coherent beliefs, despite the fact that equilibria remain partitional. Consequently, provided the transmission of information, measured by the cardinality of the partitions, is not below some threshold in both periods, reputational concerns improve the expected value of the interaction between D and E in equilibrium in the first period.

Proposition 3 holds under the assumptions that the final partitions have fixed cardinality for every history experienced in period 1. If the size of the final partition changes according to the events occurred in the initial period, then the effect of reputational concerns on the initial partition can change both in magnitude and in direction. For instance, if a final expected statistical bias greater than the prior one were followed by the termination of the relationship between E and D, then E would take into account only the gains in reputation from his announcements. And reputational concerns would strengthen the incentives of E towards inflated messages. In this sense, tougher punishments for poor performance can result in worse communication.

Finally, the impact of reputational concerns on the equilibrium initial partitions positively depends on the variance of the prior distribution of the unknown statistical bias, because that prior variance affects the extent to which the reputation of E can change across periods according to the history of the game. In this sense, fixed the initial expected systematic error $\overline{\eta}_1$, the more uncertain D is about the competence of E, the stronger the influence of reputational concerns will be.

6 Conclusions

The paper is concerned with the repeated interaction between a decision-maker and an expert of uncertain reliability. The distinctive assumption of the paper is that the decision-maker makes a subjective assessment of the statistical bias affecting the honest forecasts of the expert.

The paper shows that strategic communication in the final period will occur because E is interested in inducing D to believe what he genuinely believes. That interest leads E to make announcements more extreme than his true forecasts in the final period. Despite the absence of an upper bound to the size of the equilibrium partitions in the final period, perfect communication cannot be achieved even if the size of the partitions goes to infinity.

Instrumental reputational concerns are related to the future estimate of the systematic error affecting the predictions of E. In the first period instrumental reputational concerns will affect the equilibrium partition in period 1. When E is interested in his future reputation only, the initial reports of E can be more uncertain than the true predictions of E, in order to decrease the risk of a loss in reputation. When in the first period E is interested also in the current action, then instrumental reputational concerns can mitigate the extent of strategic communication towards inflated announcements. They can improve welfare in the first period by changing the distribution of the size of the elements in the equilibrium initial partitions of the true forecasts of E.

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8 Appendix A

Proof. of Lemma 1

From (1)-(2) best responses, $\alpha^*_{h_1}$ and $\mu^*_{h_1}$, can be written as follows:

$$\alpha^*_{h_1}\left(w_{h_1}\left(m_2\right)\right) = 1 - 2w_{h_1}\left(m_2\right) \tag{15}$$

$$\begin{aligned} m_2' &\in Supp\left(\mu^*_{h_1}\left(\cdot \mid p_2\right)\right) \text{ only if} \\ \left|p_2 - w_{h_1}\left(m_2'\right)\right| &\leq \left|p_2 - w_{h_1}\left(m_2\right)\right| \quad \forall m_2 \neq m_2' \end{aligned} \tag{16}$$

given that:

$$\left(1 - 2p_2\right) = \arg\max_A \pi^E_2\left(a_2 \mid p_2\right)$$

Under Property $\widehat{H}$, $\widehat{\alpha}_{h_1}$ and $\widehat{\mu}_{h_1}$ must satisfy conditions (15) and (16).

Suppose that a proper interval $[\underline{w}, \overline{w}]$, with $\frac{1}{2}\overline{\eta}_{h_1} < \underline{w} < \overline{w} < 0.5$, belongs to the set $\hat{w}_{h_1}$. Consider some value $\widetilde{w}$ in $(\underline{w}, \overline{w})$. Suppose that $\widetilde{w}$ is the best belief that E can induce when his true forecast is $\tilde{p}_2$. It follows that $\tilde{p}_2 = \widetilde{w}$. Hence, the set $\widetilde{M}_2$ of all the final messages inducing belief $\widetilde{w}$ is non empty. Consider some $m_2'' \in \widetilde{M}_2$, and let $P_{m_2''}$ denote the subset of true forecasts of E conditional on which m_2'' is sent with positive probability. Given (16), $P_{m_2''}$ must be singular, but in that case $\widetilde{w}$ cannot be coherent according to (5). ■

Proof. of Proposition 1

The following preliminary steps are instrumental to the proof.

