Incorporating Safety-First Constraints in Linear Programming Production Models

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A recent survey indicated that many producers view risk in a safety-first context. Traditional methods used to impose safety-first constraints in optimization models have often been difficult to implement. This is particularly true when endogenous decisions affect the distribution of the chance-constrained random variable. This paper presents a method whereby probabilistic constraints can be easily imposed upon finitely discrete random variables. The procedure uses a linear version of the lower partial moment stochastic inequality. The resulting solutions are somewhat conservative but are less so than the results using the previously published mean income-absolute deviation stochastic inequality.

Key words: chance-constraints, lower partial moments, safety-first, stochastic inequalities.

Identifying and modeling the processes that decision makers use to address and control risk continue to stimulate the research efforts of agricultural economists. Several models have been commonly employed including the expected utility model and the safety-first models. The expected utility model is widely accepted as a risk model due, at least in part, to its axiomatic choice foundations. Recently several researchers have noted consistent violations of one or more of the choice axioms, which has led to reexamination of alternative or modified models of risky decision making (Machina).

The safety-first model is an alternative model in which the decision maker is concerned with (or constrained by) the probability of failing to achieve his income goals. The safety-first model, in general, is not consistent with the expected utility model (Pyle and Turnovsky). However, as reported by Patrick et al., the results of a recent producer survey indicate that safety-first-type concerns may be important in firm decisions.

To identify the risk perceptions and responses of producers, Patrick et al. coordinated a survey of 149 agricultural producers in twelve states. Patrick et al. report that many producers "indicated what could be interpreted as substantial 'safety-first' considerations in their decision making" (pp. 237–238). Such results indicate that the safety-first model may be worth further investigation.

Safety-first-type concepts have long been discussed as a method of decision making under uncertainty (see Shackle, Roy, Telser, Kataoka). Various safety-first criteria have been discussed. Roy proposed that the probability of income falling below critical values or goal be minimized. Telser proposed that expected income be maximized subject to satisfying probabilistic constraints upon the likelihood of low income levels. Kataoka discussed finding the maximum income level g for which the probability of income falling below g is below a prespecified level. Both Telser's and Kataoka's criterion involve enforcing probabilistic constraints of the form,

$$\Pr(Z < g) \leq 1/L^*,$$

where Pr $(\cdot)$ is the probability of event $(\cdot)$, Z

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is a random variable (income), g is a goal associated with Z, and $1/L^*$ is an upper limit on $\Pr(Z < g)$. Enforcing such probabilistic constraints in an optimization model can be difficult if (*a*) the income distribution is nonnormal or (*b*) the realized values of the income distribution are affected by endogenous model decisions. These and other difficulties are discussed by Sengupta.

This paper presents a method whereby probabilistic constraints can be enforced easily in a slightly modified Target MOTAD model. The method uses the linear lower partial moment (LPM) stochastic inequality recently presented by Atwood. The procedure is flexible in that it requires only that the random variable be finitely discrete (or approximated as such). The random variable can be a linear combination and/or transformation of finitely discrete multivariate random variables.

This paper will briefly review alternative methods of imposing probabilistic or safety-first-type constraints including the use of the lower partial moment (LPM) stochastic inequality. Modified Target MOTAD models are then presented which implement Telser's and Kataoka's criterion. An example using data from Hazell's MOTAD article is then presented and contrasted to the results of the linear stochastic inequality model discussed by Anderson, Dillon, and Hardaker. A discussion of model weaknesses and possible extensions concludes the paper.

Imposing Probabilistic Constraints

Methods to enforce safety-first or probabilistic constraints vary depending upon model assumptions. In chance-constrained programming, a common approach is to convert the probabilistic constraint into a deterministically equivalent constraint (Charnes and Cooper). However, in many applications, the distribution of the random variable is endogenously determined by the choice of activities, making the derivation of a deterministically equivalent constraint difficult or impossible. A second approach requires the assumption of multivariate normality and the ability to generate a mean-variance (E-V)-efficient set of solutions. Pyle and Turnovsky use this assumption and method to contrast expected utility maximization and safety-first solutions.

