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# An Empirical Bayes Approach to Modeling Drought

P. J. Chamberlain

This paper illustrates an alternative approach to estimating the occurrence of drought. The empirical Bayes methodology was developed because of deficiencies in time-series and regression analysis with respect to prediction of drought. This manuscript is comprised of (a) a discussion of "classical" and Bayes estimators of probability density (or mass) functions, (b) a description of the model, and (c) a comparison of the performances of the empirical Bayes and two classical estimators in predicting the elapsed time until drought. The Bayes value (incorporating both a priori and data information) was found to be superior to the traditional estimates.

*Key words:* data sensitivity, interdrought periods, marginal distribution, posterior distribution, prior distribution.

The time until a drought occurs is of particular interest to, among others, cattle ranchers in the southern United States. Sufficient rainfall supplements the supply of forage through facilitation of year-round grazing (an activity that offsets the expense of purchasing feed), while during certain dry periods insufficient rainfall can be economically disastrous for the rancher who may have to deplete cash flow to purchase feed or even destock in the absence of adequate short-term finances (Lundgren, Conner, and Pearson).

Modeling the occurrence of rainfall for use in prediction has been attempted in classical areas of statistics, such as time-series analysis, ordinary regression analysis, and stochastic processes. These approaches have met with varying degrees of success (Norwine). I have endeavored to fit more than sixty time-series models, such as AR, MA, ARMA, and AR-ARMA, none of which contained any significant parameter estimates. A prominent feature of the foregoing methodology is its failure to incorporate the researcher's or manager's ex-

perience and personal observation of the phenomenon (rainfall). In this manuscript I am proposing an alternative predictive device that is based on an empirically obtained Bayesian prior distribution.

The classical approach to decision making can often produce misleading or erroneous decisions (or estimates). Classical statistics are more useful for inference problems, where one considers repeated experimentation or "large" sample size to establish convergence. Bayesian statistics may be more appropriate for decision making based upon small or intermediate sample sizes. Traditional statistical inference is surpassed by Bayesian decision analysis in certain areas because of the former's deficiencies in the following.

(a) Inadequate consideration of loss incurred for a "bad" decision when hypothesis testing and failure to quantitatively incorporate prior knowledge into the estimation procedure.

(b) Emphasis on initial precision (i.e., size  $\alpha$  tests are prespecified before use of data in hypothesis testing) while (final) precision for performance of estimates, based on both data and prior information, is not considered. For example, an estimator obtained may converge to the actual parameter value if a large series of samples is obtained; however, because of physical or economic constraints, one may be lim-

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Texas Agricultural Experiment Station Technical Article No. TA23431.

The author wishes to acknowledge and thank J. Richard Conner for helpful comments and support.

ited to one sample of "n" and the costs of error (i.e., loss) may be large and increasing in distance between estimated value and actual value of the parameter. Hence, we need something with final precision.

(c) Hypothesis construction is inconsistent with the distribution or actual range of parameter concerned. It may be contradictory to test a point-null hypothesis if what is actually desired is a null space around the hypothesized point. If one is interested in the "closeness" of a parameter value to a given point and the actual value differs only slightly from that point, the point null hypothesis will be rejected with increasing probability as sample size increases. Hence, the investigator may be led to draw the false inference that the parameter is nowhere near the specified null point. Instead, one may actually be concerned about a parameter being in a particular range (loss may be constant for values in that interval and increasing, or at least higher, for values outside of that region).

A particular rancher's decision based on expected rainfall or time until the next drought is conditional upon prior information and a finite number of past observations. Furthermore, there is a (possibly nonconstant in error) loss associated with nonoptimal decisions. Hence, the Bayesian approach was deemed appropriate.

The Bayes approach to parameter estimation is best described in a decision-theoretic context. Assume the prior distribution  $\pi(\theta)$ ,  $\theta \in \Theta$  where  $\theta$  is unobservable, the (conditional) distribution of the observable data  $f(x|\theta)$ ,  $x \in R$ , and the posterior distribution  $\pi(\theta|x) = f(x|\theta)\pi(\theta)/M(x)$  where  $M(x) = \int_{\Theta} f(x|\theta)\pi(\theta) d\lambda(\theta)$  ( $\lambda$  signifies integration with respect to Lebesgue measure to maintain generality) will be referred to as the marginal or predictive distribution for  $X$ . The loss function  $L(\theta, \delta(x))$  is a measure of "loss" associated with the discrepancy between  $\theta$  and its estimate (decision)  $\delta(x)$ . The random variable  $\theta$  is predicted in Bayes estimation by selecting that decision rule  $\delta$  that minimizes Bayes risk,

$$r(\pi, \delta) = E^{\pi(\theta)}[E^{\lambda(x|\theta)}[1(\theta, \delta(x))]],$$

that is,  $r(\pi, \delta^*) = \inf_{\delta \in D} r(\pi, \delta)$  for decision space

$D$ . But  $r(\pi, \delta) = E^{M(x)}[E^{\pi(\theta|x)}[1(\theta, \delta(x))]]$  and  $\delta^*$  is obtained by minimizing the integrand,  $E^{\pi(\theta|x)}[1(\theta, \delta(x))]$ .

