Multiple Optimal Solutions in Quadratic Programming Models

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The problem of determining whether quadratic programming models possess either unique or multiple optimal solutions is important for empirical analyses which use a mathematical programming framework. Policy recommendations which disregard multiple optimal solutions (when they exist) are potentially incorrect and less than efficient. This paper proposes a strategy and the associated algorithm for finding all optimal solutions to any positive semidefinite linear complementarity problem. One of the main results is that the set of complementary solutions is convex. Although not obvious, this proposition is analogous to the well-known result in linear programming which states that any convex combination of optimal solutions is itself optimal.

The importance of not overlooking multiple optimal solutions in empirical studies based on linear programming (LP) models was discussed by Paris in a recent article. In the last decade, however, quadratic programming (QP) models have been used at an increasing rate for analyzing problems of choice under market and general equilibria as well as under risky environments.

While conditions leading to alternate optimal solutions in LP have been known for a long time, knowledge of the structural causes underlying multiple optimal solutions in QP, and of criteria for their detection is rather limited. The study of this subject is of recent vintage. The results obtained so far are confined either to specialized journals or unpublished papers.

The existence of either unique or multiple optimal solutions in QP models has significant consequences in the formulation of policy recommendations. Unfortunately, commercial computer programs for solving QP problems are completely silent about this aspect and leave it entirely to the enterprising researcher to find convenient ways for assessing the number of optimal solutions and their values.

Multiple optimal solutions are an appealing feature of programming models at least for two reasons. First of all, they allow greater diversification of activities representing an economic environment. In other words, all the activities specified in the model can potentially be operated at positive levels regardless of the number of constraints. Secondly, a policy maker has greater flexibility in choosing the strategy to implement knowing that he need not sacrifice economic efficiency.

For many years, references to uniqueness of solutions in QP models have been scant. A reference to a sufficient condition for uniqueness of a part of the solution vector in a QP model, namely the positive definiteness of the quadratic form, is found in Takayama and Judge (p. 164). However, it is not necessary to have positive definite quadratic forms to have unique solutions. The relevant aspect of the problem is, therefore, to know both the necessary and sufficient conditions for uniqueness. Hence, a more interesting problem can be stated as follows: If the quadratic form in a QP model is positive semidefinite (as are the quadratic forms in many empirical problems presented in the literature), how do we know whether the given problem has a unique solution

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or it admits multiple optimal solutions? This paper addresses this problem and presents an algorithmic approach to its solution. The algorithm is relatively simple and can be implemented efficiently on a computer even for large scale models. A particularly interesting result of this study is that the set of multiple optimal solutions in positive semidefinite QP models is convex. The possibility of diversified policy strategies is based upon this finding.

The paper relies heavily on numerical examples to illustrate the seemingly intricate structure associated with either uniqueness or multiplicity of solutions. After discussing the convexity of the set of multiple optimal solutions, the same algorithm is applied to a LP and a QP problem to illustrate its numerical feasibility. A remarkable feature of the discussion presented below is that for finding all multiple optimal solutions of a QP problem it is sufficient to solve an associated linear programming problem.

The Linear Complementarity Problem

One promising way to gain insight into this rather complex problem is to regard the quadratic program as a linear complementarity (LC) problem. Consider the following symmetric QP problem

$$\max \{c'x - k_x x'Ds/2 - k_y y'Ey/2\} \qquad (1)$$

subject to:

$$Ax - k_y Ey \leq b, \quad x \geq 0, \quad y \geq 0,$$

where A is an $(m \times n)$ matrix, D and E are symmetric positive semidefinite (PSD) matrices of order n and m, respectively. Parameters k_x and k_y are nonnegative scalars suitable for representing various economic scenarios, from perfect and imperfect market equilibria to risk and uncertainty problems. It can be easily shown that the necessary and sufficient Kuhn-Tucker conditions corresponding to (1) can be written in the form of the following LC problem: find an $[(n + m) \times 1]$ vector z such that

$$w = Mz + q \geq 0, \; z \geq 0 \qquad (2)$$

and:

$$z'w = 0,$$

where w is an $[(n + m) \times 1]$ vector of slack variables, $q' = [-c', b']$, $z' = [x', y']$ and $M = \begin{bmatrix} k_x D & A' \\ -A & k_y E \end{bmatrix}$ is an $[(m + n) \times (m + n)]$ PSD matrix (for any A). It should be apparent that when E is the null matrix, problem (1) represents the traditional asymmetric quadratic program, and when both D and E are null a LP problem is obtained.

