Optimality and Separable Linear Programming: An Additional Reminder

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The assumed global optimum solution obtained in linear programming is not an assumed characteristic of separable linear programming. Separable programming is nonlinear programming and must possess certain sufficient conditions for a global optimum to be obtained. The global optimum conditions for separable programming are setforth.

Separable linear programming is a method for solving nonlinear problems by using the simplex algorithm employed in linear programming. Its use in agricultural economics is illustrated by the Blakley and Kloth study of plant location and the Holland and Baritelle study of school location. However, a shortcoming of separable linear programming is the risk of not obtaining a global optimum solution. Neither of the above studies reported information on the likelihood of having obtained non-global solutions. While this problem is reasonably well documented in literature on quantitative methods, it is reexamined and illustrated in the following discussion to help assure the proper use of separable programming in applied research.

Sufficient conditions for a global optimum solution to a mathematical programming problem are predicated on a particular combination of convex and concave functions. In a mathematical programming problem where the objective is to:

(1) Maximize $f(x_1, \ldots, x_n)$

subject to:

(2) $g_i(X) \gtrless b_i$ for $i = 1, \ldots, n$; $X = (x, \ldots, x_n)$

(3) $X \geqslant 0$

the sufficient conditions for a global optimum are:

(4) $f(x_1, \ldots, x_n)$ is a concave function (convex for minimization)

and

(5) $X = \{x \mid g_i \gtrless b_i \text{ for } i = 1, 2, \ldots, n, X \geqslant 0\}$ is a convex set.

The Hessian matrix can be used to determine the concavity (convexity) of the objective function. If the function is concave (convex) the Hessian will be negative (positive) definite or semi-definite.

This test becomes unwieldly for large problems. An alternative approach is to determine if each term in the objective function is concave (convex) by using the Hessian matrix. For example, assume the functional form $f(x) \equiv \sum_{i=1}^{n} h_i(X)$. If all h_i functions are concave (convex), the summation of all terms in the objective function f(X) results in a concave (convex) function. The functions must not be quasi-concave or quasi-convex. Quasi functions do not necessarily preserve their quasi-concave or convex properties under addition [Simmons, p. 151].

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A linear programming package that does not contain a separable programming routine may be used to approximate a mathematical programming problem if the sufficient conditions for a global optimum are met. The characteristics of a separable nonlinear function are introduced into a linear program through a series of linear segments.[1] The resulting optimum is global for the linearized problem that approximates the original nonlinear problem. By increasing the number of linear segments for each function, the global optimum of the approximating problem will converge on the global optimum of the original nonlinear problem.

A linear programming package that contains a separable programming routine may be used to approximate the mathematical programming problem with or without meeting the sufficient conditions for a global optimum.[2] However, a global optimum is not assured. If the sufficient conditions are met, a global optimum will be obtained if the problem contains "a relatively small number of nonlinear constraints" [IBM, p. 238]. A "small number" is undefined. If the sufficient conditions are not met, the global optimum will remain unidentified because of the unknown number of local optimums that comprise the set from which a global optimum is chosen.

The following problem illustrates the difficulties encountered when sufficient conditions for a global optimum are not met.

(6) Maximize $z = y - 3x$

subject to:

(7) $y - 8x \leqslant -4$

(8) $y + 2x - 5x^2 + x^3 \leqslant 0$

(9) $y,x \geqslant 0.$

Equation eight is nonlinear and has both concave and convex arcs (Figure 1); therefore, the constraints do not form a convex set as required in equation 5. Two local optimums exist at points L and G with G being the global optimum. Once the separable programming algorithm has found point L, a point immediately to the right or left of point L will result in a lower objective function value. The algorithm will indicate that an optimum has been reached.

The dual should be run to verify the results of a separable programming problem when a restricted basis entry routine is used. This holds whether or not the sufficient conditions for a global optimum are met. Further verification can be accomplished by solving the problem with the special variables set at the upper (lower) bound, depending on the value used to obtain the initial solution.

If the results are not verified, parametric variations in the constraints, the objective function, or both may be used to determine if a better local optimum exists. If another area of the feasible region increases the objective function value, the basis should be saved and the coefficients of the original problem used to solve the problem from the new basis. The resulting optimum may or may not improve the previously identified local optimums. The search process may resume again until an "acceptable" solution is obtained. The global optimum will remain unidentified even if it is the "acceptable" solution. The search process may or may not exhaust all local optimums, one of which may be the global optimum.

In conclusion, separable programming is a technique through which nonlinear programming problems may be solved using the simplex method. The assumed global optimum characteristics of linear programming are not necessarily preserved in separable programming. A separable programming

[1] Data preparation instructions are found in Simmons.