Presuming that the parametric forms of the

prior and conditional distributions are known, a loss function is chosen; and, noting the foregoing discussion of estimating  $\theta$ , one need only obtain the parameters in the prior distribution,  $\pi$ . In a traditional Bayes framework these parameters are prespecified. If repeated one-step (sequential) prediction is desired, this is done in the standard Bayes framework by "updating" the posterior after a new data value,  $x$ , is observed. A variation on this theme is subjectively to assess an empirical distribution of  $\theta$  and estimate the parameters in  $\pi$  from common statistical methods as done in Bessler and Chamberlain. The important feature of the foregoing approaches is that a prior is obtained without requiring past data points,  $x$ ; this aspect is common to all traditional Bayes procedures.

An alternative approach which reduces the subjectivity in specification of the prior is the empirical Bayes method. In this procedure, estimation of the parameters of  $\pi$  proceeds from observing past values of the data of concern,  $x$ , and where possible, observed values of  $\theta$ . For the case wherein the  $\theta$ 's are not observable

for jointly generated pairs  $\{(\theta_i, x_i)\}_{i=1}^n$  one can

use the unconditional joint distribution,  $M(x)$  (vector  $x$ ), as a likelihood function to obtain MLE estimates for the parameters in  $\pi$  (if MLEs are appropriate). More thorough treatments of the empirical Bayes technique are given in Maritz and Copas.

The empirical Bayes approach is quite unsavory to both "purist" Bayesians and staunch frequentists. This is probably because of its hybridization of the Bayes philosophy concerning prior distributions and the frequentist notions surrounding the observation of past data for use in estimating parameters. This melding of the two approaches was found to be quite useful in modeling interdrought times because it permits the selection of an appropriate prior distribution, thus affording some degree of assessing final precision of an estimate (via the posterior distribution) while maintaining the desirable classical properties associated with parameter estimation in a regular (nonpathological) setting. The balance of this paper comprises a development of the empirical Bayes model, a description and test of a hypothesis concerning mean time between droughts, and a decision (estimate) under a loss function.

