Interpretations and Transformations of Scale for the Pratt-Arrow Absolute Risk Aversion Coefficient: Implications for Generalized Stochastic Dominance

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The Pratt-Arrow measure of absolute risk aversion, as defined by $r(x) = -u''(x)/u'(x)$, is well known to be invariant to linear transformations. However, this invariance property applies with respect to transformations of u and not with respect to arbitrary rescalings of the outcome variables, x. The effects of this misunderstanding has led to ambiguity in classifying attitudes by risk aversion coefficients. It is shown that inappropriate rescalings of the outcome variable can lead to inaccurate rankings produced by generalized stochastic dominance.

Key words: marginal utility, risk aversion, scale transformations, stochastic dominances.

The Pratt-Arrow absolute risk aversion coefficient (Pratt), defined as $r(x) = -u''(x)/u'(x)$, has been used in many analyses which order alternative action choices under conditions of uncertainty (Cochran, Robison, and Lodwick; Cochran et al.; Danok, McCarl, and White; Holt and Brandt; King and Lybecker; King and Oamek; Kramer and Pope; Lemieux, Richardson, and Nixon; Meyer 1977b; Rister, Skees, and Black; Tauer 1985; Wilson and Eidman 1985; Zacharias and Grube). Problems arise, particularly in applications of generalized stochastic dominance (or stochastic dominance with respect to a function—SDWRF) (Meyer 1977a) when Pratt-Arrow coefficients elicited in one study are used as secondary data in other studies with different outcome ranges. It is well known that the Pratt-Arrow measure is invariant to linear transformations (King and Robison, p. 512). However, this invariance property applies only to transformations of u and not to arbitrary rescalings of x, the outcome measure. Confusion about rescaling can lead to conflicting classifications of preferences into such fuzzy groups as slightly risk averse, moderately risk averse, etc. In table 1, a summary of commonly used risk aversion coefficients are displayed for two such classifications. Labels in the table reflect classifications provided by the authors wherever possible. However, in some cases liberties have been taken and labels have been assigned to those preference intervals most approximate to $r(x) = 0$ or to those with the greatest $r(x)$ values. Upper bounds on almost risk-neutral preferences range from .000001 to .005. It appears from this table that most coefficients are now assumed, based on certainty equivalents or on secondary data from other studies.[1] Little consistency is evident on appropriate coefficients or classifications of specific coefficient values. Furthermore, inaccurate rankings of action choices can be produced with SDWRF when inappropriate rescalings of the outcome variable have been made.

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[1] King suggests that caution be taken that the risk aversion coefficient be set such that the certainty equivalent is never smaller than the lowest observation of the outcome variable.

Table 1. Summary of Commonly Used Risk Aversion Coefficients

Study	Almost Risk Neutral	Strongly Risk Averse	Outcome Variable	Source of $r(x)$
1) Holt and Brandt	.005 to .005	.02 to .04	Hog prices ($/cwt)	Assumed based on Kramer and Pope
2) Meyer 1977b		6.0	% Annual return on mutual funds	Assumed based on C.E.
3) Cochran, Robison, and Lodwick		.0015	Annual income from 10-acre block	Elicited
4) Lemieux, Richardson, and Nixon	−.00001 to .00001	.000015	After-tax NPV (10 year)	Assumed
5) Tauer, 1985		.0002 to .0003	$100,000 farm purchase	Assumed based on King and Robison
6) Love and Robison	−.00001 to .0002	.0025 to ∞	After-tax annual income	Elicited
7) Rister, Skees, and Black	−.00001 to .00001	.00004 to .00008	Annual returns to grain storage	Assumed based on C.E.
8) Wilson and Eidman 1983	−.0001 to .0001	.0002 to .001	After-tax annual farm income	Elicited
9) Zacharias and Grube	−.0000001 to .000001	.000042 to .0035	Annual farm income	Assumed threshold
10) King and Oamek	−.00001 to .00001	.00005 to .0001	Annual farm income	Elicited
11) Danok, McCarl, and White		.1	Annual farm income	Assumed
12) King and Lybecker	−.0001 to .0001	.0003 to .0006	Annual income from 1,000 cwt dry beans	Assumed
13) Kramer and Pope	.000 to .00125	.02 to .03	Annual farm income	Assumed based on C.E.
14) King and Robison	−.0001 to .0001	.001	Annual income	Elicited
15) Cochran et al.	−.0001 to .0001	.001	Annual farm income	Assumed based on Love and Robison; Cochran, Robison, and Lodwick
16) Tauer 1986	−.0001 to .001	.001 to ∞	Annual farm income	Elicited
17) Greene et al.	.0 to .00125	.005 to .0075	Annual farm income	Assumed

