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# Portfolio Selection Under Exponential and Quadratic Utility

Steven T. Buccola

The production or marketing portfolio that is optimal under the assumption of quadratic utility may or may not be optimal under the assumption of exponential utility. In certain cases, the necessary and sufficient condition for an identical solution is that absolute risk aversion coefficients associated with the two utility functions be the same. In other cases, equality of risk aversion coefficients is a sufficient condition only. A comparison is made between use of exponential and quadratic utility in the analysis of a California farmer's marketing problem.

A large number of utility functional forms have been proposed for use in selecting optimal portfolios of risky prospects [Tsaing]. For purposes of both theoretical and applied economic research, attention has primarily been directed to the quadratic and exponential forms. After enjoying widespread use in the 1950s and 1960s, the quadratic form came under criticism by Pratt and others and lost much of its reputability. In contrast the exponential form has become increasingly popular, due in large measure to its mathematical tractability [Attanasi and Karlinger; O'Connor]. However, Anderson, Dillon, and Hardaker (pp. 94-95) have recently defended the quadratic, and it continues in use [Lin, Dean, and Moore; Hanoch and Levy; Kallberg and Ziemba]. The objective of the present paper is to explore conditions under which use of quadratic utility results in optimal choices identical to, or different from, those resulting from use of exponential utility. The analysis makes clear that under certain conditions, the choice between quadratic and exponential utility makes very little

difference. Under other conditions, the choice has a significant impact on optimal behavior.

Suppose an analyst is studying a continuously divisible set of alternative production or marketing options offering approximately normally-distributed returns, and that he wishes to select the portfolio of these options with highest expected utility. The analyst is considering using an exponential utility function  $U = -\exp(-\lambda x)$ ,  $\lambda > 0$ , or a quadratic utility function  $U = x - vx^2$ ,  $v > 0$ ,  $x < \frac{1}{2}v$ , where  $U$  is utility and  $x$  is wealth. Corresponding to each of these functions is a unique absolute risk aversion function  $r(x) = -U''(x)/U'(x)$  from which a risk aversion coefficient can be determined at any level of  $x$ . For exponential utility, risk aversion  $r$  is in fact invariant with respect to  $x$ , whereas for quadratic utility it rises monotonically with  $x$  [Pratt, p. 132]. Thus at one, and only one, wealth level the two risk aversion functions will be equated. The wealth level at which a risk aversion coefficient is properly evaluated is the decision maker's initial wealth plus his expected return from the portfolio considered; hence the portfolio considered generally affects the risk aversion coefficient itself.

The thesis of the present paper may be stated as follows: if the optimal portfolios include each production or marketing option

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at a nonzero level and if the decision maker is assumed to operate under at most one linear resource constraint (such as an acreage or capital constraint), then necessary and sufficient conditions for selecting the same portfolio using either quadratic or exponential utility is that the absolute risk aversion coefficients be equal at the optimal point. If some options are optimally operated at zero level or if more than one resource constraint is employed, then equality of risk aversion coefficients is sufficient but not necessary for selecting the same portfolio. An implication of this argument is that in many cases the composition of optimal portfolios depends upon the utility functional form selected. Only in special circumstances can identical behavior in the face of risk be used to infer identical risk aversion, although the reverse inference can more frequently be made.

### Portfolio Problem Under Unlimited Resources

The situation first considered here is that in which an individual contemplates  $n$  alternative risky prospects  $R_i$ ,  $i = 1, 2, \dots, n$ , where  $R_i \sim N(\mu_i, \sigma_i^2)$ . The individual has access to an unlimited nonnegative number  $P_i$  of units of each prospect. Under these conditions, portfolio return is  $z = \sum_i P_i R_i$ . If  $x$  is the individual's pre-risk wealth, total wealth after realization of the random return is  $w = x + z$ . Hence  $U$  and  $r$  may be evaluated in terms of expected post-risk or terminal wealth  $\mu_w$ . Terminal wealth has mean  $\mu_w = x + \sum_i P_i \mu_i$  and variance  $\sigma_w^2 = \sum_i P_i^2 \sigma_i^2 + \sum_{j \neq i} \sum_j P_i P_j \sigma_{ij}$ .