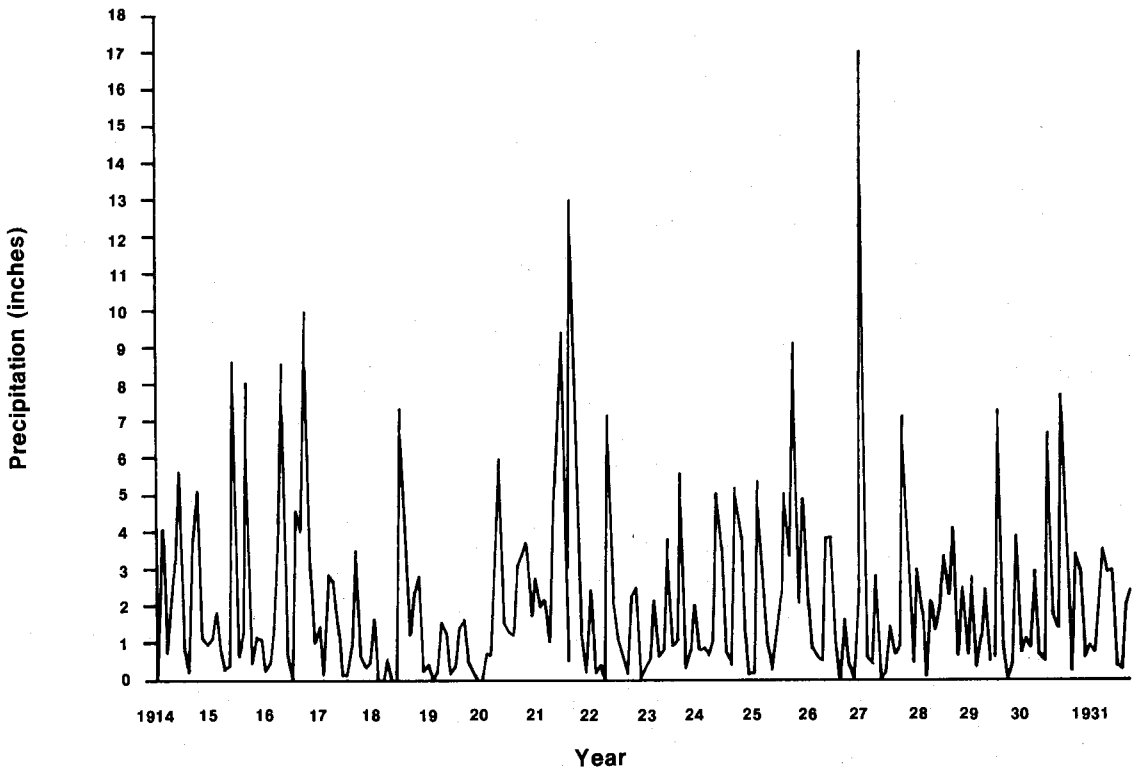


Figure 1. Alice, Texas, precipitation data, 1914-31

### Model Development

To construct our decision model, we will need a prior distribution, a distribution for time until drought, and a loss function. From these, we will obtain a posterior distribution, which will be used in hypothesis testing. Finally, we will formulate our decision model.

We will assume that the time until drought (hereafter denoted  $X$ ) is distributed as  $\exp(\theta)$ . This is felt to be appropriate because of  $X$  being a continuous-valued random variable and the exponential's memoryless property (i.e.,  $P_\theta[X \in (t, t + \Delta t)] = P_\theta[X \in (0, \Delta t)]$ ). The probability of rainfall occurring in a given time interval  $(t, t + \Delta)$ , is independent of  $t$ . Here  $\theta$  is interpreted as being  $(E_\theta^2[X])^{-1}$ . The distribution of  $X$  (given  $\theta$ ) is

$$(1) \quad f(x|\theta) = \theta e^{-\theta x}, \quad x \in (0, \infty), \quad \theta > 0.$$

Since  $\theta$  is unobservable, the choice of the prior  $\pi(\theta)$  is somewhat arbitrary. Herein the prior distribution of  $\theta$  is considered to be distributed as  $G(\alpha, \beta)$  because of both prior beliefs concerning the range of  $\theta$  and mathematical con-

venience (when  $X$  is distributed as exponential, the gamma is a conjugate family, i.e., the posterior is also gamma). The estimate of inter-drought time will be obtained from the marginal distribution (see appendix). For our example,

$$(2) \quad M(x) = \alpha\beta^\alpha / (\beta + x)^{\alpha+1}.$$

In the analysis, 864 months of rainfall data (starting in 1914) for Alice, Texas, were examined. The data are graphically presented in figures 1 through 4. Drought was defined as the event of three or more consecutive months of rainfall below the sample mean minus 10% of the sample standard deviation of each month followed by three or more consecutive months of rainfall below  $X_i - .5S_i$ ,  $i = 1, \dots, 12$ . In the monthly series we found seven drought periods with mean time until drought equaling 82.00 months and a sample standard deviation of 90.40 months. The data are as follows: 16, 11, 128, 180, 216, 14, and 9 (months), the last drought ending in September 1973.

To obtain the estimates,  $\hat{\alpha}$ ,  $\hat{\beta}$ , the likelihood equations (4) and (5) in the appendix were