It is well known that when multiple optimal solutions exist in a LP problem, their set constitutes a face of the convex polytope of all feasible solutions. This property can be extended to the LC problem (2). First of all, notice that the linear inequalities of problem (2) form a convex set of feasible solutions. Of course, we are not merely interested in the set of feasible solutions but in the set of feasible as well as complementary solutions, that is those solutions (w, z) which satisfy the feasibility conditions $w \geq 0$, $z \geq 0$ and also the complementarity condition $w'z = 0$. All complementary solutions to (2) are optimal solutions for the QP problem (1).

In LP problems, the set of optimal solutions is convex. This well known fact implies that a convex combination of any two optimal solutions is itself an optimal solution. From an empirical viewpoint this is an important result because it admits that the number of positive components of an optimal solution be greater than the number of independent constraints. Hence, when multiple optimal solutions exist, one can select a more diversified solution for policy recommendation. It turns out that, as in LP, the set of optimal solutions in QP problems is convex. To demonstrate this less known proposition it is sufficient to prove that the set of complementary solutions of problem (2) is convex. The proof requires the results of the following

Lemma: Suppose $(\hat{z}, \hat{w})$ and $(\bar{z}, \bar{w})$ are complementary solutions to problem (2). Then, $\hat{w}'\bar{z} = \bar{w}'\hat{z} = (\hat{z} - \bar{z})'M(\hat{z} - \bar{z}) = 0$.

Proof: According to (2), the definition of the $\hat{w}$ and $\bar{w}$ vectors is $\hat{w} = M\hat{z} + q$ and $\bar{w} = M\bar{z} + q$. Subtracting $\bar{w}$ from $\hat{w}$: $(\hat{w} - \bar{w}) = M(\hat{z} - \bar{z})$. Premultiplying by $(\hat{z} - \bar{z})'$ the above result gives:

$$(\hat{z} - \bar{z})'(\hat{w} - \bar{w}) = (\hat{z} - \bar{z})'M(\hat{z} - \bar{z}) \geq 0 \tag{3}$$

because M is PSD

$$\hat{z}'\hat{w} - \hat{z}'\bar{w} - \bar{z}'\hat{w} + \bar{z}'\bar{w} = -\hat{z}'\bar{w} - \bar{z}'\hat{w} \leq 0.$$

The simplification in the second row of (3) is obtained because, by assumption, $(\hat{z}, \hat{w})$ and $(\bar{z}, \bar{w})$ are complementary solutions. Furthermore, the inequality is established in the direction of nonpositivity because $\hat{z}$, $\hat{w}$, $\bar{z}$, and $\bar{w}$ are nonnegative. Hence, the two inequalities in (3) establish the conclusion of the lemma.

We can now demonstrate the following important

Theorem: The set of all complementary solutions in a PSD-LC problem is convex.

Proof: Consider any two distinct pairs of complementary solutions to problem (2), say $(\bar{z}, \bar{w})$ and $(\hat{z}, \hat{w})$. We need to show that (z, w) defined as a convex combination of $(\bar{z}, \bar{w})$ and $(\hat{z}, \hat{w})$ is also a complementary solution. Let $z = \alpha\bar{z} + (1 - \alpha)\hat{z}$ and $w = \alpha\bar{w} + (1 - \alpha)\hat{w}$ for $0 \leq \alpha \leq 1$. Then, (z, w) is a feasible solution to (2) since $z \geq 0$, $w \geq 0$ and

$$\begin{aligned} Mz + q &= M[\alpha\bar{z} + (1 - \alpha)\hat{z}] + q \\ &= \alpha M\bar{z} + (1 - \alpha)M\hat{z} + q \\ &= \alpha(\bar{w} - q) + (1 - \alpha)(\hat{w} - q) + q \\ &= \alpha\bar{w} + (1 - \alpha)\hat{w} = w. \end{aligned}$$

To show that (z, w) is a complementary solution

$$\begin{aligned} w'z &= [\alpha\bar{w} + (1 - \alpha)\hat{w}]'[\alpha\bar{z} + (1 - \alpha)\hat{z}] \\ &= \alpha^2\bar{w}'\bar{z} + (1 - \alpha)^2\hat{w}'\hat{z} + \alpha(1 - \alpha)\bar{w}'\hat{z} \\ &\quad + \alpha(1 - \alpha)\hat{w}'\bar{z} = 0 \end{aligned}$$

since $\bar{w}'\bar{z}$ and $\hat{w}'\hat{z}$ are equal to zero for being complementary solutions, while $\bar{w}'\hat{z}$ and $\hat{w}'\bar{z}$ are zero according to the lemma.

