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Part II. Economic Optimization in Agricultural and Resource Economics

This paper is dedicated to Geoffrey S. Shepherd, exemplary scholar and developer of economies and economists. Edward Sparling assisted in the preparation of the section on risk and uncertainty and the bibliography. The paper was written while the author was a visiting professor at the Mathematics Research Center, University of Wisconsin.

R. H. D.

On Economic Optimization: A Nontechnical Survey

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For at least two centuries economic principles have involved three fundamental concepts. First, individual or group behavior can be explained—at least in part—as the result of pursuing one's advantage. Second, a given system of individuals or nations may possess a kind of harmony or equilibrium when each individual or nation pursues its own advantage. Third, if the environment is properly structured, the working of an economy may bring about individual optima and group equilibria. As early as Cournot [1838], these ideas began to receive an explicit mathematical treatment. It was Cournot who first used calculus to analyze the three classical notions of optimum, equilibrium, and process in markets. The methods that he initiated dominated analytical economics for over a century. The more or less definitive form of this neoclassical, marginalist economics was established by Jevons [1871], Marshall [1890], and Walras [1874] and culminated with Hicks's *Value and Capital* [1939] and Samuelson's *Foundations of Economic Analysis* [1948].

From the vantage point of our generation it is clear that something substantial was lost in the neoclassical mathematization of classical economic thought. The issue involves alternative assumptions about the underlying structure of choices and its role in bringing about compatibility between individual optimization and group equilibrium. It is now clear, thanks to Samuelson [1949, 1959a, 1959b] and others, that some of the classical ideas—for example, the theory of rent (Malthus, West) and the theory of trade (Ricardo, Mill)—are most naturally expressed by means of linear programming models, a postneoclassical development. But this linear programming structure could not be accommodated in the "smooth" neoclassical world. With the advent of Arrow and Debreu's analysis [1954] of general equilibrium this problem was overcome. Classical linearities and inequalities could be incorporated into the general economic optimization framework. Thus, modern optimization theory not only helps to mathematize and illustrate classical ideas, it makes it possible to identify the fundamental unity in two centuries of economic thought.

The transition to the modern period began to occur even before the closing of the neoclassical system by Hicks and Samuelson. The catalysts for this include Leontief, Von Neumann, and Wald. Leontief's input-output or interindustry model [1928, 1936] and Von Neumann's growth model [1937, 1945] captured essential features of classical thought and through the use of algebra forced a shift away from the calculus of the neoclassical school. The game theory of Von Neumann [1928] and Von Neumann and Morgenstern [1944] made possible a profound new formalization of the multiperson joint optimization problem inherent in economics and introduced the axiomatic method and topology. Wald [1936, 1951] contributed the first rigorous proof of the existence of general equilibrium among economic optimizing individuals. The full impact of this reorientation came at mid-century when the duality of constrained optimization and economic valuation was established by Gale, Kuhn, and Tucker [1951] and Kuhn and Tucker [1951] and when efficient optimization algorithms were discovered. Especially because of Dantzig's simplex method [1949] for linear programming, optimization became a tool for planners as well as a theory for economists. Further background material relating optimization concepts to the history of economic thought will be found in Samuelson [1948], Koopmans [1951], Dorfman, Samuelson, and Solow [1958], and Leontief [1960].

During the past two decades modern optimization theory and methods have continued to develop, and at the same time their effective application has spread to a growing variety of important applied problems. The literature is indeed by now so vast as to preclude a comprehensive survey. In this overview, therefore, we shall present a nontechnical summary of the most important concepts involved in these developments for economic theory and applied analysis. The applications of optimization theory to problems in agricultural and resource economics are reviewed elsewhere in this volume (see "Optimization Models in Agricultural and Resource Economics" by Richard H. Day and Edward Sparling).

Optimization Models

For a very long time the mathematical development of theoretical and applied economics was severely circumscribed by the limited class of optimization model types with which it could cope. Though many fundamental barriers remain, the breakthroughs just recalled resulted in a spectrum of operational optimizing models. This spectrum can be broken down according to the components of an optimization model which we will outline. Various important examples are then illustrated. Next, the concept of infinite programming is used to show how the classical and neoclassical optimization approaches are related. Remarks on the distinction between problems and models conclude the chapter.

The following more or less standard definitions will facilitate our discussion: "Optimizing" is finding a best choice among possible or feasible alternative choices. An "optimization model" is a specific formalization of a problem in terms of its comparable alternatives, the criterion for comparing alternatives, and the feasible alternatives. A "mathematical optimization model" consists of a "choice space," which is the set of comparable alternatives, an "objective function," which describes how alternatives are to be compared, and a "feasible region," which is a subset of the choice space and contains those alternatives that are eligible for choice. The feasible region is usually—though not always—defined by equations or inequality constraints.

The choice space. Virtually all economic optimization models involve real linear spaces in which each comparable choice may be represented by a vector of real variables. If the dimension of the choice space is finite, so that the number of choice variables is finite, we have a finite-dimensional optimization model. Otherwise the model is called infinite-dimensional. If each variable or component in the choice space can take on any real value, we have "continuous" or "real" optimization; if each variable may take on only discrete values, we have "discrete" or "integer" optimization. If some variables are discrete while others are continuous, we have "mixed integer" optimization. According to the type of choice space, then, we may distinguish six types of optimization models as summarized in the following outline.

Optimization Models by Type of Choice Space

1. Finite-dimensional optimization
   - 1.1 Continuous or real variables
   - 1.2 Discrete or integer variables (integer programming)
   - 1.3 Continuous *and* discrete variables (mixed integer programming)
2. Infinite-dimensional optimization
   - 2.1 Continuous variables
   - 2.2 Discrete variables
   - 2.3 Continuous *and* discrete variables

The linear and quadratic programming problems and the neoclassical optimizing fall in category 1.1. The transportation problem is an example of category 1.2. Models including increasing returns to scale utilize category 1.3. Optimal control and dynamic programming models are often defined for an infinite future and provide examples of category 2.1. Categories 2.2 or 2.3 would arise if "lumpy" or discrete capital goods were incorporated into the infinite optimization, though to date this does not appear to have been done.

The objective (criterion, utility, payoff) function. In mathematical optimization alternatives are compared by means of their real value as given by some real valued function. This function defines a preference ordering on the alternatives in the choice space. Objective functions may be classified according to their mathematical properties: smoothness or continuity properties, concavity or convexity properties, separability or interdependence properties, and special forms. Thus we have the following outline which gives some of the relevant distinctions.

Function Characteristics

1. Continuity properties
   - 1.1 Semicontinuous, upper or lower (allows for step functions)
   - 1.2 Continuous functions
   - 1.3 Differentiable functions
   - 1.4 Twice-differentiable functions
   - Etc.
2. Concavity properties
   - 2.1 Concave (convex)
   - 2.2 Strictly concave (convex)
   - 2.3 Pseudo concave (convex)
   - 2.4 Quasi concave (convex)
   - 2.5 Strictly quasi concave (convex)
   - Etc.
3. Separability properties
   - 3.1 Partially separable
   - 3.2 Completely separable
4. Special functional forms
   - 4.1 Linear
   - 4.2 Quadratic
   - 4.3 Power
   - Etc.
5. Monotonicity properties
   - 5.1 Nondecreasing (increasing)
   - 5.2 Strictly increasing (decreasing)

In discrete or mixed optimization problems the objective function is usually defined on the continuous space within which the choice space is imbedded. The preference ordering is then defined for all continuous choices even though only discrete ones are allowed.

The feasible region. If the choice is unrestricted in the choice space, the optimization model is called "unconstrained." In this case the feasible region is the entire choice space. Otherwise, when the feasible region is a proper subset of the choice space, it is called "constrained." Feasible regions are classified according to various criteria: closedness (containing limit points), boundedness, or more generally by their compactness or noncompactness; convexity properties; and special functional forms. If the feasible region is defined by an equation, or a set of equations, then we have an "equality-constrained optimization model." If it is defined by inequalities, it is called an "inequality-constrained problem." Because an equation can be expressed by two inequalities, the latter contains the former as a special case. Nonetheless, because mathematical techniques employed in each differ markedly, equality and inequality cases should be regarded as separate categories. In either event it is necessary to define constraint functions. For each type of constraint function we get a specific type of optimization problem. Thus the classes of functions enumerated above are relevant from this point of view too. The following outline summarizes the most important criteria for determining types of feasible regions.

Feasible Region Characteristics

1. Unconstrained optimization: The feasible region is the entire space.
2. Equality-constrained optimization: The feasible region is defined by equations. See the table of function characteristics above.
3. Inequality-constrained optimization
   - 3.1 Compactness or noncompactness
   - 3.2 Geometric properties
     - (1) nonconvexity
     - (2) convexity
     - (3) strictly convexity
     - (4) polyhedral form (as in linear programming)
   - 3.3 Constraint function types. See the table of function characteristics given earlier.

The basic questions of optimization theory must be posed for each optimization model or class of models: (1) Do solutions exist? (2) How many are there? (3) How can solutions be characterized? (4) How can solutions be found? The answers and the methods used to obtain them depend of course on the characteristics of each model type. The neoclassical economists rarely concerned themselves with the possibility of multiple optima, indeed, they used the calculus of smooth functions to structure models which possessed unique solutions although they were characteristically vague about, or even ignored, the exact model characteristics to justify their results. The implications of these mathematical issues for economic theory are of greater importance than is usually recognized. Indeed, with the appreciation of function characteristics in optimization theory (especially convexity properties) has come the significant realization that what was once assumed to be true of all private or public ownership economies could in fact be proven true only for economies with very special and not too realistic constraint and preference structures.

Modern optimization theory addresses itself very often to situations in which many "best" solutions exist such as in linear programming. In this example the set of solutions forms a simplex, a "polyhedral face" generated by its extreme points. The location of such extreme points was found by Dantzig to involve sequences of straightforward algebraic calculations. This shows how theoretical characteristics yield insights leading to answers to the question: How can solutions be found? Another example, brilliantly expounded in Samuelson's classic "Market Mechanisms and Maximization" [1949] shows how the duality properties of constrained optimization can be used to guide a sequence of relatively simple adjustments to the constrained optimum thus, in effect, mimicking the market process. We shall return to these issues later. However, at this point we illustrate a few of the most important optimization models in a way that brings out some of their distinctive features.

In figure 1 the isoquants of an objective function are illustrated by more or less concentric, somewhat irregular curves. The arrows normal to these isoquants indicate the direction of locally steepest ascent of the objective function. Point A is the optimizer of the unconstrained problem. The curve in the

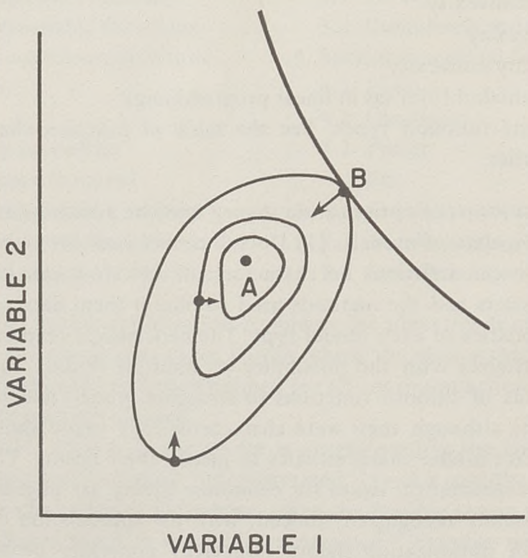


Figure 1. Unconstrained and equation-constrained optima

upper right part of the diagram illustrates an equation constraint to which the choice would be confined for an equality-constrained problem, in which case the optimizer is the point B, as point A is no longer feasible. Figure 2 shows how the mathematical programming model varies in its structure according to changes in the choice space and in the type of objective and constraint func-

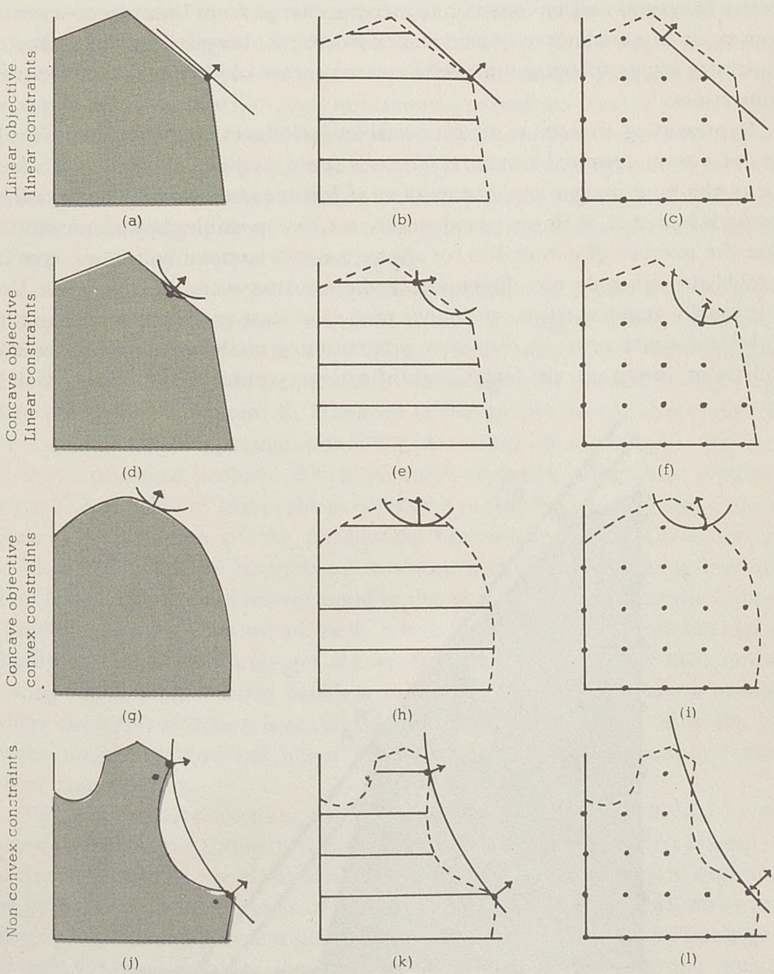


Figure 2. Some types of mathematical programming models

tion. Diagram (a) is the linear programming model, (b) the mixed integer, and (c) the integer linear programming model. In the first column the choice variables can vary continuously; the feasible region is the shaded area on the diagram. In the middle column variable 2 must be an integer while variable 1 can be continuous; the feasible region consists of the parallel lines. In the last column only integer variables are allowed; the feasible region is represented by the dots. The rows show how problems in these three categories change as objective functions and/or constraint functions change from linear to concave or convex or to nonconcave or nonconvex functions. Diagram (l), for example, illustrates integer programming with quasi concave objectives and nonconvex constraints.

Representing choices as integer variables introduces mathematical difficulties of a most formidable nature. There is some intellectual irony in this fact for in the pure integer case the number of feasible alternatives, if the feasible region is bounded, is finite; an exhaustive search is possible. In the continuous case the number of contenders for choice is nondenumerably infinite, even in a problem with only one dimension, and exhaustive search is impossible. Yet it is usually easier to solve continuous models at least approximately than it is to solve discrete ones. In the linear programming model where objective and constraint functions are linear, an unfortunate consequence of this fact is

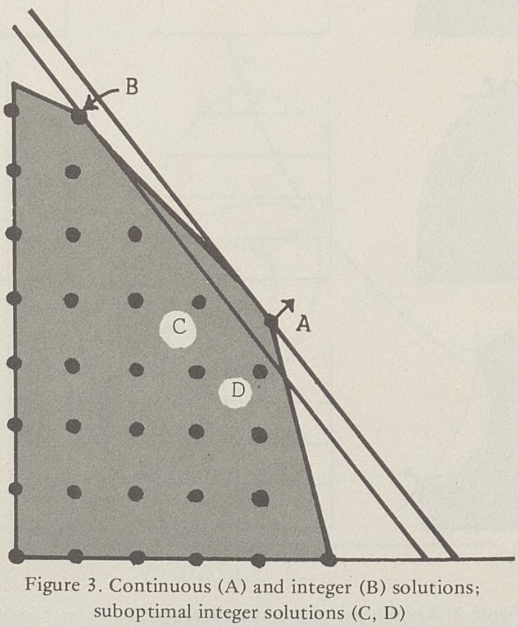


Figure 3. Continuous (A) and integer (B) solutions; suboptimal integer solutions (C, D)

easily illustrated. Figure 3 shows the true feasible region consisting of integer valued variables (dots) inside the shaded convex feasible region where the variables are assumed to be continuous. The continuous variable optimum is A, a point very far removed from the ("true") integer optimum B. If we round the continuous solution A to its nearest integer value, we get C or D, points also far removed from the optimum.

Infinite-dimensional programming. Optimizing over an infinite horizon arises in economic theories of capital and growth. When formalized, these theories lead to programming models in which the choice space is infinite-dimensional. (We shall take up this class of models later.) What is scarcely appreciated by economists, though fundamental in mathematics, is the extremely close relationship between infinite-dimensional and finite, continuous problems. We touch on this point next because it affords an opportunity to show how concepts from mathematical optimization theory can be exploited to reveal the underlying unity of various schools of economic thought to which we referred in the first section of this chapter.