Step 1. I want to show that, provided every conditional p.d.f. $\hat{\mu}\left(m_2 \mid p_2\right)$ is uniform and supported on $\hat{Y}_{i,n_2|h_1}$ if $p_2 \in \left(\hat{y}_{i-1,n_2|h_1}, \hat{y}_{i,n_2|h_1}\right)$, a final partition $\widehat{Y}_{n_2|h_1}$, with $n_2 > 1$, will satisfy Property $\hat{H}$ if and only if:

$$\left(1 - \overline{\eta}_{h_1}\right) \frac{\widehat{y}_{i-1,n_2|h_1} + \widehat{y}_{i+1,n_2|h_1}}{2} + \overline{\eta}_{h_1} - \left(1 + \overline{\eta}_{h_1}\right)\widehat{y}_{i,n_2|h_1} = 0 \tag{17}$$

for every i, with $i = 1, ..., (n_2 - 1)$. Moreover, every induced belief $\widehat{w}_{i,n_2|h_1}$ will be such that:

$$\widehat{w}_{i,n_2|h_1} = \left(1 - \overline{\eta}_{h_1}\right) \frac{\widehat{y}_{i-1,n_2|h_1} + \widehat{y}_{i,n_2|h_1}}{2} + \frac{1}{2}\overline{\eta}_{h_1} \tag{18}$$

Proof of step 1:

given (15), under Property $\hat{H}$, every subinterval $\widehat{Y}_{i,n_2|h_1}$ must satisfy the following condition:

$$\widehat{Y}_{i,n_2|h_1} = \left\{ \begin{array}{c} p_2 \mid \frac{\widehat{w}_{i,n_2|h_1} + \widehat{w}_{j,n_2|h_1}}{2} \leq p_2 \leq \frac{\widehat{w}_{i,n_2|h_1} + \widehat{w}_{k,n_2|h_1}}{2} \\ \text{for every } \widehat{w}_{j,n_2|h_1} < \widehat{w}_{i,n_2|h_1} < \widehat{w}_{k,n_2|h_1} \end{array} \right\} \tag{19}$$

Since $\tilde{w}_{i,n_2|h_1}$ is an ordered set, then condition (19) is equivalent to:

$$\widehat{Y}_{i,n_2|h_1} = \left\{ p_2 \mid \frac{\widehat{w}_{i,n_2|h_1} + \widehat{w}_{i-1,n_2|h_1}}{2} \leq p_2 \leq \frac{\widehat{w}_{i,n_2|h_1} + \widehat{w}_{i+1,n_2|h_1}}{2} \right\}$$

Let $P_{i,n_2|h_1}$ denote the set of true forecasts inducing belief $w_{i,n_2|h_1}$, i.e.:

$$P_{i,n_2|h_1} = \left\{p_2 \mid \mu\left(m_2 \mid p_2\right) > 0 \text{ for some } m_2 \in M_{i,n_2|h_1}\right\}$$

When $y_{i,n_2|h_1} = 0.5\left(w_{i,n_2|h_1} + w_{i+1,n_2|h_1}\right)$, every p_2 in $\left(y_{i-1,n_2|h_1}, y_{i,n_2|h_1}\right)$ will be such that $\mu\left(m_2 \mid p_2\right) = 0$ if $m_2 \notin M_{i,n_2|h_1}$.

Coherent beliefs imply that:

$$\widehat{w}_{i,n_2|h_1} = \left(1 - \overline{\eta}_{h_1}\right) \int_{\widehat{P}_{i,n_2|h_1}} \frac{p_2}{\int_{\widehat{P}_{i,n_2|h_1}} dy} dp_2 + \frac{1}{2}\overline{\eta}_{h_1}$$

Hence, under Property $\hat{H}$, (18) must hold. It follows that every $\widehat{y}_{i,n_2|h_1}$ must satisfy (17).