An important corollary to this theorem is that the number of solutions to a PSD-LC problem is either 0, 1, or ∞. This is so because either the problem has no solution, or has a unique solution, or if it has more than one solution, by convexity it has an infinite number of them.

Determining the Number of Solutions

Judging from the empirical literature, almost never has it been a concern of authors to state whether a QP problem possesses either a unique or multiple optimal solutions.[1] It is difficult, however, to downplay the importance of this aspect in empirical studies. To turn the tide around, referees and journal editors ought to make it a definite point to require information about uniqueness of the solution in all mathematical programming analyses submitted to them. Admittedly, this additional piece of information requires additional computations over and above those necessary to obtain an optimal solution. In econometrics, computational requirements have rarely been regarded as a deterrent for achieving a correct and complete analysis. There is no reason to suppose that they should deter a mathematical programmer.

To reduce as much as possible these additional computations a two-stage procedure seems convenient. After achieving

[1] von Oppen and Scott (p. 440) present a rare passing reference of solution uniqueness of their QP model. They do not state, however, whether the associated quadratic form is positive definite or semidefinite, nor how the uniqueness of the solution was determined.

any optimal solution of the QP (LC) problem, determine the number of solutions by means of a recent suggestion presented by Kaneko. If the results of the algorithm indicate that the solution is unique, stop. If the number of solutions is infinite, it is possible to proceed to find all the extreme point optimal solutions (finite in number) of the QP problem through the combination of results obtained by Adler and Gale and by Mattheiss.

The algorithm suggested by Kaneko is simple. As already stated, its objective is to determine the number of solutions of the PSD-LC problem, not to find those solutions. The first step is to solve the LC problem (corresponding to the QP problem) by means of any suitable algorithm, for example, Lemke's complementary pivot algorithm. At this point, let $\rho = \{j\}$ be the set of all the j indexes for which $\bar{w}_j = \bar{z}_j = 0$, $j = 1, \ldots m + n$, where $(\bar{z}, \bar{w})$ is a solution to (2). In other words, consider all the degenerate components of the complementary solution. If ρ is empty, $\rho = \emptyset$, stop because the solution is unique. Otherwise, let $\bar{M}$ be the transformation of M in the final tableau of the Lemke's algorithm and solve the following PSD-QP problem.

$$\text{minimize } R = u'\bar{M}_{\rho\rho}u/2 \tag{4}$$

subject to:

$$s'u \geq 1, \; u \geq 0$$

where s is a vector of ones. This QP problem corresponds to the following PSD-LC problem:

$$Lv + d \geq 0, \quad v \geq 0 \tag{5}$$

$$v'(Lv + d) = 0$$

$$\text{where } L = \begin{bmatrix} \bar{M}_{\rho\rho} & -s \\ s' & 0 \end{bmatrix}, \; d = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ and } v = \begin{bmatrix} u \\ 2R \end{bmatrix}.$$

Kaneko has demonstrated that if no solution exists or if a solution is found such that $R > 0$, then the solution to the original QP (LC) problem is unique. On the contrary, if a solution exists such that $R = 0$, then the number of solutions to the original QP (LC) problem is infinite. In other words, the admissibility of multiple optimal solutions requires that the matrix $\bar{M}_{\rho\rho}$ be positive semidefinite. Notice that the dimensions of the $\bar{M}_{\rho\rho}$ matrix depend on the number of degeneracies present in the first optimal solution found in step 1. In many instances $\bar{M}_{\rho\rho}$ is a rather small matrix also for large scale models and problem (4) is easy to solve.

The rationale of Kaneko's algorithm is based on the fact that a degenerate solution of the LC problem opens the way for the linear dependence of the vectors in a submatrix, $\bar{M}_{\rho\rho}$, of the final optimal tableau of problem (2). The constraint of problem (4) defines a convex combination, while the objective function tests the linear dependence (or independence) of the subset of vectors associated with the degenerate components of the original optimal solution to problem (1). Hence, degeneracy of an optimal solution is a necessary but not sufficient condition for multiple optimal solutions: degeneracy and linear dependence of the associated submatrix are necessary and sufficient.