Let us consider the purely competitive optizimizing problem of the price-taking firm that produces an output in amount y, using an input in amount x according to a production function illustrated by the smooth curve in figure 4. If the profit isoquants are parallel to the straight line marked $\pi = P_y - Q_x$, then the optimum is point E. If instead of the smooth neoclassical curve f(x) we used the linear approximation OBD, we would obtain an approximation to the neoclassical problem which we could represent using linear programming. The solution of this problem is point B in figure 4. By choosing a better linear approximation of the production function—say, OABCD—point A, which is closer to the neoclassical solution, is chosen. By making finer and finer linear approximations we could in this way come as close to the smooth optimum solution E as we pleased, just as a circle can be approximated as closely as we like by a polygon. As we do so, the dimension of the approximating linear programming problem increases, going in the limit to infinity where the approximation is perfect. In this way we see a type of duality between infinite-dimensional linear programming and finite-dimensional nonlinear programming.

The mathematical duality just illustrated is analogously reflected in the history of economic thought. It is well known (we recall our earlier references to Samuelson) that the classical theory of production, most clearly expounded by Ricardo in his exegesis of the Malthus-West theory of rent, involves a linear programming problem like that illustrated by the piece-wise linear production function in which the input variable is interpreted to be the amount of land with different qualities and in which the yield declines as more land is brought into cultivation. By increasing the number of qualities of land we see

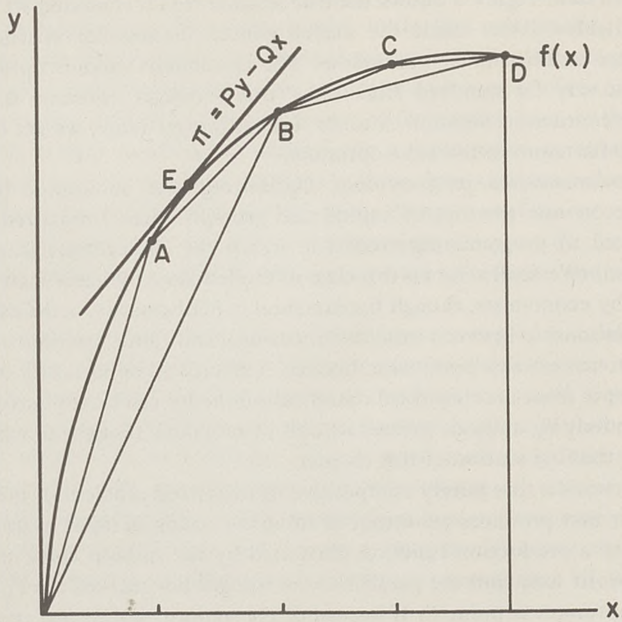


Figure 4. Linear programming as an approximation of neoclassical optimizing

the convergences of the classical linear programming to the neoclassical smooth production function point of view developed many decades later.

The logical duality of the classical-neoclassical points of view should now alert us to a need for care in how we interpret the term "approximation." Whether we regard the classical linear programming model as an approximation of the neoclassical smooth optimization model or vice versa is a matter not of logic but of relevance, convenience, or interpretation in a particular application. Either may be used as an approximation to some real optimization problem, and one may be preferred to the other on empirical or computational grounds, depending on the nature of the problem at hand. A few economists still seem to think that neoclassical economics is *economics* whereas other forms of optimization theory are methods of operations research of no intrinsic economic interest and useful only in computation settings. Nothing could be further from the truth, as the above exercise demonstrates. Indeed, the neoclassical framework is of no more or less interest or relevance than its classical predecessor, and the very much more general formulation of modern optimization theory encompasses both and establishes their underlying unity.

It is also important to distinguish between model and problem optima. For example, we often use models involving continuous variables when, clearly, many economic variables are discrete in nature (machines, factories, farm buildings). Actual choice situations therefore must often distinguish among "lumpy" alternatives. The continuous optimization *model* must then be thought of as an approximation to an underlying discrete optimization *problem*. That the approximation may not be close is a possibility we have already illustrated in figure 3.

The interested reader should become acquainted with the following texts, which among them cover all of the major optimization model types. We list them in (roughly) ascending order of difficulty: Heady and Candler [1958], Hadley [1962], and Gale [1960] cover linear programming; Hu [1969] is concerned with integer and mixed-integer models; Hadley [1964], Intriligator [1971], Mangasarian [1969], and Karlin [1959] among them cover nonlinear programming, dynamic programming, and optimal control; Canon, Cullum, and Polak [1970] and Leuenberger [1969] give a unified treatment of programming, programming in infinite spaces, and optimal control. Aubin [n.d.] provides an advanced synthesis of optimization and game theory emphasizing duality relationships. There are also several excellent expository pieces by Dorfman [1953] on linear programming, Dorfman [1969] on optimal control, and Baumol [1958].

Parametric Programming and Comparative Static Analysis

In both theoretical and applied economics the study of how optima change in response to changes in the situation of the decision maker is of extreme interest. In optimization theory this study is called parametric programming or perturbation analysis. In economic theory it is called comparative statics. By means of it economists have constructed special theories of consumer demand, of producer supply, and of derived producer demand. Moreover, the careful mathematical study of optimizing behavior plays a central role in modern general equilibrium theory. This is because the theoretical analysis of the existence and properties of general equilibria depend on how well-behaved or smooth optimal sets are in their response to market situations.

In neoclassical models in which unconstrained or equality-constrained optimizations are specified, functions are assumed to be sufficiently smooth to make possible application of ordinary calculus. Equations are defined by setting the gradient of the objective function, or of the Lagrangian (in the equality-constrained case), equal to zero. Any optimum must satisfy these equations. These so-called first-order conditions are then interpreted as implicit functions which can be solved to give the decision variables as functions of the parameters and exogenous variables of the problem. Even if this explicit functional dependence cannot be derived practically, it is often possible to infer its qualitative character such as "an increase in price will cause a fall in demand" and so forth. Econometricians are especially interested in those models for which the equations can be solved, for then the parameters of the optimization model may be estimated in reduced form.

As we have already noted, the classical models did not have sufficient regularity to make possible the application of calculus, and no doubt largely for that reason interest in them waned until the modern era, when the tools for inequality-constrained optimizations were perfected. Efficient algorithms for parametric programming made possible a reconsideration of step supply and demand functions and the kinked total cost functions of the classical production theory. They also made it possible to conduct traditional comparative static analysis for a vastly expanded range of economic problems.

The achievement was not without cost, however, for the neoclassical equilibrium and welfare theory completed by Hicks, Lange, and Samuelson did not cover the more general optimizations used in practical decision making. The methods for studying the modern optimization models in the general equilibrium setting, however, were not long in coming. Arrow and Debreu [1954] and McKenzie [1955] showed how topological methods and convex analysis could be used to extend the results on existence and efficiency of competitive equilibria to an economy made of modern (and classical) mathematical programmers.

The supply and demand functions that emerge from the modern point of view include, in addition to the traditional smooth neoclassical variety of Marshall's principles as shown in figure 5 (b), the classical step functions and the modern multivalued mappings or correspondences of the kind illustrated in figure 5 (a), (b), and (c). In (a) we find several prices at which the underlying optimizing behavior can take on any one of several possible supplies or demands. In (c) this indeterminance is continuous. In (d) optimizing behavior of a consumer also becomes indeterminant after some *income* level is reached. Any quantity within a given range inside the shaded area might be picked. The incorporation of the integer and mixed integer cases into the main stream of economic theory was given an impressive beginning by Charles Frank [1969]. But a complete comparative static treatment of it as needed for general equilibrium theory remains a task for future contributors. The kinky step function and correspondences that derive from modern parametric programming often have more complex qualitative appearances than their neoclassical counterparts and can indeed seldom be expressed in mathematically closed form. Instead they must be derived computationally and except for

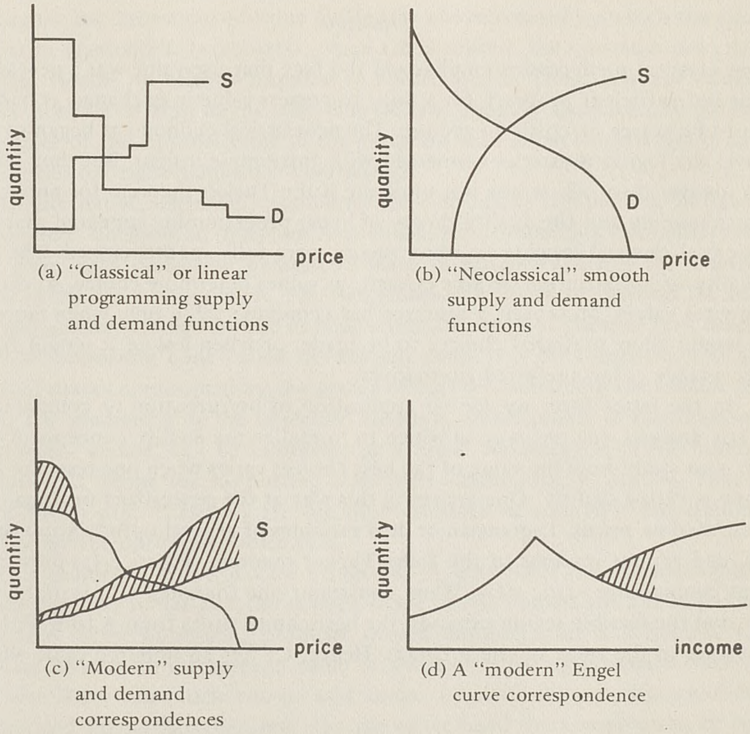


Figure 5. Comparative statics

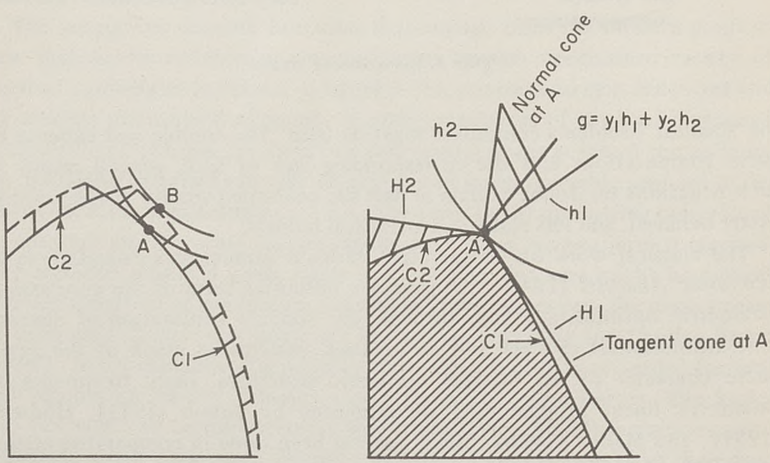
the smallest problems computers must be used. The trouble and expense of these computations and the corresponding lack of hard, general results in such situations no doubt explain in part the continued vitality of the simpler, better behaved, and less realistic neoclassical models.

The classical work on comparative statics is Samuelson's *Foundations of Economic Analysis* [1948]. He gave the definitive form of the neoclassical parametric optimization and, through the copious exploration of discrete (not infinitesimal) changes and inequalities, anticipated much of the qualitative character of the modern economic structures. Early treatments of parametric linear programming are presented by Simon [1951], Hildreth [1957], and Manne [1956]. Much less has been done in comparative statics for nonlinear and infinite programming, although the very recent work of Araujo, Chichilnisky, and Kalman [1973] promises to provide a breakthrough in this area.

Duality

Our classical predecessors emphasized the fact that use-value was a necessary but not sufficient property for a thing to possess value in exchange. It had also to be scarce or costly to acquire. The neoclassical economists began to unravel the logical mysteries connected with this simple insight. But they failed to unravel them all. It was not until the Kuhn-Tucker theorem for nonlinear programming and the duality theory of linear programming appeared that the essential classical insights on value theory were fully mathematized. The full duality of optimization became evident: as values determine choice, so choice imputes values. Moreover, a resource has economic value only when more of it would allow preferred choices to be made, or when less of it would force acceptance of less preferred alternatives.

In the latter form we see an application of perturbation or comparative static analysis, for one way in which to formalize the duality concepts of value is to study how the value of the best choices varies when one resource at a time is varied slightly. One arrives in this way at the generalized marginal values, shadow prices, Lagrangian or dual variables of general optimization theory, and various versions of the Kuhn-Tucker theorem. Figure 6 (a) illustrates this comparative static view. When constraint one (denoted C1) is perturbed so that the feasible region expands, the best choice shifts from A to B with an increase in the value of the program. Hence, C1 has an imputed value which



(a) Constraint perturbation

(b) Tangent and normal cones

Figure 6. Two views of duality

is roughly the increment of value divided by the increment by which the constraint is augmented. In contrast, when C2 is shifted, the optimum does not change. Hence, no value is imputed to constraint C2. These imputed values, either positive or zero as the case may be, are, roughly speaking, the partial derivative of the optimal value of the paragram with respect to changes in the limitations or "right-hand-side" coefficients.

An alternative, essentially geometric view of duality is illustrated in figure 6 (b). Here we show an optimum A at which *both* constraints are binding. At this point each constraint possesses a plane of support which is the tangent plane at the optimum point A. The two planes of support are denoted H1 and H2 and are determined by the normal vectors h_1 and h_2. These in turn determine a supporting cone called the tangent cone. It is the intersection of all the half spaces determined by the planes of support containing the feasible region. The gradient g of the objective function, which points in the direction of steepest ascent, can be expressed as a linear combination of the normal vectors that define the supporting cone with weights, say, y_1 and y_2. That is $g = y_1 h_1 + y_2 h_2$. These y's are the dual variables or economic imputations implied by the optimum choice at A.

This geometric point of view brings out in stark relief the relationship between imputed values and the convex shape of the constraints and objective functions of the optimizing problem. Indeed imputed value is difficult to determine or even to interpret in some of the less regular optimization models. In these latter cases little can be said about the possibility or efficacy of decentralized market mechanisms. On the other hand the computation of optima is likewise difficult so that central planning may still be difficult or impossible to carry out in such cases. Procedures more or less the same as trial and error must be invoked.

The references given at the end of the section on optimization models all have good discussions of duality. Much of the contemporary work in duality theory stems from Rockafellar's *Convex Analysis* [1970]. Balinsky and Baumol [1968] supply an elaborate economic exegesis of duality in nonlinear programming, and Leuenberger [1969] gives a good advanced treatment. Aubin [n.d.] provides an extremely general abstract development.

Algorithms

It is often said that modern optimization concepts were given their great impetus by the electronic computer and George Dantzig's simplex method, for it is one thing to know that an optimum exists and quite another to know how to find one economically. The simplex method for linear programming was extended to various quadratic programming models, to mixed integer programming, and to other examples. Very quickly thereafter various gradient methods appeared for nonlinear programming when the functions were convex or concave. Gradient methods (or methods of steepest ascent) of various kinds were suggested, some of which were built directly on the Kuhn-Tucker theorem and some of which were geometrically motivated.

Implicit in every optimization model whose numerical solution is sought, is the question, "What is the optimum way to find the optimum?" That is, "How can the cost of using a given optimizing model be minimized?" One of the very early discoveries connected with the new simplex algorithm was its astonishing efficiency for general classes of problems. Yet, no one has ever shown it to be the best algorithm for general linear programming problems. Indeed, new modifications and improvements continue to appear, and better ways of finding optima for special types of linear programming models are found in a seemingly unending progression.

The technical issues involved can be illustrated by a smooth, unconstrained minimization model that has a geometric analog, the finding of a lowest point, A, in a valley. Now imagine that the diagrams in figure 7 are the contour maps of this valley. A ball could be released at point 0. If it were propelled solely by gravity, it would presumably follow the path of most rapid descent, a smooth curve as shown emanating from the initial point and minimizing its elapsed time of arrival to the optimum point A. This would be an optimum way of finding the minimum if we evaluate cost as time elapsed. But this path involves a continuous adjustment to the local gradient as the latter varies continuously. And it assumes away inertia. Because of the latter the ball would wander off the optimum path, then veer back and forth across it as shown by the dotted line. The ball would not in fact follow the path of steepest descent but a more or less suboptimal one. Practical numerical methods are somewhat similar to the latter kind of path. Indeed, computation algorithms must be blind to the situation as a whole. They proceed for a time in a given direction generally downward, mistakenly move up, then correct the error and determine a new locally best (but globally suboptimal) direction of descent. The path for such an algorithm is illustrated in figure 7 (b).

It can be stated categorically that optimal algorithms are rare and experience must be used to infer how good a given procedure is and under what conditions a given algorithm works well. This is partly because algorithms for digital computers always involve sequences of relatively simple computations based on purely local information. They begin at some initial, perhaps arbitrary starting point, compute some purely local information that indicates a direction in which a new guess may be chosen to improve on the initial guess, and calculate how far to go in that direction. A new guess is chosen and new local information about neighboring alternatives is computed and the

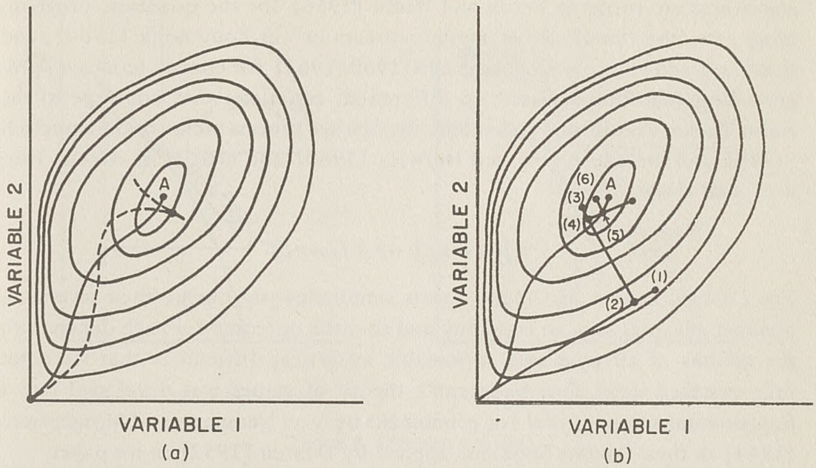


Figure 7. (a) The "optimal" path to A in the absence of friction and inertia is indicated by the solid line, with inertia it is indicated by the dashed line; (b) the path of a typical computer or learning algorithm

process continues. In this way a sequence of suboptimizations is generated which under favorable conditions converges to a final best solution. The reader familiar with the behavioral economics of Simon [1957] and Cyert and March [1963] should note here the striking similarity between optimizing algorithms and behavioral economics.