A third approach to imposing probabilistic constraints involves the use of stochastic inequalities to generate sharp upper bounds upon probabilities. The use of such inequalities requires the knowledge of certain population parameters such as the mean and variance and usually generates conservative probability bounds. Several authors have discussed the use of stochastic inequalities. Examples include papers by Roy; Telser; Kennedy and Francisco; Anderson, Dillon, and Hardaker; Gabriel and Baker; Sengupta; and Berck and Hihn. These studies (with the exception of Berck and Hihn) use the well-known Chebychev inequality to impose probabilistic constraints upon random variables. Berck and Hihn use a nonlinear version of the lower partial moment inequality to be discussed below.

Commonly known stochastic inequalities, such as Chebychev's mean-standard–error inequality, usually generate conservative probability bounds. Being nonlinear, directly implementing probabilistic constraints with Chebychev's inequality can be difficult (see Sengupta).[1] Alternative linear Chebychev-type inequalities are available which generate less conservative probability bounds than the general Chebychev inequality. One such inequality using lower partial moments was recently presented by Atwood using continuous distributions.

The stochastic inequality presented by Atwood uses a parameter which has been termed a lower partial moment (see Nantell, Price, and Price). For discrete populations the lower partial moment LPM can be denoted as $R(a, t)$, where

$$R(a, t) = \sum_{z_i \leq t} (t - z_i)^a f_i. \tag{1}$$

In (1), $R(a, t)$ is the lower partial moment, t is a reference level below which deviations are measured, z_i is the value of Z should state i occur, $a > 0$ is the power to which deviations below t are raised, and f_i is the probability that state i occurs. The above LPM can be used to generate the following stochastic inequality (Atwood):

$$\Pr(Z \leq t - pQ(a, t)) \leq (1/p)^a, \tag{2}$$

[1] Linear versions of Chebychev's inequality exist. Anderson, Dillon, and Hardaker use a mean-absolute deviation version to impose probabilistic constraints upon solutions. Although the E-A inequality can be used in a linear model, the resulting solutions tend to be quite conservative.

where $Q(a, t) = [R(a, t)]^{1/a} > 0$ and p is a constant greater than zero. If p is defined as $p \equiv (t - g)/Q(a, t)$ with $t > g$ and $Q(a, t) > 0$, then (2) can be written as

$$\text{(3)} \quad \Pr(Z \leq g) = \Pr(Z \leq t - pQ(a, t)) \leq [Q(a, t)/(t - g)]^{a}.$$

If either $Q(a, t) = 0$ or $t = g$, the stochastic inequality is inapplicable as p in (2) or $Q(a, t)/(t - g)$ in (3) is undefined.

The reader will note that any power of $a > 0$ (in particular, $a = 1$) can be used in the above inequality. For the remainder of this paper, the linear lower partial moment ($a = 1$) will be used and denoted as $Q(t) = Q(1, t) = R(1, t)$. This allows (3) to be rewritten as

$$\text{(4)} \quad \Pr(Z \leq g) = \Pr(Z \leq t - pQ(t)) \leq Q(t)/(t - g)$$

if $Q(t) > 0$ and $t > g$. Using (4), Atwood stated that enforcing the following constraint in an optimization model is sufficient to guarantee $\Pr(Z < g) \leq 1/L^*$ with discrete populations. The sufficiency constraint is

$$\text{(5)} \quad t - L^*Q(t) \geq g.$$

A demonstration that (5) is only sufficient for $\Pr(Z < g) \leq 1/L^*$ and not $\Pr(Z \leq g) \leq 1/L^*$ is presented in appendix 1.

The LPM and the Linear Model

The above sufficiency constraint (5) can be easily enforced with a set of linear constraints by slightly modifying the linear LPM model known as Target MOTAD (see Held, Watts, and Helmers; Tauer; and Watts, Held, and Helmers). To demonstrate this, note that the Target MOTAD model can be written as

$$\begin{array}{lll}
\text{(6)} & \text{maximize} & E(Z) = E_y'x \\
\text{(6a)} & \text{subject to:} & Ax \leq b, \\
\text{(6b)} & & Yx - 1t + Id \geq 0, \\
\text{(6c)} & & r'd \leq k, \\
\text{(6d)} & & t = g, \\
 & & x, d, \geq 0,
\end{array}$$