Step 2. I want to show the following: consider the real interval $[c, 1-c]$, with c in $[0,1)$, and a constant τ in $(0,1)$. When $n > 1$, the unique partition $Z_{n|c,\tau}$ of the interval $[c, 1-c]$, with cardinality n and representative subinterval $Z_{i,n|c,\tau} = \left[z_{i-1,n|c,\tau}, z_{i,n|c,\tau}\right]$, will satisfy:

$$\left(1-\tau\right) \frac{z_{i-1,n|c,\tau} + z_{i+1,n|c,\tau}}{2} + \tau - \left(1+\tau\right) z_{i,n|c,\tau} = 0 \tag{20}$$

for every i, with $i = 1, ...(n-1)$, if:

$$\begin{aligned} z_{i,n|c,\tau} &= \frac{1}{2} - \left(\frac{1}{2} - c\right) \frac{\zeta_\tau^{n-i}\xi_\tau^i - \zeta_\tau^i \xi_\tau^{n-i}}{\zeta_\tau^n - \xi_\tau^n} \qquad \forall i, i = 0, ..., n \\ \zeta_\tau &= 1 + \sqrt{\tau} \qquad\qquad \xi_\tau = 1 - \sqrt{\tau} \end{aligned} \tag{21}$$

Proof step 2:

(20) can be written in the following way:

$$z_{i+1,n|c,\tau} - \frac{2\left(1+\tau\right)}{1-\tau} z_{i,n|c,\tau} + z_{i-1,n|c,\tau} = -\frac{2\tau}{1-\tau} \tag{22}$$

(22) is a second order linear difference equation with constant coefficient and constant term. From Melumad and Shibano (1991), the solution to (22) is:

$$z_{i,n|c,\tau} = B_1 \beta_1^i + B_2 \beta_2^i + \frac{1}{2}$$

where $\beta_1 = \frac{1+\sqrt{\tau}}{1-\sqrt{\tau}}$ and $\beta_2 = \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}$ solve:

$$\beta^2 - \frac{2\left(1+\tau\right)}{1-\tau}\beta + 1 = 0$$

and $\frac{1}{2}$ is the solution to:

$$e - \frac{2\left(1+\tau\right)}{1-\tau} e + e = -\frac{2\tau}{1-\tau}$$

Given the boundary condition:

$$z_{0,n|c,\tau} = B_1 + B_2 + \frac{1}{2}$$

and

$$z_{1,n|c,\tau} = B_1\beta_1 + B_2\beta_2 + \frac{1}{2}$$

then:

$$z_{i,n|c,\tau} = -z_{1,n|c,\tau}\frac{\beta_1^i - \beta_2^i}{\beta_1 - \beta_2} + \theta_i$$

with $\theta_i = \left[-\frac{\left(\beta_2\beta_1^i - \beta_1\beta_2^i\right)}{\beta_1-\beta_2} z_{0,n|c,\tau} - \frac{1}{2}\left(\frac{(1-\beta_2)\beta_1^i-(1-\beta_1)\beta_2^i}{\beta_1-\beta_2} - 1\right)\right]$. (21) follows from:

$$z_{n,n|c,\tau} = 1 - z_{0,n|c,\tau} = -z_{1,n|c,\tau}\frac{\beta_1^n - \beta_2^n}{\beta_1 - \beta_2} + \theta_n$$

(7) follows from (21).

Now, I can consider the following cases:

Case $n_2 = 1$: the unique, consistent induced belief $\widehat{w}_{1,1|h_1}$ is equal to 0.5, and E is indifferent to the message m_2 he can send.