### Optimal Portfolio Selection

If the decision maker has an exponential utility function with parameter  $\lambda$  and returns are normally distributed, Freund has shown that the certainty equivalent of expected terminal wealth is  $CE_e = \mu_w - (\lambda/2)\sigma_w^2$ .<sup>1</sup> Sub-

stituting the above mean and variance into the latter formula gives

$$(1) \quad CE_e = x + \sum_i P_i \mu_i - \lambda \left( \sum_i P_i^2 \sigma_i^2 + \sum_{j \neq i} \sum_j P_i P_j \sigma_{ij} \right) / 2.$$

The optimal quantity  $P_{i,e}^*$  allocated by the decision maker to the  $i$ th prospect is found by satisfying the Kuhn-Tucker conditions in which resource constraints are not specified, that is by satisfying  $\partial CE_e / \partial P_i \leq 0$ ,  $(\partial CE_e / \partial P_i) P_i = 0$ ,  $P_i \geq 0$  (Intriligator, p. 50). It is assumed at the outset that  $P_i > 0$ , so it is only required that  $\partial CE_e / \partial P_i = 0$ , all  $i$ . That is,

$$\partial CE_e / \partial P_i = \mu_i - \lambda (P_i \sigma_i^2 + \sum_{j \neq i} P_j \sigma_{ij}) = 0.$$

Solving for the optimal  $P_{i,e}^*$  gives<sup>2</sup>

$$(2) \quad P_{i,e}^* = \frac{(\mu_i / \lambda) - \sum_{j \neq i} P_j \sigma_{ij}}{\sigma_i^2}.$$

Expected utility reaches a maximum at this point because the matrix of second derivatives  $\| \partial^2 CE_e / \partial P_i \partial P_j \|$ , which equals  $\| -\lambda \sigma_{ij} \|$  for all  $i, j$ , has a negative semi-definite form.

Corresponding first order conditions for the quadratic utility function are derived by maximizing the expected utility function  $EU_q = \mu_w - v\mu_w^2 - v\sigma_w^2$  with respect to portfolio quantity  $P_i$ . If  $P_i$  is positive, in the optimum it is required that

$$\partial EU_q / \partial P_i = \mu_i (1 - 2vx) - 2v(P_i \mu_i^2 + \mu_i \sum_{j \neq i} P_j \mu_j + P_i \sigma_i^2 + \sum_{j \neq i} P_j \sigma_{ij}) = 0,$$

<sup>2</sup>Conditions (2) would be solved simultaneously for all  $P_i$ ,  $i = 1, 2, \dots, n$ . It is not necessarily true in (2) that  $\partial P_{i,e}^* / \partial \lambda = -\mu_i / \lambda^2 < 0$  because quantity  $P_{j \neq i}$  in the numerator of  $P_{i,e}^*$  is, at the optimum, also a negative function of  $\lambda$ . Hence in the simultaneous solution,  $\partial P_{i,e}^* / \partial \lambda \leq 0$  for any particular risky prospect  $R_i$ . For the two-variable case, it may be shown that increases in risk aversion coefficient  $\lambda$  decrease the proportion optimally allocated to the prospect with higher mean.

<sup>1</sup>All e subscripts refer to the exponential utility function and q subscripts to the quadratic utility function. Subscripts w refer to terminal wealth.

so that optimal quantity  $P_{i,q}^*$  is

$$(3) \quad P_{i,q}^* = \frac{\mu_i[(1/2v) - x] - (\mu_i \sum_{j \neq i} P_j \mu_j + \sum_{j \neq i} P_j \sigma_{ij})}{(\mu_i^2 + \sigma_i^2)}.$$

Second order conditions for a maximum are again satisfied because the matrix of second derivatives  $\| \partial^2 E U_q / \partial P_i \partial P_j \|$  is equivalent to  $\| -2v(\mu_i \mu_j + \sigma_{ij}) \|$ , a negative semi-definite form.