Interpretation in Terms of Changes in Marginal Utility

The Pratt-Arrow measure of absolute risk aversion can be defined in several equivalent ways:

$$r(x) = -\frac{u''(x)}{u'(x)},$$
$$= -\frac{(du'/dx)}{u'},$$
$$= -\frac{d}{dx}\log u',$$
$$= -\frac{(du'/u')}{dx}.$$

The last of the above expressions suggests that the coefficient can be interpreted as the percent change in marginal utility per unit of outcome space. Therefore, r has associated with it a unit, the reciprocal of the unit with which the outcome space is measured. For instance, suppose with outcomes measured in dollars, r is elicited as .0001. Such a value is actually .0001/$ and would more properly be specified with its unit intact. It indicates that near the outcome level at which the elicitation was made, the decision maker's marginal utility is dropping at the rate of .01% per dollar change in income. Similarly, $r(x) = -.00005$/$ implies that around the outcome value x, marginal utility is rising at the

Table 2. Marginal Utility as a Function of Income and Constant Absolute Risk Aversion

r (/$)	Marginal Utility (Relative to u' ($0) = 1)[a]			
	$50	$250	$10,000	$50,000
−.0001	1.005	1.03	2.71	148
−.00001	1.0005	1.002	1.11	1.65
.0	1.0	1.0	1.0	1.0
.00001	.99995	.998	.90	.61
.00005	.998	.99	.61	.08
.0001	.995	.98	.37	.01
.0002	.99	.95	.14	.00005
.0003	.985	.93	.05	3×10^{-7}
.0004	.98	.90	.018	2×10^{-9}
.001	.95	.78	.00005	2×10^{-22}
.01	.61	.08		
.1	.007	1×10^{-11}		

Values are obtained from $u'(x) = \exp(-rx)$

[a] In using the table with an interval representation of r, the bounds on the marginal utility are precisely the marginal utilities of the bounds.

rate of .005% per dollar change in income. Likewise, if r is known only to lie in the interval (.00004/$, .00006/$), then marginal utility is falling at a rate between .004% and .006% per dollar.

Conversion to Marginal Utilities

To get a feel for plausible values of r, it is illuminating to convert quoted values of r from the literature into marginal utilities at various outcome levels. For simplicity, constant absolute risk aversion will be assumed, so that $u'(x) = \exp(-rx)$, where $u'(0) = 1$ (see table 2). The values in the table, therefore, represent the worth of the next dollar, given that the first dollar is worth 1 unit.

What is surprising is the rapid falloff in marginal utility for what has been considered moderately risk averse behavior, such as .0002/$. Even at allegedly risk-neutral levels (Cochran et al.; Holt and Brandt; King and Lybecker; Wilson and Eidman 1983), r = .00005/$, the value of a dollar at $50,000 is just one-seventh the value of a dollar at $10,000.