To illustrate this point and the working of Kaneko's algorithm, two numerical examples of asymmetric quadratic programs will be discussed. Example 1 illustrates the necessary aspect of degeneracy (but not its sufficiency) for the existence of multiple optimal solutions. Example 2 shows that degeneracy of an optimal solution must be accompanied by linear dependence of the submatrix, $\bar{M}_{\rho\rho}$, for the existence of multiple optimal solutions. Familiarity with the complementarity pivot algorithm of Lemke will be assumed throughout.

Example 1

$$\max \{c'x - x'Dx/2\}$$

subject to:

$$Ax \leq b, \; x \geq 0$$

where $c' = [12 \;\; 8 \;\; 11/2]$, $\;b' = [18 \;\; 12]$

TABLEAU 1. Initial Tableau of Example 1.

w_1	w_2	w_3	w_4	w_5	z_1	z_2	z_3	z_4	z_5	z_0	q	Basic Variables
1					−3	−2	−3/2	−6	−4	−1	−12	w_1
	1				−2	−4/3	−1	−4	−3	−1	−8	w_2
		1			−3/2	−1	−3/2	−2	−1	−1	−11/2	w_3
			1		6	4	2	0	0	−1	18	w_4
				1	4	3	1	0	0	−1	12	w_5

$$A = \begin{bmatrix} 6 & 4 & 2 \\ 4 & 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 2 & 3/2 \\ 2 & 4/3 & 1 \\ 3/2 & 1 & 3/2 \end{bmatrix}.$$

The matrix D is PSD of rank 2. To formulate and solve this QP problem as a LC problem we must set up a tableau following Lemke's instructions and having the structure ($Iw - Mz - sz_0$; q), where s is a vector of ones and z_0 is the associated artificial variable. All the other components of the problem are defined as in (2). The layout of Example 1 is given in Tableau 1. The final Tableau exhibiting a complementary solution is given in Tableau 2. The complementary solution of Tableau 2 translates into an optimal QP solution as $z_1 = x_1 = 3$, $z_4 = y_1 = ½$ while all the other x and y variables are zero. The optimal value of the QP objective function is 22.5.

Degeneracy appears in three pairs of complementary variables $\bar{w}_j = \bar{z}_j = 0$ for j = 2, 3, 5. Hence, Kaneko's index set is $\rho = \{2, 3, 5\}$. This index set corresponds to the following $-\bar{M}_{\rho\rho}$ matrix:

$$-\bar{M}_{\rho\rho} = \begin{bmatrix} 0 & 0 & -1/3 \\ 0 & -5/6 & 1/3 \\ 1/3 & -1/3 & 0 \end{bmatrix}.$$

To determine the uniqueness or the multiplicity of solutions according to Kaneko one must solve problem (4), alternatively problem (5). We choose problem (5) and Tableaux 3 and 4 give the corresponding initial and final layouts.

From Tableau 4 it can be observed that $\bar{v}_4 = 2R = 2/15 > 0$, and hence, in spite of its extended degeneracy, the problem in Example 1 has one complementary solution, the one presented in Tableau 2. Correspondingly, it can be observed that the matrix $\bar{M}_{\rho\rho}$ is positive definite. Of course, with a small matrix it may be easier to determine its definiteness directly by means of evaluating its minors and determinant. But as soon as the dimensions of $\bar{M}_{\rho\rho}$ become respectable, say greater than 6 or 7, solving Kaneko's problem (5) is definitely easier.

Example 2

In this example another QP problem is considered with the following coefficients:

$$c' = [12 \;\; 8 \;\; 4], \quad b' = [18 \;\; 12]$$

$$A = \begin{bmatrix} 6 & 4 & 2 \\ 4 & 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4/3 & 2/3 \\ 1 & 2/3 & 1/3 \end{bmatrix}.$$

The matrix D is PSD of rank 1. The initial and final Tableaux corresponding to this problem are presented in Tableaux 5 and 6, respectively.

The index set of degenerate complementary variables is again $\rho = \{2, 3, 5\}$ and the corresponding $-\bar{M}_{\rho\rho}$ matrix is:

$$-\bar{M}_{\rho\rho} = \begin{bmatrix} 0 & 0 & -1/3 \\ 0 & 0 & 1/3 \\ 1/3 & -1/3 & 0 \end{bmatrix}.$$

The matrix $\bar{M}_{\rho\rho}$ is, obviously, singular and, thus, PSD. Hence, we can conclude that the QP in Example 2 has multiple optimal solutions. However, for sake of completeness and for familiarization with Kaneko's algorithm and its interpretation, the full computations are presented in Tableaux 7 and 8.

