Choice of Flexible Functional Forms: Review and Appraisal

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Choice between alternative flexible functional forms has received little explicit treatment in many empirical agricultural studies. Theoretical criteria and empirical techniques for choosing between flexible functional forms are reviewed. Theoretical topics include definitions of flexibility, mathematical expansions, separability, and regular regions. Empirical techniques examined are Monte Carlo analysis, parametric modeling, bayesian inference, and nonnested hypothesis testing. Comparison of the full range of theoretical and empirical aspects may provide more credible and reliable empirical estimates when consumer or producer duality assumptions are appropriate in agricultural applications.

Key words: duality, flexible functional forms, mathematical expansions, nonparametric tests.

Choice of functional form has been a pressing issue for empirical production and consumer studies since the pioneering work of Douglas (Cobb and Douglas, Douglas) and Stone. Tests of the classical theory of the firm based on the restrictive Cobb-Douglas production function have been thoroughly criticized (Samuelson, Simon). The development of flexible functional forms was driven by the search for functional forms which imposed fewer maintained hypotheses. The econometric limitations of the Cobb-Douglas functions (Hoch), for example, provided impetus for derivation of the CES and other functional forms (Zellner and Revankar). With the subsequent formalization of the notion of flexibility (Diewert 1971), a large set of flexible functional forms (FFF) has become available to the empirical researcher (see appendix).

In some empirical studies, the reasons for choosing a particular flexible functional form are not explicitly stated. Recent agricultural production duality applications of FFF, for example, have not addressed in detail the issue of choice among alternative FFF (Antle; Lopez 1980; Shumway; Sidhu and Baanante; Weaver

1983). Advocating that the choice of functional form should be treated explicitly in empirical research, Griffin, Montgomery, and Rister have identified and evaluated criteria for choosing between competing functional forms in production function analysis.

Following the Griffin, Montgomery, and Rister prescription for treating the choice of functional form explicitly, the focus of this paper is on choice of FFF in producer and consumer duality settings where cost, profit, or indirect utility functions or systems of equations derived from these functions are to be estimated. The scope of the paper is limited to FFF because of their recent popularity in applied studies which use duality theory. Theoretical characteristics of FFF are first discussed, compared, and assessed. Empirical techniques for choosing among FFF are then reviewed and appraised. Conclusions regarding the application of an empirical procedure for choosing among FFF follow.

Pros and Cons of FFF

Duality theory advances have ushered in the widespread use of FFF for a number of reasons. First, with the satisfaction of regularity conditions such as convexity (concavity), monotonicity, and homogeneity, duality re-

sults preclude the need for self-dual functions. Further, the use of derivative properties—Hotelling's and Shephard's lemmas and Roy's Identity—allows for derivation of demand and supply (or share) functions without solving analytically for those functions. Comparative statics are easily derived from the properties of the parent indirect functions. Finally, FFF have gained popularity because of the enhanced capacity of nonlinear estimation procedures for nonlinear-in-parameters equation systems.[1]

Empirical use of FFF has certain drawbacks: collinearity due to numerous terms involving transformations of the same variables and interaction among variables; failure to satisfy the regularity conditions over the entire range of sample observations; and, less important, difficulty in interpreting initial parameter estimates. Estimation of nonlinear-in-parameters systems may involve problems with convergence and statistical theory (Lau 1986), while interpretation of FFF as approximations to arbitrary functions also may cause bias from an estimation standpoint (White, Byron and Bera).

Difficulties in Comparing FFF

Aside from the relative advantages and disadvantages of FFF, choices between alternative FFF are seldom based on a comparison of a full range of theoretical and empirical criteria. Systematic comparison of FFF is beset by many offsetting and sometimes conflicting theoretical and empirical criteria. Consider, for example, the conflict between parameter parsimony (Fuss, McFadden, and Mundlak) and order of expansion. A third-order expansion of the utility function permits empirical tests of propositions relating to partial strong separability, whereas second-order expansions yield ambiguous test results (Hayes). Yet the number of estimated parameters for a third-order, translog indirect utility function could be intractably large for all but the case of a few goods. Thus, the theoretically desirable ability to test more generalized notions of separability may result in empirically undesirable phenomena such as collinearity, reduced degrees of

freedom, and difficulty in interpreting individual parameter estimates.

Evaluation and comparison of FFF are complicated further by use of FFF in consumer and producer applications. Most empirical criteria, such as parameter parsimony and ease of interpretation, may not differ across applications; however, theoretical criteria are not always consistent across consumer and producer applications. For example, in production applications an FFF which implies that variable inputs are used when no output is produced may not be acceptable if no production lags are posited. Yet the same FFF may be employed in consumer theory to assure consistent aggregation across households (Lopez 1985, p. 596). Hence, an FFF which may be restrictive in the producer context can be useful for consumer applications.

Although no single FFF is unequivocally superior with respect to all theoretical and empirical criteria, systematic consideration of the relative advantages of each may provide more compelling grounds for choosing functional forms. The list of theoretical and empirical criteria discussed in the following sections is not exhaustive, but it is a collection of various criteria which appear not to have been considered together. The criteria can serve as a checklist for the applied researcher to consider in light of the particular empirical problem to be analyzed. Note that FFF have been utilized almost exclusively in duality applications. Hence, the theoretical criteria are discussed without restricting the implications solely to consumer or producer duality.