where $E(Z)$ is expected aggregate income; E_y' is a transposed vector of per-unit expected income levels; x is a vector of activity levels; A is a matrix of technical coefficients; b is a vector of right-hand-side coefficients; $Y = [y_{ij}]$ is a matrix of possible per-unit income levels with y_{ij} the income of activity j should outcome i occur. Yx is thus a vector of possible aggregate income states;[2] 1 is a vector of ones; t is a reference level for aggregate income; I is an identity matrix; d is a vector with the ith element equal to the deviation below t if aggregate income in state i falls below t and zero if income exceeds t; r' is a transposed vector of probability levels with $r'd$ equal to $Q(t)$; k is an upper limit on $Q(t) = r'd$; and g is the aggregate income goal of concern. The Target MOTAD model determines a feasible choice vector x, which maximizes expected aggregate income while requiring that probability-weighted deviations below $t = g$ not exceed k. Tauer demonstrated that solutions to system (6), if unique, are second-degree stochastically efficient. The inequalities in (6) can be easily modified to compute $Q(1, t)$ for any t-level and to impose sufficiency constraint (5) on the system. The following system maximizes expected aggregate income subject to $\Pr(Z < g) \leq 1/L^*$. The system is

$$\begin{array}{lll}
\text{(7)} & \text{maximize} & E(Z) = E_y'x \\
\text{(7a)} & \text{subject to:} & Ax \leq b, \\
\text{(7b)} & & Yx - 1t + Id \geq 0, \\
\text{(7c)} & & r'd - Q(t) = 0, \\
\text{(7d)} & & t - L^*Q(t) \geq g, \\
 & & x, d, \geq 0.
\end{array}$$

System (7) differs from (6) in several ways. An activity (column) has been included in (7) to compute and transfer $Q(t) = r'd$ from (7c) and subtract $L^*Q(t)$ from (7d). The deviation reference level t is no longer required to equal g but is allowed to be endogenously set at any level which satisfies (7d) [or (5)]. Simultaneously, (7b) and (7c) compute the corresponding $Q(t)$ level. If (7d) is constraining, the level of t selected will be the least constraining level possible while satisfying (7d). As stated earlier, this corresponds to the endogenous selection of the least constraining linear lower partial moment from the set of lower partial moments for which $t - L^*Q(t) \geq g$.

System (7) selects an activity mix x which maximizes $E(Z)$ while simultaneously enforcing probabilistic constraints upon the possible outcomes with the least constraining linear

[2] The vector Yx can be viewed as a univariate vector z upon which deviations are computed and probabilistic constraints imposed. The univariate vector z may not be aggregate income. The methods presented can thus be used to impose probabilistic constraints upon non-income random variables.

Table 1. Example Tableau of Telser's Criterion—Max E_Z s.t. $\Pr(Z < g) \leq 1/L^*$

Rows	x_1	x_2	x_3	x_4	t	d_1	d_2	d_3	d_4	d_5	d_6	$TQ(t)$	TOBJ	RHS
Obj. Fcn.													1	Maximize
Land (ac.)	1	1	1	1										≤200
Labor (hrs.)	25	36	27	87										≤10,000
Rotate (ac.)	−1	1	−1	1										≤0
y_1 ($)	292	−128	420	579	−1	1								≥0
y_2 ($)	179	560	187	639	−1		1							≥0
y_3 ($)	114	648	366	379	−1			1						≥0
y_4 ($)	247	544	249	924	−1				1					≥0
y_5 ($)	426	182	322	5	−1					1				≥0
y_6 ($)	259	850	159	569	−1						1			≥0
$Q(t)$						.1667	.1667	.1667	.1667	.1667	.1667	−1		=0
SUFCON					1							$-L^*$	0[a]	≥g[b]
Exp. Inc ($)	253	443	284	516									−1[c]	≥0

[a] To implement Kataoka's criterion, the coefficient in this cell is −1.
[b] To implement Kataoka's criterion, the entry in this cell is 0.
[c] To implement Kataoka's criterion, the entry in this cell is 0.

LPM stochastic inequality. By redefining the vector $Yx = z$, (see footnote 2), the above system can be used to constrain any finitely discrete univariate random variable. The random variables so constrained can be a linear transformation and/or combination of one or more finitely discrete random variables. In the above case (Telser's criterion), the random variable Z (aggregate income) is a linear combination of k multivariate income random variables. As will be noted later, the random variables need not be income.