Early computational experience with Dantzig's simplex method [1949] is discussed in an interesting manner by Orchard-Hays [1956]. The concept of an optimum algorithm and many examples involving unconstrained problems with one or only a few variables will be found in Wilde [1968]. The sequences of suboptimizations involved in most algorithms would appear to be analogous to the behavior of decision makers in complex organizations and in market economies. This suggests that the study of such algorithms should have considerable interest for economists. The formal mathematical study of algorithms was initiated by Zangwill [1969]. A recent contribution is by Fiacco [1974]. The relationship between otpimizing algorithms and behavioral economics was pointed out in Day [1964] and developed in the context of the theory of the firm in Day and Tinney [1968]. Related articles were prepared by Baumol and Quandt [1964] and Alchian [1950]. The reader interested in computational algorithms for various of the optimization models should find the following references of interest. A complete exegesis of the simplex method is given in Orchard-Hays [1961]. Important early nonlinear programming algorithms are those of Frank and Wolfe [1956] for the quadratic programming case, the "methods of feasible directions" in Zoutendijk [1960], and Rosen's gradient projection methods [1960, 1961] for convex (concave) programming. Algorithms based on differential equations that converge to the Kuhn-Tucker conditions and mimic the market process stem from Samuelson [1949] and include Arrow and Hurwicz [1960] and articles in Arrow, Hurwicz, and Uzawa [1958].

Efficiency and Games

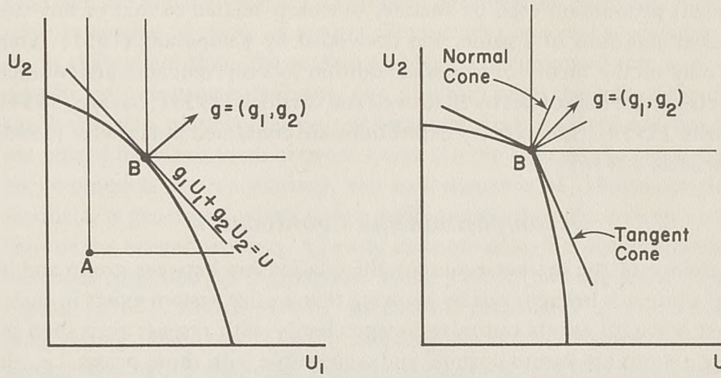
The classical notion that many agents simultaneously pursue their several individual advantages in an economy and that the outcome for each depends on the actions of all possessed formidable analytical difficulties that were not fully resolved until Von Neumann's theory of games was developed into a fundamental working tool for economists by Von Neumann and Morgenstern [1944] in their famous book and applied by Debreu [1952] in his paper.

In this theory not just one but many utility or objective functions guide choices so that the optimizing theory as we have reviewed it so far is inadequate. Indeed, the notion of "optimum" must be expanded. This has been done in various ways, but the one central to most work in economics rests on Pareto's concept of an "efficient" or "Pareto optimal" set of actions in which no one agent can choose a preferred action without forcing another player in the game to choose a less preferred alternative.

The theory of games made possible a deeper understanding of many forms of market competition, as developed, for example, in Shubik's *Strategy and Market Competition* [1959]. It also became a basic tool in studying the theory of risky decisions. Games against nature were constructed to formalize the problem facing a single agent when he could only guess what state his environment might take. The application to statistical inference, the scientific counterpart of this theory, was developed very early by Wald [1945].

But in spite of the extension of optimizing concepts involved in the theory of games, the close relationship to conventional optimizing theory became increasingly evident. For example, it was seen that every two-person, zero-sum game was equivalent to a linear programming model. Kuhn and Tucker [1951] showed that Pareto optimal solutions to a class of multiobjective optimization problems could be characterized by the optimum of a linear combination of those objectives. This quite general duality is at the heart of welfare economics which shows the efficiency properties of competitive equilibria.

The basic idea in this relationship between Pareto optima and conventional optimization is captured in the diagram shown in figure 8 (a) already familiar



(a) The "neoclassical" case (b) The "modern" case

Figure 8. Efficient choices

to generations of economists. If more of U_1 or U_2 is better than less, then all vectors in the cone emanating from the point A are "Pareto better" or more efficient than the point A itself. But no point in the cone emanating from the point B is feasible except B. Hence B is a Pareto optimum or efficient point with respect to the variables U_1 and U_2.

The set of attainable utility combinations is supported at B by the plane H represented by the vector $g = (g_1, g_2)$. Hence B optimizes a linear combination $g_1U_1 + g_2U_2$. Now if U_1 and U_2 are considered to be the satisfaction levels for agents one and two, respectively, then we see how the Pareto optimum B is represented or "supported" by an ordinary optimum. One may also interpret g as the gradient of a social welfare function $\varphi(U_1, U_2)$, which is optimized at point B.

In figure 8 (b) the situation is shown where the set of attainable (U_1, U_2) combinations is convex as in (a) but not smooth. This "modern" case is analogous with the duality diagram for nonlinear programming shown in figure 6 (b). Here the normal cone at point B gives a set of weights (g_1, g_2) so that maximizing $g_1U_1 + g_2U_2$ will give back point B as a solution to the implied convex programming problem.

A good discussion of modern optimization and welfare economics is offered in Dorfman, Samuelson, and Solow [1958, chapter 14]. They bring out the point suggested in figure 8 that the efficient or Pareto optimal solutions to the "game" or multioptimization problem can be obtained by means of parametric programming. An early application of this technique is Manne's study [1956] of the United States petroleum refining industry. The concept of efficient production used by Manne, so closely related to that of the Pareto optimal solutions of a game, was developed by Koopmans [1951]. Standard works on the theory of games in addition to Von Neumann and Morgenstern's classic are the studies by Blackwell and Girshick [1954], Savage [1954], and Karlin [1959]. Elementary expositions are contained in Hurwicz [1945] and Marschak [1946].

Decomposition and Coordination

In the theory of the market economy the relationship between group and individual optima is brought out by showing that a price system exists in such a way that when all agents optimize independently with respect to it then the resulting actions are Pareto optimal and compatible with those prices, i.e., the markets are cleared and all firms and households survive. Koopmans [1957, part I] provides a classic modern nontechnical discussion. The problem of finding the "best" social choice might then be viewed as finding a price adjustment process that will guide a sequence of suboptimizations to an efficient or Pareto optimal point. Such adjustment processes are called tâtonnement processes and represent one general means by which the problem of decomposition of social choice and coordination of individual choices is studied. To be compatible with the requirement of leading to Pareto better solutions, such a process would have to lead to a choice lying in the cone emanating from the initial starting point. Arrow and Hahn [1971] discuss tâtonnement-type models of market processes and Arrow and Hahn [1971] cite earlier work in bibliographical notes.

A second setting in which the relationship between group and individual optima is studied is illustrated by Robinson Crusoe, subject of the most famous parable of the centrally planned economy. A Robinson (or a socialist state) is decomposable into Robinson the consumer and Robinson the producer by means of a price system so that Robinson the consumer can achieve the highest feasible utility at minimum cost and Robinson the producer can maximize his profit of production. This analogy, fully developed by Koopmans [1957], shows that with sufficiently convex technology and preference both the competitive market economy and the socialist economy share the same social equilibria. From this point of view the market is seen to be a device for decomposing the economy's overwhelmingly complex problem of resource allocation into a host of relatively simple, individual suboptimizations which are coordinated by the price system to achieve allocations that are efficient. The idea that marketlike computational procedures could be developed for carrying out central planning was developed by Arrow and Hurwicz [1960].

With this background it is hardly surprising that some of the computer algorithms for solving large-scale complex optimization problems have characteristics similar to those of market tâtonnement or socialist planning processes. In these algorithms the master problem is decomposed into a set of much simpler optimization submodels, one of which plays the role of the coordinator, helmsman, planning bureau, or whatever. Each is solved and the solutions are passed back and forth between them. On the basis of the new information the submodels are reoptimized, and so a sequence of suboptimizations with feedback is generated which, when well conceived, will converge to the solution of the master problem. An early example of such a decomposition procedure was proposed by Dantzig and Wolfe [1961] and applied by Kornai and Liptak [1965]. Kornai [1967] discussed it thoroughly in the national planning setting. Malinvaud [1967] prepared an excellent general discussion of several alternative planning procedures. Lasdon [1970] developed a quite comprehensive text from the computational point of view.

Two fundamental problems complicate the theory and impede progress in its development. One is the formalization of data processing, decision making, and administrative costs and the determination thereby of the optimal level of decentralization. The second is the problem of incentives. Decentralized procedures must be coordinated by an appropriate system of incentives and/or constraints to bring about a compatibility between decentralized optima and the central optimum. Some progress in the former has been made by the developers of team theory, J. Marschak and Radner [1972], who exploit concepts from decision and game theory to formalize the problem of determining optimal decisions and information networks in organizations. This work stems from Marschak's early concern with developing an economic organization theory. Attention to some of the dynamic aspects of such theory is found in T. Marschak [1959, 1968]. The incentive problem has been tackled by Groves [1973].

Multiple Goals

Increasing attention is being paid to the decision problem in which many goals or objectives are pursued. Formally, the problem is much like the n-person game theory in which many objective functions are simultaneously optimized. Not unexpectedly, then, one way of approaching the problem is by means of the efficiency or Pareto optimality concept with which we have already been concerned in several different settings. In particular, the Kuhn-Tucker efficiency theorem mentioned in the section on efficiency and games serves as the basis for an interactive planning procedure involving a planner who has several measurable goals. In this procedure, developed by Geoffrion, Dyer, and Feinberg [1972], a decision maker is asked to specify an initial set of "weights" g_1 and g_2. An efficient point is found if his goal functions are concave. He is then asked to choose new weights and a new efficient point is found. If the decision maker has a sufficiently regular utility function combining the separate goals or objective functions into a single overall goal, then this iterative sequence will lead to the optimal fulfillment of all the goals. If not, he is left with a number of efficient or Pareto optimal possibilities.

Another approach, suggested by Georgescu-Roegen [1954] , is that of lexicographic orderings in which the several goals or objective functions are arranged in a hierarchy. Each is maximized or satiated one after the other until no further scope remains for choice. The relationship of such a procedure to rational choice axioms was investigated by Chipman [1960] . Encarnacion proposed various applications (for example, see Encarnacion [1964]). Day and Robinson [1973] established sufficient conditions for such choice models to be compatible with the requirements of general equilibrium theory. A comprehensive collection involving these and other approaches was assembled by Cochrane and Zeleny [1973] .

Risk and Uncertainty

Although the formal study of risk began during the classical era of economics with Bernoulli and Laplace, and though its importance was recognized and accounted for in the neoclassical period by Marshall and Walras and later by Knight [1921] and Hart [1942] , its formal treatment by means of optimization theory is of modern origin, at least so far as economic theory is concerned. Early attempts to study the problem mathematically in an economic setting were made by Makower and J. Marschak [1938] and by Tintner [1941] . But it is in Von Neumann and Morgenstern's seminal game theory book and in Savage's fundamental work on decision theory [1954] that the decision-theoretic foundations were definitively established. The Von Neumann-Morgenstern approach is that of *expected utility* and makes it possible to study the "best choice" which accounts for risk using conventional optimization theory.

In brief, probabilities (assumed usually to be subjective in nature) are assigned to states of the world. The utility is then conceived to be a random variable whose expected value is to be maximized by choosing an appropriate act or decision, subject to the constraints of the problem. Risk-averting, risk-preferring, and risk-neutral individuals can be represented in this way and the propensity to hedge, to carry portfolios, or to gamble can be explained. Application in economics are by now widespread and, depending on the specific form of the underlying spaces, the risky decision problem is converted into a linear, quadratic, or more general nonlinear optimization problem. Arrow

[1951] prepared a very comprehensive early survey, and Van Moeseke [1965] published an excellent discussion using modern nonlinear programming theory. Markowitz [1958] took as his subject the analysis of "portfolio" type behavior using quadratic programming. Dillon [1971] wrote a comprehensive review of various approaches.

Bayesian decision theory represents an extension of the Von Neumann-Morgenstern approach to the dynamic setting in which the decision maker faces a sequence of choices. At each stage he may modify his subjective probabilities on the basis of current information. An optimal choice can then be made. This approach represents a true formalization of Knight's concept of uncertainty as opposed to risk, for it explicitly treats the probabilities as unknown. The dynamic nature of the Bayesian point of view is brought out lucidly in J. Marschak's "On Adaptive Programming" [1963]. Various applications to problems in econometrics are developed in Zellner [1971]. Cyert and De Groot [1975] examine its application to the theory of the household.

So far the approaches summarized involve properly accounting for the possibility of doing better or worse than one expects by incorporating into the objective function terms that account for risk and uncertainty. A different component that deserves equal attention is the feasible region, for one outcome of a risky or uncertain decision is the impossibility of carrying out the desired choice. At worst this situation spells disaster, at best it forces a new choice. Again, the specific ways in which this problem has been formalized are numerous, but one must mention all of the "safety-first" and risk-programming procedures—for example, those developed by Charnes and Cooper [1959] and Roy [1952] and reviewed by Sengupta [1969]. Day, Aigner, and Smith [1971] provided an exposition of this approach in the setting of the elementary theory of the firm; in their paper three variants of the approach are discussed—one which minimizes the probability of disaster or maximizes the safety margin (safety), one which maximizes expected utility given a fixed probability of disaster (safety-fixed), and one which maximizes expected utility given that a minimum level of safety (probability of survival) has been reached (safety first). The last approach leads to a lexicographic ordering of survival probabilities and expected utility.

Another approach to the study of decision making under uncertainty is that of game theory where the agent is characterized by a game against nature or where two or more agents, represented by two or more persons in the game, account for the most damaging strategy against them. This approach, founded by Von Neumann [1928], was developed by Von Neumann and Morgenstern [1944]. More recent texts include those by Blackwell and Girshick [1954], Savage [1954], and Karlin [1959]. Elementary expositions are provided by Hurwicz [1945] and J. Marschak [1946].

In all of the above approaches probabilities are explicitly involved. A general principle that need not make explicit use of probability is the "principle of cautious optimizing" outlined elsewhere by Day [1970]. In this approach the decision maker being modeled optimizes in the usual way except that he limits his choices to alternatives "close enough" to a safety zone. The region of "safe enough" solutions can be based directly on a safety metric or "danger distance" instead of a probability of disaster. It therefore generalizes and places on a behavioral footing the idea of safety-first decision making. An alternative but closely related way of modifying the feasible region to account for uncertainty is Shackle's idea of focus loss [1949]. In the form developed by Boussard and Petit [1967] for a firm with a linear programming choice structure, the agent has a focus on loss or disaster level associated with each activity. In addition, the firm has an allowable level of loss usually associated with some minimal survival income. Each activity has an allowable proportion of the total allowable loss, and each activity adds to the allowable loss by increasing the total expected income.

Dynamic Optimization

The role of foresight in decision making can be illustrated by means of a diagram which incorporates several of the fundamental ingredients of dynamic optimization theory. This is done in figure 9. "States of the world" are represented by axis s and acts or decisions (we do not distinguish between the two here) are represented by axis a. Associated with each state is a feasible interval of choices. The interval is determined by a correspondence (see figure 5) in the upper right quadrant. For example, the set of feasible choices associated with the initial state s_0 is that part of the vertical line through the point s_0 that lies in the shaded graph representing the feasibility correspondence. To each act on the upper vertical axis is associated a payoff or outcome as determined by the concave payoff function in the upper left quadrant. For example, π_0 is the payoff associated with the act a_0. The environmental transition is represented in the lower left quadrant and shows how the state changes in response to each act. Thus, if the agent chooses a_0, the succeeding state will be s_1. By measuring the act on the same scale as the payoff we can project the act chosen onto the left horizontal axis, in this way generating a dynamic process.

Now suppose we begin with a rational but myopic decision maker. He knows the feasible region given the state s, and he knows the payoff function. But suppose that he does not know or try to estimate the environmental transition function. Given that he does the best he can in the given situation beginning at s_0, he chooses a_0, which leads to s_1, at which point he picks a_1.