The fourteen FFF selected for comparison appear in the appendix. The generalized Leontief and translog have been the most often used FFF in empirical studies. The quadratic mean of order rho, the generalized Cobb-Douglas, and the generalized square root quadratic may be categorized as embellishments in the spirit of second-order approximations. The minflex Laurent forms and the Fourier form were introduced more recently in the literature to deal with difficulties in approximation and estimation. The generalized McFadden, generalized Barnett, and generalized Fuss functions are the most recent additions to the FFF menu which were proposed for their ease in imposing global curvature conditions. Although some of the more recent FFF nest the old FFF as special cases, the newer

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[1] Nonlinear-in-parameters equation systems are common in consumer analysis. Nonlinear systems are less common in production applications. (See Just, Zilberman, and Hochman for a production example.)

forms have been proposed to deal with deficiencies in the older FFF.

Theoretical Criteria

Theoretical criteria for choosing among FFF are distinguished by their *ex ante* role in the choice of functional form (Lau 1986). In econometric parlance, theoretical criteria are a source of a priori restrictions regarding choice of functional or algebraic forms for estimation. Although microeconomic theory is nearly always augmented by the researcher's familiarity with the empirical application in forming a priori restrictions on functional form, theoretical criteria are discussed per se in the following sections for expository purposes.

Definitions of Flexibility

The most fundamental comparison of FFF can be made according to the two commonly used definitions of flexibility. Griffin, Montgomery, and Rister offer a comprehensive treatment of flexibility criteria. The following brief discussion of flexibility definitions is intended to highlight the differences between notions of local and global flexibility. Diewert (1971) formalized the notion of flexibility in functional forms by defining a second-order approximation to an arbitrary function. In descriptive terms, Diewert's definition of a flexible functional form requires that an FFF have parameter values such that the FFF and its first- and second-order derivatives are equal to the arbitrary function and its first- and second-order derivatives, respectively, for any particular point in the domain. Thus, Diewert's flexibility definition refers to a local property. Any FFF which satisfies this definition may be denoted Diewert-flexible. Of course, the derivative function of any indirect function (indirect utility, cost, or profit function) will be approximated only up to its first derivative evaluated at any point (Chambers).

Gallant (1981) has proposed the Sobolev norm as a more attractive measure of flexibility than Diewert's definition. The appeal of the Sobolev norm is due to its measure of average error of approximation over a chosen order of derivatives. Sobolev-flexibility is a global property, and any functional form displaying this property will yield elasticities closely approximating the true ones (Elbadawi, Gallant, and Souza).

Sobolev-flexibility is attractive because it confers on the empirical model nonparametric properties: (*a*) small average bias approximations (Gallant 1981); (*b*) consistent estimators of substitution elasticities (Elbadawi, Gallant, and Souza); and (*c*) asymptotically size α testing procedures (Gallant 1982). Gallant maintains that Sobolev-flexibility asymptotically removes the augmenting hypothesis that the true model be a member of the family of models used in the approximation analysis because the Fourier form has desirable nonparametric properties.

In mathematical and statistical terms, Sobolev-flexibility appears to be a more attractive criterion than Diewert-flexibility. Yet the relative complexity of specifying and estimating a Fourier form has probably contributed to its use in relatively few applications.[2] For estimating the Fourier, theoretical issues regarding choice of sample size rules for specifying the number of parameters to estimate as well as order of expansion are not clearly settled. Difficulties in the calculation of standard errors for Fourier parameters also may cause some reluctance to use the Fourier form. Thus Diewert-flexibility is the more widely applied definition primarily because of the ease in using FFF which satisfy Diewert's definition (column 1 of table 1).

Mathematical Expansions

Comparison of many FFF may be made on the basis of the class of mathematical expansions from which they are derived. None of the definitions of flexibility limits FFF to functional forms derived from an underlying mathematical expansion. However, many widely used FFF may be treated as mathematical expansions about some arbitrary point.

The Taylor, Laurent, and Fourier expansions each have been used to derive FFF. The generalized Leontief, normalized quadratic, and the transcendental logarithmic (translog) functional forms may be interpreted as second-order Taylor-series expansions about different

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[2] Gallant, Chalfant and Gallant, Chalfant, Wohlgenant (1983, 1984) and Ewis and Fisher, appear to be the only readily accessible studies during the period 1980 to 1985 which utilize the Fourier form.

Table 1. Theoretical Properties of Flexible Functional Forms

Flexible Functional Form[a]	(1) Definition of Flexibility	(2) Type of Expansion	(3) Minimum Number of Parameters for Flexibility	(4) Separability Approximation	(5) Separability True	(6) Global Regular Regions[h]
1) Normalized quadratic	Diewert	Taylor	Yes	$WS \geqq SS$[e]	$SS\ (LFFF)$[g]	Leontief
2) Generalized Leontief	Diewert	Taylor	Yes	$WS \geqq SS$	$SS\ (LFFF)$	Cobb-Douglas
3) Translog	Diewert	Taylor	Yes	$WS \geqq SS$	$SS\ (NLFFF)$	Leontief
4) Quadratic mean of order rho	Unknown	Taylor	Yes	$WS \geqq SS$	$SS\ (LFFF)$	Cobb-Douglas
5) Generalized Cobb-Douglas	Unknown	None	Yes	Strongly separable		Cobb-Douglas
6) Generalized square root quadratic	Unknown	Taylor[b]	Yes			Leontief
7) Generalized Box-Cox	Unknown	Taylor	Yes			
8) Generalized McFadden	Diewert	None	No			May be imposed
9) Generalized Barnett	"Quasi-Diewert"	None	Yes			May be imposed
10) Generalized Fuss	Diewert	None	Yes			
11) Biquadratic	Diewert	Laurent[c]	Yes			May be imposed
12) Minflex Laurent generalized Leontief	Diewert	Laurent	Yes[d]			
13) Minflex Laurent translog	Diewert	Laurent	Yes			
14) Fourier	Diewert/Sobolev	Fourier	Not applicable	Homothetic separable[f]		May be imposed