One of the hazards of working with numbers as tiny as these is to underestimate the distinction between them. A pair of decision makers exhibiting seemingly close values of r such as .0002/$ and .0003/$, respectively, would disagree on the value of the 10,001st dollar by a factor of three and on the value of the 50,001st dollar by a factor of 160. Furthermore, some studies have often lumped "reasonable" values of r, i.e., all those between .0001 and −.0001 in the same interval (Cochran et al.; King and Lybecker; Kramer and Pope; Love and Robison; Tauer 1986; Wilson and Eidman 1985), but the interval must be considered as representing a less-than-homogenous group at some values of X. At some values of x, the preferences may be similar, but at x = 50,000 the weight at $r(x)$ = .00005 is eight times as large as the weight at $r(x)$ = .0001. Such a wide interval at large values of x may be very weak in its discriminatory abilities and may result in a large type II error.

Change of Outcome Scales

The need for the explicit specification of the unit of r might arise when elicited values are used outside the context of the original study. If a risk aversion coefficient elicited over an outcome space measured in one unit is later applied over outcomes measured in another unit, it must be converted by the appropriate factor. An approximation of such a conversion is often trivial, as the following three examples illustrate. First, two theorems are introduced to guide the approximation to necessary conversions. The theorems are proven in the appendix.

THEOREM 1. *Let $r(x) = -u''(x)/u'(x)$. Define a transformation of scale on x such that $w = x/c$, where c is a constant. Then $r(w) = cr(x)$.*

In other words, if the outcome scale is contracted by a factor of c, then a value of r over the old scale must be multiplied by the same scaling factor c to be meaningful over the new scale. This result may seem surprising in light of the common notion that the coefficient is invariant to linear transformations. However, this invariance property applies only to transformations of the utility function (which can be arbitrarily scaled) and not to transformations of x itself.[2]

It should be noted that although r is changed by a contraction or expansion of the outcome scale, it is unaltered by a shift (or translation) of the scale upward or downward:

THEOREM 2. *If $v = x + c$, where c is a constant, then $r(v) = r(x)$. Therefore, the magnitude of the risk aversion coefficient is unaffected by the use of incremental rather than absolute returns (or vice versa).*

Example 1: *Change of Currencies.* Suppose that $r = .0001/\$$ (U.S.) is to be used as an upper bound of risk aversion in modeling decision making among Australian farmers. To become a meaningful risk measure in Australian currency, r must be converted by the appropriate exchange rate. At 1.5 Australian dollars per U.S. dollar,

$$r(x) = .0001/\$\text{ (U.S.)} * \frac{1\$\text{ (U.S.)}}{1.5\$\text{ (Australian)}} = .0000667/\$\text{ (Australian)}.$$

Change in the Spatial or Temporal Dimension of the Outcomes

In the example given above, the underlying attitude towards risk was not affected by the transformation of scale; the risk measure was merely expressed in different units. The situation may be more complex when the change of scale is brought about by a change in the temporal or spatial dimension of the outcomes. The following examples illustrate the potential problems:

Example 2: *Per Acre Analyses.* Suppose that r is elicited as .0001/\$ as a measure of aversion to annual income risk and is to be applied to returns expressed on a per acre basis. It is common to neglect the distinction between per acre risk and total farm income risk, yet it is easy to show that the two values are not the same. A .01% falloff in marginal utility per farm income dollar is very different from a .01% falloff in marginal utility per acre income dollar; the latter is a far more risk-neutral description. The necessity for a distinction arises because two different outcome scales are involved: total returns and per acre returns. If the crop under study represented the entire income of the farmer and the farm size were known, the conversion factor between the two scales would be the number of acres. Then for a 100-acre farm, $r(x) = .0001/\$$, where x represents annual income dollars, would be equivalent to $r(w) = .01/\$$ where w is in acre income dollars. However, if other crops contribute to the farm income, the farmer might exhibit differing attitudes toward risk for each crop.