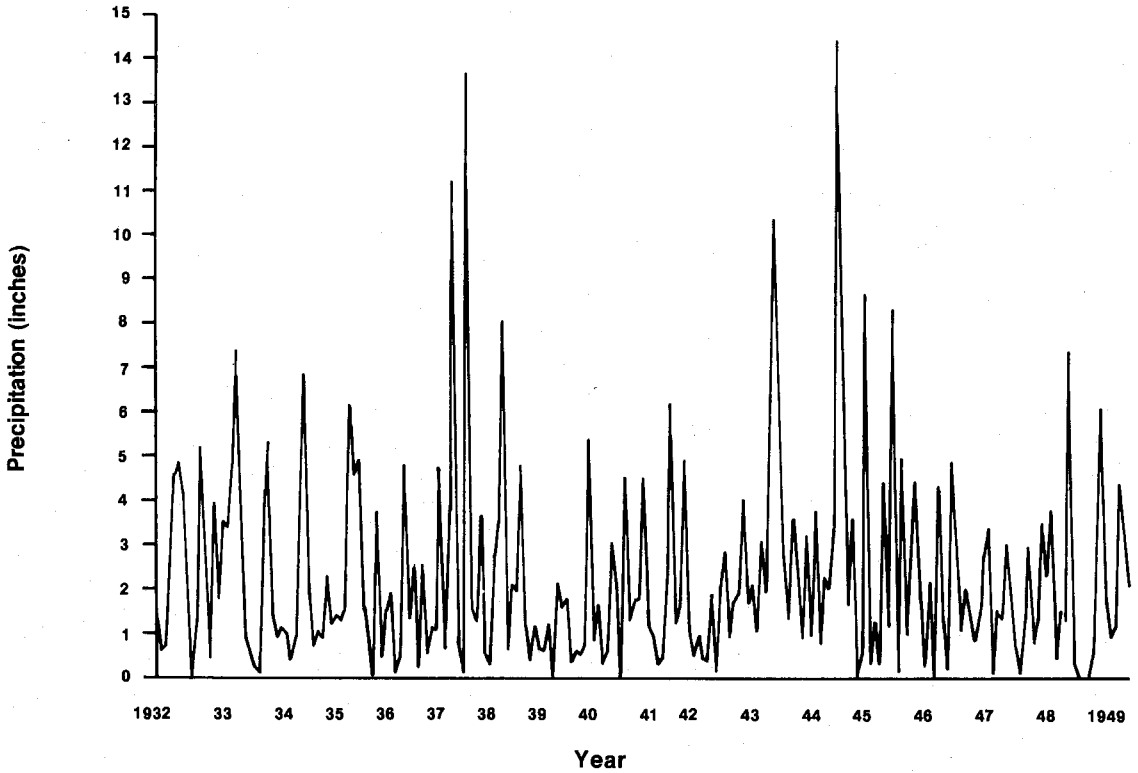


Figure 2. Alice, Texas, precipitation data, 1932-49

solved using the steepest descent method. A starting point was chosen based on the method of moments estimates and the equalities

$$E^{M(x)}[X] = E^*[E^X[X|\theta]], \text{ and}$$

$$\text{var}^{M(x)}[X] = E^*[\text{var}^X[X|\theta]] + \text{var}^*[E^X[X|\theta]].$$

Letting the left-hand side of the above equations equal  $\bar{x}$  and  $s^2$ , respectively, and computing the right-hand side (equalling  $\beta/(\alpha - 1)$  and  $\alpha\beta^2/(\alpha - 1)^2(\alpha - 2)$ , respectively).

The starting point has  $\alpha_0 = 2s^2/\bar{x}^2 - (\bar{x})^2$ ,  $\beta_0 = \bar{x}(\alpha_0 - 1)$ . Hence  $(\alpha_0, \beta_0) = (11.29, 843.47)$ . The MLE  $(\hat{\alpha}, \hat{\beta})$  was found to be (2.42, 131.11).

The predicted value for time until the next drought is then  $\hat{\beta}/\hat{\alpha} - 1 \approx 91.78$  months. Our Bayes-credible region for the time until the next drought (given  $k = .1$ ) is (0, 207.98).

The hypothesis  $H_0: \theta \in [1/84, \infty)$  versus  $H_1: \theta \in (0, 1/84)$  was tested after observing  $\bar{x}_{n+1} = 120$  using  $\hat{\alpha}$  and  $\hat{\beta}$  above to find that the expression in (6) of the appendix is .52 and that in (8) is 1.09.  $H_0$  was rejected since  $.52 > .50$ .

### Classical Competitors

Two classical estimators for  $E^{f(x|\theta)}[X]$  were developed. One was obtained directly by transformation, the other was derived by invoking a central limit theorem for i.i.d.-random variables. The following are derivations of each.

Since time between droughts,  $X \sim \exp(\theta)$ , it can be shown that  $\sum_{i=1}^n X_i \sim G(n, \theta)$ . Furthermore, since  $2\theta \sum_{i=1}^n X_i \sim \chi_{2n}^2$ , we have the probability statement  $P_{\theta_0}[2\theta \sum_{i=1}^n X_i \geq \chi_{2n, 1-a}^2] = a$ , where  $a$  is the test. Hypothesizing  $H_0: \theta \in [1/84, \infty)$  versus  $H_1: \theta \in (0, 1/84)$  and noting that  $f(x|\theta)$  is of the form  $\exp\{c(\theta)T(x) + d(\theta) + S(x)\}$ , the test that rejects  $H_0$  when  $2\theta \sum_{i=1}^n X_i \leq \chi_{2n, a}^2$  is uniformly most powerful among all tests of size  $a$  (Kendall and Stuart). Since this is a composite hypothesis, we select the null-space parameter such that the probability of rejecting the  $H_0$  (under  $H_0$ ) was greatest, hence  $\theta_0 = 1/84$ . With  $n = 7$ ,  $\sum_{i=1}^n X_i = 574$ , and  $2\theta \sum_{i=1}^n X_i = 13.67 > 7.79 = \chi_{14, .1}^2$ , hence we fail to reject  $H_0$  and conclude that  $\theta \in [1/84, \infty)$  at  $a = .1$ . Thus  $E^{f(x|\theta)}[X_8] = 84$ . The confidence statement is  $P_{\theta_0}[1 \leq 2\theta \sum_{i=1}^n X_i \leq u] = 1 - a$ ,