Case $n_2 > 1$: the result of step 2 applies. ■

Proof. of Corollary 1

Given the symmetry of every equilibrium partition, consider every integer i from 1 to than $n_2/2$. The size of every interval, $\Delta_{i,n_2|h_1} = \widehat{y}_{i,n_2|h_1} - \widehat{y}_{i-1,n_2|h_1}$, decreases in i, since the underlying conflict of opinion decreases as p_2 tends to 0.5. However, the ratio between the sizes of adjacent intervals, $\Delta_{i,n_2|h_1}/\Delta_{i+1,n_2|h_1}$, is always greater than 1, even for n_2 going to infinity since $lim_{n_2\to\infty}\frac{\Delta_{i,n_2|h_1}}{\Delta_{i+1,n_2|h_1}} = \frac{1}{\gamma_{h_1}}$. In particular, the size $\Delta_{1,n_2|h_1}$ of the first interval will never be smaller than $\frac{1}{2}\left(1-\gamma_{h_1}\right)$. ■

Proof. of Corollary 2

Given a partition $Y_{n_2|h_1}$ and the corresponding set of coherent induced beliefs, the payoff expected by agent r, denoted by $\pi_2^r\left(n_2 \mid h_1\right)$, is equal to:

$$\begin{aligned} &\pi_2^E\left(n_2 \mid h_1\right) \\ &= -2\left\{\begin{array}{c} \sum_{i=1}^{n_2}\left(y_{i,n_2|h_1} - y_{i-1,n_2|h_1}\right) \\ \left[\begin{array}{c} \left(1-w_{i,n_2|h_1}\right)^2\left(y_{i-1,n_2|h_1} + y_{i,n_2|h_1}\right) \\ +w_{i,n_2|h_1}^2\left(2 - y_{i-1,n_2|h_1} - y_{i,n_2|h_1}\right) \end{array}\right] \end{array}\right\} \end{aligned} \tag{23}$$

$$\begin{aligned} &\pi_2^D\left(n_2 \mid h_1\right) \\ &= -2\left\{\begin{array}{c} \sum_{i=1}^{n_2}\left(y_{(i,n_2)|m_1,\omega_1} - y_{(i-1,n_2)|m_1,\omega_1}\right) \\ \left[\begin{array}{c} \left(1-w_{i,n_2|h_1}\right)^2\left[\left(1-\overline{\eta}_{h_1}\right)\left(y_{i-1,n_2|h_1} + y_{(i,n_2)|h_1}\right) + \overline{\eta}_{h_1}\right] \\ +w_{i,n_2|h_1}^2\left[\left(1-\overline{\eta}_{h_1}\right)\left(2 - y_{i-1,n_2|h_1} - y_{i,n_2|h_1}\right) + \overline{\eta}_{h_1}\right] \end{array}\right] \end{array}\right\} \end{aligned} \tag{24}$$

(8) – (9) follow from applying the telescoping property of finite sums, given a partition $\widehat{Y}_{i,n_2|h_1}$ from Proposition 1.

From (23) and (24):

$$\frac{\vartheta\pi_2^r\left(n_2 \mid h_1\right)}{\vartheta y_{i,n_2|h_1}} = 0 \ \text{ if } \ y_{i,n_2|h_1} = \frac{y_{i-1,n_2|h_1} + y_{i+1,n_2|h_1}}{2}$$

so that (23) and (24) are maximized if $\left(y^*_{i,n_2|h_1} - y^*_{i-1,n_2|h_1}\right) = \frac{1}{n_2}$ for every i with:

$$y^*_{i,n_2|h_1} = \frac{i}{n_2}$$

as if $\overline{\eta}_{h_1} = 0$.