Example 3: *Ten-Year Horizon.* Suppose that $r(x) = .0001/\$$ is now to be applied to returns expressed on a ten-year net present value basis. In analogy with the previous example, risk per annual income dollar must be distinguished from risk per ten-year NPV dollar. The r over the new ten-year NPV scale would be obtained by dividing the old r by the ten-year NPV of a dollar. Once again, we are confronted with a problem. Will the marginal utility curve retain its shape over varying time horizons? Or, put in another way,will utility for wealth have the same properties as utility for annual income? Sinn suggests that with the passage of time, the degree of relative risk aversion will move towards unity. Both Pratt and Tsiang indicate that the Pratt-Arrow risk aversion measure is stable as long as the gamble is small in relation to total wealth. The NPV of a ten-year income stream may not fall within this range. Empirical evidence is lacking to make more than a tentative conclusion at this point (we are unaware of any efforts to date to elicit preferences over ten-year NPV distributions), but concern must be expressed. Caution must also be exercised in the discounting process itself to handle appropriately the traditional risk premium component of the discount rate.

Comparison to the Relative Risk Aversion Coefficient

The relative risk coefficient r^* is defined as $r^* = r^*x$, where x is an element of the outcome space. While r measures the percent change of marginal utility per unit change of the outcome space, r^* measures the same marginal utility change per percent change of the outcome

[2] Examples of similar scalings can be found in Tauer 1985 and Zacharias and Grube.

Table 3. Comparison of Strategy Rankings with and without Appropriate Output Scaling

Risk Interval	Rankings of Strategies[a] Without Conversion	With Conversion
−.0008 to −.0001	1) DYNCP2[b]	1) TENSCP50
	2) TENSCP50	2) DYNCP2
	3) DYNCP3	3) TENSCP60
−.0001 to .0001	1) DYNCP2	1) TENSCP50**[c]
	2) TENSCP50	1) TENSCP55**
	3) DYNCP3	1) TENSCP60**
	4) TENSCP55	1) TENSCP65**
	5) TENSCP45	1) DYNCP2**
		1) DYNCP3**
.0001 to .0004	1) DYNCP2	1) TENSCP65
	2) TENSCP50**	2) TENSCP60
	2) DYNCP3**	3) TENSCP55
	4) TENSCP55	4) DYNCP3
.0004 to .001	1) DYNCP2	1) TENSCP65
	2) TENSCP50**	2) TENSCP60
	2) DYNCP3**	3) TENSCP55
	3) TENSCP55	4) DYNCP3

[a] All returns were expressed in terms of per acre net revenue while risk aversion coefficients are associated with after-tax net farm income. Scaling was performed by multiplying the per acre returns by the number of acres in an average size farm.
[b] TENSCP45 = −.45 atm from first square to 8 weeks past first bloom; TENSCP50 = −.50 atm from first square to 8 weeks past first bloom; TENSCP55 = −.55 atm from first square to 8 weeks past first bloom; TENSCP60 = −.60 atm from first square to 8 weeks past first bloom; TENSCP 65 = −.65 atm from first square to 8 weeks past first bloom; DYNCP2 = −.3 atm to −.45 atm from first square to 6 weeks past first bloom, followed by −.46 atm to −.55 atm during the 6 to 8 week period past first bloom; DYNCP3 = −.3 atm to −.45 atm from first square to 3 weeks past first bloom, followed by −.46 atm to −.55 atm during the 4 to 8 week period past first bloom.
[c] Double asterisk denotes strategies appearing in the same efficient set.

space. The relative risk aversion coefficient is therefore the elasticity of the marginal utility function and is unitless. However, it is unitless only when the spatial and temporal dimensions of the underlying marginal utility function are consistent with those of x itself. Therefore, while the relative risk aversion coefficient is not subject to the scaling problems of types presented in examples 1 and 2, it is still susceptible to the problem of the marginal utility and returns being expressed in incommensurable units.