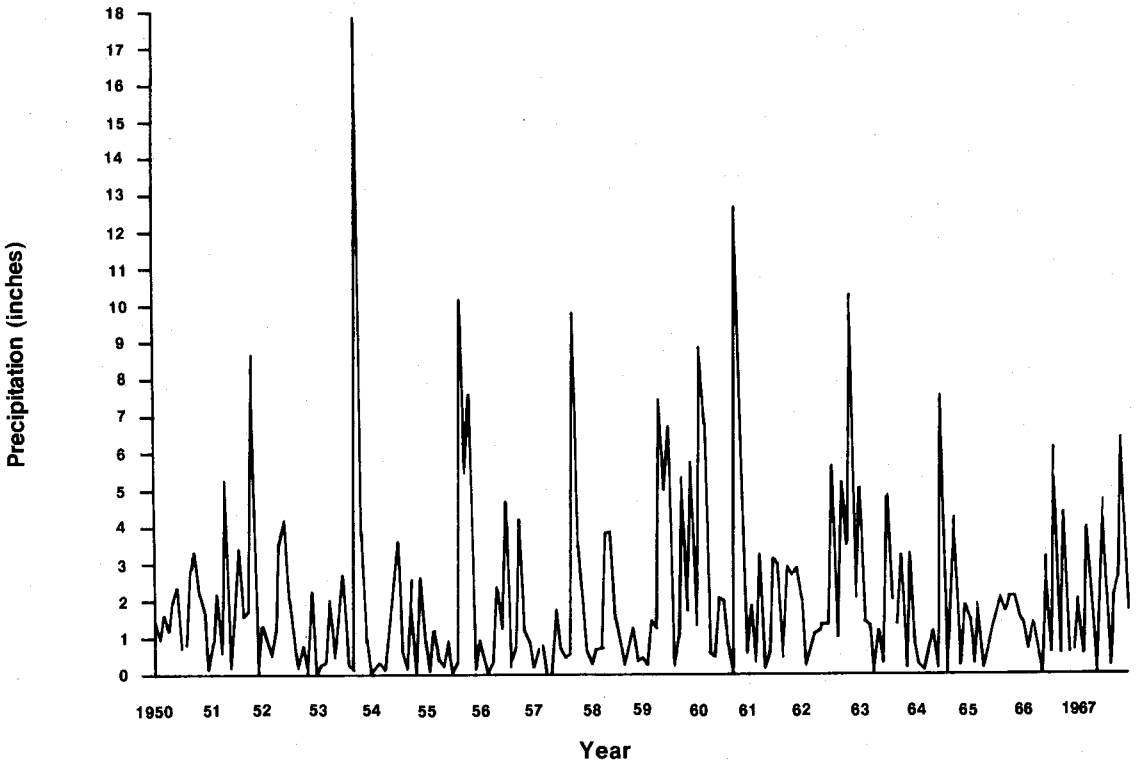


Figure 3. Alice, Texas, precipitation data, 1950-67

where  $l, u$  are respectively lower and upper confidence limits—we desire an interval of shortest length. The minimum-length confidence interval (Mood, Graybill, and Boes) is  $[5.60, 22.05]/2(574) \approx [1/206, 1/52]$ . The second estimator was the MLE for  $\theta, 1/\bar{x}$ . Since  $\{X_i\}_{i=1}^n$  were independent  $\exp(\theta)$  with mean  $1/\theta$  and variance  $1/\theta^2 < \infty$ , we can invoke the

Lindeberg-Levy CLT; viz.  $\sqrt{n}(\bar{x} - 1/\theta) \xrightarrow{d} N(0, 1/\theta^2)$ . Letting  $\hat{\theta} = 1/\bar{x} = h(\bar{x})$ ,  $\text{var}[\bar{x}] = \sigma_n^2 = 1/n\theta^2 \xrightarrow{n \rightarrow \infty} 0$  hence (Serfling, Thm. 3.1.A),  $\{h(\bar{x}) - h(\mu)\}/\sigma h'(\mu) \xrightarrow{d} N(0, 1)$ ,  $h'(\mu) = 0$  we have the statement  $P_\theta[\sqrt{n}(1/\bar{x} - \theta)/\theta \leq -1.28] = \Phi(-1.28) = .1$  where  $\Phi(z_1)$  is the normal cumulant for the quantile  $z_1$ . Again testing  $H_0: \theta \in [1/84, \infty)$  versus  $H_a: \theta \in (0, 1/84)$  with  $n = 7$  and  $\bar{x} = 82$ , our test statistic was  $\sqrt{7}(1/82 - 1/84)(84) \approx -.064$ ; leading us to fail to reject  $H_0$  ( $\alpha = .1$ ) and conclude that  $E^{X(x|\theta)}[X_8] = 84$ . Since MLEs are asymptotically efficient,  $H_0$  was tested using  $\sigma_n = 1/\sqrt{\{nE_{\theta_0}[(\partial \log f(x|\theta)/\partial \theta)]\}} = \theta/\sqrt{n}$  in lieu of that given above the test failed to reject  $H_0$  (the test statistic was also  $-.064$ ); however a conclusion based upon