[a] See appendix for the notation of each functional form.
[b] For suitable parameter values, the FFF collapses to functional forms which can be derived from Taylor-series expansions.
[c] Minflex Laurent functional forms are special cases of the full Laurent expansion.
[d] Minflex Laurent functional forms have the same degree of parametric freedom because the extra parameters are nonnegatively constrained.
[e] Weak separability (WS) implies strong separability (SS) for the case of more than two goods or inputs.
[f] Gallant (1981, sec. 3.2) has devised a statistical test for homothetic separability which is not an application of the usual parameter restrictions resulting from the differentiable separability definition $[(\partial f/\partial x_i)/(\partial f/\partial x_j)]\partial x_k$.
[g] Linear FFF ($LFFF$) are characterized by linear transformations of the dependent variable (Lopez). Nonlinear FFF ($NLFFF$) have nonlinear transformations of the dependent variable.
[h] Global regular regions indicate the special case of the functional form which is globally regular.

points with different transformations of the variables (Fuss, McFadden, and Mundlak; Blackorby, Primont, and Russell).

Recognizing deficiencies in the Taylor-series expansion as a generating function for FFF, Barnett proposed the Laurent expansion for which the remainder term varies less over the interval of convergence (Barnett 1983). The principal mathematical advantage of the Taylor-series expansion is that its remainder term converges to zero within the interval of convergence as the number of expansion terms increases. Although the Laurent remainder term may not converge to zero at any point within the interval of convergence, the Laurent remainder term varies less within the same interval than does the Taylor-series remainder term (for Laurent and Taylor series of the same fixed order of expansion).

The Fourier approximation, which provides a global rather than a local approximation, is the basis for a functional form proposed by Gallant (1981). Although the logarithmic Fourier form is composed of a second-order translog portion plus a trigonometric approximation, the Fourier offers fundamentally different mathematical properties. Other types of mathematical expansions, such as the Muntz-Szatz (Barnett and Jonas), have been developed theoretically but remain to be implemented empirically. The type of mathematical expression from which some FFF are derived is summarized in column 2 of table 1.

Although each type of expansion possesses some desirable characteristics, the Taylor-series expansion results in FFF which have fewer unconstrained parameters than do the Laurent or Fourier.[3] The minflex Laurents have the same degree of "parametric freedom" as the translog and generalized Leontief because the additional parameters are subject to inequality restrictions (Barnett and Lee). The Fourier functional form is not readily comparable because the number of estimated parameters must be chosen according to a sample size rule to assure consistent estimation (Elbadawi, Gallant, and Souza) (column 3 of table 1).

Solely on the basis of mathematical expansions, no particular FFF emerges as the most attractive. Both Taylor- and Laurent-series expansions are Diewert-flexible (Barnett 1983),

whereas the Fourier is Sobolev-flexible.[4] From a strictly mathematical standpoint, Laurent and Taylor series may not provide accurate numerical approximations if the data lie outside the interval of convergence. The Fourier, in contrast, provides a global approximation in the sense that it minimizes average bias across all data points. Regardless of the type of expansion, the order of the expansion required to obtain a "good" approximation is not known; truncation error is a possible source of error for Taylor-series, Laurent, and Fourier expansions alike (Weaver 1983, 1984). Accordingly, third-order functions have been advocated and estimated (Dalal, Hayes). Hence, the theoretical ability of each class of FFF to approximate satisfactorily—whether locally or globally—an arbitrary function is not guaranteed.

One caveat on comparing FFF in terms of mathematical expansions is that not all FFF are derived directly from a mathematical expansion. The generalized McFadden and generalized Barnett functions are not derived solely from an underlying mathematical expansion (Diewert and Wales). Neither of these two FFF can be directly compared in terms of expansion remainder terms even though each has been proved Diewert-flexible.

Approximations vis-à-vis True Functions

In empirical studies, FFF have been treated both as approximations to some unknown true function and as true functions despite the fact that Diewert-flexibility and Sobolev-flexibility are notions based on approximations. However, interpretation of FFF as approximations rather than as exact functions has generated criticism in three areas: estimation, hypothesis testing, and separability properties. The potential bias of estimating FFF parameters with ordinary least squares (OLS) on the basis of approximations made at a particular point has caused contention (White, Gallant 1981, Byron and Bera). Difficulties in making statistical inferences on the basis of approximations also occur in consumer demand applications (Hayes, Simmons and Weiserbs). Separability restrictions and tests for separability differ de-

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[3] The comparison of number of parameters is for second-order expansions.

[4] Barnett (1983) notes that Gallant's Fourier model has not been proved to satisfy Diewert's definition of flexibility. Gallant, however, asserts that the Fourier is Diewert-flexible in certain cases (1981, p. 220).

pending on whether FFF are treated as approximations or true functions.

Separability

The potential restrictions imposed by the separability properties deserve attention in empirical studies utilizing FFF. Whether FFF models are interpreted as exact functions or approximations, separability restrictions are essential for consistent aggregation, sequential optimization, and specification of marginal substitution relationships.