The above system can easily be modified to directly solve Kataoka's safety-first criterion. Since $t - L^*Q(t) \geq g$ is equivalent to requiring $t - L^*Q(t) - g \geq 0$, a system which implements Kataoka's criterion can be written as

(8) maximize g

(8a) subject to: $Ax \leq b,$

(8b) $Yx - 1t + Id \geq 0,$

(8c) $r'd - Q(t) = 0,$

(8d) $t - L^*Q(t) - g \geq 0,$

$x, d, \geq 0.$

Numerical Example

Data from Hazell's original MOTAD article will now be used to demonstrate the implementation of Telser's and Kataoka's criteria.[3] Table 1 presents the tableau which implements Telser's criterion. (The footnotes indicate changes required for Kataoka's criterion.) The system in table 1 is identical to system (7) except that an accounting row and transfer column have been added to compute and transfer expected aggregate income into the objective function.[4] In table 1, the selection of carrot (x_1), celery (x_2), cucumber (x_3), and pepper (x_4) activity levels are constrained by the land, labor, and rotational constraints of Hazell's example. The y_{ij} entries under the x_j activities correspond to gross income levels in state i for crop j and are obtained from Hazell's table 1.

[3] Hazell's data has now been used in several risk models. The reader may wonder which is "best" for risky decision models. The answer will likely depend upon the objectives of the study. Should the researcher feel that the E-V model is appropriate, the original MOTAD model may be appropriate. The Target MOTAD model can be used to generate stochastically efficient solutions. The model of this paper enforces safety-first decisions. It is likely that no single model can be used exclusively in modeling risky decision making.

[4] The income accounting row and transfer column simplify the changes required in the tableau to implement Kataoka's criterion.

Table 2. Lower Partial Moment Safety-First Solutions—Telser's Criterion

Income Goal g	Probability Constraint $1/L^*$	Mean Income E_Z ($)	Activity Levels x_1	x_2	x_3	x_4	Actual Probabilities Pr $(Z < g)$	Pr $(Z \leq g)$	Endogenously Selected t	$Q(t)$
50,000	0	74,054	100	39	0	61	0	.167	50,000	0
	.25	77,996[a]	0	27	100	73	.1667	.1667	NA[b]	NA
	.30	77,996[a]	0	27	100	73	.1667	.1667	NA	NA
	.35	77,996[a]	0	27	100	73	.1667	.1667	NA	NA
55,000	0	71,003	111	41	0	48	0	.333	55,000	0
	.25	73,526	33	28	81	57	.1667	.1667	73,497	4,624
	.30	77,996[a]	0	27	100	73	.1667	.1667	NA	NA
	.35	77,996[a]	0	27	100	73	.1667	.1667	NA	NA
60,000	0	65,818	89	31	50	29	0	.5	60,000	0
	.25	66,002	82	30	58	30	.1667	.1667	61,546	386
	.30	66,895	76	30	61	33	.1667	.1667	62,963	889
	.35	77,996[a]	0	27	100	73	.1667	.1667	NA	NA

[a] Sufficiency constraint nonbinding.
[b] Non-constraining with multiple t and $Q(t)$ feasible.

Tables 2, 3, and 4 present solutions for the example problem. Table 2 presents the solutions with g levels (income goals) set at \$50,000, \$55,000, and \$60,000, and probability limits of 0, .25, .3, and .35. The reader will note that for all solutions presented, the probability constraint $\Pr(Z < g)$ is satisfied. As stated above, however, the use of the LPM stochastic inequality often results in conservative solutions. As an example, when g = \$60,000 and $1/L^*$ = .25, the optimal activity mix has an expected income of \$66,002 with $\Pr(Z < 60{,}000) = .1667$. The non-risk-constrained expected profit-maximizing solution has an expected return of \$77,996 but also only has $\Pr(Z < 60{,}000) = .1667$.[5] Should the researcher not be willing to tolerate such conservative solutions, he may be required to attempt other chance-constrained methods as discussed above.