Since $\left(1 - \frac{\gamma^i_{h_1} - \gamma^{n_2-i}_{h_1}}{1-\gamma^{n_2}_{h_1}}\right)$ is increasing in $\overline{\eta}_{h_1}$, and has limit $\frac{2i}{n_2}$ for $\overline{\eta}_{h_1} \to 0$, then:

$$\hat{y}_{i,n_2|h_1} > y^*_{i,n_2|h_1} \quad \begin{cases} \text{for every } i < n_2/2 \ \text{ when } n_2 \text{ is even} \\ \text{for every } i \leq \left(n_2 - 1\right)/2 \ \text{ when } n_2 \text{ is odd} \end{cases}$$

■

Proof. of Lemma 2:

$$\begin{aligned} & \pi_2^E\left(p_1, m_1\right) \\ = \ & p_1\pi_2^E\left(\bar{\eta}_1 - \sigma^2\frac{m_1 - \frac{1}{2}}{\left(1-\bar{\eta}_1\right)m_1 + \frac{1}{2}\bar{\eta}_1}\right) + (1-p_1)\pi_2^E\left(\bar{\eta}_1 + \sigma^2\frac{m_1 - \frac{1}{2}}{\left(1-\bar{\eta}_1\right)\left(1-m_1\right) + \frac{1}{2}\bar{\eta}_1}\right) \end{aligned}$$

Since $\frac{\vartheta^2\pi_2^E}{\vartheta m_1^2} < 0$ and $\frac{\vartheta^2\pi_2^E}{\vartheta m_1 \vartheta p_1} > 0$, $\tilde{m}_1\left(p_1\right) \in \arg\max \pi_2^E\left(p_1, m_1\right)$ is well defined and continuous in p_1 for $p_1 \in \left[\dot{p}_1, \frac{1}{2}\right]$, where $\dot{p}_1$, strictly positive, is the true prediction such that $\frac{\vartheta\pi_2^E\left(\dot{p}_1, m_1\right)}{\vartheta m_1} = 0$ if $m_1 = 0$.

Hence, for every p_1 lower than $\dot{p}_1$, the initial message sent by E will be 0.

Consider the following implicit function:

$$\begin{aligned} L(p_1, m_1) \quad &= \quad p_1\frac{\vartheta\pi_2^E\left(\bar{\eta}_1 - \sigma^2\frac{m_1-\frac{1}{2}}{m_1+\frac{1}{2}\bar{\eta}_1}\right)}{\vartheta m_1} + (1-p_1)\frac{\vartheta\pi_2^E\left(\bar{\eta}_1 + \sigma^2\frac{m_1-\frac{1}{2}}{m_1+\frac{1}{2}\bar{\eta}_1}\right)}{\vartheta m_1} = 0 \\ \text{when } \ p_1 \quad &\in \quad \left[\dot{p}_1, \frac{1}{2}\right] \end{aligned}$$

Since $\frac{\vartheta L}{\vartheta m_1} < 0$ and since $(p_1, m_1) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is a point of L, L defines m_1 as a continuously differentiable function of p_1 around $\left(\frac{1}{2}, \frac{1}{2}\right)$. Since, around $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\frac{dm_1}{dp_1} \in (0,1)$, then, at $p_1' = \frac{1}{2} - \varepsilon$, $m_1' \approx \frac{1}{2} - \frac{dm_1}{dp_1}\varepsilon > \frac{1}{2} - \varepsilon$.

Alternatively, $E\left[\bar{\eta}_2 \mid m_1\right] = \bar{\eta}_1 - \sigma^2\left(m_1 - \frac{1}{2}\right)\frac{p_1 - m_1 + \bar{\eta}_1\left(m_1 - \frac{1}{2}\right)}{\left[(1-\bar{\eta}_1)m_1 + \frac{1}{2}\bar{\eta}_1\right]\left[(1-\bar{\eta}_1)(1-m_1) + \frac{1}{2}\bar{\eta}_1\right]}$. Since $\pi_2^E\left(\bar{\eta}_{h_1}\right)$ is concave in $\bar{\eta}_{h_1}$, then, in correspondence to some $(p_1, m_1) = \left(\frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon,\right)$ there will be a profitable deviation for E towards $m_1 = \frac{1}{2}$. ■