### Conditions for Identical Optimal Portfolios

To derive the necessary conditions under which the composition of the optimal portfolio would not depend upon utility functional form, optimal quantity  $P_{i,e}^*$  in (2) is equated with optimal quantity  $P_{i,q}^*$  in (3). To simplify notation, let  $\sigma_{io} = \sum_{j \neq i} P_j \sigma_{ij}$  and  $\mu_o = \sum_{j \neq i} P_j \mu_j$ , where o refers to alternatives other than the  $i$ th. Then

$$(4) \quad \frac{(\mu_i/\lambda) - \sigma_{io}}{\sigma_i^2} = \frac{\mu_i[(1/2v) - x] - (\mu_i \mu_o + \sigma_{io})}{(\mu_i^2 + \sigma_i^2)}.$$

Solving for the exponential utility's constant risk aversion coefficient gives

$$(5) \quad \lambda = \frac{(\mu_i^2 + \sigma_i^2)}{[(1/2v) - x] \sigma_i^2 - (\sigma_i^2 \mu_o - \mu_i \sigma_{io})}.$$

It now remains to be shown that the quadratic utility's risk aversion coefficient  $r_q$  equals  $\lambda$  in (5).

To show that this is so, recall from the risk aversion coefficient formula  $r = -U''(\mu_w)/U'(\mu_w)$  that  $r_q = 2v/(1 - 2v\mu_w) = 1/[(1/2v) - (x + \mu_z)]$ . Coefficient  $r_q$  varies with  $\mu_z$  and thus with the particular portfolio selected. Define  $\mu_z^*$  as the mean portfolio return associated with the optimal portfolio quantity  $P_{i,q}^*$ ; that is

$$(6) \quad \mu_z^* = \mu_o + P_{i,q}^* \mu_i.$$

The quadratic utility's risk aversion coefficient at this optimal point is

$$(7) \quad r_q^* = 1/[(1/2v) - (x + \mu_z^*)].$$

Substituting from  $\mu_z^*$  in (6) and from  $P_{i,q}^*$  in (3):

$$(7)' \quad r_q^* = \frac{1}{(1/2v) - (x + \mu_o) - \left[ \frac{\mu_i^2[(1/2v) - x] - \mu_i(\mu_i \mu_o + \sigma_{io})}{(\mu_i^2 + \sigma_i^2)} \right]}$$

Multiplying numerator and denominator of (7)' by  $(\mu_i^2 + \sigma_i^2)$  gives

$$r_q^* = \frac{(\mu_i^2 + \sigma_i^2)}{(\mu_i^2 + \sigma_i^2) / 2v - (x + \mu_o)(\mu_i^2 + \sigma_i^2) - \mu_i^2 [(1/2v) - x] + \mu_i(\mu_i \mu_o + \sigma_{io})}.$$

Cancelling and collecting terms in the denominator,

$$(7)'' \quad r_q^* = \frac{(\mu_i^2 + \sigma_i^2)}{[(1/2v) - x] \sigma_i^2 - (\sigma_i^2 \mu_o - \mu_i \sigma_{io})},$$

which is identical to the risk aversion coefficient (5) for the exponential utility function.

Thus, if resources are unlimited and each portfolio option is optimally held at a nonzero level, a necessary condition for the optimal quantities  $P_i^*$ ,  $i = 1, 2, \dots, n$ , to be the same regardless of functional form is that the absolute risk aversion coefficients be the same at expected terminal wealth. To prove that equality of risk aversion coefficients is sufficient for agreement on  $P_i^*$ , it is only required that  $\lambda$  be set equal to (7) or equivalently to (7)''; this is the same as equation (5), which is a rearranged form of (4) where the optimal portfolio quantities are identical.