The solutions presented in table 2 demonstrate that (5) or (7d) does not guarantee $\Pr(Z \leq g) \leq 1/L^*$ when $Q(t = g) = 0$. All solutions for which $1/q^* = 0$ generated $Q(t) = 0$ with $t = g$. With g = 50,000, (7d) is satisfied with t = 50,000, and $Q(t) = 0$. Although $\Pr(Z < 50{,}000) = 0$, $\Pr(Z \leq 50{,}000) = .167 > 0$. Similarly $\Pr(Z \leq g = 55{,}000)$ and $\Pr(Z \leq g = 60{,}000)$ are .333 and .5 which exceed $1/L^*$ = 0. These results are consistent with the discussion of inequality (5) presented in appendix 1. Also consistent with these discussions is the fact that if $\Pr(Z < g) > 0$, (7d) is satisfied only with $Q(t) > 0$ and $t > g$.

Table 3 presents solutions of Kataoka's criterion obtained with system (7) and the E-A stochastic inequality approach. (Appendix 2 presents the required modifications of the Anderson, Dillon, and Hardaker E-A safety-first method.) The goal obtained with the LPM method equals \$60,456 for all probability levels examined. The increased conservativeness of the solutions obtained with the E-A stochastic inequality is apparent from table 3. If $1/L^*$ = 0, the only feasible solution is the origin for which $\Pr(Z \leq 0) = 1$. This results from the fact that with MOTAD models the risk reference point is mean income. With Hazell's example, no solutions existed with $E(Z) > 0$ and $A = 0$. When $1/L^*$ = .25, .30, or .35, the E-A method generated the solution $(x_1, x_2, x_3, x_4) = (72, 27, 84, 17)$. For this solution $E(Z)$ = \$62,769, which is lower than that obtained with the LPM inequality. In all cases, the reported goal g was also lower than that obtained with the linear LPM inequality.

Table 4 contrasts the results of using the LPM versus the E-A inequalities when implementing Telser's criterion. In all cases, the results obtained with the LPM method generate higher expected incomes than with the E-A method. As an example, when g = \$52,000,

[5] Although such solutions remain conservative, they are generally much less so than those generated with the E-A stochastic inequality method as will be discussed shortly.

Table 3. Lower Partial Moment and E-A Safety-First Solutions—Kataoka's Criterion

Probability Constraint $1/L^*$	Maximum Goal g	Mean Income E_Z	Activity Levels x_1	x_2	x_3	x_4	Actual Probabilities Pr $(Z < g)$	Pr $(Z \leq g)$
LPM method								
0	60,456	65,316	86	31	56	27	0	2/3
.25	60,456	65,316	86	31	56	27	0	2/3
.30	60,456	65,316	86	31	56	27	0	2/3
.35	60,456	65,316	86	31	56	27	0	2/3
E-A method								
0	0	0	0	0	0	0	0	1
.25	51,755	62,769	72	27	84	17	0	0
.30	53,590	62,769	72	27	84	17	0	0
.35	54,902	62,769	72	27	84	17	0	0

the LPM method generates $E(Z) = \$73{,}201$ and \$77,896 when $1/L^* = 0$ and .25, respectively. At these probability levels, no feasible solutions are available with $g = \$52{,}000$ for the E-A method. When $1/L^*$ is increased to .3, the mean income levels are \$77,996 and \$64,328 for the LPM and E-A methods, respectively. In all solutions reported in table 4, the excess conservatism of the E-A Chebychev inequality results in relatively large sacrifices of expected income when contrasted to the corresponding LPM solution.

Summary and Conclusions

Results recently reported by Patrick et al. indicate that the safety-first model may be useful in explaining producer behavior. Traditional methods to impose safety-first or probabilistic constraints in an optimization model may be difficult to implement when producer decisions influence outcome distributions.

This paper has demonstrated that a linear version of the lower partial moment inequality presented by Atwood can be used to enforce probabilistic or safety-first constraints in linear models.[6] The methods presented require that the constrained distributions be finitely discrete. The procedures are flexible and require only that the vector of potential events can be computed with linear operations. If a vector of potential events can be so computed, then linear inequalities can be used to impose safety-first upon solutions. This is accomplished while simultaneously allowing the model to select the least constraining linear lower partial moment which satisfies the stochastic inequality. While the resulting solutions are often conservative, they are usually much less conservative than when using the linear mean income–absolute deviation inequality discussed by Anderson, Dillon, and Hardaker.

In conclusion, the safety-first model has been theoretically discussed for a number of years. The linear lower partial moment methods presented in this paper enable the researcher to impose probabilistic constraints in a more complex decision setting and with less conservative probability limits than was previously possible. As a result, the agricultural researcher should be able to investigate more thoroughly and analyze the potential of the safety-first model.

