Interpretations and Transformations of Scale for the Pratt-Arrow Absolute Risk Aversion Coefficient: Implications for Generalized Stochastic Dominance: Comment

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Raskin and Cochran (RC) recently presented a useful paper dealing with several items relating to the Pratt-Arrow risk aversion coefficient (RAC). One aspect of that paper deals with the way that the RAC changes with respect to arithmetic manipulations of the outcome variable. In particular, the authors show the effects of multiplying and adding constants. This note takes issue with RC's statements regarding the addition of constants, i.e., their theorem 2. In turn, the implications of the criticisms in terms of RC's conclusions are explored.

Theorem 2 in the RC article states that, given a utility function $u(w)$ and its associated risk aversion function $r(w)$, "If $v = x + c$ where c is a constant, then $r(v) = r(x)$. Therefore, the magnitude of the risk aversion coefficient is unaffected by the use of incremental rather than absolute returns. . ." (p. 207). The first part of the statement of the theorem is accurate and properly proven in the appendix; however, the basic contention of the second half of the theorem is that the RAC at income level x is equal to the RAC at wealth level $v = x + c$. This is not equivalent to saying $r(v) = r(x)$ as RC state in the first part of the theorem but rather one must show that

$$-\frac{\dfrac{d^2u(x+c)}{dx^2}}{\dfrac{du(x+c)}{dx}} = -\frac{\dfrac{d^2u(x)}{dx^2}}{\dfrac{du(x)}{dx}}.$$

However, this is not generally true, as can be demonstrated with a counter-example. Consider the commonly posited polynomial utility form, specifically (and purely for expository purposes), the quadratic

$$u(x) = a + bx + dx^2;$$

then if

$$v = x + c,$$

$$u(v) = u(x + c) = a + b(x + c) + d(x + c)^2,$$

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Table 1. Utility Functions Used

Number	Source	Utility Function
1	Lin, Dean, and Moore subject 1[a]	$U = 44.52 + 1.96W - .0099W^2$
2	Lin, Dean, and Moore subject 3	$U = 55.74 + 1.27W - .0031W^2$
3	Lin, Dean, and Moore subject 5	$U = -54.01 + 9.67W - .19W^2 + .0012W^3$
4	Lin, Dean, and Moore subject 6	$U = 70.01 + 1.30W - .0064W^2$
5	Kaufman[b]	$U = -263.61 + 22.093 \ln(W + 150{,}000)$

[a] The Lin, Dean, and Moore functions are drawn from their table 2 on page 504. W was elicited in thousands of dollars of net farm income. Linear functions were omitted as they exhibit zero RAC's.
[b] The Kaufman function exhibits a decreasing Pratt-Arrow coefficient over its whole domain.

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Table 2. Risk Aversion Coefficients at Various Wealth Levels

Size of Increment[a] (%)	Utility Function				
	1	2	3	4	5
−50	1.352E−02	5.561E−03	8.264E−02	1.306E−02	5.714E−06
−10	1.852E−02	6.256E−03	−4.000E−01	1.768E−02	5.128E−06
−1	2.021E−02	6.438E−03	−7.396E−02	1.921E−02	5.013E−06
−.1	2.039E−02	6.456E−03	−6.189E−02	1.938E−02	5.001E−06
0	2.041E−02	6.458E−03	−6.061E−02	1.939E−02	5.000E−06
+.1	2.043E−02	6.460E−03	−5.934E−02	1.941E−02	4.999E−06
+1	2.062E−02	6.479E−03	−4.836E−02	1.958E−02	4.988E−06
+10	2.273E−02	6.674E−03	4.706E−02	2.148E−02	4.878E−06
+50	4.168E−02	7.702E−03	−1.127E−01	3.765E−02	4.444E−06

[a] Relative to initial wealth.

the risk aversion coefficient of the function $u(x)$ is

$$\frac{-2d}{b + 2dx},$$

while the risk aversion coefficient of $u(x + c)$ is

$$\frac{-2d}{b + 2d(x + c)}.$$

Here these coefficients are equal only if c equals zero. This is clearly a counter-example to the second part of RC's theorem 2, rendering it invalid.