### Portfolio Problem Under Fixed Resources

The above demonstration may be extended to the case in which the decision maker operates under a single linear resource con-

straint. A typical constraint is that the decision maker has access to only  $A$  units of all risky prospects  $R_i$ . This total is allocated to the proportions  $P_i$ ,  $i = 1, 2, \dots, n$ , such that  $A \sum_i P_i = A$  and  $P_i \geq 0$ , all  $i$ . The Lagrangian is  $\bar{L} = F(P) - y(A - A \sum_i P_i)$ , where  $y$  is the Lagrange multiplier and  $F$  is replaced by  $CE_e$  for exponential utility and  $EU_q$  for quadratic utility. Kuhn - Tucker conditions for the primal problem are  $\partial F / \partial P_i - yA \leq 0$ ,  $(\partial F / \partial P_i - yA)(P_i) = 0$ ,  $P_i \geq 0$ , all  $i$ . Under the assumption that all  $P_i$  are strictly positive, this reduces to simply  $\partial F / \partial P_i - yA = 0$ . The latter criterion may be satisfied by including the resource constraint as part of the wealth equation and then optimizing as in the unconstrained case. Specifically, letting terminal wealth be  $w' = x + A [P_i R_i + (1 - \sum_{j \neq i} P_j R_j)]$ , new objective functions  $CE_e'$ ,  $EU_q'$  are derived and conditions  $\partial CE_e' / \partial P_i = \partial EU_q' / \partial P_i = 0$  satisfied to derive the optimal  $P_{i,e}^*$  and  $P_{i,q}^*$ . Using the same methods as employed earlier, the latter portfolio proportions are shown equal to each other iff  $\lambda = r_q^*$ . This proof is very lengthy for the  $n$ -option case, but proof of the two-option case is shown in the Appendix.

If the decision maker operates under more than a single linear resource constraint, or if some portfolio options are optimally held at zero level, equality of risk aversion coefficients is no longer necessary for identical risky choice. Suppose the optimal solution occurs at a corner of the constraint set such that at least two constraints are satisfied as equalities. This could occur where one or more of the nonnegativity restrictions are satisfied as strict equalities, implying they are optimally held at zero level. In such a situation, a wide number of gradients  $\partial P_j / \partial P_i$  on the criterion surface, each corresponding to different risk aversion coefficients, could satisfy the optimality condition. However, even in this case, equality of risk aversion coefficients is sufficient for identical risky choice because Pratt (pp. 125, 128) has proven that equality of risk aversion coefficients implies equality of certainty equiva-

lents for a given prospect.<sup>3</sup> Individuals who agree upon the certainty equivalent for each risky prospect will always agree on how to rank these prospects.

### Applications to a Portfolio Problem

It would seem at first glance that a situation in which there is only one linear resource constraint and in which all activities are optimally at nonzero levels would rarely be encountered in research. Certainly the situation would not be characteristic of whole-farm planning models; these problems realistically require a large number of constraints and involve many potential production activities, some of which are frequently not adopted in the optimal solution. For such situations, only the sufficient conditions stated above are generally applicable. But analysis of financial or marketing portfolios often are adequately characterized by the assumption of a single capital, acreage, or tonnage constraint. Furthermore, the number of feasible marketing alternatives is often severely restricted by available sales or contract opportunities. In these cases, optimal marketing strategy will, if positive price correlations are not too great, frequently call for simultaneous use of each alternative. For such a case, both the necessary and sufficient conditions outlined above will usually apply.