Proof. of Proposition 2:
When X_{n_1} is an equilibrium partition, then every element must be such that:

$$X_{i,n_1} = \left\{ p_1 \mid \left\{ \begin{array}{c} p_1 \left[\rho^E \left(\overline{\eta}_{j,n_1|-1} \right) - \rho^E \left(\overline{\eta}_{i,n_1|-1} \right) \right] - \\ (1-p_1) \left[\rho^E \left(\overline{\eta}_{i,n_1|+1} \right) - \rho^E \left(\overline{\eta}_{j,n_1|+1} \right) \right] \end{array} \right\} \leq 0 \right\} \tag{25}$$

Because of first order stochastic dominance:

$$\begin{aligned} \rho^E \left(\overline{\eta}_{j,n_1|-1} \right) - \rho^E \left(\overline{\eta}_{i,n_1|-1} \right) &> 0 \text{ and} \\ \rho^E \left(\overline{\eta}_{i,n_1|1} \right) - \rho^E \left(\overline{\eta}_{j,n_1|1} \right) &> 0 \qquad \text{if } j > i \end{aligned} \tag{26}$$

Hence, provided the following condition is satisfied:

$$\left\{ \begin{array}{c} x_{i,n_1} \left[\rho^E \left(\overline{\eta}_{i+1,n_1|-1} \right) - \rho^E \left(\overline{\eta}_{i,n_1|-1} \right) \right] \\ -(1-x_{i,n_1}) \left[\rho^E \left(\overline{\eta}_{i,n_1|1} \right) - \rho^E \left(\overline{\eta}_{i+1,n_1|1} \right) \right] = 0 \quad \text{for every } i \end{array} \right. \tag{27}$$

an optimal message rule will be such that:

$$\begin{aligned} \mu(m_1 \mid p_1) &= 0 \quad \text{if } m_1 \in M_{i,n_1} \text{ and } p_1 > x_{i,n_1} \\ \mu(m_1 \mid p_1) &= 0 \quad \text{if } m_1 \in M_{i+1,n_1} \text{ and } p_1 < x_{i,n_1} \end{aligned}$$

Consequently, $\mu(m_1 \mid p_1)$ can be supported on M_{i,n_1} if and only if $p_1 \in (x_{i-1,n_1}, x_{i,n_1})$, and (25) can be written as follows:

$$X_{i,n_1} = \left\{ p_1 \mid \left\{ \begin{array}{c} p_1 \left[\rho^E \left(\overline{\eta}_{i+1,n_1|-1} \right) - \rho^E \left(\overline{\eta}_{i,n_1|-1} \right) \right] - \\ (1-p_1) \left[\rho^E \left(\overline{\eta}_{i,n_1|1} \right) - \rho^E \left(\overline{\eta}_{i+1,n_1|1} \right) \right] \end{array} \right\} \leq 0 \right\}$$

A message rule where $\mu(m_1 \mid p_1)$ is uniform, supported on X_{i,n_1} if $p_1 \in (x_{i-1,n_1}, x_{i,n_1})$, is optimal. Uniformity guarantees that $g(\eta \mid m_1, \omega_1)$ and $g(\eta \mid m_1', \omega_1)$ are equal for every m_1, m_1' in M_{i,n_1}. The expected values in (12) are coherent. Finally, when condition (27) is satisfied for i, it will be satisfied for $(n_1 - i)$, given $p_1 = 1 - x_{i,n_1}$. Hence, the equilibrium partition is symmetric at 0.5 and:

$$\begin{aligned} \overline{\eta}_{n_1-i,n_1|-1} &= \overline{\eta}_{i,1} \\ \overline{\eta}_{n_1-i,n_1|1} &= \overline{\eta}_{i,-1} \end{aligned}$$