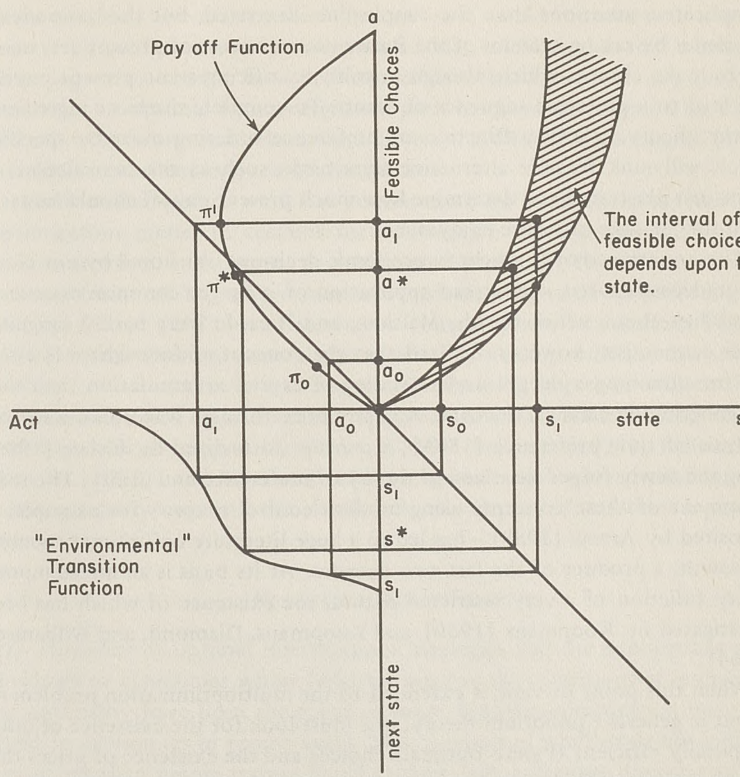


Figure 9. "Short-sighted" and "far-sighted" optimizing

This leads (coincidentally!) back to s_0 and to choice a_0 again. Subsequently an oscillation between a_0 and a_1 occurs. The sequence of payoffs is $[\pi_0, \pi_1, \pi_0, \pi_1, \pi_0, \ldots]$.

Now consider a not-so-myopic individual who knows and takes account of the environmental impact of his actions. Beginning at s_0 he chooses a_0 as before but instead of a_1 at s_1 he picks a^*, realizing that if he picks a_1 as his myopic counterpart did, he will be prevented, because of the environmental feedback, from achieving such a good gain in the next period. His "far-sighted" choice yields a payoff level π^* that can be maintained in perpetuity. Because π^* lies above the chord connecting π_0 and π_1 the average payoff yielded by this far-sighted strategy is better than the myopic strategy.

Of course intertemporal optimization theory encompasses much more complicated situations than the simple one illustrated, but the basic idea is the same: by taking account of the future consequences of present acts one is led to make choices which, though possibly sacrificing some present payoff, will lead to a preferred sequence of events. To complete the basic ingredients of the theory, a utility function or preference ordering must be specified which will rank feasible alternative time paths such as the two alternative paths just illustrated and determine how much present payoff should be sacrificed for the sake of future enjoyment.

The importance of foresight in economic decisions was noted by our classical predecessors, but widespread application of it by the common man was a possibility about which Smith, Malthus, and Ricardo were hardly sanguine. Later economists, however, realized that the concept of foresight was essential for obtaining a deeper understanding of capital accumulation than they had inherited in classical doctrine. An early breakthrough was Böhm-Bawerk's analysis of time preference [1884], a concept formalized by Fisher [1906] using the newly forged neoclassical theory of preference and utility. The redevelopment of these concepts using modern control theory—for example, as exposited by Arrow [1968] —has led to a huge literature on optimal economic growth, a product of the last two decades. At its basis is an intertemporal utility function of a very restricted nature, the existence of which has been investigated by Koopmans [1960] and Koopmans, Diamond, and Williamson [1964].

When this point of view is extended to the multioptimization problem inherent in general equilibrium theory, one must look for the existence of intertemporally efficient (Pareto optimal) choices and the existence of prices that would permit individual intertemporal optimizations, using these prices to achieve the efficient solutions. The role of such intertemporal efficiency prices has been investigated by Malinvaud [1953]. The neoclassical version of general equilibrium theory from the point of view of intertemporally optimizing firms and households was worked out by Hicks [1939], but his focus was that of temporary rather than dynamic equilibrium. A contemporary line of development that extends the optimal control point of view to the game situation is the work on differential games. A recent collection of studies of this kind (Kuhn and Szegö [1971]) includes papers by Berkovitz, Blaquiere, Friedman, Rockafellar, and Varaiya. An example is by Simaan and Takayama [1974].

The utility aspects of decision making are omitted altogether in on important line of optimal growth theory, namely, the line emanating from Von Neumann's general equilibrium model [1945]. In this theory only the technology of the economy is specified. No time preferences enter the argument. Instead, a technologically maximal rate of growth is defined and its existence is determined. The existence of "prices" that would support such a rate of growth by profit-maximizing individuals is also established so that a behavioral analog exists in part. Indeed, the possibility that a real economy could follow such an optimal balanced growth path has been investigated by Tsukui [1968]. Dorfman, Samuelson, and Solow [1958] and Koopmans [1964] provide an excellent exegesis of this theory.

Much contemporary work has concentrated on generalizing the individual optimization model to account for information costs, uncertainty and the joint estimation, and control or dual control problems. This line of work had led to a fusion of Bayesian statistical decision theory as developed, for example, by Zellner [1971] and stochastic control theory as described by Aoki [1967]. An influential control theoretic study dealing with the dual control problem of simultaneously deciding and obtaining improved information is Fel'dbaum [1965]. The Bayesian approach involves dynamic programming techniques as developed by Karlin [1955], Bellman [1957], Bellman and Dreyfus [1962], and Blackwell [1967]. A good review of control theory, dynamic programming, and the closely related calculus of variations is given by Intriligator [1971].

Recursive Optimization

The existence of optimal intertemporal strategies and the implications on individuals or economies whose behavior satisfies the conditions of intertemporal optimality have been the focus of most dynamic optimization theory applied to economics in recent years. The question of whether or not individuals of less than heroic stature could in their daily enterprise discover such behavior has only recently begun to receive attention. One way to approach the problem is to break the complex intertemporal optimization problem down into a sequence of much simpler, possibly myopic or relatively short-sighted suboptimizations with feedback. The decision maker does not know the environmental transition equations but merely approximates them, or more simply he forecasts relevant information on the basis of past observations without trying to estimate the structure of the system as a whole. Then, protecting himself from blunders of short-run overcommitment by rules of caution or uncertainty or risky decision making, he optimizes the current situation. When new observations are available, he reestimates and forecasts the relevant information variables and optimizes anew. Thus a sequence of optimizations with feedback is generated which explains actual behavior and which, if the true environment is well behaved, may converge to a path that is intertemporally optimal in some sense, just as a sequence of tâtonnementlike adjustments may lead to a general equilibrium that is efficient or Pareto optimal.

On the other hand, such a convergence may not occur, as in the example with which we introduced the idea of dynamic optimization in the preceding section. It is probably not hard to convince oneself that such convergence does not always occur and perhaps only rarely occurs in the real world.

Sequences of optimizations with feedback are called recursive programs. We have seen that such systems arise not only in the attempt to develop a formal theory of adaptive behavior as just outlined but also in a variety of seemingly quite different settings. We have, for example, observed that mathematical programming algorithms have this structure. We have observed that tâtonnement and decentralized decision processes have this character also. The explicit mathematical representation of economic behavior using recursive optimization originated with Cournot, who used it to investigate the behavior of competing duopolies. Later variants of duopoly theory that preceded the theory of games and Chamberlin's monopolistic competition theory [1948] used an essentially similar type of model. A growth model based on such a principle was stated by Leontief [1958], and a general class of recursive programs was developed by Day and Kennedy [1970] and Day [1970]. Various applications to quantitative modeling of industrial sectors and agricultural regions have been undertaken by Day and others, some of which have been collected in Takayama and Judge [1973] and a number of others in Day and Cigno [in preparation]. Applications to general equilibrium theory are supplied by Cigno [in preparation] and Allingham [1974]. These applications are based on the premise that economic agents' decisions are best characterized by local suboptimizations of partial models of the economy as a whole which are updated and re-solved period after period in response to new information about what other agents have done and what the economy as a whole has done. Like their counterparts in the field of optimization algorithms the recursive optimization models usually cannot be shown to be the best way for the agents to suboptimize. However, some progress has been made in showing that recursive programs based on plausible behavioral hypotheses may converge to the results obtained from the optimal control point of view.

A special and very limited class of recursive programs arises when an optimal strategy can be derived from the dynamic programming point of view which shows how, on the basis of current information and past choices, the next decision can be decided on in the best way. A special group of models falling in this category that have been widely applied in econometrics is the linear decision rules of Holt, Modigliani, Muth, and Simon [1960] and Theil [1964].

On the Normative Content of Optimizing

In reviewing the theory and application of optimization concepts one is struck by the contrasting interpretations given to mathematical programming and game theory models. On the one hand, an optimization model is formulated to find the "best" solution of some problem. On the other hand, it may be used to characterize the behavior of a real world decision maker. The fallibility of the latter, however, is all too evident to each of us. Moreover, what is "best" clearly rests on a subjective basis—namely, what the agent thinks he likes and wants and what he thinks he can do at the time. These subjective constituents of optimal choice may change whenever something new is learned. How to learn in the optimal way is a problem shrouded in mystery, despite progress in decision theory. What is true of the agents in an economy in general is also true of the model builder and theorist in particular: a solution to an optimizing model is contingent on the structure of the model, something that ultimately rests on the subjective perceptions of mind and on current scientific theories and models which must always be approximations to the real world itself. We thus come to the conclusion that optimality is essentially a logical property of model solutions. Any normative content attributed to optimal solutions must be subjective in character.

Indeed, if one takes into account one's mortal existence and the problem of accommodating the unknown and unknowable preferences of generations yet unborn, one wonders what meaning, if any, the notion of intertemporal optimality has. As the problems associated with global economic development become better known, and as the very-long-run implications of present industrial activity receive increasing attention, this question too is bound to receive more and more attention.

Concluding Remarks

If we were to model our world system microeconomically using optimization theory, we should have to conceive of a game with some three or four billion players, a number growing at a rate of some five thousand per hour. Moreover, the collection of players and its organization into groups of various kinds such as families and firms are variables determined in a complex way by the evolution of the process as a whole. We know also that nature—man's environment—consisting of our nonhuman neighbors, who likewise are evolving in complex living systems, and the physical world must be considered as a player in this game. But what a player! With an uncountable number of strategies at his disposal.

The fact is that none of us takes into account the actions of very many players in this ultimate game. Even if we should like not to do so, we ignore, because we must, all but a few chosen friends and enemies and acquaintances about whom we care or become aware. We account for tiny facets of the universe in our specialized thoughts. And when we turn inward to explore our preferences, we find opening up before us a mystery as infinitely varied and as unfathomable as interstellar space. We make simplified models of ourselves, solve them, and act. But we throw these models away and start over, or we modify them over and over again. If we fail to do so, our humanity withers and we become like automata in our consistency.

Thus, optimization theories can never yield a complete theory of being or becoming. The man who finds his only poetry in mathematical programs, games, and marginal calculations, who fails to listen to his hunches and feel his senses to the full, who ignores the pleadings of the spirit as it cries out from the works of poets and prophets and painters and from dreams, is less a man.

Still optimization models help us in our battle to create order. The simple clarity of their insights is poetry! Possibly even, their concepts belong to the a priori properties of mind by which, according to Kant, thought's content is defined, by which thought's possibilities and limits are demarcated like fiery poles fixing emblazoned zones in the night. In this case it would be futile for some new doctrine of economy to try to get along without them, just as it would be futile for man to try to get along without science and art in general.

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Optimization Models in Agricultural and Resource Economics

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The application of optimization concepts to the economics of agriculture and resource use has a history as long as mathematical economics itself. It was in the context of agriculture in an "isolated state" that von Thünen [1966] in 1826 developed his own concept of gain and loss at the "margin" and used it to develop a theory of relative economic value and spatial diversity in the use of land, labor, and capital. Indeed, we have it on the good authority of Marshall [1890] that von Thünen, the first agricultural economist among economists, along with Cournot, provided the initial inspiration for marginalist economics.

On the other hand, von Thünen and his classical predecessors Smith, Malthus, and Ricardo were also employing concepts that are most effectively represented by the use of linear programming theory. It was not until the modern era that the full unity underlying these different classical and neoclassical optimization approaches could be brought out by means of a single mathematical structure which incorporated both points of view as special cases—namely, the Kuhn-Tucker theorem (Kuhn and Tucker [1951]).

The balanced blend of analytical reasoning and careful empirical observation that characterized von Thünen's work was evident in the work of later economists who specialized in agriculture. An important example is provid-

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ed by "budgeting." Its development and widespread application in agricultural economics occurred in the first quarter of this century. Not only was it an extension of von Thünen's early studies, but it played a central role in the education of a generation of agricultural economists, thereby helping to prepare the discipline for the rapid adoption of modern optimization methods. Indeed, in the hands of its best practitioners, budgeting was more than a trivial special case of mathematical programming. It was an explicit arithmetic procedure for obtaining approximate optima of simple constrained optimization problems and for exploring the broader implications at the regional and national levels of economic behavior in response to changing economic conditions and policy controls.

The budgeting era may be said to have reached its culmination in 1951, for in that year Mighell and Black's masterly exercise in budgeting, *Interregional Competition in Agriculture*, was published. It is something of an irony that the modern optimization methods introduced in the same year effectively rendered obsolete that splendid monument to good economic thinking and patient arithmetic. While no one would think of doing it that way any more, it is clear that the modern approach has formalized economic optimization and has eased the computational burdens of using it but has added few if any insights into the nature of the problem not already fully appreciated in the economic literature.

The first specific application of modern optimization to agricultural economics was by Hildreth and Reiter in 1951, and the application to the spatial problems that had dominated much of von Thünen's original work came with Fox's study [1953] of the feed-livestock economy. But the rapid adoption and widespread application of modern optimization methods to the economic analysis of agriculture and resource use may have been largely the result of the extensive and varied examples produced by Earl Heady and his associates in the 1950s. From that period on the application of optimization concepts to the formulation and solution of substantive problems in agriculture and resource economics has led to a literature so vast that a comprehensive survey is impossible. Consequently, this survey is restricted to selected contributions (primarily from the American literature) that are of seminal importance from a historical point of view, that are representative of an important research area, or that are of contemporary interest. The references cited in the text are supplemented in the bibliography by a few key survey papers on research that could not be covered in this paper.

Modern optimization methods and their application in agriculture and resource management are of interest to specialists in many fields, and as a result articles published in the professional journals for general economics, engineering, operations research, and so on are relevant. Moreover, economists must be aware of the bulletins emanating from various federal agencies, state experiment stations, world organizations such as the United Nations, and the International Bank for Reconstruction and Development, as well as journals published in other countries. With these observations in mind perhaps the reader may find it possible to forgive the authors for any oversights that occur in this paper and for the somewhat arbitrary nature of the selected references.

The literature reviewed is divided into categories of food and diet, farm and agribusiness management, farm firm development, production response, interregional and spatial economics, natural resources, and agricultural development problems. For convenience in researching the literature table 1, which follows the text of this chapter, classifies the references according to these categories.

Throughout the discussion "neoclassical optimizing" refers to maximizing smooth, unconstrained, or equation-constrained functions using the basic tools of marginal analysis, i.e., traditional calculus. "Classical optimizing" is used here to include the linear programming problem that underlies the classical rent and trade theories and the budgeting arithmetic of von Thünen and later economists. "Modern optimizing" refers to the maximization of objective functions constrained by inequalities or equalities, requiring generalized Lagrange techniques and including the classical and neoclassical approaches as special cases.

Food and Diet

The diet problem seems an appropriate subject with which to begin a review of the literature on applications of modern optimization theory in agriculture and resource economics. Obviously, the efficient use of food resources is a goal of growing importance in our finite world with its rapidly growing, often ill-fed population. The diet problem is that of determining the least cost combination of foods that will meet dietary standards. According to Dorfman, Samuelson, and Solow [1958], Jerome Cornfield was the first to formulate this problem in an unpublished memorandum in 1941. Stigler presents a careful statement and discussion of the problem in "The Cost of Subsistence" [1945]. It is interesting to note that his solution of the problem was not cast explicitly in the modern mathematical programming framework, but instead involved a careful application of the arithmetic budgeting procedures known in agricultural economics for decades.

A thorough and illuminating explicit linear programming treatment of the problem (which cites an unpublished 1947 paper by Dantzig and Laderman) is given by Dorfman, Samuelson, and Solow [1958]. An elaborate empirical study developed for the interesting and important problem of protein supply in a developing economy is found in the work of V. E. Smith [1974].

The economic dietary (food-mix) problem for animal production is logically the same as the one for humans, and Waugh [1951] was the first to exploit linear programming in its explication. The budgeting framework was effectively and independently applied to the same problem at about the same time by Christenson and Mighell [1951]. By way of contrast to the linear programming work, a neoclassical optimizing approach was the basis of Heady's analysis of least cost dairy cow rations [1951] and hog rations (Heady et al. [1953]). An ingenious by-product was the "pork costulator" that allowed farmers to take advantage, without great computational effort, of the marginalism that economic theorists had long supposed to be descriptive of farmers' behavior. Briefly, Heady and his collaborators statistically estimated production functions for animals using experimental data, solved the least cost feed-mix problem for various input-output price combinations, and in this way located points on the derived economic demand function for various feed inputs. The input demand schedule was then represented by means of an inexpensive plastic circular slide rule.

Farm and Agribusiness Management

In 1951 a simplified version of the optimal crop rotation problem using modern optimization methods was published. This was Hildreth and Reiter's contribution [1951] in the famous Koopmans volume. There appears to have been a lag between this seminal application and the widespread adoption of linear programming as a standard working tool in the profession, but the lag was short. With characteristic pragmatism and innovation agricultural economists were quick to see the utility in the new approach. A flood of effective studies in farm management appeared in the mid-fifties. Very early studies were made by King and Freund [1953] and King [1953], Swanson and Fox [1954], and Bowlen and Heady [1955]. Quick to follow were studies by Bishop [1956], Heady, McAlexander, and Schrader [1956], Swanson [1956], and Coutu [1957]. In 1958 Heady and Candler published one of the first comprehensive texts on applied linear programming. Comparative static analyses using parametric linear programming algorithms also appeared at this time (for example, McPherson and Faris [1958] and the elegant piece by Hildreth [1957b]). Somewhat more recent applications are discussed by Krenz, Baumann, and Heady [1962] and Bolton [1964]. This early work is still of interest, and indeed it is worth serious reconsideration for much of it deals with the economics of soil conservation practices which have not been in vogue for some time but which are receiving renewed attention.