When common second-order FFF are interpreted as approximations to an unknown true function rather than as exact functions, Blackorby, Primont, and Russell (1977) have proved that such FFF are "separability-inflexible." For those second-order FFF, weak separability implies strong separability and, more important, their weakly separable forms (see column 4 of table 1) cannot provide second-order approximations about an arbitrary point. If the FFF is interpreted as an exact function, then testing for the existence of an aggregate input is also a test of homotheticity for the aggregator function.

For production applications, Lopez indicates that those second-order FFF which are distinguished by a linear transformation of the dependent variable, such as profit, imply a special type of additive separability and quasi-homotheticity restrictions (see column 5 of table 1). Even when weak separability is not imposed by these linear dependent variable FFF, the restrictions hold.[5] Nonlinear FFF, which are characterized by some nonlinear transformation of the dependent variable, do not impose these restrictions.

When FFF are considered as approximations to some arbitrary function characterized by particular properties, the implied separability restrictions are modified. For the second-order translog approximation to an arbitrary production function, Denny and Fuss prove that the tests for weak separability do not imply a test of strong separability. Hayes extends the precision of the separability tests by deriving them from a third-order translog utility function. A test for distinguishing between weak separability and general partial strong

separability is possible with Hayes' third-order model.

The separability properties of more recently developed FFF, such as the minflex Laurents, generalized Barnett, generalized McFadden (Diewert and Wales), generalized Fuss (Diewert and Ostensoe), and biquadratic (Diewert), have not been examined. The general quadratic function separability results from Blackorby, Primont, and Russell cannot be directly applied to all these newer FFF because they are not solely quadratic functions. Rather, some of these FFF may be considered as non-negative sums of concave functions.

Global Regularity Conditions

Satisfaction of regularity conditions provides another theoretical criterion for judging whether alternative FFF conform to the properties of microeconomic theory. All but one of the FFF considered here do not satisfy global convexity (concavity) conditions. Only the normalized quadratic is capable of satisfying global convexity (concavity) restrictions without additional constraints in estimation. The capability of different FFF to satisfy regularity conditions can be measured by regular regions. Regular regions of different FFF are calculated by fixing relevant elasticity values at some chosen level and then determining the parameter values for which the regularity conditions hold. In general, the larger is the FFF's regular region for a given elasticity of substitution, the more theoretically compatible is the FFF. If a priori information exists about the magnitude of the elasticities of substitution for a particular application, size of the regular regions provides a means for discriminating between FFF. More usefully, over a wide range of substitution elasticity values, a given FFF may have a larger regular region than other FFF.

In what follows, regularity conditions are referred to solely in terms of the indirect utility function because all of the previous studies have focused on this consumer case. Regularity conditions are (*a*) monotonicity and (*b*) quasi-convexity; homogeneity is customarily imposed. Barnett, Lee, and Wolfe (1985, 1987) have extended Caves and Christensen's original work to consider the three-good, nonhomothetic case for the generalized Leontief, translog, and minflex Laurent generalized Leontief and minflex Laurent translog indirect utility functions. The minflex Laurent models

[5] In profit function applications, quasi-homotheticity implies that the marginal rate of substitution between inputs is invariant to output level.

generally possess larger regular regions than either the generalized Leontief or translog.[6] The generalized Leontief and translog both have large regular regions in the neighborhood of their globally regular special cases—Leontief and Cobb-Douglas, respectively—but their regular regions diminish rapidly for elasticity values diverging from these special cases. The minflex translog tends to possess a larger regular region than the minflex generalized Leontief except in cases where there is limited substitutability between pairs of goods (Barnett, Lee, and Wolfe).

No studies have examined regular regions for the consumer case of more than three goods nor have production applications been explored. While the two-dimensional consumer cases have an intuitively appealing interpretation with indifference curves (see Caves and Christensen), generalization of the regular region notion to three-space requires volume measures which are envisioned less easily. Extensions to higher dimensions would require higher-order volume measures.

Empirical procedures can be employed to impose regularity conditions in the estimation of FFF parameters. Lau (1978a) demonstrated a procedure for imposing global concavity (convexity) conditions with econometric estimation. Lau also suggested that monotonicity may be imposed by squaring the appropriate parameters. Jorgenson and Fraumeni used Lau's method of imposing concavity by restricting the elements of the Cholesky factorization of the matrix of share elasticities derived from a price function. The imposition of concavity, however, resulted in setting a large number of the elasticities equal to zero. Moreover, imposing negative semidefiniteness on the matrix of own and cross price elasticities of the translog cost function can bias the resulting estimated elasticities (Diewert and Wales).

More recently, other procedures for imposing global concavity on FFF have been advanced by Gallant and Golub, and Diewert and Wales. Development of the generalized McFadden, generalized Barnett, and generalized Fuss forms by Diewert and Wales was motivated by the need to impose globally convexity (or concavity). The techniques of assuring satisfaction of global curvature condi-

tions may be classified as system estimation, possibly nonlinear, subject to restrictions.[7] Gallant and Golub suggest a two-stage optimization technique for imposing curvature conditions at every data point. Diewert and Wales propose a restricted estimation technique equivalent to Lau's restrictions on the Cholesky factorization of the relevant hessian matrix.[8]

Imposition of curvature restrictions for some FFF may provide more credible elasticity estimates over the entire range of the sample data. Clearly, theoretical primal-dual mappings may be invoked in empirical studies if the regularity—curvature, monotonicity, and homogeneity—conditions are imposed. From an estimation standpoint, more efficient parameter estimates are obtained from estimation subject to restrictions; imposition of erroneous constraints would result in biased or inconsistent estimates, however.