Consequently, RC cannot claim "the magnitude of the risk aversion coefficient is unaffected by the use of incremental rather than absolute returns" (p. 207). This would be the case with certain very restrictive functional forms, i.e., linear and constant risk aversion negative exponential functions, or zero wealth. Furthermore, the Pratt-Arrow RAC [$r(w)$] has been proposed as a risk measure in terms of wealth (w). If RC's theorem were valid, one could abandon the wealth concept and look only at wealth increments (i.e., income). The invalidity of this theorem leaves wealth as the item of focus.

The above findings have implications for RC's examples 2 and 3. RC contend in examples 2 and 3 that when the size of the risky prospect is changed, the RAC should be changed in a reciprocal fashion; i.e., in example 3 (p. 207) RC state that when deciding on annual income vis-à-vis ten-year net present value, "The r over the new ten-year [period] . . . would be obtained by dividing the old r by the ten-year NPV." In example 2, RC indicate when going from a whole farm to a single-acre basis, r should be divided by the reciprocal of the number of acres. In both cases, the rules are strictly correct only if wealth is zero or is divided by the same amount.

This may again be illustrated through example. Table 1 shows utility functions reported in Lin, Dean, and Moore; and Kaufman (p. 178 or as reported in Keeney and Raiffa, p. 205). Evaluating the associated RAC's at an initial wealth level of \$50 for utility functions 1–4 and \$50,000 for number 5 as well as at wealth plus and minus an increment equaling .1%, 1%, 10%, and 50% of initial wealth yields the data in table 2. Dividing the resultant RAC by the RAC at initial wealth leads to the data in table 3.

Table 3. Comparison of Proportional Change in the Risk Aversion Coefficient With Those Predicted by the Raskin and Cochran Formula

Size of Increment[a] (%)	Proportional Change by Utility Function[b]					Change Predicted by Raskin and Cochran Formula
	1	2	3	4	5	
−50	.66	.86	−1.36	.67	1.14	2.00
−10	.91	.97	6.60	.91	1.03	1.11
−1	.99	1.00	1.22	.99	1.00	1.01
−.1	1.00	1.00	.98	1.00	1.00	1.00
+.1	1.00	1.00	.98	1.00	1.00	1.00
+1	1.01	1.00	.80	1.01	1.00	.99
+10	1.11	1.03	−.78	1.11	.98	.91
+50	2.04	1.19	1.86	1.94	.89	.67

[a] Relative to initial wealth.
[b] Formed by dividing the RAC at terminal wealth by the RAC at initial wealth.

The last column of table 3 gives RC's formula evaluated at the change in the risk aversion coefficient when wealth is not divided by the increase in the bet size; i.e., one over the proportional change. Thus, if the income level is raised by 1.5, then RC would forecast 1/1.5 or .67 of that before. The data show that $r(w)$ potentially does change as the incremental income gets large relative to initial wealth. However, as can be seen from table 3, RC's forecasts are not very accurate. Under the increasing risk aversion functions (1, 2, and 3), the forecast is in the wrong direction, while, with the decreasing RAC function (5) RC forecast, it is too large a change.

In summary, Raskin and Cochran properly conclude that the units of r and x are inversely related but improperly conclude that risk aversion coefficients are unaffected by the addition of constants. Furthermore, the RAC cannot be considered solely with respect to the size of the risky prospect at RC's discussion implies, but rather wealth must also be considered. Consequently, when there is nonzero wealth, the magnitude of the risk aversion coefficient does not vary in a reciprocal relationship with the size of the risky prospect unless wealth is also scaled accordingly.

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References

Kaufman, G. M. *Statistical Decision and Related Techniques in Oil and Gas Exploration.* Englewood Cliffs NJ: Prentice Hall, 1963.

Keeney, R. L., and H. Raiffa, *Decisions with Multiple Objectives: Preferences and Value Tradeoffs.* New York: John Wiley & Sons, 1976.

Lin, W., G. W. Dean, and C. V. Moore. "An Empirical Test of Utility vs. Profit Maximization in Agricultural Production." *Amer. J. Agr. Econ.* 56(1974):498–508.

Raskin, R., and M. J. Cochran. "Interpretations and Transformations of Scale for the Pratt-Arrow Absolute Risk Aversion Coefficient: Implications for Generalized Stochastic Dominance." *West. J. Agr. Econ.* 11(1986):204–10.