TABLEAU 2. Final Tableau of Example 1 (after Reordering of rows and columns).

$\bar{z}_1$	$\bar{w}_2$	$\bar{w}_3$	$\bar{z}_4$	$\bar{w}_5$	$\bar{w}_1$	$\bar{z}_2$	$\bar{z}_3$	$\bar{w}_4$	$\bar{z}_5$	$\bar{q}$	Basic Variables
1					0	2/3	1/3	1/6	0	3	$\bar{z}_1$
	1				−2/3	0	0	0	−1/3	0	$\bar{w}_2$
		1			−1/3	0	−5/6	1/12	1/3	0	$\bar{w}_3$
			1		−1/6	0	1/12	−1/12	2/3	1/2	$\bar{z}_4$
				1	0	1/3	−1/3	−2/3	0	0	$\bar{w}_5$

Tableau 8 shows that $\bar{v}_4 = 2R = 0$ and we conclude that the QP problem of Example 2 possesses an infinite number of optimal solutions.

Determining All Basic Complementary Solutions

Once it has been determined that the number of solutions of a given QP (LC) problem is infinite, it is of interest to find all the basic complementary solutions associated with the vertices of the corresponding convex set. Recall that such a set constitutes a face of the convex set of feasible solutions of the given LC problem. Adler and Gale have demonstrated that this face is defined by the following systems of inequalities and equations

$$\bar{M}_{\cdot\rho} z_\rho + \bar{q} \geq 0, \quad z_\rho \geq 0 \tag{6}$$

$$(\bar{M}_{\rho\rho} + \bar{M}_{\rho\rho}')z_\rho = 0 \tag{7}$$

where $\bar{M}$ is the complementary transform of the given LC problem obtained in the final Tableau of the Lemke's algorithm; ρ is the index set of subscripts corresponding to degenerate complementary pairs of variables; $\bar{M}_{\cdot\rho}$ is the submatrix of $\bar{M}$ with the columns defined by the index set ρ; $\bar{M}_{\rho\rho}$ is the submatrix of $\bar{M}$ with both rows and columns defined by ρ; $\bar{q}$ is the transform of q in the final Tableau.

Any solution to (6) and (7) constitutes a complementary solution to the original LC problem. At this point an algorithm is required for enumerating all vertices of problem (6) and (7). The work of Mattheiss provides such an algorithm that is both elegant and efficient.

Consider the system of linear inequalities $Ax \leq b$, which must also include all nonnegative constraints. Let A be an $(m \times n)$ matrix, $m > n$. Let K be the n-convex set of solutions of the given system of inequalities. K is embedded in a one-higher-dimensional space forming the convex $(n + 1)$ polytope C, which is the set of feasible solutions of the following linear program:

$$\text{maximize } Z = y \tag{8}$$

$$\text{subject to: } Ax + ty + Is = b, \quad y \geq 0, \quad s \geq 0$$

where x is an $(n \times 1)$ vector variable, y is a scalar variable, s is a $(m \times 1)$ vector of slack variables and t is a $(m \times 1)$ vector of coefficients defined as:

$$t_i = \left(\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}, \quad i = 1, \ldots, m.$$

TABLEAU 3. Initial Tableau for Problem 5, Example 1.

w_1	w_2	w_3	w_4	v_1	v_2	v_3	v_4	v_0	q	Basic Variables
1				0	0	−1/3	1	−1	0	w_1
	1			0	−5/6	1/3	1	−1	0	w_2
		1		1/3	−1/3	0	1	−1	0	w_3
			1	−1	−1	−1	0	−1	−1	w_4

TABLEAU 4. Final Tableau for Problem 5, Example 1 (Reordered).