The latter situation is illustrated here for the case of a California producer of processing tomatoes who, at planting time, has the choice of securing a contract to sell all the produce from a portion of his acreage on cost-plus terms; produce from the remaining acreage would be sold at harvest-time spot market prices. A critical factor of the cost-plus

<sup>3</sup>Given normal distributions, the certainty equivalent of a random prospect  $z$  is  $\mu_z - r(x + \mu_z)\sigma_z^2/2$ , that is the mean of the prospect minus the decision maker's risk premium for the prospect. Since equality of risk aversion coefficients implies equality of risk premiums when moments  $\mu_z$  and  $\sigma_z^2$  are constant, the former also implies equality of certainty equivalents when moments  $\mu_z$  and  $\sigma_z^2$  are constant.

contract is the mark-up the buyer offers over the seller's variable costs. Let  $A$  be total acreage planted,  $R_1$  be per-acre variable costs,  $k$  be the multiplicative mark-up over variable costs,  $R_2$  be per-acre market value of the tomatoes,  $P$  be the fraction of acreage allocated to cost-plus contracts, and  $F$  be per-acre fixed costs (assumed nonstochastic). Terminal wealth is then  $w = x + A [PkR_1 + (1 - P)R_2 - R_1 - F]$ . First and second order conditions for an expected utility optimum are found by selecting a utility functional form and applying the suitably transformed versions of equations (3) or (4) in the Appendix. An interesting feature of this example is that the optimal cost-plus proportion of sales,  $P^*$ , is a function of the cost-plus markup  $k$ . Because  $AP^*$  is the quantity of acreage optimally offered on cost-plus terms and  $kR_1$  is a price, the function  $P^* = P^*(k)$  is equivalent to the individual's supply of product under contract.

Figure 1 presents a comparison of such portfolio supply functions for a particular tomato farmer assuming exponential, then quadratic, utility. Except for the utility functional forms, all probability moments and covariances used to generate the functions were the same. Furthermore, both utility functions were estimated from the same set of farmer utility response data.<sup>4</sup> Note that as per-unit fixed costs  $F$  were altered, the quadratic decision maker's supply curve shifted but the exponential decision maker's remained fixed.

When fixed costs were set at a low level, the supply curve constructed under the assumption of exponential utility did not intersect the supply curve constructed under the assumption of quadratic utility, meaning that portfolio agreement was not reached. As shown in Table 1, the exponential decision maker allocated approximately 13 percent of his acreage to cost-plus contracts when the cost-plus markup was 132 percent of variable costs. The quadratic decision maker allocated

nearly 25 percent of his acreage in response to the identical markup. The two sets of portfolios converged or diverged according as  $r_q^*$  converged to or diverged from  $\lambda$ .

When fixed costs were set at a higher level, the supply curve of the exponential decision maker twice intersected the supply curve of the quadratic decision maker. This would not be an unusual phenomenon in portfolio analysis: as markup  $k$  increases, expected return  $\mu_z$  first falls, then rises in response to changing portfolio allocation  $P^*$ . Thus also, the quadratic individual's risk aversion  $r_q^*$  first falls, then rises, permitting risk aversion equalities  $r_q^* = \lambda$  to occur at two points. In subsequent solutions using alternative utility function parameters and probability moments, portfolio supply curves constructed under exponential utility continued to be more linear than those constructed under quadratic utility. No pair of exponential-based and quadratic-based supply curves intersected in more than two places.

## Conclusions

Given normally-distributed returns, the choice between quadratic and exponential utility will have no effect on optimal portfolio selection if (and in certain situations only if) the corresponding absolute risk aversion coefficients at the optimal solution are equal. Because the risk aversion functions corresponding to exponential and quadratic utility intersect only once, optimal solutions will often diverge widely depending upon the functional form assumed. This will especially be the case in those (primarily marketing) situations where risk aversion equality is necessary as well as sufficient for portfolio agreement.