The relationship between classical economic (budgeting) thinking and modern optimization theory is reflected in pieces by Mighell [1955] and Kottke [1961] that point out the similarity or equivalence between the budgeting and linear programming approaches. Edwards [1966] gives a lucid exposition of this relationship, and Swanson [1961] notes that linear programming logic has had a profound impact on budgeting procedures. It is also worthwhile to point out that workers in agriculture were quick to find the intellectual intrigue in simplex and parametric programming algorithms and contributed expository pieces as well as methodological wrinkles of considerable ingenuity. Heady's economic interpretation of the simplex algorithm [1954] is a prime example, as are the studies by Hildreth [1957b], Puterbaugh, Kehrberg, and Dunbar [1957], and Candler [1956, 1957, 1960].

The potential usefulness of integer and mixed integer programming in farm management has been recognized for some time. Edwards [1963] suggested a number of possible applications of the techniques to farm problems using Gomory's integer programming algorithm [1958]. There are several integer and mixed integer programming algorithms available, some of which were surveyed in Maruyama and Fuller [1964]. The most current and complete survey of integer programming algorithms is by Geoffrion and Marsten [1972]. Maruyama and Fuller [1964] proposed an "RHS" ("right-hand-side") method, which was essentially a computerized complete enumeration method. Candler and Manning [1961] and Musgrave [1962] used parametric linear programming to deal with decreasing costs and increasing returns. Giaever and Seagraves [1960] and Yaron and Heady [1961] used integer, mixed integer, and nonlinear programming to investigate decisions involving economies of scale.

Marketing analysis of agricultural commodities naturally involves transportation costs and spatial efficiency, aspects of optimization to be considered later in this paper. Several studies, however, may appropriately be mentioned here. Stollsteimer [1963] developed a linear programming model which determines the number, size, and location of plants processing a fixed amount of a single raw material into a single output. Economies of scale were dealt with through the use of parametric programming. King and Logan [1964] attacked the same problem and added an iterative, partly heuristic method to handle economies of scale. Candler, Snyder, and Faught [1972] dealt with a more general problem involving several raw materials and multiple outputs using a concave programming algorithm. The algorithm, a mechanized version of the King-Logan algorithm, is equipped to solve multiple local optima problems. Bressler and Hammerberg [1942] and Hammerberg, Parker, and Bressler [1942] used budgeting to specify optimal route organization and truck sizes. Bressler [1952] applied the same methods to develop an efficient system of city milk distribution in Connecticut.

By the late 1950s rapidly developing computer technology was expanding the scope for sophisticated programming techniques. Bellman's dynamic programming [1957] was one approach which accordingly found increased application to management decisions in various industries; agriculture was no exception. The earliest agricultural applications of Bellman's approach were to optimal replacement problems, as described by White [1959], Faris [1960], and Halter and White [1962]. Burt and Allison [1963] applied the method to a Markov process in choosing wheat rotations, Minden [1968] proposed the use of dynamic programming as a tool for farm investment decisions, and Hinrichs [1972] discussed a recent application in West German agriculture. One of the most attractive features of dynamic programming is the facility with which stochastic parameters may be incorporated (for example, Burt [1965]). After a first flash of excitement induced by the flexibility and potential of Bellman's approach as typified by Throsby [1964], applications have been limited to rather simple subsystems of total farm systems. The explanation (Throsby [1968]) lies partly in the formidable computational requirements of dynamic programming. This is particularly true of allocation problems such as multiperiod farm investment. When both inputs and outputs are multiple, dynamic programming is beset by the "curse of dimensionality," because computational burdens increase exponentially with the number of outputs or inputs considered.

The significance of risk and uncertainty in the farm environment is underscored by the numerous efforts of agricultural economists to embody these concepts in decision models. Freund [1956] made the first application of active stochastic programming to a farm management problem. This technique is essentially the same as that of Markowitz's portfolio selection technique [1952]; the resultant problem is a quadratic program. Examples of applications of active stochastic programming are provided by McFarquhar [1961], Merrill [1965], and S. R. Johnson, Tefertiller, and Moore [1967]. Compared with linear programming, quadratic programming algorithms make heavy computational demands. Hazell [1971a] develops a technique leading to a linear programming problem which incorporates the mean absolute deviation of the objective function parameters, and Thomson and Hazell [1972] report a Monte Carlo study which indicates that Hazell's method gives results which are quite close to quadratic programming results. Chen's remarks [1971] and Hazell's reply should be read with Hazell [1971a]. A separable programming approach which also approximates Markowitz's E,V method was employed by Thomas et al. [1972]. A particularly important method for incorporating uncertainty is the focus-loss approach introduced into agricultural economics by Boussard and Petit [1967]. Boussard [1969] later showed that the descriptive power of the model was at least as good as that of alternative models. In 1955 Dantzig offered a model of sequential programming under uncertainty which combined the merits of linear programming and sequential analysis; Cocks [1968], Rae [1971a, 1971b], and Yaron and Horowitz [1972] have applied this model and its extensions to problems of farm management.

Dillon's expository article [1971] reviews thoroughly the application of subjective probability theory to agriculture. This includes as special cases many of the approaches, mentioned above, including E,V analysis.

Early application of game theory to farm management were restricted to "games against nature." Probably the first example was Schickele's [1950] application to climatic uncertainty. Later, Swanson [1957] suggested application of game theoretic frameworks to the same problem. In a series of analyses Dillon and Heady [1961] applied the Wald, Laplace, and Savage criteria to farmers' choices of enterprises and found a poor descriptive fit. In an extensive application to weather uncertainty, Walker et al. [1960] showed how the various criteria suited different financial situations and attitudes toward risk. In 1962 Dillon wrote his excellent survey article of game theory applied to agriculture, detailing both suggested and actual applications. His conclusion that the use of game theory had nearly run its course was premised on the continued use of ordinary games against nature. More recently, however, McInerney [1967] suggested the use of constrained games against nature. Several theoretical works have followed, notably McInerney [1969], Hazell [1970], Maruyama [1972], and Kawaguchi and Maruyama [1972], and it appears that practical applications of constrained games may be in sight. In private correspondence Professor Maruyama informed the authors of this paper that constrained games were applied in Japanese agricultural economics literature as early as 1966 (see, for example, Imamura [1966]).

Interesting work involving modern techniques of farm management is not always reported in the professional literature, or it may appear in relatively obscure outlets, working memoranda, and so on. An example is the computer-aided real-time farm management advisory service under the direction of John Schmidt of Wisconsin. Similar systems are operating at Purdue and Michigan State. Candler, Boehlje, and Saathoff [1970] outlined the problems of development and implementation of the top farmer program at Purdue. Nonetheless operational developments attest to the practical relevance of what might otherwise be thought of as elegant toys for mathematicians and economic theorists.

The above references only scratch the surface of a vast body of literature, but we hope they provide a sample adequate for illustrating the variety of uses to which modern optimization methods have been put in the study of optimal farm management. Before proceeding to other major areas of application, it would be in the spirit of the present undertaking to comment briefly on the role of the more traditional neoclassical marginal analysis in the farm management setting. The tradition goes back to von Thünen; its definitive modern statement is in Black's *Introduction to Production Economics* [1926] and Heady's *Economics of Agricultural Production and Resource Use* [1952]. Modern developments in statistics have made possible the quantitative exploitation of the neoclassical point of view. The early examples not surprisingly came from the Ames School with a focus on crop-nutrient response (for example, Tintner [1944], Heady [1946], and Heady, Pesek, and Brown [1955]). Some of Glenn Johnson's work at Kentucky and at Michigan State in the 1950s, described in Bradford and Johnson [1953], Haffnar and Johnson [1966], and Johnson and Quance [1972], was also based on this model. Of course, the optimal feed-mix and feed-ration problems solved in either the linear programming way or the neoclassical way are also an important aspect of farm management. (Early work of this kind was mentioned in the section on food and diet.)

Farm Firm Development

Economic development in agriculture (in the absence of a geographical frontier) usually involves the growth of some farm firms and the decline or abandonment of others. From the managerial point of view, in which firm policies to enhance growth are sought, and also from the production response point of view, in which aggregate implications of development and agricultural policy are the focus, farm growth and decline are of interest. Studies of farm development have much in common with the farm management studies already reviewed, and it is not always possible to categorize a model into one class or another unambiguously. For example, the early multiperiodic linear programming studies of farm growth and investment such as Swanson [1955] or Loftsgard and Heady [1959] had managerial, production response, and farm growth aspects. However, this field of application is important enough to consider separately, as is indicated by Irwin's review of various methods for farm growth modeling [1968].

Irwin and Baker [1962] marked the beginning of a noteworthy series of farm growth models which have emphasized financial aspects of farm firm growth. Martin and Plaxico [1967] report on a polyperiod model of farm growth with investment, capital markets, and consumption all considered in some detail. Johnson, Tefertiller, and Moore [1967] apply Monte Carlo techniques to a firm growth model with stochastic crop yields. White [1959] expands the Martin-Plaxico and Johnson-Tefertiller-Moore models by incorporating investment, credit, production, and consumption matrices. In two articles [1968a, 1968b] Baker extended and generalized the Irwin-Baker model. The work of Baker and his protégés is of particular interest as a behavioral approach to modeling firm growth. These studies focus on financial constraints or rules of thumb. Barry and Baker [1971], for example, use reservation prices on credit to infer attitudes toward uncertainty.

One problem common to multiperiod linear programming models of farm growth has been matrix size. Given a single period submatrix of any detail, the multiperiod model presents formidable problems in construction and in computation of solutions. J. M. Boussard has made significant strides on both of these problems: his matrix generating program GEMAGRI (Boussard [1972]) automatically generates a multiperiod linear program on punched cards from standard farm records, and his clever application of a turnpike theorem to the multiperiod linear programming model of farm growth derives a practical method for finding the "optimal" horizon for such a model (Boussard [1971]).

Heidhues [1966] focuses the recursive programming approach on the study of farm growth and decline in an analysis of West German farms. His study incorporates considerable technological and financial detail. It was followed by Steiger's study [1968], summarized in de Haen and Heidhues [1973 and forthcoming], which developed individual recursive programming models for all farms in two villages of an area where examples of growth and decay were evident. A recently completed study by Ahn and Singh [1972] uses a similar approach to study the differential effect of development policies on farms of different sizes in a developing agriculture.

A line of work closely related to that of farm growth has long been pursued at the United States Department of Agriculture—namely, the analysis of resource requirements for achieving various income levels in various farm situations. The work of John Brewster [1957] and others must be mentioned in this context.

Production Response

One can interpret the solution of an optimizing model as being a long run equilibrium toward which the economy is tending and/or toward which it might be encouraged by various incentives and controls. If reality can be described in this way, then optimizing becomes a powerful tool for policy analysis. This idea lies behind many important applications of optimization methods in agricultural economics. Many of the regional budgeting studies that originated in the 1920s were oriented to such production response purposes. With the advent of linear programming many joint USDA and state experiment station studies were converted to the new approach. The effects of price supports, income controls, and varying technological, marketing, and pricing situations were investigated using linear programming and parametric programming techniques.

The production response work was generally conducted under the title of "adjustment," and many important examples were sponsored by the USDA. Thus we had one program of research involving the cotton states of the South, one focusing on wheat production in the West, one involving livestock and feedgrains in the Corn Belt, and two more concentrating on dairying in the Lake States and in New England. Only a small proportion of this work has ever been published, but no doubt a significant number of active agricultural economists gained their early training in part through participation in these undertakings. The work was described in general terms in Sundquist et al. [1963], Colyer and Irwin [1967], and the Northeast Dairy Adjustments Study Committee [1963].

A concern that emerged in the course of this work was the aggregation problem involving the question of how much estimates of regional responses were distorted by the use of linear programming models of whole regions or representative farms as opposed to "adding up" individual farm models. As the latter was uneconomic, the issue was one of great importance. The first analysis of the problem using the duality theory of linear programming was by Day [1963]. Further consideration was given by Miller [1966] and Lee [1966]. Buckwell and Hazell [1972] applied a clustering technique to identify groups of farms which could be legitimately aggregated according to an extension of Day's criteria. Empirical work addressing the same issues was reported by Sheehy and McAlexander [1965], Barker and Stanton [1965], and Frick and Andrews [1965].

Another problem encountered in the application of representative firm models was representation of investment and disinvestment. Glenn Johnson's fixed asset theory [1958] was an important step toward solution of this problem.

Parametric programming techniques were applied to the problem of inferring supply functions and resource allocation responses from both aggregative and representative firm models. Kottke [1967] summarizes work in this field.

A quite different point of view was taken by the developers of recursive programming. Their view, as initially applied in agriculture by Henderson [1959], was to use programming models augmented by behavioral constraints of the kind already used by Wood [1951], to estimate short-run behavior of farmers at the regional level in a disequilibrium situation. Henderson's original model was used to make a one-year forecast of the allocation of land to various crops for a hundred United States farming regions. The dynamic implications of Henderson's model were brought out by Day [1963], who then stated the general class of recursive linear programming models to which Henderson's model belonged as a special case. Day's study also gave the first example of how recursive linear programming could be used to trace out the evolution of an industry over time. Applications by Schaller and Dean [1965], Muto [1965], and Cigno [1971] followed. Nontechnical discussions of the general methodology were also contributed by Day [1961, 1962]. An ambitious application of the recursive programming approach was the national model which originated with Glen T. Barton's production response group at the USDA in 1958. Day's 1963 study was the prototype study for this undertaking, and after the follow-up test by Schaller and Dean [1965] a national model was planned and implemented. Sharples and Schaller [1968] described the project during its construction phase. The model is currently being used as an experimental working tool and is being replaced by a more complex general simulation model. An even more ambitious undertaking is Thoss's multisector, multiregional recursive programming model for short-run national planning in Germany [1970]. Henrichsmeyer and de Haen [1972] describe a "next-generation" effort that is currently in the planning stage.

Interregional and Spatial Economics

Of extreme importance in agriculture and in resource economics generally is the study of interregional or spatial efficiency and development. Going all the way back to von Thünen for its conceptual foundation, the application of modern techniques came with the development of the Hitchcock-Koopmans transportation model, a special case of linear programming for which efficient computer algorithms were developed in the early 1950s. Early applications of this model to distribution and pricing are discussed by Judge [1956], Henry and Bishop [1957], Farris and King [1961], Snodgrass and French [1958], and Stemberger [1959].

Beckmann and Marschak [1961] used the more general activity analysis framework of Koopmans and Reiter [1951] to extend the spatial distribution model to include production. Building on this work, Lefeber [1958] specified a linear programming model to determine efficient allocation and shadow prices, given the regional prices of final products and the regional endowment of primary factors. Orden's transshipment problem [1956] is a special case of the Beckmann-Marschak model which King and Logan [1964] applied to determine the optimum location, number, and size of processing plants and factor and final product flows. Judge, Havlicek, and Rizek [1965] studied the optimum location of livestock slaughter and geographical flows of live animals and meat. In Snodgrass and French [1958] an aggregate model is used to determine the optimum interregional flows of whole milk and the corresponding equilibrium prices for 1953. This general model is also applied individually to fluid milk, butter, cheese, evaporated milk, and nonfat dry milk solids. A second model minimizes transportation and processing costs in determining the location of processing plants, and a third model adds production costs to these and specifies optimal production location. In a series of well-known papers Egbert and Heady focus on the best location for producing a fixed final national bill of wheat and feed grains (Heady and Egbert [1959], Egbert and Heady [1961, 1963]). Buchholz and Judge [1966] focus on livestock using the same approach. The Egbert-Heady models were forerunners of a family of works relating to the national allocation of agricultural resources: Heady and Skold [1965], Heady and Whittlesey [1965], Eyvindson, Heady and Srivastava [1975], and Brokken and Heady [1968]. Birowo and Renborg [1965] supply an application to Swedish agriculture.

Characteristics of all of the above work were the exclusion of explicit demand functions and the treatment of prices as exogenous. Building on the theoretical work of Enke [1951] and Samuelson [1952], who showed how trade theory could be formulated in mathematical programming terms, Fox [1953] shows how interregional supply-demand equilibrium could be modeled and solved computationally. The initial model focused on livestock feed. The United States was divided into ten regions and the demand for feed was estimated for each. Using the 1949-50 figures for regional production of feed, numbers and prices of livestock, and their demand equations, Fox derives equilibrium consumption, price, and shipments of feed for each region. In a later article Fox and Taeuber [1955] extend the 1953 model to include livestock. Regional demand and supply functions for livestock are added to the previous model, and a joint equilibrium solution is derived for both feed and livestock.

Dunn [1954] broadens and applies von Thünen's theory of location to the agricultural segment of the economy. An equilibrium system which includes space is formalized and is designed to solve problems on an aggregated or industrial level. Dunn's framework takes multiple products and technological interrelationships into account. Judge [1956] uses the Enke-Samuelson formulation as a basis for determining the spatial equilibrium prices for eggs when the regional supplies of eggs are taken as fixed and the demand functions are explicitly included. He then uses the linear programming transportation model to determine the optimum geographical flows of the commodity. Judge and Wallace [1958] propose an iterative parametric solution procedure to solve for prices, consumption, supplies, and flows when regional demands are represented by functional relations and supplies are predetermined. Judge and Wallace [1959, 1960] develop an equilibrium model for beef and pork, a model which incorporates given regional supplies, transport costs, and demand equations for twenty-one regions of the United States.