$\bar{w}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_1$	$\bar{w}_2$	$\bar{w}_3$	$\bar{w}_4$	$\bar{q}$	Basic Variables
1				−8/15	4/5	−9/5	−1/15	1/15	$\bar{w}_1$
	1			4/5	−6/5	6/5	−2/5	2/5	$\bar{v}_2$
		1		1/5	6/5	−6/5	−3/5	3/5	$\bar{v}_3$
			1	3/5	−2/5	7/5	−2/15	2/15	$\bar{v}_4$

The t vector is regarded as a generalized slack activity whose purpose is to define and construct the radius of the largest sphere inscribable in the set of feasible solutions K. The idea of embedding K in C is to make the convex set K to be a face of the (n + 1) polytope C. Then, by starting at the vertex of C where (the radius) y is maximum, it is possible to reach every vertex of K by simplex pivot operations that, it is well known, lead to adjacent vertices.

Every optimal solution to the linear program (8) is characterized by all x_j variables $j = 1, \ldots, n$ and y as basic variables. Otherwise, the problem is infeasible. Also $(m - n - 1)$ slack variables will be basic while the remaining $(n - 1)$ slacks not in the basis $(s_i = 0)$ identify the set of binding constraints H_p where p is the index of the solution.

The primal tableau of a basic feasible solution has the following structure:

	Z	X	Y	SB	SNB	B
Z	1				W	Z
X		I			UX	BX
Y			1		UY	BY
SB				I	US	BS

where SB = slack variables in the basis.
SNB = slack variables not in the basis.
B = the solution column, BX is a $(n \times 1)$ block giving the values of x, BY is a (1×1) scalar giving the value of y and BS is an $[(m - n - 1) \times 1]$ block giving the solution values of the basic slack variables SB.
W = the row of dual variables.
Z = the current solution value.
U = the matrix of coefficients of the slack variables not in the basis divided in three blocks corresponding to X, Y and SB variables.

To travel from one vertex to another vertex of C requires pivot operations according to the feasibility criterion of the primal simplex algorithm. However, a pivot in the UX block of coefficient is inadmissible because it would remove some x_j from the basis, thus leaving the set of feasible solutions K. A pivot selected in the US block will exchange slack activities in the basis, providing another solution of the linear program. A pivot executed in the

TABLEAU 5. Initial Tableau of Example 2.

w_1	w_2	w_3	w_4	w_5	z_1	z_2	z_3	z_4	z_5	z_0	q	Basic Variables
1					−3	−2	−1	−6	−4	−1	−12	w_1
	1				−2	−4/3	−2/3	−4	−3	−1	−8	w_2
		1			−1	−2/3	−1/3	−2	−1	−1	−4	w_3
			1		6	4	2	0	0	−1	18	w_4
				1	4	3	1	0	0	−1	12	w_5

TABLEAU 6. Final Tableau of Example 2 (Reordered).

$\bar{z}_1$	$\bar{w}_2$	$\bar{w}_3$	$\bar{z}_4$	$\bar{w}_5$	$\bar{w}_1$	$\bar{z}_2$	$\bar{z}_3$	$\bar{w}_4$	$\bar{z}_5$	$\bar{q}$	Basic Variables
1					0	2/3	1/3	1/6	0	3	$\bar{z}_1$
	1				−2/3	0	0	0	−1/3	0	$\bar{w}_2$
		1			−1/3	0	0	0	1/3	0	$\bar{w}_3$
			1		−1/6	0	0	−1/12	2/3	1/2	$\bar{z}_4$
				1	0	1/3	−1/3	−2/3	0	0	$\bar{w}_5$

TABLEAU 7. Initial Tableau for Problem 5, Example 2.

w_1	w_2	w_3	w_4	v_1	v_2	v_3	v_4	v_0	q	Basic Variables
1				0	0	−1/3	1	−1	0	w_1
	1			0	0	1/3	1	−1	0	w_2
		1		1/3	−1/3	0	1	−1	0	w_3
			1	−1	−1	−1	0	−1	−1	w_4

TABLEAU 8. Final Tableau of Problem 5, Example 2 (Reordered).

$\bar{w}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_1$	$\bar{w}_2$	$\bar{w}_3$	$\bar{w}_4$	$\bar{q}$	Basic Variables
1				0	−3/2	3/2	−1/2	1/2	$\bar{w}_1$
	1			3/2	0	−3/2	−1/2	1/2	$\bar{v}_2$
		1		−3/2	3/2	0	0	0	$\bar{v}_3$
			1	1/2	1/2	0	0	0	$\bar{v}_4$

TABLEAU 9. Initial Tableau of Example 3.

w_1	w_2	w_3	w_4	w_5	w_6	z_1	z_2	z_3	z_4	z_5	z_6	z_0	q	Basic Variables
1						0	0	0	0	−2	−1	−1	−53/22	w_1
	1					0	0	0	0	−1	−3	−1	−39/22	w_2
		1				0	0	0	0	−5	2	−1	−5	w_3
			1			0	0	0	0	−1	−4	−1	−2	w_4
				1		2	1	5	1	0	0	−1	4	w_5
					1	1	3	−2	4	0	0	−1	0	w_6

TABLEAU 10. Final Tableau of Example 3 (Reordered).