Review of Theoretical Criteria

The theoretical criteria enumerated could lead to contradictory conclusions in the choice between currently available functional forms. The difficulty in using the theoretical criteria is complicated insofar as FFF may be interpreted as approximations or true functions. A summary of the theoretical properties in table 1 indicates the potential for contradictory theoretical prescriptions.

From a mathematical standpoint, using well-behaved expansions, such as the Laurent and Fourier, provide more desirable approximations. If curvature conditions are also a consideration, a potentially fruitful development may be FFF which are the nonnegative sums of concave functions because these functions readily allow the imposition of global curvature conditions. However, the separability properties of the newer FFF—minflex Laurent translog and generalized Leontief, generalized McFadden, and generalized Barnett—have yet to be compared systematically. Thus, as might

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[6] The minflex Laurent also displays increasing regular region volume for trended time-series data.

[7] Hazilla and Kopp imposed regularity conditions by means of nonlinear restrictions on a long-run cost function. The restrictions were imposed only at the point of approximation, however, because the cost function was interpreted as an approximation to a true function.

[8] The Cholesky techniques proposed by Lau and by Diewert and Wales were used earlier by Wiley, Schmidt, and Bramble. The Cholesky decomposition converts a constrained linear estimation problem into an unconstrained nonlinear estimation problem.

be expected, no single FFF appears unambiguously to dominate all others on theoretical grounds. However, the relative theoretical strengths and weaknesses of each are apparent.

Empirical Criteria and Techniques

Empirical criteria for judging FFF are characterized by their *ex post* role in the choice of functional forms. Empirical criteria by definition are less general than theoretical criteria because of their application in a specific data setting; different empirical criteria can be applied in both consumer and production problems, but they are contingent upon the data analyzed. Thus, nearly all empirical criteria yield data-specific conclusions.

Empirical criteria for distinguishing between alternative FFF can be catalogued into four groups: (*a*) Monte Carlo studies; (*b*) parametric models; (*c*) bayesian analysis; and (*d*) nonnested hypothesis testing. In Monte Carlo studies, the ability of an FFF to approximate a known underlying technology or preference mapping is measured. In the latter three cases, however, data-generated measures are used to compare competing functional forms where the underlying function is unknown. Knowledge of the data generating process in Monte Carlo studies permits more hypothetical consideration of the approximating abilities of the FFF than the data-generated criteria. Monte Carlo results may provide some *ex ante* indication of an FFF's comparative strengths and weaknesses, given various data sets.

Monte Carlo Studies

In the first application of Monte Carlo techniques for assessing the approximation capabilities of FFF, Wales compared the ability of translog and generalized Leontief reciprocal indirect utility functions to approximate a homothetic two-good CES utility function. The theoretical results derived from examining regular regions were corroborated: the translog performed well when substitution elasticities were near unity, whereas the generalized Leontief better approximated substitution elasticities near zero. Both functional forms violated regularity conditions—quasi-concavity and monotonicity—even though they fit the data well in terms of R^2 and closely estimated the true substitution elasticities.

Guilkey and Lovell, and Guilkey and Sickles extended the Monte Carlo technique to measure approximation of true substitution elasticities, economies of scale, and single *vis-à-vis* systems estimation techniques. In the second study, the single output, three-input cost function was used to compare the translog, generalized Leontief, and the generalized Cobb-Douglas. Generally, the translog dominated the other two forms, although Guilkey and Sickles stressed that the better performance of the translog did not imply that it was acceptable in all instances. Furthermore, the deviation of the estimated substitution elasticities from the true elasticities as measured by bias and mean absolute deviation was not substantially different for the three functional forms in some cases. In more complex cases with diverging true partial substitution elasticities and complementarity, the systems estimator (Zellner's iterative seemingly unrelated regressions) was preferred.

A recent Monte Carlo study by Chalfant and Gallant assessed the ability of the logarithmic Fourier functional form to approximate substitution elasticities generated by a three-input, homothetic generalized Box-Cox cost function. The logarithmic Fourier, which nests the translog as a special case, approximated elasticities of substitution with little bias. Experiments with different sample sizes suggested that the measured bias was due to errors-in-variables, not to specification bias. Hence, for purposes of testing economic theory, the Fourier appears to be the least ambiguous form for statistical inference.

Dixon, Garcia, and Anderson conducted Monte Carlo simulations to evaluate the usefulness of pretests in testing the behavioral assumptions and regularity conditions for the translog and generalized Leontief profit functions. While concluding that such pretests are generally not useful validation tools, they noted difficulty in choosing between the translog and generalized Leontief functional forms. The generalized Leontief consistently underestimated substitution elasticities, while the translog elasticity estimates displayed extreme variations about the true means.

The Monte Carlo results discussed are specific to the data generated and the microeconomic context analyzed. Both consumer (indirect utility function) and production (cost and profit function) applications have been considered. Thus, the results of the studies are

not directly comparable on economic grounds. Experimental design differs considerably across these Monte Carlo studies: data generation, sample size, number of replications, and stochastic disturbance specifications vary throughout.

Fruitful ground for future studies lies in the comparison of a wider range of FFF, such as the generalized Box-Cox, minflex Laurent, Fourier, and nonnegatively summed concave functions while varying assumptions regarding data generation and sample size.