This study has excluded several significant dimensions of risk analysis, including cases of non-normally distributed returns and of utility functions other than the exponential or quadratic. Subjectively-perceived return distributions of some individuals may be skewed; at least this might be inferred from the skewness of certain yield or mortality variables [Bessler]. And when skewnesses

<sup>4</sup>The exponential function estimated was  $U = -\exp[-.0012w]$  and the quadratic function was  $U = w - .000678w^2$ , where  $w$  is expressed in \$1,000 units.

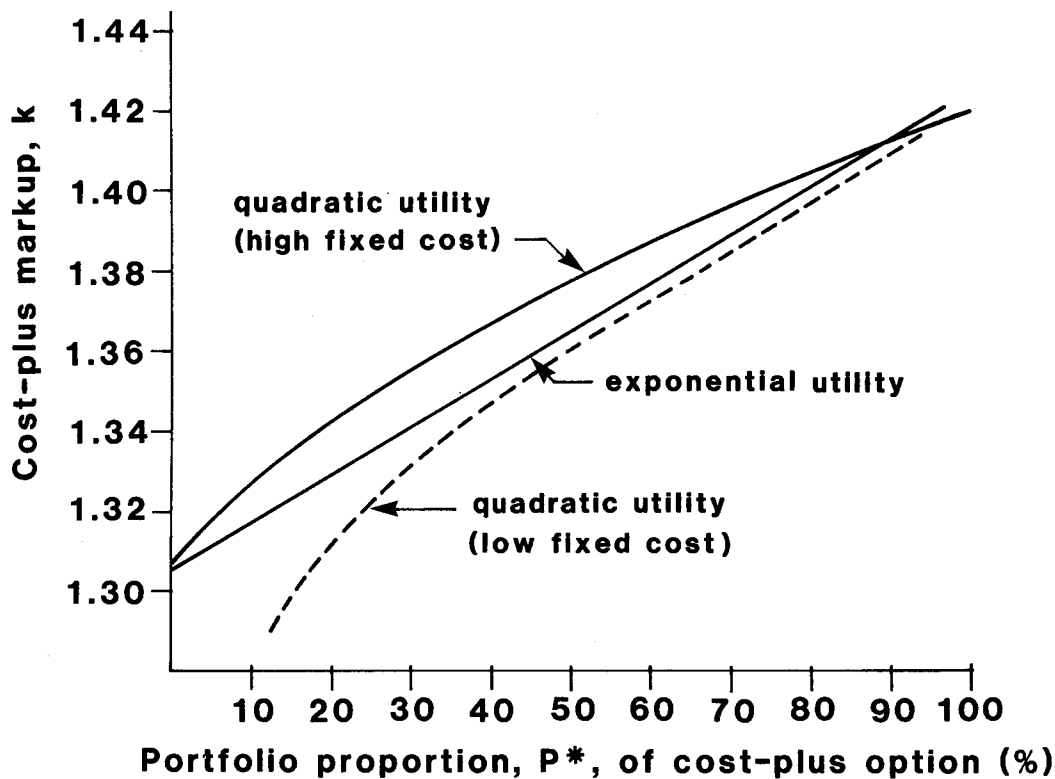


Figure 1. Supply of Acreage Contracted on a Cost-Plus Basis: Exponential Versus Quadratic Utility.

are strong, the results developed above do not necessarily hold. However, several reasons justify our continuing to pay some attention to normally-distributed prospects. First, there is evidence that many profit distributions faced by farming and (more especially) agricultural marketing and trading firms are negligibly skewed [Mandelbrot; Fama].<sup>5</sup> Specifically, there is less evidence for skewness in market prices than in farm yields, a significant fact considering that farm profits are affected by prices as well as yields,

and that marketing firms' returns are often directly affected by prices alone. Second, research resources are sometimes inadequate for constructing subjectively-based probability distributions of returns that are deemed representative of the decision-making population. In these situations, the assumption of unskewed distributions is no more, and perhaps less, arbitrary than the assumption of some particular positive or negative skewness. Third, the premise of normality permits us to derive analytical results that are much more manageable than those derivable with nonzero skewness, and these results at least provide a benchmark for understanding more complicated phenomena. Although stochastic dominance methods, for example, allow us to partially rank specific risky prospects, the methods are not suitable for generalizing about the behavior of an optimal

<sup>5</sup>Mandelbrot observed that distributions of spot market cotton prices are more thickly tailed than they would be if they were normally distributed, but that the distributions are only slightly negatively skewed. The skewness was so small as to be ignored in Mandelbrot's later work and in Fama's extensions to other commodity and security prices.