Consider the interval $[x_{i-1,n_1}, x_{i+1,n_1}]$ where $x_{i,n_1} < \frac{1}{2}$. Since $x_{i,n_1} < x_{i+1,n_1} \leq \frac{(1-\bar{\eta}_1)x_{i+1,n_1}+\bar{\eta}_1}{1+\bar{\eta}_1}$, then $E[\eta_2 \mid X_{i+1,n_1}] \leq \bar{\eta}_1$. Instead, $E[\eta_2 \mid X_{i,n_1}] < \bar{\eta}_1$ only if $x_{i,n_1} < \frac{(1-\bar{\eta}_1)x_{i-1,n_1}+\bar{\eta}_1}{1+\bar{\eta}_1}$. Since $x_{i+1,n_1} - x_{i,n_1} > x_{i+1,n_1} - \frac{(1-\bar{\eta}_1)x_{i-1,n_1}+\bar{\eta}_1}{1+\bar{\eta}_1} > \frac{\bar{\eta}_1}{1+\bar{\eta}_1}$, then the equilibrium partition X_{n_1} can be finite only. It follows that:

$$\frac{1}{2} - \left[\left\langle \frac{n_1}{2} \right\rangle - 1 \right] \frac{\bar{\eta}_1}{1+\bar{\eta}_1} > 0$$

■

Proof. of Proposition 3

The total payoff expected by E, when he has belief p_1 and sends a message inducing $\widetilde{w}(m_1)$, corresponds to:

$$\begin{aligned} & \pi_1^E\left(\widetilde{w}(m_1) \mid p_1\right) + \widehat{\pi}_2^E\left(\widetilde{w}(m_1) \mid p_1\right) \\ = & \left\{-4\left\{p_1\left[1-\widetilde{w}(m_1)\right]^2 + (1-p_1)\left[\widetilde{w}(m_1)\right]^2\right\} - 1 + q_{m_1}(n_2)\right. \\ \text{where } q_{m_1}(n_2) = & \left\{\begin{array}{c} 0 \text{ if } n_2 = 1 \\ \frac{1}{4}\left[p_1\rho^E\left(\overline{\eta}_{m_1,-1}\right) + (1-p_1)\rho^E\left(\overline{\eta}_{m_1,1}\right)\right] \quad \text{if } n_2 > 1 \end{array}\right. \end{aligned}$$

When $n_2 > 1$, an initial partition can be supported in equilibrium if and only if each element of that partition is such that:

$$\widetilde{X}_{i,n_1} = \left\{\begin{array}{c} p_1 \mid \pi_1^E\left(\widetilde{w}_{i,n_1} \mid p_1\right) - \pi_1^E\left(\widetilde{w}_{j,n_1} \mid p_1\right) \\ \geqslant \widehat{\pi}_2^E\left(\widetilde{w}_{j,n_1} \mid p_1\right) - \widehat{\pi}_2^E\left(\widetilde{w}_{i,n_1} \mid p_1\right) \end{array}\right. \quad \text{for every } j \neq i$$

i.e.:

$$\widetilde{X}_{i,n_1} = \left\{\begin{array}{cc} & 4\left(\widetilde{w}_{j,n_1} - \widetilde{w}_{i,n_1}\right)\left(\widetilde{w}_{i,n_1} + \widetilde{w}_{j,n_1} - 2p_1\right) \geq \\ p_1 \mid & \frac{1}{4}\left\{\begin{array}{c} p_1\left[\rho^E\left(\overline{\eta}_{j,n_1|-1}\right) - \rho^E\left(\overline{\eta}_{i,n_1,n_1|-1}\right)\right] - \\ (1-p_1)\left[\rho^E\left(\overline{\eta}_{i,n_1|+1}\right) - \rho^E\left(\overline{\eta}_{j,n_1|+1}\right)\right] \end{array}\right\} \\ & \forall \widetilde{w}_{j,n_1} \in \widetilde{w}_{n_1} \backslash \widetilde{w}_{i,n_1} \end{array}\right\}$$

Consider p_1 in $\left(\widetilde{x}_{i-1,n_1}, \widetilde{x}_{i+1,n_1}\right)$. Suppose that the following condition is satisfied for every X_{i,n_1}:

$$4\left(\widetilde{w}_{i+1,n_1} - \widetilde{w}_{i,n_1}\right)\left(\widetilde{w}_{i,n_1} + \widetilde{w}_{i+1,n_1} - 2\widetilde{x}_{i,n_1}\right) = \varphi_i\left(\widetilde{x}_{i,n_1}\right)$$