Parametric Modeling

Parametric methods have been proposed to assess the plausibility of different functional forms in fitting actual data. The generalized Box-Cox function has been used for parametric testing; with suitable parameter restrictions, the generalized Box-Cox nests the translog, generalized Leontief, and the generalized square-root quadratic functions as special cases (see Griffin, Montgomery, and Rister). Hence, rather than estimating any one of the special cases with the functional form as a maintained hypothesis, the functional form can be tested through hypothesis tests of the appropriate parameter restrictions.

Comparison of the parametric test results across studies using different data sets indicates that many commonly used functional forms are rejected. Using updated production data from Berndt and Christensen, Appelbaum rejected the translog and generalized Leontief in primal and dual share equation models but failed to reject the generalized square root quadratic in the dual setting. Using similar production data from Berndt and Wood, Berndt and Khaled could not reject the generalized Leontief but rejected the generalized square root quadratic and probably rejected the translog for approximating the cost function.[9] Using aggregate agricultural input data, Chalfant rejected all three functional forms in a cost function context.

The primary limitation of the generalized Box-Cox as previously formulated is that para-metric tests can only discriminate among a subset of FFF. Laurent, Fourier, generalized McFadden, and generalized Barnett functions cannot be tested as special cases of the generalized Box-Cox.[10] Further, the usual caveats regarding power of hypothesis tests are applicable with the use of the generalized Box-Cox model.

Bayesian Analysis

An alternative to parametric testing is bayesian analysis by which a posteriori comparisons of FFF can be made. The attraction of this method is that it allows comparison of fundamentally different models on the basis of actual data; nonnested models can be compared on the basis of diffuse priors.

In a consumer application, Berndt, Darrough, and Diewert compared the translog, generalized Leontief, and generalized Cobb-Douglas in estimating market demand shares for Canadian consumption data. With and without symmetry restrictions imposed, the translog was preferred a posteriori while the generalized Leontief and generalized Cobb-Douglas had nearly identical log likelihood values in both cases. For U.S. time-series food consumption data, Wohlgenant estimated two-good demand functions with the generalized Leontief, translog, and Fourier forms. The Fourier form dominated the translog and generalized Leontief forms with posterior odds ratio of 9.83:1 and 41.95:1, respectively. The Fourier also performed favorably when own-price and income elasticities of demand were compared at each sample point. In a production study using the Berndt and Wood data on aggregate U.S. manufacturing, Rossi compared the translog and logarithmic Fourier for a three-input cost share system exclusive of the cost function. Two methods to calculate the posterior odds ratio were used, and the logarithmic Fourier was preferred by a ratio of approximately 3:2.

Whether applied in a consumer or production setting, bayesian analysis affords a convenient means for discriminating among competing FFF on the basis of actual data. Yet, there exists potential for conflict between mi-

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[9] The Berndt and Khaled formulation of the generalized Box-Cox does not allow for direct testing of the translog because portions of the likelihood function of the generalized Box-Cox become degenerate when parameter values yield the translog special case (Berndt and Khaled, p. 1227). Tests for parameter values approaching those of the translog lead to rejection of those functional forms.

[10] The logarithmic Fourier nests the translog, the minflex translog nests the translog, and the minflex generalized Leontief nests the generalized Leontief. Hypothesis tests of subsets of the parameters in each of these models could be used to test the special cases. However, none nests a wide range of alternative models.

croeconomic theoretical results and the esti-
mated parameters of the FFF chosen a pos-
teriori. The functional form with the highest
posterior odds ratio could possess, for exam-
ple, own-price elasticities which are positive
or violate other regularity conditions. The ap-
plicability of microlevel theoretical results in
aggregate or market studies would have to be
assessed in light of the bayesian results.

Nonnested Hypothesis Testing

Nonnested hypothesis testing allows the re-
searcher to make pair-wise comparisons be-
tween competing models. A conceptual differ-
ence between nonnested testing and bayesian
analysis is that nonnested testing allows for all
proposed models to be rejected on the basis of
the data. For FFF interpreted as approxima-
tions, nonnested tests provide the possibility
of rejecting all approximations to the unknown
underlying function. With bayesian analysis,
in contrast, the models would be ranked on
the basis of posterior odds ratios so that some
alternative model would probably be deemed
the most plausible.

Although myriad nonnested tests have been
proposed recently, the Cox test for nonnested,
nonlinear equations systems (Pesaran and
Deaton) is the most general for testing alter-
native FFF in budget or cost (profit) share
equation systems. The only apparent limita-
tion for the Cox test is that all competing func-
tional forms must be specified with the same
transformations of the dependent variables.
The Cox statistic has been developed for test-
ing linear and log-linear single equation regres-
sion models (Aneuryn-Evans and Deaton).
However, empirical applications of the Cox
test for linear and logarithmic dependent vari-
ables have not been made to either single-
equation nonlinear regressions or multiple
equation regressions. Share equation and de-
mand/supply equation systems which have dif-
ferent transformations of the dependent vari-
ables (e.g., shares vs. single variables) could
not be used directly to perform a nonnested
test with the Cox statistic.

Few empirical studies have used the Cox
statistic to compare functional forms (Pesaran
and Deaton; Deaton). Alternative functional
forms, not flexible functional forms, have been
tested in the single-equation agricultural pro-
duction function applications (Ackello-Ogutu,
Paris, and Williams). The extent to which non-

nested testing results coincide with bayesian
results is a potential methodological and em-
pirical question.