Tramel and Seale [1959] develop a reactive programming procedure for determining the competitive prices and flows for the Enke-Samuelson problem. This procedure was applied by Maruyama and Yoshida [1960] in Japan and then was developed into two interrelated sets of interregional quadratic programming models by Takayama and Judge [1964a, 1964b] and Maruyama and Fuller [1964, 1965]. The framework for the quadratic version of the modified Beckmann-Marschak interregional activity analysis model is contained in two 1964 articles by Takayama and Judge. Subsequent articles by Plessner and Heady [1965], Yaron, Plessner, and Heady [1965], and Plessner [1972] contributed to the development of the quadratic programming model and investigated approaches to the problem when market demand functions fail to satisfy the integrability condition. Applications of the Takayama and Judge model include an interregional analysis by Buchholz and Judge [1966] of the United States feed-livestock economy and a spatial equilibrium analysis by Hall, Heady, and Plessner [1968] of the field crop sector of United States agriculture. Applications to other areas: Louwes, Boot, and Wage [1963] apply quadratic programming to the solution of the problem of optimal use of milk in the Netherlands when there are monopolistic tendencies in the market; Bawden [1966] shows how multicommodity international trade problems may be solved by exploiting the quadratic programming model of Takayama and Judge [1964a, 1964b]; Plessner [1967] carries out purely theoretical work designed to show how these operational spatial models fit into the general equilibrium theory. There have been a number of large-scale applications of the Maruyama-Fuller model in the Japanese literature. For example, the studies by Maruyama [1967] and Muto [1965] both had direct impact on Japanese government policy. Dynamic interregional equilibrium using concepts of intertemporal optimality and multihorizon programming has been treated formally by Judge and Takayama [1973], although empirical applications have yet to be achieved.

Interregional economics with a focus on disequilibrium and comparative dynamics instead of equilibrium and comparative statics is proposed by Day [1967] and is given theoretical treatment by Day and Kennedy [1970]. Bawden's spatial model [1966] constitutes an interesting example of this recursive programming approach to the interregional equilibrium problem. He represents regional production by econometric equations that depend on prices which are determined by a transportation model that optimizes short-run trade patterns. It may be regarded as a complex type of cobweb approach to supply-demand interactions as opposed to the equilibrium theory following the Samuelson-Enke formulation. A more recent study by Schmitz and Bawden [1973] applies this methodology to the world wheat market. Quite similar to Bawden's approach is Kottke's application [1970] of a recursive version of the Takayama-Judge model to an imperfectly competitive dairy industry. We have already mentioned the related work by Thoss [1970] and by Henrichsmeyer and de Haen [1972].

There are at least two good survey articles on spatial equilibrium models: Bawden [1964], and Weinschenck, Henrichsmeyer, and Aldinger [1969]. In addition, Takayama and Judge [1971] and Judge and Takayama [1973] furnish extensive bibliographies, exposition of theory and methodology, and examples of applications of spatial and temporal price allocation models.

Natural Resources

Recent applications of quantitative optimization techniques to allocation of natural resources have been numerous, and in particular Bellman's dynamic programming principle has been extensively applied. Underlying this recent work is the general economics of extractive resources. Hotelling's [1931] pioneering application of the calculus of variations to the theory of nonreplenishable resources was perhaps the earliest contribution to this theory. Subsequently, numerous works by S. V. Ciriacy-Wantrup and others laid further theoretical groundwork for the application of sophisticated optimization techniques during the 1960s.

Economic models of commercial fishing have played an important role in developing an approach to replenishable resources. Two seminal works are provided by H. S. Gordon [1954] and Scott [1955]. These neoclassical models were applied by Crutchfield and Zellner [1962] and Quirk and Smith [1969]. Optimal control was used to good effect by Clark [1973]. In 1968 V. L. Smith proposed a general economic model of production from natural resources, and in 1970 Burt and Cummings utilized Bellman's dynamic programming framework to state an even more general theory of production and investment for natural resources. Most of the applied work has been focused on water resources. We shall first review contributions here and then briefly consider pollution studies.

Moore [1961] was one of the first to identify the problems of allocation of water over time. An important subset of the water conservation (temporal allocation) problem is the use of groundwater. In a series of works Burt [1964a, 1964b, 1966, 1967a, 1967b, 1970b] developed an approach to the groundwater problem using dynamic programming and employing stochastic state variables to represent stochastic elements in the supply of groundwater. Closely related to the groundwater problem is the allocation of irrigation water. Burt [1964b] and de Lucia [1969] both treat the case of conjunctive use of groundwater and surface water, and Biere and Lee [1972] treat the case of reservoir water used to recharge groundwater in dynamic programming frameworks, but most of the studies of irrigation water are related to the management of reservoirs for water used directly in irrigation. The decision environment of the reservoir managers includes several elements of uncertainty including the weather and the demand for water. The authors of several articles (R. L. Anderson [1968], R. L. Anderson and Maass [1971], Butcher [1971], and W. A. Hall, Butcher, and Esogbue [1968]) apply stochastic dynamic programming, assuming the supply of water to be stochastic and the demand determinant; Burt and Stauber [1971] assume a given inflow and a stochastic demand; de Lucia [1969], Dudley, Howell, and Musgrave [1971a, 1971b, 1972], Dudley [1970, 1972], and Dudley and Burt [1973] assume both stochastic supply and stochastic demand. The series of articles by Dudley alone and in collaboration with others culminates in the 1973 article by Dudley and Burt, which outlines a general stochastic dynamic programming model to determine optimal levels of intertemporal water application rates, intraseason irrigated acreage, and preseason acreage to be planted.

There have been other approaches to the problem of optimum reservoir management. One is the application of chance-constrained programming to single-purpose reservoirs by Eisel [1970, 1972], Loucks [1970], Joeres, Leibman, and Revelle [1971], and Nayak and Arora [1971]. Guise and Flinn [1970] employ a Takayama-Judge spatial equilibrium model to derive optimal prices for a water system. In an early application of stochastic linear programming Manne [1962] employed Markov process optimization to management of a multipurpose reservoir. Young [1967] was perhaps the first to apply linear decision rules to reservoir management.

Several studies have concentrated on selection, sequencing, and timing of investments in water resource projects. Jacoby and Loucks [1972] have described a technique combining simulation models of river basins and optimization routines to select and assess possible patterns of investment. Cummings and Winkelmann [1970] and Regev and Schwartz [1973] apply the dynamic programming framework of Burt and Cummings [1970] to the problem of interregional investment and allocation of water. Regev and Schwartz are also concerned with economies of scale and therefore apply mixed-integer programming. Young and Pisano [1970] apply nonlinear programming to minimize investment costs in water projects; Butcher, Haimes, and Hall [1969] and Morin and Esogbue [1971] propose special dynamic programming algorithms for sequencing and scheduling of water supply projects; and Erlenkotter [1973] formulates a dynamic programming model to minimize costs of developing a given hydroelectric capacity in a river basin. R. A. Young and Bredenhoft [1972] use a two-stage optimization model to simulate reactions of economic decision makers in a river basin. A current study being conducted under the auspices of Heady at Iowa State is concerned with the allocation of water resources between regions in the United States and the environmental effects of these allocations. A part of this study is reflected in the recent application of the Heady-Egbert regional adjustment model by Heady et al. [1973].

One "natural resource" that is currently in the public eye is the capacity of the environment to absorb society's pollution residuals. An imaginative approach to this problem is illustrated by d'Arge [1971] in his use of a parable of an astronaut irretrievably lost in space. To determine the astronaut's optimal pattern of consumption over time, d'Arge uses optimal control theory. On a more mundane level engineers and economists are developing models which will help to determine "optimal" levels of pollution. At the University of Illinois Earl Swanson and his colleagues have been conducting interdisciplinary work to determine the sedimentation effects of various cropping systems using linear programming models. Narayanan and Swanson [1972] report the results of a parametric linear programming study of the trade-offs between sedimentation and farm income. A similar work was undertaken by Seay [1970]. Graves, Hatfield, and Whinston [1972] outline an approach which employs nonlinear programming to determine optimal methods of water quality control for the Delaware River Estuary. Davidson and Bradshaw [1970] employ Pontriyagin's minimum principle to the treatment of polluted streams, and Hass [1970] proposed the Dantzig-Wolfe decomposition algorithm as a basis for a decentralized method of arriving at optimal water pollution taxes. It is certain that many more such applications will follow as the quality of data and the understanding of environmental systems improves.

Agricultural Development Problems

National planning has been the predominant setting in which optimization techniques have been applied to problems in economic development. For computational reasons these studies have until quite recently been limited to linear programming methods. One of the earliest (and best-known) examples of such a model is that of Sandee [1960]. Some of the more frequently cited works in this field are those by Manne [1966], Manne and Weiskopf [1969], Eckaus and Parikh [1968], Chenery and MacEwan [1966], and Bruno [1967]. Many of these focus on the optimal resource allocation between agriculture and other sectors when such national goals as foreign exchange maximization are pursued. Often they build on and incorporate previous Leontief-style input-output models of the economy in question. This underlying input-output work is summarized in a series of conference proceedings beginning in 1951, and continuing through Barna [1963], Carter and Brody [1972a, 1972b], and Brody and Carter [1973]. An illuminating example focusing on agriculture is the study by Fox, Sengupta, and Thorbecke [1966], who proposed imbedding an input-output model in a more general multisector analysis.

More recent planning models reflect advances in computer technology both in their increased attention to detail and in their use of more difficult optimizing techniques such as mixed-integer programming and decomposition. A number of important examples by such authors as Barraza, Bossoco, Duloy, Norton, Kutcher, Winkelmann, and others will be found in Goreux and Manne [1973]. Dynamic programming and mixed-integer programming techniques have begun to find application in sectoral or single-industry planning models; for example, Manne [1967] applies both techniques to several industries of the Indian economy, and Westphal [1971a] applies mixed-integer programming to the economy of South Korea.

Application of optimization techniques to farm management in less developed countries to date has been limited. McFarquhar and Evans [1957] provide an early application of linear programming to combinations of enterprises in tropical agriculture. More recently Heyer [1971] has applied linear programming to the problem of allocating peasant resources in a small rural area of Kenya. In a second work Heyer [1972] extends her original model to account for uncertainty through the use of a game theoretic framework. Spencer [1973] has applied linear programming to a study of the allocation of labor resources to rice production in Sierra Leone. His study was based on farm management survey data, and its objective was to improve interregional allocation of labor resources. Baker [1973] employs linear programming in an analysis of the role of credit in smallholder farming. Probably nowhere will one find a greater output of useful optimization studies to problems of less developed agriculture than at the Punjab Agricultural University in Ludhiana. Most of these studies are by S. S. Johl and A. S. Kahlon (for example, Johl and Kahlon [1967]) and various of their students. This demonstrates the need for scholars to research local journals and experiment station reports for applied studies relevant to their special problems.

General systems simulation models such as those described by Halter, Hayenga, and Manetsch [1970] are of growing importance in less developed countries. The reason is that they make possible the systematic study of a model economy when data are inadequate, or when goal specification is difficult, or when the economy is simply too complex to optimize with existing algorithms and computers. They also are useful when, as a prelude to systematic planning, one wants to understand how the economy works and how it is likely to respond to policy controls.

General systems simulation includes, as a special category, models in which given components are represented by optimizing submodels. This category also belongs to the class of recursive programming systems of Day and Kennedy [1970]. Examples of the recursive programming approach to the problem of tracking a developing agricultural economy include Singh [1969, 1971], and Ahn and Singh [1972]. Thoss [1970] focuses on multisector development using this technique.

Table 1. Categorization of Survey References into Eight Branches of Agricultural Economics Research

Citations preceded by an asterisk (*) indicate survey articles or substantial bibliographical sources.

1. *Introduction.* Day [1977] ; Fox [1953] ; Hildreth and Reiter [1951] ; Kuhn and Tucker [1951] ; Marshall [1890] ; Mighell and Black [1951] ; and von Thünen [1966].
2. *Food and Diet.* Christenson and Mighell [1951] ; Heady [1951] ; Heady, Woodworth, et al. [1953] ; V. E. Smith [1974] ; Stigler [1945] ; and Waugh [1951].
3. *Farm and Agribusiness Management.* Agrawal and Heady [1968] ; *Agrawal and Heady [1972] ; J. R. Anderson and Hardaker [1972] ; Babbar [1955] ; Bellman [1957] ; Bishop [1956] ; Black [1926] ; Bowlen and Heady [1955] ; Bradford and Johnson [1953] ; Bressler [1952] ; Bressler and Hammerberg [1942] ; Brewster [1957] ; Burt [1965] ; Burt and Allison [1963] ; Byerlee and Anderson [1969] ; Candler [1956] ; Candler [1957] ; Candler [1960] ; Candler [1972] ; Candler, Boehlje, and Saathoff [1970] ; Candler and Manning [1961] ; Candler, Snyder, and Faught [1972] ; Charnes [1953] ; Charnes and Cooper [1959] ; Chen [1971] ; Cocks [1968] ; Conner, Freund, and Godwin [1972] ; Coutu [1957] ; Dantzig [1951] ; Dantzig [1955] ; Dantzig and Wolfe [1960] ; *Dillon [1962] ; *Dillon [1971] ; Dillon and Heady [1961] ; Doll [1972] ; Dorfman, Samuelson, and Solow [1958] ; Edwards [1963] ; Edwards [1966] ; Faris [1960] ; Freund [1956] ; Geoffrion and Marsten [1972] ; Giaever and Seagraves [1960] ; Halter and Dean [1971] ; Halter and White [1962] ; Hammerberg, Parker, and Bressler [1942] ; Hazell [1970] ; Hazell [1971a] ; Hazell [1971b] ; Heady [1946] ; Heady [1951] ; Heady [1952] ; Heady [1954] ; Heady [1971] ; Heady and Candler [1958], Heady and Egbert [1964] ; Heady, McAlexander, and Schrader [1956] ; Heady and Pesek [1954] ; Heady, Pesek, and Brown [1955] ; Heady, Woodworth, et al. [1953] ; Hildreth [1957a] ; Hildreth [1957b] ; Hildreth and Reiter [1951] ; Hinrichs [1972] ; Hitchcock [1941] ; Hutton [1963] ; *Hutton [1965] ; Imamura [1966] ; G. L. Johnson [1952a] ; G. L. Johnson [1952b] ; G. L. Johnson [1955] ; G. L. Johnson and Haver [1953] ; G. L. Johnson and Quance [1972] ; Kawaguchi and Maruyama [1972] ; G. A. King and Logan [1964] ; R. A. King [1953] ; R. A. King and Freund [1953] ; Kottke [1961] ; Langham [1963] ; Loftsgard and Heady [1959] ; McFarquhar [1961] ; McInerney [1967] ; McInerney [1969] ; McPherson and Faris [1958] ; Markowitz [1952] ; Markowitz [1959] ; Maruyama [1972] ; Maruyama and Fuller [1964] ; Maruyama and Yoshida [1960] ; Merrill [1965] ; Mighell [1955] ; Minden [1968] ; Musgrave [1962] ; Officer and Halter [1968] ; Peterson [1955] ; Puterbaugh, Kehrberg, and Dunbar [1957] ; Rae [1971a] ; Rae [1971b] ; Roy [1952] ;

Table 1. Categorization of Survey References into Eight Branches of Agricultural Economics Research (Cont.)

Citations preceded by an asterisk (*) indicate survey articles or substantial bibliographical sources.

Schickele [1950] ; B. J. Smith [1973] ; Stollsteimer [1963] ; Swanson [1955] ; Swanson [1956] ; Swanson [1957] ; Swanson [1961] ; Swanson [1966] ; Swanson and Fox [1954] ; Thomas et al. [1972] ; Thomson and Hazell [1972] ; Throsby [1964] ; *Throsby [1968] ; Tintner [1944] ; Tintner [1955] ; Walker et al. [1960] ; Waugh [1951] ; *Weinschenck, Henrichsmeyer, and Aldinger [1969] ; White [1959] ; Yaron and Heady [1961] ; Yaron and Horowitz [1972] .

4. *Farm Firm Development.* Baker [1968a] ; Baker [1968b] ; Baker [1973] ; Barry and Baker [1971] ; Boehlje [1967] ; Boehlje and White [1969] ; Bolton [1964] ; Boussard [1969] ; Boussard [1971] ; Boussard [1972] ; Boussard and Petit [1967] ; Day and Cigno [forthcoming] ; de Haen and Heidhues [1973] ; de Haen and Heidhues [forthcoming] ; *Irwin [1968] ; Irwin and Baker [1962] ; S. R. Johnson, Tefertiller, and Moore [1967] ; J. R. Martin and Plaxico [1967] ; *Renborg [1970] ; Steiger [1968] ; Yaron and Horowitz [1972] .