**TABLE 1. Portfolio Supply Curves When the Alternative Marketing Options are Cost-Plus and Market Price: Exponential Versus Quadratic Utility.<sup>a</sup>**

Cost-Plus Markup	Portfolio Proportions		Expected Return Per Acre	Standard Deviation of Return Per Acre	Absolute Risk Aversion Coefficient <sup>b</sup>
	Cost-Plus	Market Price			
	(%)	(%)	(\$)	(\$)	
<u>Exponential Utility</u>					
1.31	4.3	95.7	31.33	570.13	.0012
1.32	12.6	87.4	13.03	537.69	.0012
1.33	21.0	79.0	-3.50	500.22	.0012
1.34	29.4	70.6	-17.11	457.34	.0012
1.35	37.7	62.3	-23.33	408.92	.0012
1.36	46.0	54.0	-29.33	355.25	.0012
1.37	54.3	45.7	-24.32	297.25	.0012
1.38	62.6	37.4	-9.17	236.73	.0012
1.39	70.8	29.2	17.70	176.67	.0012
1.40	79.1	20.9	57.17	121.65	.0012
1.41	87.3	12.7	108.91	79.17	.0012
1.42	95.5	4.5	171.21	60.53	.0012
<u>Quadratic Utility</u>					
1.31	19.8	80.2	154.08	682.51	.00144
1.32	24.3	75.7	96.79	621.86	.00139
1.33	29.5	70.5	51.40	561.34	.00135
1.34	35.4	64.6	17.86	500.96	.00132
1.35	42.1	57.9	-3.88	440.75	.00129
1.36	49.6	50.4	-13.83	380.76	.00129
1.37	57.6	42.4	-12.07	321.09	.00130
1.38	66.1	33.9	1.40	261.90	.00133
1.39	74.7	25.3	25.51	203.55	.00140
1.40	82.8	17.2	63.24	146.97	.00151
1.41	90.3	9.7	111.53	95.21	.00169
1.42	96.2	3.8	171.35	61.68	.00196

<sup>a</sup>The quadratic decision maker's supply curve represented here corresponds to the low-fixed-cost (dotted line) curve in Figure 1.

<sup>b</sup>Assumes money is expressed in \$1,000 units.

solution in response to selected parameter changes.

Similarly, it would be useful to analyze other utility functions, especially those exhibiting decreasing absolute risk aversion. But some of these, such as  $U = ax - \gamma \exp(-\delta x)$ ,  $a, \gamma, \delta > 0$ , cannot be explicitly optimized when expressed in expected utility form. Others, such as  $U = (x + c)^d$ ,  $c, x > 0$ ,  $0 < d < 1$ , involve complex expected utility formulations even when returns are normally distributed. Exploration of such alternative utility specifications will require the use of numerical methods.

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## Appendix

Where there are two portfolio options and a single resource constraint of the form  $AP_1 + AP_2 \leq A$ , portfolio return  $z$  has mean and variance

$$(1) \quad \mu_z = A [P_1\mu_1 + (1 - P_1)\mu_2]$$

$$(2) \quad \sigma_z^2 = A^2 [P_1^2\sigma_1^2 + (1 - P_1)^2\sigma_2^2 + 2(P_1 - P_1^2)\sigma_{12}]$$