The left hand side represents the difference in the expected initial payoffs of E, conditional on $\widetilde{m}_{i,n_1}$ and $\widetilde{m}_{i+1,n_1}$, when the true forecast of E is $p_1 = \widetilde{x}_{i,n_1}$. Under coherent beliefs, from (14), the function $\chi_i\left(\widetilde{x}_{i,n_1}\right)$ can be written as follows:

$$\begin{aligned} \chi_i\left(\widetilde{x}_{i,n_1}\right) = & \; 2(1-\overline{\eta}_1)\left(\widetilde{x}_{i+1,n_1} - \widetilde{x}_{i-1,n_1}\right) \\ & \left[\frac{1-\overline{\eta}_1}{2}\left(\widetilde{x}_{i+1,n_1} + \widetilde{x}_{i-1,n_1}\right) + \overline{\eta}_1 - (1+\overline{\eta}_1)\widetilde{x}_{i,n_1}\right] \end{aligned}$$

From (26):

$$\chi_i(p_1) > \varphi_i(p_1) \quad \text{if } p_1 < \widetilde{x}_{i,n_1} \tag{28}$$

The inequality in (28) is reversed when $p_1 > \widetilde{x}_{i,n_1}$. Consequently, an initial partition can be supported in equilibrium if and only if (13) holds.

a) Case $n_1 = 1$.

From (12), $\overline{\eta}_{1,1|-1} = \overline{\eta}_{1,1|1} = \overline{\eta}_1$[26]. Since E is indifferent to the initial message he can send, an initial partition with $\widetilde{X}_{1,1} = [0,1]$ is consistent with the equilibrium conditions.

b) Case $n_1 > 1$.

b.1) When $\psi_i\left(\widetilde{x}_{i,n_1}\right) = 0$ for every i from 1 to $(n_1 - 1)$, then (13) is equivalent to (17) and, in that case, the equilibrium initial partition $\widetilde{X}_{n_1}$ can be derived from Proposition 1 resulting in:

$$\begin{aligned}
\widetilde{x}_{i,n_1} &= \frac{1}{2} - \frac{1}{2}\frac{\zeta^{n_1-i}\xi^{i} - \zeta^{i}\xi^{n_1-i}}{\zeta^{n_1} - \xi^{n_1}} \\
\zeta &= 1 + \sqrt{\overline{\eta}_1}; \quad \xi = 1 - \sqrt{\overline{\eta}_1}
\end{aligned}$$

for every i, with $i = 0, ..., n_1$.

Consider $n_1 = 2$. Given $\widetilde{x}_{1,2} = 0.5$, $\psi_i\left(\widetilde{x}_{1,2}\right) = 0$ for every n_2.

Consider $n_2 = 1$. In that case $\psi_i\left(\widetilde{x}_{i,n_1}\right) = 0$ for every $n_1 > 1$.

b.2) Consider $n_1 > 2$ and $n_2 > 1$. When $\chi_i\left(x_{i,n_1}\right) = 0$ for every i, with $x_{i,n_1} \leq 0.5$, then $\psi_i\left(x_{i,n_1}\right) > 0$ because $E\left[\eta_2 \mid x_{i,n_1}\right] > \overline{\eta}_1 > E\left[\eta_2 \mid x_{i+1,n_1}\right]$. Hence, in equilibrium $\widetilde{x}_{i,n_1}$ must be greater than it would be under reputational cheap talk, because of $\chi_i\left(\cdot\right)$, and lower than it would be under an exclusive interest of E in the initial action, because of $\psi_i\left(\cdot\right)$. ■

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[26] In case of a babbling equilibrium in period 1, the reputation of E cannot change from the initial to the final period, since every announcement is made with equal probability for every true initial prediction E can have.

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