this test is inappropriate due to the “small” sample size. Since the normal distribution is symmetric, the minimum length confidence limits were equal in absolute value. The interval obtained from the region  $[z_{-.95} \leq \sqrt{7}(1/82 - \theta)/\theta \leq z_{.95}]$  was  $[1/132.98, 1/31.02]$ .

**Bayes versus Classical Estimators**

Comparing  $E^{M(x)}[X_8]$  with both estimates,  $E^{X(x|\theta)}[X_8]$ , the first was found to be 91.8 months and both that obtained from the chi-squared test and the MLE to yield 84 months and 82 months, respectively. The next drought was in 1984; a period of about 120 months had elapsed until its occurrence. Hence, the Bayesian prediction was closer than either of the traditionally developed estimates. The credible region,  $[0, 207.28]$ , was wider than both the confidence interval of the  $\chi^2$  estimate  $[52, 206]$  and that of the MLE  $[31.02, 132.98]$ . This is due to the final precision of the Bayes HPD region which reflects uncertainty associated with the estimation of parameters in the marginal distribution. In fact, only in the case of the prior and conditional distribution (given  $\theta$ ) being

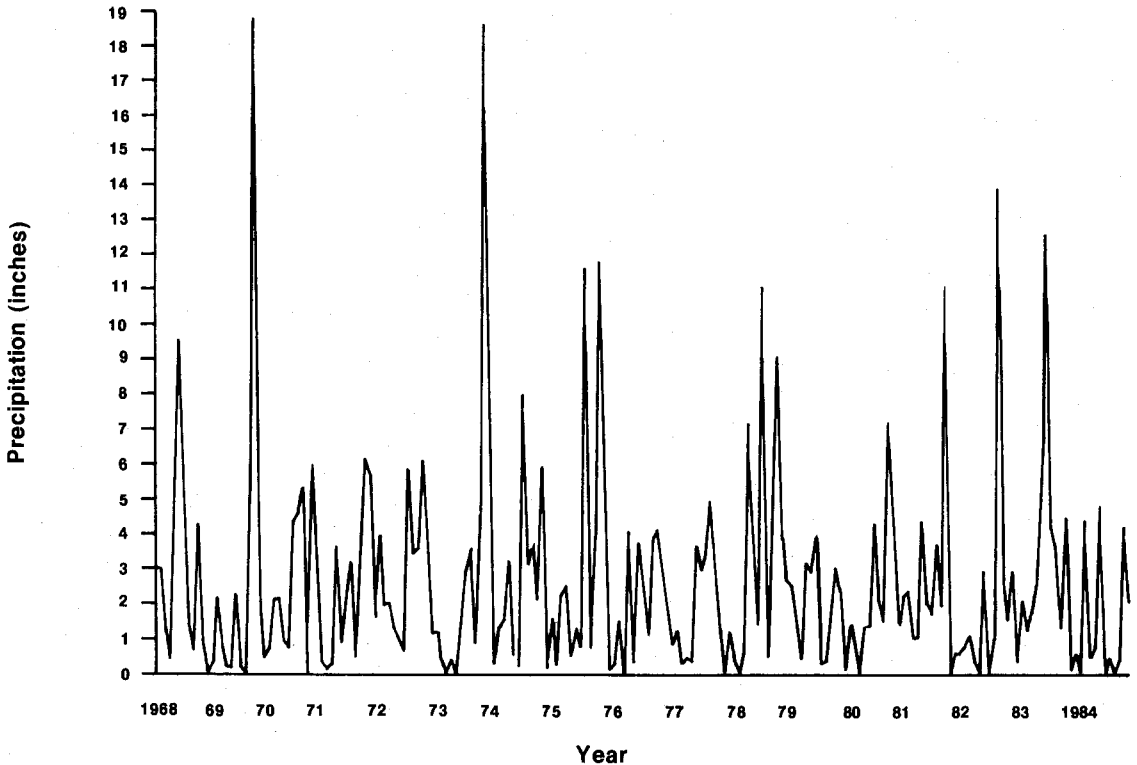


Figure 4. Alice, Texas, precipitation data, 1968-84

normal do the classical confidence regions agree with the Bayes HPD region.