To simplify notation, let  $a = \mu_1 - \mu_2$  and  $b = \sigma_1^2 - \sigma_2^2 - 2\sigma_{12}$ . As before, expected terminal wealth is  $\mu_w = x + \mu_z$ . Then, for exponential utility, the certainty equivalent  $CE_e$  of a risky prospect is  $\mu_w - (\lambda/2)\sigma_w^2$ , or

$$CE_e = x + A(\mu_2 + aP_1) - A^2\lambda(\sigma_2^2 + 2\sigma_{12}P_1 + bP_1^2)/2.$$

If nonzero, the optimal proportions are

$$(3) \quad P_{1,e}^* = \frac{a - \lambda A \sigma_{12}}{\lambda A b}$$

$$P_{2,e}^* = 1 - P_{1,e}^*$$

with second order conditions  $\partial^2 CE_e / \partial P_1^2 = -\lambda A^2 b \leq 0$  as required for a utility maximum.

Corresponding optimum under quadratic utility is found by maximizing  $EU_q = \mu_w - v\mu_w^2 - v\sigma_w^2$ . Substituting (1) and (2) into  $EU_q$  and solving for a maximum with respect to  $P_1$  gives the optimal portfolio proportions

$$(4) \quad P_{1,q}^* = \frac{a(1 - 2xv) - 2vA(\sigma_{12} + \mu_2 a)}{2vA(b + a^2)}$$

$$P_{2,q}^* = 1 - P_{1,q}^*$$

Second order conditions are  $\partial^2 EU_q / \partial P_1^2 = -2vA^2(b + a^2) \leq 0$ , as required for a utility maximum.

The conditions under which the exponential and quadratic solutions are identical are given by equating  $P_{1,e}^*$  and  $P_{1,q}^*$ :

$$\frac{a - \lambda A \sigma_{12}}{\lambda A b} = \frac{a(1 - 2xv) - 2vA(\sigma_{12} + \mu_2 a)}{2vA(b + a^2)}$$

Solving for the exponential utility's risk aversion coefficient in terms of the probability moments and quadratic utility parameter yields

$$(5) \quad \lambda = \frac{2v(b + a^2)}{b(1 - 2xv) - 2vA(b\mu_2 - a\sigma_{12})}$$

When operating at the optimum portfolio, the decision maker with quadratic utility faces mean terminal wealth  $\mu_w = x + A[P_{1,q}^*\mu_1 + (1 - P_{1,q}^*)\mu_2] = x + A[\mu_2 + aP_{1,q}^*]$ . At the optimum, his risk aversion

coefficient is evaluated at this expected terminal wealth point:

$$(6) \quad r_q^* = \frac{2v}{1 - 2v(x + A\mu_2 + AaP_{1,q}^*)}.$$

Substituting from (4),

$$\begin{aligned} r_q^* &= \frac{2v}{1 - 2vx - 2vA\mu_2} \\ &\quad - 2vAa \left[ \frac{a(1 - 2xv) - 2vA(\sigma_{12} + \mu_2 a)}{2vA(b + a^2)} \right] \\ &= \frac{2v}{1 - 2vx - 2vA\mu_2} \\ &\quad - \left[ \frac{a^2(1 - 2xv) - 2vAa(\sigma_{12} + \mu_2 a)}{b + a^2} \right] \\ &= \frac{2v(b + a^2)}{(b + a^2) - 2vx(b + a^2) - 2vA\mu_2(b + a^2)} \\ &\quad - \frac{a^2(1 - 2xv) + 2vAa(\sigma_{12} + \mu_2 a)}{(b + a^2) - 2vx(b + a^2) - 2vA\mu_2(b + a^2)} \\ (7) \quad &= \frac{2v(b + a^2)}{b(1 - 2xv) - 2vA(b\mu_2 - a\sigma_{12})} \end{aligned}$$

which equals (5), the risk aversion coefficient corresponding to the exponential utility function. This establishes the necessary conditions. Sufficient conditions are established by observing that we could just as well reason from equation (7) backwards to equation (5).