5. *Production Response.* Barker and Stanton [1965] ; Barry and Baker [1971] ; Bolton [1964] ; Boussard [1969] ; Boussard [1971] ; Boussard [1972] ; Boussard and Petit [1967] ; Brokken and Heady [1968] ; Buchholz and Judge [1966] ; Buckwell and Hazell [1972] ; Cigno [1971] ; Colyer and Irwin [1967] ; Cowling and Baker [1963] ; Day [1961] ; Day [1962] ; Day [1963] ; Day [1967] ; Day and Kennedy [1970] ; de Haen [1973] ; Egbert and Heady [1963] ; Eyvindson, Heady and Srivastava [1975] ; Frick and Andrews [1965] ; Heady and Skold [1965] ; Heady and Whittlesey [1965] ; Heidhues [1966] ; Henderson [1959] ; Henrichsmeyer and de Haen [1972] ; G. L. Johnson [1955] ; G. L. Johnson [1958] ; G. L. Johnson and Haver [1953] ; Kottke [1967] ; Kottke [1970] ; Krenz, Baumann, and Heady [1962] ; Lee [1966] ; Maruyama and Fuller [1965] ; Miller [1966] ; Miller [1972] ; Muto [1965] ; Northeast Dairy Adjustments Study Committee [1963] ; Plessner and Heady [1965] ; Schaller [1968] ; Schaller and Dean [1965] ; Schmitz and Bawden [1973] ; Sharples and Schaller [1968] ; Sheehy and McAlexander [1965] ; Sundquist et al. [1963] ; Thoss [1970] ; Wood [1951] .

6. *Inter-regional and Spatial Economics.* *Bawden [1964] ; Bawden [1966] ; Bawden, Carter, and Dean [1966] ; Beckmann and Marschak [1961] ; Birowo and Renborg [1965] ; Bressler [1952] ; Bressler and Hammerberg [1942] ; Brokken and Heady [1968] ; Buchholz and Judge [1966] ; Candler, Snyder, and Faught [1972] ; Day [1962] ; Dunn [1954] ; Egbert and Heady [1961] ; Egbert and Heady [1963] ; Enke [1951] ; Eyvindson, Heady, and Srivastava [1975] ; Farris and King [1961] ; Fox [1953] ; Fox and Taeuber [1955] ; Guise and Flinn [1970] ; Haffnar and Johnson [1966] ; Hall, Heady, and Plessner [1968] ; Heady and Egbert [1959] ; Heady and Skold [1965] ; Heady and Whittlesey [1965] ; Heidhues [1966] ; Henrichsmeyer and de Haen [1972] ; Henry and Bishop [1957] ; Hitchcock [1941] ; Judge [1956] ; Judge, Havlicek, and Rizek [1965] ; Judge and Takayama [1973] ; Judge and Wallace [1958] ; Judge and Wallace [1959] ; Judge and Wallace [1960] ; G. A. King and Logan [1964] ; Koopmans [1949] ; Koopmans and Reiter [1951] ; Kottke [1970] ; Lefeber [1958] ; *Leuthold and Bawden [1966] ; Louwes, Boot, and Wage [1963] ; Maruyama [1967] ;

Table 1. Categorization of Survey References into Eight Branches of Agricultural Economics Research (Cont.)

Citations preceded by an asterisk (*) indicate survey articles or substantial bibliographical sources.

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Agricultural Production Function Studies

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Following World War II, agricultural economists made a sustained effort to improve methodology and develop applications in quantifying agricultural production relationships mathematically and in using this knowledge to determine economic attributes of the production process. These studies involved calculus and incorporated such recent developments in statistics as more efficient design of experiments, multiple regression, and tests of significance. Perhaps more important from the standpoint of applied economics, the work used production principles based on marginal analysis and equilibrium conditions.

In 1939 Sune Carlson in his classic book, *A Study on the Pure Theory of Production*, defined the production function as the relationship between the variable productive services and the output under the assumption that the plant or fixed services remained constant. He said that this relationship could be most conveniently expressed in mathematical form, writing the amount of output as a function of the different variable services. He also defined marginal productivity, the production surface, isoquants, isoclines, ridge lines, the expansion path, isocosts, and other properties with economic implications derived from the production function.

Several contributions to agricultural economics literature synthesized the advances of Carlson [1939], Hicks [1946], and others, relating the theory of the firm to the applied field of agricultural production economics. The well-known text by Black and his associates [1947], *Farm Management*, and

Heady's "Elementary Models in Farm Production Economics Research" [1948] were important contributions in the immediate postwar period. Later, Heady's *Economics of Agricultural Production and Resource Use* [1952a] and Bradford and Johnson's *Farm Management Analysis* [1953] became the basic references for the new orientation in agricultural production economics. During this same period Allen's *Mathematical Analysis for Economists* [1953] and Tintner's *Econometrics* [1952] were widely used texts for mathematical and econometric models and methods.

Earlier work in quantifying production relationships sometimes involved a continuous relationship with or without a mathematical expression of the relationship between input and output. Mitscherlich was perhaps the first to suggest a nonlinear production function relating fertilizer use to crop output [1928]. Spillman [1933] also utilized an exponential yield curve with similar characteristics.

The USDA Technical Bulletin 1277, *Input as Related to Output in Farm Organization and Cost of Production Studies*, by Tolley, Black, and Ezekiel [1924] stimulated much interest in production function analyses of farm enterprises from farm data. Examples of production relationships include a tabular production surface of daily gain for steers as related to daily corn and hay consumption and output of pork showing diminishing marginal feed productivity.

During the World War II period three USDA studies, stimulated by John D. Black of Harvard, were published. They related output of milk, pork, and beef to total feed consumed (Jensen et al. [1942], Atkinson and Klein [1945], and Nelson [1945]).

A variety of applications has been made involving the production function approach with and without reference to agriculture. Of special interest to agricultural economists are those concerning the farming and agribusiness industries, groups of farms, production of specific crops or livestock, and other rural applications. This review primarily relates to production functions of agricultural crop and livestock enterprises from experimentally derived input-output data. Prior to this main area of concentration, however, a brief discussion of other applications of the production function approach in agriculture is presented.

The pre-World War II industry studies (Douglas [1934], Douglas and Gunn [1942]) using cross-section or time series data, provided some of the methodology for more recent work including use of the exponential function generally known as the Cobb-Douglas. Logarithmic transformations have been widely used because of their convenience in interpreting elasticities of production, minimal requirements for degrees of freedom, and its simplicity of computation.

Aggregate Production Functions

A series of whole-farm production function studies have been conducted in which different farms were used to get different levels and combinations of inputs, and farm income was used as the dependent variable. The best-known early applications in the United States are those presented by Tintner and Brownlee [1944] and Heady [1946].

Bradford and Johnson [1953] analyzed TVA test-demonstration farm records in Marshall County, Kentucky. Marginal value productivities were derived for acres of land, months of labor, investment in forage and livestock, and current expense. They concluded that a larger investment in livestock and forage, a lower machinery investment, and a reduced relative labor input would be needed to equate marginal value productivities with costs. Studies by Heady and Shaw [1954] and Heady and Baker [1954] were concerned with productivity in four farming areas in Montana, Alabama, and northern and southern Iowa. Heady [1955] compared resource productivity and imputed shares between landlord and tenant for a sample of rented farms. Heady and Swanson [1952] compared marginal productivities of farm resources for five areas of Iowa. In 1956 Heady, Johnson, and Hardin edited *Resource Productivity, Returns to Scale, and Farm Size*, a collection of studies concerned with a variety of concepts, procedures, and problems of production function analysis using cross-sectional farm data.

Hildebrand [1960] reported results from Kansas using farm record data for different years and with variations in the model used. An important finding of his research was the wide variability of results from year to year and from model to model in spite of the fact that nearly all of the correlation coefficients were significant at the one-percent probability level.

These and later studies note important implications for the allocation and productivity of resources in agriculture. Their major limitations relate to the great heterogeneity of conditions from farm to farm, the complete or relative absence of control or measurement of variables not included in the function, and the real possibility of multicollinearity among variables. The literature in economics journals contains many articles on the limitations and possible sources of bias in production function research with cross-section or time series data. Plaxico [1955] warned against use of this research for making adjustments on individual farms. Griliches [1957] showed how lack of specification of a management variable could bias the productivity estimates for capital upward and returns to scale downward. Similarly, lack of quantification of the quality of labor could increase the elasticity of capital and decrease the elasticity of labor. (See also Bronfenbrenner [1944], Mundlak [1961], and Reder [1943].)

Other studies have used time series or cross-sectional data in a Cobb-

Douglas analysis of various policy issues for the United States. Two examples are D. G. Johnson's analysis [1960] of output implications of a declining farm labor force and Griliches's study [1957] of the sources of productivity growth using sixty-eight regions as observations and including levels of education as an input.

In the latter part of the 1950s researchers began to utilize linear programming as a way of synthesizing production relationships without having to rely on time series or cross-sectional data from existing farms. Early work includes that of McPherson and Faris [1958] to derive milk output as a function of the price of milk. Martin, Coutu, and Singh [1960] analyzed levels of capital and management on small farms. O'Neal [1959] studied resource productivity in north Georgia using data from linear programming to obtain income estimates for different levels and combinations of resources. Several regional projects, such as the Southern Farm Management Research Committee S-42 work, were conducted to determine the effects of alternative prices and programs on farm adjustments and output.

Applications of functional analysis in the agricultural processing and marketing industries developed in the 1940s were also important forerunners to the crop and livestock production function work which followed. Bressler's approach [1945] in synthesizing cost curves for milk plants using budgeting and industrial engineering techniques resulted in great interest within the profession. Nicholls [1948] used weekly time series data from fourteen departments of a midwestern meat packing firm to predict the number of hogs processed as a function of total man-hours and labor per person per week.

Few production function studies are reported relating to rural development. Undoubtedly the difficulty of specifying outputs has inhibited research in this and related fields. While water supply, sewage treatment, and refuse removal can be quantified relatively easily, many other services have no physical unit of measure. One recent example deals with functions for student achievement in rural high schools by Bieker and Anschel [1974a, 1974b]. For a review of applications in the field of public finance, see Shoup [1969] and Hirsch [1970].

The literature in recent years contains numerous articles on alternative or modifying forms for the Cobb-Douglas production equation to change the assumptions on the elasticity of substitution, marginal products, and returns to scale. Examples include Zellner and Revankar [1969], and Dobell [1968]. Halter, Carter, and Hocking [1957] showed how modifications could allow for all three phases of the production relationship.

Production Functions for Crop Production

In about 1950 production economists, inclined toward the new emphasis on production economics research, started investigations in the economics of fertilizer use. Interest among professional workers developed rapidly, and investigations and assessments were under way at agricultural experiment stations and by the regional farm management research committees, the USDA, the TVA, and private industry.

In assessing the present state of knowledge researchers pointed out that recommendations to farmers traditionally had been the responsibility of physical scientists (Dorner [1954], Hutton [1955]). As a result, criteria of physical response rather than economic response was generally used. Also, experiments were relatively inefficient for quantifying the economic range of the production surface. Examination of agronomic data revealed that rates of application were generally not at high enough levels to permit identification of the economic optimum. The reliance on testing for significant differences in yield for different levels of a fertilizer nutrient typically resulted in research designs where nutrient levels were spaced geometrically, whereas characterization of response as a continuous relationship with treatment levels evenly spaced is more efficient for estimating functional relationships. Reporting only averages of locations and years obscured or concealed economically important variables. Physical and economic interrelationships among nutrients and other important variables were unknown or of uncertain validity. Questions were raised about what effect optimizing N, P, and K simultaneously would have on economic optima compared with determining optimum levels of each nutrient separately with the others at a constant level.

During 1954 formal multidisciplinary studies involving agronomists, economists, and statisticians were under way in several states including Iowa, Kentucky, Michigan, North Carolina, Virginia, Idaho, Indiana, Texas, and Vermont and also at the USDA (National Academy of Sciences [1961]). The fertilizer industry was providing substantial support for projects on the economics of fertilizer use. The TVA was supporting projects and held the first of several annual seminars bringing together production economists, agronomists, and statisticians.

The extent and magnitude of multidisciplinary cooperation which developed was remarkable. Glenn L. Johnson [1957] stated that these "evidences of cooperation on the part of agronomists make it inappropriate to continue the protestations long made by economists, that the design of agronomic experiments does not permit economic interpretation of experimental results."

Ibach and Mendum [1953] wrote a USDA report showing procedures for calculating the most profitable combinations of N, P, and K using the exponential yield curve. At Iowa State Heady and Shrader [1953] delved into the interrelationships of agronomy and economics in research and in making recommendations to farmers. The multifactor experiments at Iowa conducted by Heady and his associates were reported in a series of journal articles and station bulletins. The initial work on corn in 1952 involved a 9 by 9 incomplete N-P factorial replicated twice in a completely randomized design. This type of design was used to include a wide range of nutrient inputs without making the experiment too large. The wide range of nutrient applications was selected to ensure that the most profitable rates derived from marginal analysis would fall within the limits of the experiment and for efficiency in estimating the production surface.

Data presented by Heady, Pesek, and Brown [1955] include several types of regression equations estimated by least squares regression procedures including the Cobb-Douglas, the quadratic cross-product, and the quadratic square root equations. Isoquants were calculated to estimate combinations of nutrients to produce given yields, and the marginal properties were derived to obtain the least cost combinations of nutrients to produce a given yield and the combinations and levels of nutrients to maximize profits per acre for given sets of prices. A series of experiments followed for different crops and sections of the state and involved other variables such as rotations, initial levels of nutrients, and seeding rates. (See Heady, Doll, and Pesek [1958], Heady, Pesek, and McCarthy [1963], and Heady, Pesek and Rao [1966].)

Several important contributions during this period resulted from projects at North Carolina State University. Initially, work involved alternative procedures for analyzing existing data. These included analysis of alternative continuous functions, use of a price map to simplify presentation of optimum rates for alternative prices, and the development of a discrete model less restrictive than the traditional continuous function but still subject to a diminishing returns restriction. The involvement of statisticians Richard L. Anderson and D. D. Mason resulted in methodological developments over the years. (See P. R. Johnson [1953], Stemberger [1957], and C. G. Hildreth [1954].)

Nine years of cooperative agronomic-economic research in Michigan, conducted by Glenn L. Johnson and his associates, were summarized in Hoffman and Johnson [1966]. This report traces the attempts of researchers to characterize the response from fertilizer use under conditions where response is often obscured by other factors. When experiments involving complete rotations and "conventional" small-plot techniques began in 1954, the results generally showed a high unexplained within-treatment variation. Researchers became increasingly concerned about the universe for which the results would apply. These two problems became the central focus of experimentation.

A system was developed for using large plots on randomly selected farm fields that met selective soil and management conditions. These "controlled survey" experiments had a common check plot on each field so that between-farm differences could be accounted for, and the number of plots on any one farm was reduced to four, including the check plot. Comparable data were obtained from a farmer survey and from small-plot experiments. The authors concluded that the "controlled survey" technique was a more reliable way of getting input-output data than past efforts and that future investigators should be encouraged to define explicitly the population about which they hoped to make inferences. The authors indicated a belief that this technique should be considered as an approach in developing countries in attempting to get maximum research and extension information from a given outlay of funds.

The work and cost required for multifactor experiments caused research workers to develop and try new designs. A composite design developed by Box [1954] for industrial research was used at North Carolina (Hurst and Mason [1957], Mason [1956, 1957]). This design required a minimum of 15 treatment combinations per replication compared with 125 for the 5^3 complete factorial. Tramel [1957a] developed a modification called the triple cube design, requiring 31 treatment combinations, and C. G. Hildreth [1957] proposed an interlaced factorial design. The designs were compared by B. P. Havlicek, Smith, and J. Havlicek [1962] in a greenhouse experiment using a 5^3 factorial as a standard of comparison. The authors concluded that the composite designs are useful when successfully centered on the point of maximum yield but that miscentering resulted in biased production functions. In agricultural crop studies the location of the maximum varies with moisture conditions and other factors, often resulting in an observed maximum different from a generalized predicted one.

Interest by statisticians in improving methodology has continued. Recently Anderson and Nelson [1975] explored techniques using intersecting straight lines as an alternative to conventional curvilinear forms of curve fitting. Other important contributions are also being made by statisticians overseas (Gomes [1970], Yates [1967]).

Several studies have incorporated water from irrigation as a variable. Moore [1961] dealt with a general analytical framework. Hexem, Heady, and Caglar [1974] derived production functions relating water and nitrogen to yield for corn, wheat, cotton, and sugar beets from seventy experiments in five western and southwestern states.

A significant number of research reports have dealt with the variations of yield curves over time. Involved are variations in weather, an important factor over which farmers have little or no control, and the accumulation or depletion of nutrients in the soil. Economists have used several approaches to this problem. Brown and Oveson [1958] discussed year-to-year variations in the response of spring wheat to nitrogen applications for ten years. Orazem and Herring [1958] analyzed the effects of soil moisture at seeding time and rainfall during the growing season as related to nitrogen response by grain sorghum in Kansas. Knetsch and Smallshaw [1958] calculated a response relationship to nitrogen and drought on millet for a Tennessee location. Smith and Parks [1967] extended this analysis by simulating results over many years with a computer simulation technique and long-term weather records. They calculated a probability distribution of different outcomes from using alternative levels of nitrogen.

Swanson, Taylor, and Welch [1973] used three decision models for analyzing the year-to-year variation in corn response to nitrogen for eight locations and for five seasons in Illinois: (1) to maximize the average return, (2) to maximize the minimum return, and (3) to minimize the maximum regret or loss from not choosing the correct rate given the season.

Researchers have given attention to the importance of varying fertilizer use with changing crop-fertilizer price ratios. Hutton and Thorne [1955] pointed out that for the 1953 Iowa corn experiment it would take a substantial change in the ratio to make a difference of $4 in per-acre income and that the difference would be less than $1 based on the historical annual price ratios of the 1951-54 period. They also pointed out that using N and P_2O_5 in a one-to-one ratio instead of the optimum ratio would decrease income $0.11 to $0.33 per acre for 1951-54 annual prices.