### Conclusion

This paper has been an attempt to illustrate the methodological development and empirical formulation of Bayesian posterior, unconditional distributions (with respect to  $\theta$ ), and the distributions that are conditional on  $\theta$ . The utility of this approach in prediction of inter-drought times was also exhibited.

The classical estimates were then obtained, one by hypothesis testing and under direct distribution failing to reject  $H_0: \theta = 1/84$ , the second from the maximum likelihood method. The empirical Bayes prediction of  $X_8$  was closer to the actual than either of the classical approaches (under squared-error loss). This may be indicative of both the Bayes' ability to improve estimation (by the approximate prior density) and/or the inability of hypothesis testing and maximum likelihood estimation to provide adequate results for few data points (Bessler and Chamberlain).

The empirical Bayes result could be consid-

ered sensitive to the extreme values in data since the HPD region was wider than either (classical) confidence region (to reflect uncertainty arising from parameter estimation). Also the Bayes approach resulted in the rejection of  $H_0: \theta_8 \in [1/84, \infty)$  and concluding  $\theta_8 \in (0, 1/84)$ , while the other approaches led to failing to reject  $H_0: \theta \in [1/84, \infty)$ . This is an important illustration of the final precision of testing under the empirical Bayes framework [discussed in (b) in the introduction].

A potential extension of modeling inter-drought time is to consider the conditional distribution of time until the next drought, given that time  $t \geq t_p$ , where  $t_p$  is the lapsed time since the last drought. This would enable one to update the estimate (time until the next drought) while being in an interdrought period.

Caution is advisable when selecting both prior and conditional distribution, and one should ascertain that the foregoing choices of distributions are consistent with a priori knowledge. Furthermore, one should examine the means, variances, etc., of each for the necessity of theoretical or logical constraints on parameter spaces.

In summary, the empirical Bayes approach to prediction may be a viable alternative to classical estimators. Consideration of this approach may be particularly useful when data are not abundant.

[Received April 1986; final revision received March 1988.]

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**Appendix**

The posterior distribution is defined as

$$(1) \quad \pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{\int_0^\infty f(x|\theta) dF^{*(\theta)}} \quad \chi \in (0, \infty), \quad \theta \in \Theta,$$

$$\int_0^\infty f(x|\theta) dF^{*(\theta)} = \sum_0 f(x|\theta)\pi(\theta), \quad \theta \text{ discretely distributed}$$

$$\int_0^\infty f(x|\theta)\pi(\theta) d\theta, \quad \theta \text{ continuously distributed, henceforth the latter will be used.}$$

The realizations  $(\theta_1, x_1), (\theta_2, x_2), \dots, (\theta_n, x_n)$  are i.i.d. from a bivariate density

$$(2) \quad h(x, \theta) = f(x|\theta)\pi(\theta), \quad \theta > 0, \quad x > 0.$$

The marginal distribution of  $X_1, \dots, X_n$  is

$$M(x) = \int_0^\infty \dots \int_0^\infty h(x, \theta) d\theta_1 \dots d\theta_n = \prod_{i=1}^n \int_0^\infty f(x_i|\theta_i)\pi(\theta_i) d\theta_i, \quad \text{for } x \in (0, \infty) \times \dots \times (0, \infty).$$

Recall that  $\theta_i$ 's are unknown and  $x_i$ 's are observed.

We have

$$(3) \quad M(x) = \prod_{i=1}^n \int_0^\infty \frac{\beta^\alpha \theta_i^\alpha e^{-\theta_i(\beta+x_i)}}{\Gamma(\alpha)} d\theta_i$$

$$= \prod_{i=1}^n \frac{\alpha \beta^\alpha}{(\beta + x_i)^{\alpha+1}} \int_0^\infty \frac{(\beta + x_i)^{\alpha+1} \theta_i^\alpha e^{-\theta_i(\beta+x_i)}}{\Gamma(\alpha + 1)} d\theta_i$$

$$= \prod_{i=1}^n \frac{\alpha \beta^\alpha}{(\beta + x_i)^{\alpha+1}}.$$

Because the empirical Bayes procedure is to estimate the parameters of the prior (whose parameters are also parameters of the marginal distribution of  $X$ ) instead of to specify them, as with the classical Bayes approach, using the marginal

distribution (Cox and Hinkley), we will find the MLEs  $\hat{\alpha}, \hat{\beta}$ . Hence, we obtain  $\alpha, \beta$  such that

$$M(x, \hat{\alpha}, \hat{\beta}) = \text{Sup}_{(\alpha, \beta) \in (0, \infty) \times (0, \infty)} M(x).$$