Direct Elicitation Procedures

A way around the scaling problems inherent in examples 2 and 3 would be to elicit directly the aversion to per acre or ten-year NPV risk. Such a procedure would stray from the typical after-tax net farm income questioning commonly used in the past and focus directly on preferences for per acre (or ten-year) returns before taxes and unrelated fixed expenses. Some interesting empirical questions could potentially be answered concomitantly. How do attitudes toward risk change (if at all) as the time horizon is varied (but the time origin remains fixed)? Do we become less prone to risk taking as wealth rather than short-term income is at stake? Are our attitudes toward risk identical for each crop in a multicrop farm? Or can we even correctly measure risk for a single crop without the knowledge of the other income sources? Little work has been carried out to answer the above questions.

Converting Values of *r*

In the absence of direct elicitations, the conversion from an r over one spatial-temporal scale to another should be viewed only as an approximation. The appropriate scaling factor to use would be that which brings these returns in one scale to the approximate level of the other. Therefore, if per acre returns are to be used with a per farm value of r, the factor that brings the per acre returns up to the level of the per farm returns (perhaps the farm size) would provide the desired approximation.

Inaccurate Rankings from SDWRF

Inaccurate rankings of alternative action choices can be generated with SDWRF when scaling problems are uncorrected. By using Pratt-Arrow coefficients elicited at after-tax, whole-farm income levels to evaluate action choices described in terms of per acre net returns, an efficient set is identified for a class of decision makers which is considerably less risk averse than the one intended. The probability of large type I errors can be expected to increase.

An example of such inaccurate rankings is presented in table 3. Action choices for ten cotton irrigation strategies in Arkansas were ranked, first without an appropriate rescaling and then with a rescaling which converted the risk aversion measures from whole-farm to per acre levels. The outcome variables were expressed as per acre net returns. The strategies identify a threshold tensiometer value which triggers an irrigation application. Probability distributions were generated with COTCROP-A, a computer simulation model. Further details on the strategies and the simulations can be found in Cochran et al. It is important to note that in all four risk intervals, the efficient sets vary, implying that inappropriate scalings have produced inaccurate rankings. In the interval defined by −.0001 to .0001, strategies which were not ranked in the top five (TENSCP60 and TENSCP65) without the conversion, now are in the efficient set, a major type I error. This occurs because the class of decision makers defined without the conversion is much less risk averse than the one with the conversion.

Summary

In summary, the units of r and x must always be reciprocal to one another. If x is expressed in another unit, r can be converted to the reciprocal unit of x by the appropriate conversion factor. Unfortunately, risk attitudes may change over varying temporal or spatial scales, and in such cases conversion can be viewed only as approximations.

In general, it appears desirable to state explicitly the unit over the space with which r has been estimated. This is necessary because the value represents the percent change in marginal utility per outcome unit. Where an estimated value of r represents anything other than an aversion to annual income risk (e.g., per acre, per month, ten-year present value, etc.), and has not been converted, this should also be explicitly stated. When a value of r from one study is to be applied in another study, care must be taken that an implicit transformation of the outcome scale has not occurred. That is, per acre returns should not be used directly with an r that represents an aversion to annual income risk. This could very well lead to inaccurate rankings being generated by SDWRF. In addition, categorization of preferences and the use of descriptors such as "moderately risk averse" must take into consideration the units of the outcome variable. A value of r may have an entirely different interpretation than what it will have at some other level of x.

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Appendix

Proof of Theorem 1

If $w = x/c$, then by the chain rule,

$$\frac{du}{dw} = \frac{du}{dx}\frac{dx}{dw} = c\frac{du}{dx},$$

and

$$\frac{d^2u}{dw^2} = c\frac{d^2u}{dx^2}\frac{dx}{dw} = c^2\frac{d^2u}{dx^2}.$$

Therefore,

$$r(w) = \frac{-c^2 d^2u/dx^2}{c\ du/dx} = cr(x).$$

Proof of Theorem 2

Since

$$dv = dx,$$
$$du/dv = du/dx$$

and

$$d^2u/dv^2 = d^2u/dx^2;$$

therefore,

$$r(v) = r(x).$$