Review of Empirical Criteria

Overall assessment of the compatibility of dif-
ferent FFF with empirical data does not lead
to the acceptance of any clearly superior func-
tional form. Monte Carlo studies offer the most
general empirical means for choosing among
FFF by comparing their abilities to approxi-
mate underlying partial substitution elastici-
ties. However, no studies of a wide range of
prospective FFF have been published. The rel-
ative attractiveness of FFF in both production
and consumer Monte Carlo applications may
also yield useful results for applied researchers.

Dixon, Garcia, and Anderson shed doubt on
the usefulness of pretests in assessing main-
tained behavioral hypotheses. Nonparametric
alternatives to the parametric tests analyzed
by Dixon, Garcia, and Anderson might be a
useful pretest for maintained behavioral hy-
potheses. Varian (1982, 1984), for example,
has developed nonparametric procedures
whereby data can be tested for consistency with
utility-maximizing, cost-minimizing, or prof-
it-maximizing behavior. If the nonparametric
tests did not reject the behavioral hypotheses,
the parametric analysis with FFF could be pur-
sued with more confidence (see Barnhart and
Whitney).

Unless a more general composite model than
the generalized Box-Cox is found, the most
promising techniques for testing alternative
FFF are bayesian inference and nonnested hy-
pothesis testing. In practice, calculation of pos-
terior odds may require fewer, less complicat-
ed operations than are necessary for deriving
a Cox statistic. When appropriate, nonnested
tests based on instrumental variable esti-
mators could also be used (see Godfrey and
Pesaran). The fundamental methodological
advantage of nonnested testing is that no par-
ticular functional form is necessarily accepted
on the basis of the data; all prospective FFF
may be rejected in pair-wise tests. Whether
bayesian or nonnested techniques are used, the
ability to test models with different transfor-
mations of the dependent variable—logarith-
mic versus linear, for example—is necessary
for discriminating between a wide range of cur-
rently available FFF.

Conclusions

An extensive array of theoretical criteria and empirical techniques for choosing among FFF has been reviewed. Ample opportunity exists in many empirical studies to compare competing FFF more systematically on the basis of these theoretical and empirical measures. Although any comparison of all available functional forms by all these measures may not yield an unambiguous choice, the relative advantages of competing FFF in the particular empirical problem should provide more compelling grounds for choice of flexible functional forms.

As Griffin, Montgomery, and Rister have remarked, the choice of functional form should be included explicitly in empirical studies because nearly all econometric model specifications may be considered as approximations of some unknown underlying data generation process. One avenue for formalizing the selection process of FFF in duality modeling might be the following empirical testing procedure:

(*a*) Test the behavioral assumptions of the duality model (utility maximization, cost minimization, or profit maximization) using the nonparametric testing procedures from Varian. Choice among alternative flexible functional forms is valid conditional upon the behavioral assumptions being appropriate. If the data are not consistent with the behavioral assumptions, one would have to judge whether the inconsistencies are caused by measurement error or to cross-sectional and time-series heterogeneity (Hanoch and Rothschild). In the case of measurement error inconsistencies, the violating data points might be adjusted or the sample might be censored to purge the inconsistent data points (see Barnhart and Whitney). Cross-sectional and time-series heterogeneity, such as differing firm endowments, regional differences, and technical progress, might call for alternative models which account for the inconsistencies.

(*b*) If the data are consistent with the behavioral assumptions, test other appropriate theoretical properties of the data such as returns to scale, homotheticity, and separability using similar nonparametric tests. The appropriate theoretical properties to be tested will, of course, vary according to the particular problem.

(*c*) Choose flexible functional forms which can embody the behavioral assumptions and properties not rejected by the nonparametric tests. Parametric tests of the relevant theoretical properties for each flexible functional form may then be performed. If the parametric tests do not reject the theoretical properties, the properties can then be imposed in subsequent estimation.

(*d*) Choose among flexible functional forms using one or both of the following techniques:

(*i*) Bayesian analysis: compare the alternative FFF on the basis of posterior odds. The principal advantage of bayesian analysis is that FFF with different transformations of the dependent variables may be compared.

(*ii*) Nonnested hypothesis tests: perform pair-wise comparisons of the alternative FFF using classical statistical techniques. The main disadvantage is that systems of equations with different transformations of the dependent variables may not be compared with current formulations of the nonnested tests. With nonnested tests, no particular model might emerge as the best. Note that bayesian and nonnested test results might conflict.

(*e*) Check the robustness of the economic measures, such as price and partial substitution elasticities calculated from the flexible functional form(s), to determine the sensitivity of the measures to the choice of flexible functional form.

Presumably, more credible parameter and elasticity estimates may be obtained by these explicit comparisons of flexible functional forms when the maintained hypotheses of duality analysis are appropriate. The empirical selection procedure also appears to be be consistent with this journal's new policy of making econometric specification searches more explicit. Whether the benefits of such a formal selection procedure would outweigh the costs in the context of a data set of questionable quality is a judgment that the researcher must make.