[2] Special variables are used to approximate a nonlinear function by using a series of linear segments. The basis entry of special variables is restricted in order to incorporate the characteristics of the nonlinear function. For example, assume a nonlinear function with 10 special variables. The special variables have the following restrictions: (1) each variable must be bounded between zero and a finite number, (2) only one special variable may be in the basis with a value between zero and the upper bound, and (3) the special variables preceding the basis variable equal the upper bound and those following the basis variable equal zero.

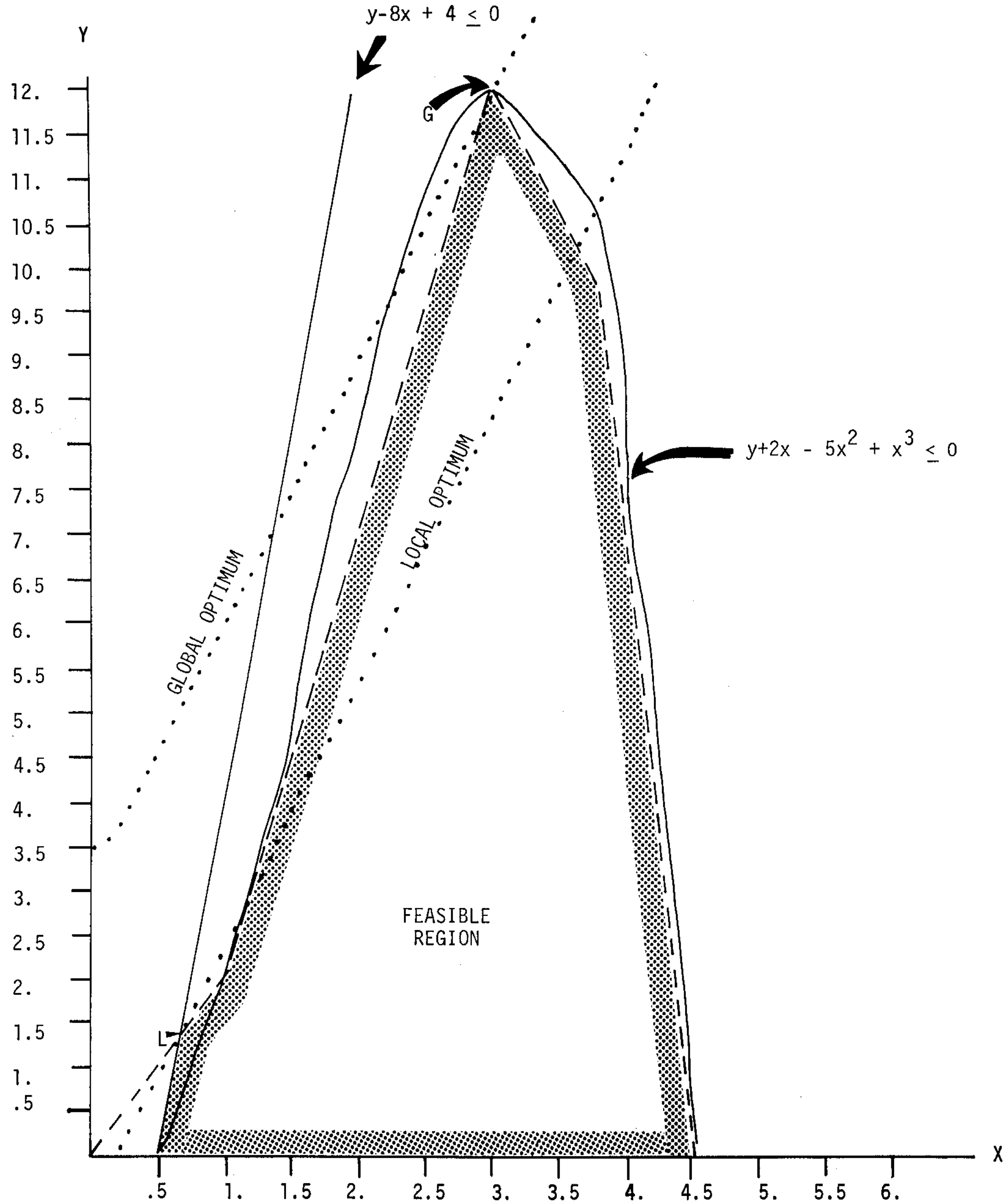

Figure 1. Global and local optima

problem must meet specific sufficient conditions in order to obtain a global optimum. If the sufficient conditions are met, separable programs should be solved without using a restricted basis separable programming option unless "a relatively small number of nonlinear constraints" [IBM, p. 238] can be defined. Finally, articles dealing with separable programming must establish the optimality characteristics of the solution obtained.

References

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