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[6] Modifications of the above system can be used in general chance-constrained applications as well. Interested readers can contact the authors for a more detailed discussion of general chance-constrained applications.

References

Anderson, J. R., J. C. Dillon, and B. Hardaker. *Agricultural Decision Analysis,* pp. 211–12. Ames: Iowa State University Press, 1977.

Atwood, J. A. "Demonstration of the Use of Lower Partial Moments to Improve Safety-First Probability Limit." *Amer. J. Agr. Econ.* 67(1985):787–93.

Berck, P., and J. M. Hihn. "Using the Semivariance to Estimate Safety-First Rules." *Amer. J. Agr. Econ.* 64(1982):298–300.

Charnes, A., and W. W. Cooper. "Chance-Constrained Programming." *Manage. Sci.* 6(1959):73–79.

Gabriel, S. C., and C. B. Baker. "Concepts of Business

Table 4. Lower Partial Moment and E-A Safety-First Solutions—Telser's Criterion

Income Goal g	Probability Constraint $1/L^*$	LPM Method: Mean Inc. E_Z	LPM Activity Levels x_1	x_2	x_3	x_4	LPM Pr $(Z < g)$	LPM Pr $(Z \leq g)$	E-A Method: Mean Inc. E_Z	E-A Activity Levels x_1	x_2	x_3	x_4	E-A Pr $(Z < g)$	E-A Pr $(Z \leq g)$
50,000	0	74,054	100	39	0	61	0	.167			Infeasible				
	.25	77,996[a]	0	27	100	73	.167	.167	64,001	68	27	84	22	0	0
	.30	77,996[a]	0	27	100	73	.167	.167	66,287	59	27	83	30	0	0
	.35	77,996[a]	0	27	100	73	.167	.167	69,467	48	28	83	42	0	0
52,000	0	73,210	100	50	0	50	0	.33			Infeasible				
	.25	77,896	3	27	98	73	.167	.167			Infeasible				
	.30	77,996[a]	0	27	100	73	.167	.167	64,328	67	27	84	23	0	0
	.35	77,996[a]	0	27	100	73	.167	.167	66,733	58	27	83	32	0	0
54,000	0	71,735	107	44	0	49	0	.33			Infeasible				
	.25	75,031	23	28	86	63	.167	.167			Infeasible				
	.30	77,996[a]	0	27	100	73	.167	.167			Infeasible				
	.35	77,996[a]	0	27	100	73	.167	.167	64,001	68	27	84	22	0	0

[a] The sufficiency constraint $t - L^* Q(t) \geq g$ is nonbinding.

and Financial Risks." *Amer. J. Agr. Econ.* 62(1980): 560–64.

Hazell, P. B. R. "A Linear Programming Alternative to Quadratic and Semivariance Programming for Farm Planning Under Uncertainty." *Amer. J. Agr. Econ.* 53(1971):53–62.

Held, L. J., M. J. Watts, and G. A. Helmers. "Deriving a More Efficient Risk-Income Frontier: The MONID Approach." Paper presented at GPC-10 meetings, Manhattan KS, 25–26 May 1982 (SR 1169, Wyoming Agr. Exp. Sta.).

Kataoka, S. "A Stochastic Programming Model." *Econometrica* 31(1963):181–96.

Kennedy, J. O. S., and F. M. Francisco. "On the Formulation of Risk Constraints for Linear Programming." *J. Agr. Econ.* 25(1974):129–45.

Machina, Mark. "New Theoretical Developments: Alternatives to EV Maximization." *Proceedings of Seminar Sponsored by Southern Regional Project S-180,* pp. 1–40. Dep. Agr. Econ. Staff Pap. No. 85-85, Michigan State University, 1985.

Nantell, T. J., K. Price, and B. Price. "Mean-Lower Partial Moment Asset-Pricing Model: Some Empirical Evidence." *J. Finan. and Quant. Anal.* 7(1982):763–82.

Patrick, G. R., P. N. Wilson, P. J. Barry, W. G. Boggess, and D. L. Young. "Risk Perceptions and Management Responses: Producer-Generated Hypothesis for Risk Modeling." *S. J. Agr. Econ.*, no. 2(1985), pp. 231–38.