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Appendix

Flexible Functional Forms

(1) Quadratic Form

$\phi(y) = a_0 + a'x + \frac{1}{2}x'Bx$

    (*a*) Normalized Quadratic: (Lau 1978a, p. 194)

        $\phi(y) = y$ $a' = [a_1, \ldots, a_n]$ $x' = [x_1, \ldots, x_n]$

        $B = \{b_{ij}\}$ $b_{ij} = b_{ji}$ $\forall$ij

    (*b*) Generalized Leontief: (Diewert 1971)

        $\phi(y) = y$ $a' = 0$ $x' = [x_1^{1/2}, \ldots, x_n^{1/2}]$

        $B = \{b_{ij}\}$ $b_{ij} = b_{ji}$ $\forall$ij

    (*c*) Translog: (Christensen, Jorgenson, and Lau)

        $\phi(y) = \ln y$ $a' = [a_1, \ldots, a_n]$ $x' = [\ln x_1, \ldots, \ln x_n]$

        $B = \{b_{ij}\}$ $b_{ij} = b_{ji}$ $\forall$ij

(2) Quadratic Mean of Order Rho: (Denny, Kadiyala)

$$y = \left(\sum_i \sum_j b_{ij} x_i^{\rho/2} x_j^{\rho/2} \right)^{1/\rho}$$

$$y = \left(\sum_i a_i x_i^{2\rho} + \sum_i \sum_j b_{ij} x_i^{\rho} x_j^{\rho} \right)^{1/2\rho}$$

(3) Generalized Cobb-Douglas: (Diewert 1973b)

$$y = \prod_i \prod_j (\tfrac{1}{2}x_i + \tfrac{1}{2}x_j)^{b_{ij}}$$ $b_{ij} = b_{ji}$ $\forall'$ij

(4) Generalized Square Root Quadratic: (Diewert 1971)

$$y = (x'Bx)^{1/2}$$ $x' = [x_i, \ldots, x_n]$ $B = \{b_{ij}\}$ $b_{ij} = b_{ji}$ $\forall$ij

(5) Generalized Box-Cox: (nonhomothetic) (Berndt and Khaled, Applebaum)

    $\gamma(\delta) = \{1 + \lambda G(X)\}^{1/\lambda}$

where

    $G(X) = a_0 + \sum_i a_i P_i(\lambda) + \frac{1}{2} \sum_i \sum_j b_{ij} P_i(\lambda) P_j(\lambda)$

and

    $P_i(\lambda) = P_i^{(\lambda/2)-1}\lambda/2$

(6) Symmetric Generalized McFadden: (Diewert and Wales)

$$\phi(y) = g(x) + \sum_i b_i x_i + \left(\sum_i c_i x_i \right)^2$$

    $g(x) = \frac{1}{2}x'Bx/\theta'x$ $\theta > 0$ $x' = [x_1, \ldots, x_n]$ $B = \{b_{ij}\}$ $b_{ij} = b_{ji}$ $\forall$ij

(7) Generalized Barnett: (Diewert and Wales)

$$\phi(y) = g(x) + \sum_i b_i x_i + \left(\sum_i c_i x_i \right)^2$$

$$g(x) = \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} b_{ij} x_i^{1/2} x_j^{1/2} - \sum_{\substack{i=2 \\ i \neq j}}^{N} \sum_{j=2}^{N} d_{ij} x_1^2 x_1^{-1/2} x_j^{-1/2}$$

$$- \sum_{i=2}^{N} \sum_{j=2}^{N} e_{ij} x_1^{-1/2} x_i^{-1/2} x_j^2$$
    $b_{ij} = b_{ji} \geq 0$
    $d_{ij} = d_{ji} \geq 0$
    $e_{ij} > 0$

(8) Generalized Fuss: (Diewert and Ostensoe)

    $y = \frac{1}{2}\alpha' z x'Axx_1^{-1} + \frac{1}{2}\beta' xz'Bzz_1^{-1} + x'Cz$

        $+ \beta' xb'zz_1^{-1} + \frac{1}{2}\beta' xb_0z_1^{-1} + x'c$

    $\alpha' > 0$ $\beta' > 0$ $A = \{a_{ij}\}$ $a_{ij} = a_{ji}$ $\forall$ij

                                   $a_{1j} = a_{1i} = 0$

$$b' = [0, b_2, \ldots, b_M] \qquad B = \{b_{ij}\} \qquad b_{ij} = b_{ji} \quad \forall ij$$
$$b_{1j} = b_{1i} = 0$$

$$c' = [c_1, \ldots, c_N] \qquad C = \{c_{ij}\}$$

(9) <u>Biquadratic</u>: (Diewert 1986)

$$y = \sum_{n=1}^{N} \alpha_n x_n + \tfrac{1}{2} \sum_{n=1}^{N-1} \sum_{i=1}^{N-1} b_{ni} x_n x_i x_n^{-1}$$

$$b_{ni} = b_{in} \qquad 1 < n < i < N$$

(10) <u>Minflex Laurents</u>: (Barnett 1983, 1985)

$$\phi(y) = a_0 + 2 \sum a_i x_i + \sum a_{ii} x_i^2 + \sum (a_{ij}^2 x_i x_j - b_{ij}^2 x_i^{-1} x_j^{-1})$$

    (*a*)   Minflex Laurent Generalized Leontief:

$$\phi(y) = y \qquad x_i = x_i^{\nu_2}$$

    (*b*)   *Minflex Laurent Translog*:

$$\phi(y) = \ln y \qquad x_i = \ln x_i$$

(11) <u>Fourier</u>: (Gallant 1981)

$$y = v_0 + b'x + \tfrac{1}{2}x'Bx + \sum_{a=i}^{A} v_{0a} + 2 \sum_{j=i}^{J} [v_{ja}\cos(j\lambda k_a'x) - v_{ja}\sin(j\lambda k_a'x)]$$

and

$$B = -\sum_{a=1}^{A} v_{0a}^{\lambda 2} k_a k_a'.$$