The log likelihood equations are

$$(4) \quad \frac{\partial \text{Ln } M(x)}{\partial \alpha} = \frac{n}{\alpha} + n \text{Ln} |\beta| - \sum_{i=1}^n \text{Ln} |\beta + x_i| = 0,$$

$$(5) \quad \frac{\partial \text{Ln } M(x)}{\partial \beta} = \frac{n\alpha}{\beta} - (\alpha + 1) \sum_{i=1}^n \frac{1}{(\beta + x_i)} = 0.$$

Since a closed-form expression for  $\hat{\alpha}, \hat{\beta}$  cannot be achieved, we must use a numerical algorithm. We desire to predict  $X_{n+1}$  given  $X_1, \dots, X_n$ . Since  $\theta_{n+1}$  is not known, under squared-error loss, actually  $E^{M(x)}[X_{n+1}]$  is needed (since  $X_i$  are independent). Using

$$M(x_{n+1}) = \frac{\alpha\beta^\alpha}{(\beta + x_{n+1})^{\alpha+1}}, \quad E^{M(x)}[X_{n+1}] = \frac{\beta}{\alpha - 1},$$

hence

$$E^{M(x)}[X_{n+1}] = \frac{\hat{\beta}}{\hat{\alpha} - 1} \quad (\alpha > 1).$$

A prediction interval for  $X_{n+1}$  in the Bayesian context is called a Bayes credible region for  $X_{n+1}$  (Berger). It is the interval  $[a, b]$  such that  $\int_a^b \hat{M}(x) dx \geq 1 - K$ , where  $K$  is the probability of  $X$  not being in the (fixed) interval  $[a, b]$ . We desire to minimize interval length while maintaining the inequality. Hence, we desire to minimize length,  $L = b - a$  subject to  $\int_a^b \hat{M}(x) dx \geq 1 - K$ . Since  $M(x)$  is declining for any fixed interval  $[a, b]$ , interval length is minimized for the interval  $(0, b]$ , where  $b$  is implicitly given by  $\frac{\hat{\beta}^{\hat{\alpha}}}{(\hat{\beta} + b)^{\hat{\alpha}}} - K = 0$ ; therefore,  $b = \frac{\hat{\beta}}{K^{1/\hat{\alpha}}} - \hat{\beta}$ .

Based on expert opinion, it was also of interest to find the region which would most likely include  $\theta_{n+1}$ . We hypothesized as follows:  $H_0: \theta \in [1/84, \infty)$ ,  $H_a: \theta \in (0, 1/84)$  hence,  $\theta$  was partitioned such that  $\Theta = \Theta_0 \cup \Theta_1$ . We would reject  $H_0$  if the posterior "odds" ratio

$$(6) \quad \frac{\int_{\theta_0} dF^{\pi(\theta|x)}}{\int_{\theta_1} dF^{\pi(\theta|x)}}$$

was less than some prespecified value (i.e., such as 1). Since  $x_{n+1}$  was not known, we use  $E^{M(x)}[X_{n+1}]$  for  $x_{n+1}$ . The posterior was

$$(7) \quad \hat{\pi}(\theta_{n+1} | x_{n+1}) \propto \frac{\hat{\beta}^{\hat{\alpha}} \theta_{n+1}^{\hat{\alpha}} e^{-\theta(x_{n+1} + \hat{\beta})}}{\Gamma(\hat{\alpha})} \propto \frac{(\hat{\beta} + \hat{x}_{n+1})^{\hat{\alpha} + 1} \theta_{n+1}^{\hat{\alpha}} e^{-\theta(\hat{\beta} + \hat{x}_{n+1})}}{\Gamma(\hat{\alpha} + 1)},$$

where  $\hat{x}_{n+1} = E^{M(x)}[X_{n+1}]$ . Furthermore, one may consider the relative losses of misspecifying the alternative intervals in which  $\theta_{n+1}$  is realized:

$$(8) \quad \frac{\int_{\theta_0} (\theta_{n+1} - \hat{\theta}_{n+1})^2 dF^{\pi(\theta_{n+1} | E^{M(x)}[X_{n+1}])}}{\int_{\theta_1} (\theta_{n+1} - \hat{\theta}_{n+1})^2 dF^{\pi(\theta_{n+1} | E^{M(x)}[X_{n+1}])}},$$

where  $\hat{\theta}_{n+1} = E^{\pi(\theta_{n+1} | \hat{x}_{n+1})}[\theta_{n+1}] = \frac{\hat{\alpha} + 1}{\hat{\beta} + \hat{x}_{n+1}}$  and reject  $H_0$  for suitable values this ratio (i.e., sufficiently greater than 1).