Pyle, D. H., and S. J. Turnovsky. "Safety-First and Expected Utility Maximization in Mean-Standard Deviation Portfolio Analysis." *Rev. Econ. and Statist.* 52(1970):75–81.

Roy, A. D. "Safety-First and Holding of Assets." *Econometrica* 20(1952):431–49.

Sengupta, J. K. "Safety-First Rules Under Chance-Constrained Linear Programming." *Oper. Res.* 17(1969): 112–31.

Shackle, G. L. S. "Decision Order and Time." *Human Affairs.* London: Cambridge University Press, 1969.

Takayama, Akira. *Mathematical Economics.* Cambridge: Cambridge University Press, 1985.

Tauer, L. W. "Target MOTAD." *Amer. J. Agr. Econ.* 65(1983):606–10.

Telser, L. G. "Safety First and Hedging." *Rev. Econ. Stud.* 23(1955):1–16.

Watts, M. J., L. J. Held, and G. A. Helmers. "A Comparison of Target MOTAD to MOTAD." *Can. J. Agr. Econ.* 32(1984):175–86.

Appendix 1

A Discussion of Inequality (5)

To demonstrate that $t - L^*Q(t) \geq g$ (5) guarantees only $\Pr(Z < g) \leq 1/L^*$ and not $\Pr(Z \leq g) \leq 1/L^*$, two situations must be examined. First, assume that $\Pr(Z < g) > 0$. In this case, since (5) requires $t \geq g$, $Q(t = g) > 0$ and (5) can be satisfied only if $t > g$. Hence, if $\Pr(Z < g) > 0$, (5) implies $Q(t)/(t - g) \leq 1/L^*$ with $t > g$ and $Q(t) > 0$. Thus, (4) and (5) imply $\Pr(Z \leq g) \leq Q(t)/(t - g) \leq 1/L^*$, which implies $\Pr(Z < g) \leq 1/L^*$.

A second condition can exist if $\Pr(Z < g) = 0$. In this situation, t can equal g with $Q(t = g) = 0$ [even if $\Pr(Z = g)$ should be as high as 1] and thus satisfy (5). However, if (5) is satisfied with $Q(t = g) = 0$, it is obvious that Z never falls below t or $\Pr(Z < t) = 0$ and hence $\Pr(Z < g) = 0 \leq 1/L^*$. Thus, when a population is discrete, enforcing (5) as a constraint in an optimization model is only sufficient for $\Pr(Z < g) \leq 1/L^*$.

Appendix 2

Chance-Constraints and the E-A Chebychev Inequality

Anderson, Dillon, and Hardaker (ADH) discuss the use of a mean income-absolute deviation version of Chebychev's inequality to impose probabilistic constraints. The inequality is

$$\Pr[|Z - E(Z)|] \leq M/K, \tag{B.1}$$

where M is the mean absolute deviation about E_Z and K is a positive constant. If g is a goal level for the random variable, ADH show that (B.1) can be used to obtain the following inequality:

$$\Pr(Z \leq g) \leq M/[E(Z) - g]. \tag{B.2}$$

ADH imply that the modeler can guarantee $\Pr(Z \leq g) \leq 1/L^*$ by enforcing

$$M/[E(Z) - g] \leq 1/L^*, \tag{B.3}$$

which can be rearranged to give

$$E(Z) - L^*M \geq g. \tag{B.4}$$

[By a process similar to that of appendix 1, it can be shown that B.4 actually guarantees only $\Pr(Z < g) \leq 1/L^*$ and not $\Pr(Z \leq g)$.] A system which implements Telser's criterion while using (B.4) can be written as

(B.5)	maximize	$E(Z) = t$
(B.5a)	subject to:	$Ax \leq b$,
(B.5b)		$Yx - 1t + Id \geq 0$,
(B.5c)		$r'd - M^- = 0$,
(B.5d)		$t - L^*2M^- \geq g$,
(B.5e)		$E_y'x - t = 0$,
		$x, d, \geq 0$,

where M^- is the mean absolute deviation below $t = E(Z)$ and the remaining parameters and variables are as defined in the text. In (B.5e), the reference level t must now equal the mean since the risk parameter used in (B.1) is M (the average absolute deviation about the mean). Since M^- (or average deviations below the mean) is equal to ½ M and (B.4) requires the use of M, M^- is modified in a manner similar to the modifications between systems (7) and (8) in the text.