Using North Carolina data, J. Havlicek and Seagraves [1962] found similar net income consequences for corn. The highest cost of a wrong decision with corn prices varying between $0.75 and $1.75 and nitrogen at $0.11 per pound was $2.90 per acre. Similarly, Knetsch [1961], using Tennessee data for corn, found that nitrogen rates could be varied by fifty or sixty pounds in either direction from the optimum with very small profit losses. Swanson, Taylor, and Welch [1973] came to similar conclusions for corn from nitrogen fertilizer studies at eight locations in Illinois. Taking one location as an example, they concluded that a drop in the corn-nitrogen price ratio from thirty to ten would require less than a twenty-pound decrease per acre in the economically optimum level of nitrogen application.

Other studies have shown a higher economic consequence for use of nonoptimum rates. For example, a Georgia study by Woodworth et al. [1957] for Coastal Bermuda hay in an unfavorable season shows a loss of $7 per acre if the hay value is $30 per ton when using an optimum rate for $20 hay. For a favorable season, the loss is $21 per acre. High economic consequences were also found for Bahiagrass hay (Beaty et al. [1961]).

The problems of determining the population to which a given response function would be applicable and relating results from agricultural experiment stations to given populations of farmers have been especially troublesome to economists concerned with crop production. Ibach, of the USDA

Economic Research Service, developed a generalized response concept and used it to make estimates of responses to fertilizer for major crops by states. The specific estimates were made by researchers in the state experiment stations and published as USDA Agricultural Handbook 68 [1954]. These basic data were used by Ibach and others for examining the outcomes of alternative policy proposals. Ibach [1957] developed estimates of land and fertilizer combinations to produce the United States corn crop. The generalized relationships were revised and published by Ibach and Adams [1968] as USDA Statistical Bulletin 431. The revised publication contains estimates for the agricultural subregions of each state. They represent an interpretation of experimental evidence, farmers' experiences, and also the distribution of the crop by soil type, cropping patterns, and levels of management.

Taylor and Swanson [1974], dealing with the economic effects of imposing per-acre restrictions on nitrogen fertilizer in Illinois, compared results from research on experiment stations with the Ibach-Adams [1968] generalized response functions and with farmers' yields for eight subregions of the state. They concluded that while the Ibach-Adams response functions do not agree exactly with actual average yield, they seem much closer than experimental functions.

Some of the more interesting applications of production function research relate to specialty crops. Eidman, Lingle, and Carter [1963], working with cantaloupe production in California, identified the relationship between fertilizer use as a function of time of ripening and total yield. Nitrogen delayed maturity while phosphate tended to hasten maturity. They developed a procedure for handling multiharvest periods. Many publications have combined production function analyses with budgeting or other techniques. A useful example is the Woolf, Sullivan, and Phillips [1967] study of cotton production, which includes production functions relating to irrigation, fertilizer, and plant population per acre. By budgeting costs, they compared irrigated and nonirrigated net returns.

Several publications summarize aspects of production function research in crop production. Heady and Dillon [1961] present the Iowa research and include a chapter from other countries. Dillon [1968] contains chapters on concepts, procedures, and applications from the United States and other countries.

Publication 918 of the National Academy of Sciences [1961] summarizes basic economic, design, and statistical analysis concepts, with sections on historical development, examples of practical application, and an extensive bibliography.

In two books resulting from TVA-sponsored seminars Baum, Heady, and Blackmore [1956] and Baum, Heady, Pesek, and C. G. Hildreth [1957] document the principal developments in methodology and application in articles by authors from economics, agronomy, and statistics. They are useful in describing research needs and problems as seen by the authors at that time. A journal article by Munson and Doll [1959] gives an excellent overview of concepts and research experiences from a number of states. For practical applications of economic principles in fertilizer use based on research analysis, see North Central Regional Publication 54 (North Central Farm Management Research Committee [1954]) and Southern Farm Management Extension Publication Number 10 (North Carolina Agricultural Extension Service [1962]).

Production Functions for Animal and Poultry Production

Production functions for animal and poultry production date back to USDA technical bulletins by Jensen et al. [1942], A. G. Nelson [1945], and Atkinson and Klein [1945]. These studies carried out by the USDA and several agricultural experiment stations in collaboration have been widely quoted and stand as landmarks in the field of production economics and farm management. They were concerned primarily with optimum marketing rates rather than the estimation of marginal rates of substitution between feeds, and succeeded in developing interdisciplinary cooperation and data appropriate for some aspects of marginal analysis.

In the early 1950s Heady and Olson [1951, 1952] and (independently) Redman [1952] at Kentucky published exploratory studies estimating isoproduct and marginal rate of substitution relationships for grain-forage feed for milk production. Heady and Olson used selected treatment from the Jensen study, and Redman also used existing data.

Each set of authors indicated that a basic purpose in conducting the research was to contribute to the national goals of soil conservation and the interests of many in agriculture to conserve grain and use more forage. Each was concerned with developing appropriate methodology for delineating new knowledge and exploring the interrelationships between the physical and economic implications of grain-forage feeding relationships in milk production.

Redman, in relating his research to the field of feeding standards, concluded: "It was undoubtedly necessary in the earlier stage of development to make such simplifying assumptions as perfect substitutability of feeds and constant returns of milk per unit of feed input in order to derive more useful knowledge about feeding for milk production. However, the time has now arrived for relaxing these assumptions of linear relationships and for incorporating the concept of changing marginal rates of substitution implied by the law of diminishing returns."

These studies were exploratory and uncertainties still remained about the true nature of the response surfaces and substitution rates. Opposing views were expressed in the *Journal of Farm Economics* (Mighell [1953a, 1953b], Heady and Olson [1953]). See H. R. Jensen [1977] for further comment.

In May 1957 a symposium on the nutritional and economic aspects of feed utilization was held at Michigan State University. It was sponsored by that university, the North Central Dairy and Farm Management Research Committees, the Farm Foundation, and the USDA. This meeting brought together research and extension workers in dairy nutrition, production economics, animal breeding, statistics, and agronomy to focus attention on interdisciplinary opportunities for improving knowledge concerning feed utilization by dairy cattle. The proceedings were published as a book edited by Hoglund et al. [1958].

Starting in 1956, a series of research reports document interdisciplinary research at Iowa involving experiments specifically designed to estimate the production surface for milk output. These experiments used four levels of hay-to-concentrate ratios and three levels of intensity of feeding. In the analysis of these data logarithmic, quadratic, and square root functions were derived by least squares regression as alternative means of specifying the production function. Heady, Jacobson, et al. [1964] include an analysis incorporating other variables such as different characteristics of cows (maturity, ability, inbreeding, and weight) and environmental conditions so that optimum feeding ratios and level of milk production can be estimated for more specific conditions of production. Also, point estimates are supplemented by confidence regions.

The existence of diminishing marginal rates of substitution in feeding has been confirmed by other researchers. Coffey and Toussaint [1963] pointed out that from analysis of the Iowa experiments the most profitable rations lie near the stomach capacity limit for most historical prices of hay, grain, and milk and that returns do not vary much over a fairly wide range of feeding levels. Dean [1960] reported on a California experiment in which rations for some treatments were changed after each twenty-eight-day period to measure carry-over effects. Hoover et al. [1967] used Kansas experiments carried out from 1956 to 1961 and concluded that the resulting production surfaces were similar to the Iowa study and that the general forms of the equations of best fit were similar. Paris et al. [1970] reported on dairy production functions where yield was alternatively measured as whole milk, fat, 4-percent milk, and skim milk.

Feeding trials to determine concentrate roughage production relationships in beef feeding have been conducted in Oklahoma (Plaxico and Pope [1959]) and in Iowa (Heady, Carter, and Culbertson [1964]). Plaxico and his associates concluded that comparisons of the Oklahoma beef study with the Iowa dairy results imply a greater curvature of the isoproduct contours for feedlot beef animals compared with milk production and that the economic incentive to adjust rations to price ratios may be greater. Also, under certain price relationships substantial savings might be made by feeding steers and heifers different rations.

Studies of corn-soybean meal feed substitution relationships for hogs in drylot feeding were reported from Iowa (Heady, Woodworth, et al. [1953, 1954]). The experimental trials included three experiments with treatments ranging from 10-percent to 20-percent protein. The data derived were used to specify: (1) least cost rations for different price relationships; (2) rations to get hogs to market weight in minimum time; (3) maximum profit rations based on historical prices for two weaning dates; and (4) optimum marketing weights. Methodological aspects included alternative equations and the use of three weight intervals as well as the whole-weight range to allow greater flexibility in substitution rates. Least cost rations resulted in a higher net return per pig compared with least time rations in fifteen years of a sixteen-year period for a November 1 weaning date and when marketed at 225 pounds. The difference was $1.00 or more per pig in five of the years and $5.82 in one year. A second publication in this series (Heady, Catron, et al. [1958]) reported results from feeding hogs corn and soybean meal for hogs produced on pasture instead of drylot.

A number of studies have been concerned with feed-weight of bird relationships in broiler production. Fellows and Judge [1952] were concerned in a Connecticut bulletin with marginal costs and returns from feeding broilers to different weights. Budgeting was used to relate this to total costs, total returns, and net returns. A special "slide rule" device made it possible for the producer to find maximum profit marketing weights for various prices. In a Washington State University study Baum and Walkup [1953] analyzed the feed-weight of bird relationships for high energy feeds compared with other rations. Heady, Balloun, and McAlexander [1956] analyzed the results of an experiment in which chicks were fed protein levels varying from 16 percent to 26 percent. Data are presented to determine least cost and least time rations as well as optimum marketing weights for specified protein levels. Heady, Balloun, and Dean [1956] published similar data for turkeys.

Assessments

Accomplishments. A wide range of experiments to characterize farm production relationships and to derive economic implications has been conducted since 1950. More studies have been concerned with crop than animal production owing in part to higher costs of large-scale animal experiments. These experiments have provided a test of the practical application and importance of principles of production economics and of plant and animal science. The literature contains evidence of a considerable advance in methodology as well as many practical applications. The results serve as a reminder to agricultural workers and farmers that most inputs are not combined in fixed proportions as point recommendations imply, but that combinations and levels can be changed as prices and other conditions warrant. They serve as a conceptual guide for making recommendations to farmers. In one state a single-level recommendation for fertilizer was changed to three levels for alternative management situations based on production function studies. Most states that had agronomic-economic studies in the 1950s and early 1960s changed recommendations as a result of the work. Generally, the change was to increase the level of application.

These studies were a useful source of input-output data for farm adjustment studies, either directly or as a basis for making judgments from all available data, of the most appropriate alternative levels of inputs and associated production. The methodology, or at least the less complex aspects, had a very important application for guiding research and development in increasing food production in developing countries. In this case, the higher costs of inputs such as fertilizer, restrictions in foreign exchange, supply restrictions, and a pressing need for increased food supply multiply the importance of efficient use of scarce resources.

A climate has evolved from this research that demonstrates the potential accomplishments of interdisciplinary approaches to problem solving. Undoubtedly, it has been a crucial factor in developing the awareness on the part of production specialists and administrators that production economists can make important contributions in planning and analysis of production experiments. Physical scientists have become more aware of the need to design experiments using the production function approach, involving design and analysis of experiments for continuous relationships as opposed to discrete responses. In recent years increasing numbers of plant and animal scientists have been conducting their own experiments on this basis—a healthy trend which helps to provide production economists with useful input-output data and at the same time allows more effort in economic analysis compared with time spent in obtaining physical data.

Limitations. Greater difficulties are experienced in quantifying biological response relationships than would be encountered in most industrial processes. Soil variability, weather, insects, diseases, residual fertilizer in the soil, nutrients in the soil, and previous crop history frequently conspire to confound researchers with unpredictable results. Response to applied P and K has been particularly uneven because of the accumulation from previous fertilization. In multivariate crop experiments in several states responses to these nutrients have not been statistically significant, reducing economic analysis to responses to applied N only. Perhaps these results should reinforce the view that precision crop production is a long way in the future. A second view would be that advances in plant science knowledge in terms of response prediction are needed before full utilization can be made of marginal relationships in crop production.

In the past decade many production economists have expressed the opinion that production function research from controlled experiments was of limited consequence, pointing to the very modest differences in net income associated with a fairly wide range of application rates for selected production functions or along certain isoproduct curves. At the same time interest waned because of continuing crop surpluses and relatively low prices for feedgrain, protein, and fertilizer. Researchers went on to new problems and approaches.

There appears to be adequate evidence that for corn and similar crops the profit consequences of not adjusting fertilizer rates optimally for normal year-to-year changes in prices are small or inconsequential. A finding that the nature of some or many response relationships does not make it worthwhile to change the levels or combinations of inputs from year to year for usual variations in price ratios is important knowledge. Finding the conditions and commodities for which it is worthwhile is also important knowledge. Discussions in the literature on this issue are based on only a few experiments, mostly for corn, and for the price variations arising in the 1950s and early 1960s.

In justifying, planning, and conducting projects economists frequently have overemphasized the importance of economic optima. The factor-product equilibrium is an appropriate guide for economic decision making but cannot be applied with the precision implied by the theoretical model. Discovery of a precise optimum may not in itself be valuable knowledge if a wide range of application rates makes little difference in net income. Rauchenstein [1953] observed that choice of forage production systems and how they fit into the whole-farm business were far more important than feed substitution possibilities to the economic health of dairy farms. (See also Coffey and Toussaint [1963].)

Many of the multinutrient fertilizer experiments were designed without sufficient basic knowledge of yield response patterns. For some of these, no meaningful economic analyses were possible because of nonsignificant or erratic responses. Where meaningful responses were obtained, they varied greatly from year to year because of weather differences. Clearly much more needs to be known about factors which affect response to P and K before elaborate

3-factor experiments to obtain economic optima can be routinely justified. The longer range aspects of N and P buildup in the soil have economic implications for farmers and for society, yet these have received scant attention in the literature on production functions and on the economics of fertilization. The general assumption of a variable resource (fertilizer) applied to a fixed resource (land) does not have universal application. Other models should be considered—for example, when land is not fixed for the individual farmer and idle or rented land can be substituted for fertilizer to produce a given level of output, when risk is an important factor, or when it is appropriate to consider an animal as the fixed plant rather than land.

Future Outlook and Research Needs

The desirability of quantifying production relationships in agriculture will continue in the future. While linear programming has become the dominant methodology for obtaining the most profitable farming systems, partial analyses based on production function studies have merit in analyzing numerous policy and farm-level decisions when interrelationships with other aspects of the farm organization are of secondary importance. In addition to that, the results of production function studies are useful in selecting data for linear programming studies.

Recent events should remind us not only that price ratios do not remain indefinitely within prescribed limits but that restrictions in the supply of fertilizer or feed can occur. Farmers and farm magazines have again raised questions about how much fertilizer to use or how to minimize feed costs. Important policy issues have again been raised about how threatened shortages and higher costs of fertilizer and energy could affect total production and how fertilizer and energy could be used more efficiently to minimize increases in the costs of food to consumers. Production function studies provide useful insights into these and such other national goals as reducing energy requirements and minimizing detrimental effects on the environment.

Additional assessment is needed of the conditions under which it could be desirable to change the rates and levels of inputs as price ratios change. This should be done systematically for a variety of crops. Similarly, additional assessments are needed of the production and income implications of restrictions in the use of inputs which may be scarce or subject to environmental controls.

In the future production function research will be needed to update existing information on production relationships as new technology and other conditions change and to provide new information in several priority areas. Much of the production function research now available in the literature was conducted a decade or more ago and is probably out of date. One area of needed research involves alternative processes in beef production utilizing forages. There is a general lack of knowledge concerning production relationships needed in selecting a forage system from the many possible combinations, the selection of fertilizer rates, and the effect of these variables on quality of beef and on net income. The rising relative cost of water along with the fact that water costs are being more closely associated with the level of water use in irrigated farming areas will increase the benefits from pinpointing optimum water application rates.

For farmer decision making improvements are needed in specifying the population for which the relationships apply. In crop production this means more research carried out on farms rather than on experiment stations. Also, a variety of economic models and research techniques should be used to provide more useful information and to duplicate better the decision making models most appropriate for farmers in different circumstances.

The production function approach needs to be an integral part of the training of physical scientists with more of the needed projects designed and carried out by physical scientists themselves. At the present time many experiments are being carried out by physical scientists with objectives that suggest the production function approach as the most efficient but that utilize more traditional, less efficient procedures. If the production function approach were used by these researchers in the future, much more data on production processes would be available for economic analysis.

In many developing countries production function studies have a higher value than in the United States economy. Typically, there is a critical need to increase food production, but foreign exchange may be required for the importation of fertilizer and the cost of the fertilizer to farmers may be high. Policy issues involve the provision of adequate incentives for using fertilizer efficiently. In a controlled economy this may require setting the prices of fertilizer and product so that the desired production can be accomplished with a minimum of foreign exchange. If the use of fertilizer is a relatively new technology in the country, neither farmers nor agricultural workers have the historical experience to determine the best rates of use except by costly trial and error. Economic studies to determine optimum rates of fertilizer use can make a major contribution under these circumstances.

Many advances in methodology will probably be made by physical scientists and statisticians or will involve them in some way. In crop production there is a great need for a greater understanding of response relationships and of nontreatment variation. Advances in this direction could lead to improved criteria for selection of functional relations and to the development of new models with the desired characteristics. Advances in knowledge which result in increased statistical efficiency, lower cost, and greater reliability for decision making could result in much greater utilization in partial or complete farm decision making models and in models for the analysis of related policy issues.

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