$\bar{w}_1$	$\bar{w}_2$	$\bar{z}_3$	$\bar{z}_4$	$\bar{z}_5$	$\bar{z}_6$	$\bar{z}_1$	$\bar{z}_2$	$\bar{w}_3$	$\bar{w}_4$	$\bar{w}_5$	$\bar{w}_6$	$\bar{q}$	Basic Variables
1						0	0	−7/22	−9/22	0	0	0	$\bar{w}_1$
	1					0	0	−1/22	−17/22	0	0	0	$\bar{w}_2$
		1				7/22	1/22	0	0	2/11	−1/22	8/11	$\bar{z}_3$
			1			9/22	17/22	0	0	1/11	5/22	4/11	$\bar{z}_4$
				1		0	0	−2/11	−1/11	0	0	12/11	$\bar{z}_5$
					1	0	0	1/22	−5/22	0	0	5/22	$\bar{z}_6$

UY block eliminates y from the basis and projects C onto some vertex of $C \cap K$, one of the desired vertices.

The description of the algorithm provided by Mattheiss is complete but also rather elaborate. Some numerical examples should be of help in following and understanding the thread of reasoning and the required computations which generate all the complementary solutions to a given LC problem. Of course, a careful reading of Mattheiss' paper will provide valuable insights and indispensible details.

Two numerical examples will be discussed. The first example is a linear program with multiple optimal solutions. We desire to enumerate all the basic optimal solutions using Adler and Gale and Mattheiss results. Since it is possible to obtain, rather simply, all the basic optimal solutions by other more traditional procedures, this example will help in understanding Adler, Gale and Mattheiss' algorithm in a way that is useful for more complex problems. The second example is Example 2 of the previous section where a QP problem was detected to possess multiple optimal solutions.

Example 3

Consider the following LP problem:

$$
\begin{aligned}
\max\ & (53/22)x_1 + (39/22)x_2 + 5x_3 + 2x_4 \\
\text{subject to}\ \ & 2x_1 + x_3 + 5x_3 + x_4 \le 4 \\
& x_1 + 3x_2 - 2x_3 + 4x_4 \le 0 \\
& x_j \ge 0 \quad j = 1, \ldots, 4.
\end{aligned}
$$

Although Lemke's algorithm is not the most convenient computational procedure to solve a LP problem, we choose this method to maintain uniformity throughout the paper. Tableaux 9 and 10 present the initial and the optimal Tableaux of the above LP Example 3.

The first primal optimal solution is $\bar{z}_3 = \bar{x}_3 = 8/11$, $\bar{z}_4 = \bar{x}_4 = 4/11$, $\bar{x}_1 = \bar{x}_2 = 0$. The dual optimal solution is $\bar{z}_5 = \bar{y}_1 = 12/11$, $\bar{z}_6 = \bar{y}_2 = 5/22$. The index set of degenerate pairs of complementary variables is $\rho = \{1, 2\}$. The systems of inequalities and equalities corresponding to the face of the convex set of multiple optimal solutions and given by (6) and (7) are, respectively,

$$
\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -7/22 & -1/22 \\ -9/22 & -17/22 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} +
\begin{bmatrix} 0 \\ 0 \\ 8/11 \\ 4/11 \\ 12/11 \\ 5/22 \end{bmatrix}
\ge
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\tag{6'}
$$

$$
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\tag{7'}
$$

Hence, system (7′) is vacuous, while system (6′) can be reduced to the two central inequalities. Mattheiss' algorithm can thus be applied to the following reduced system expressed in the $Ax \le b$ form:

$$
\begin{bmatrix} 7/22 & 1/22 \\ 9/22 & 17/22 \\ -1 & \\ & -1 \end{bmatrix}
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \le
\begin{bmatrix} 8/11 \\ 4/11 \\ 0 \\ 0 \end{bmatrix}
\tag{8'}
$$

Prior to analyzing system (8) algebra-

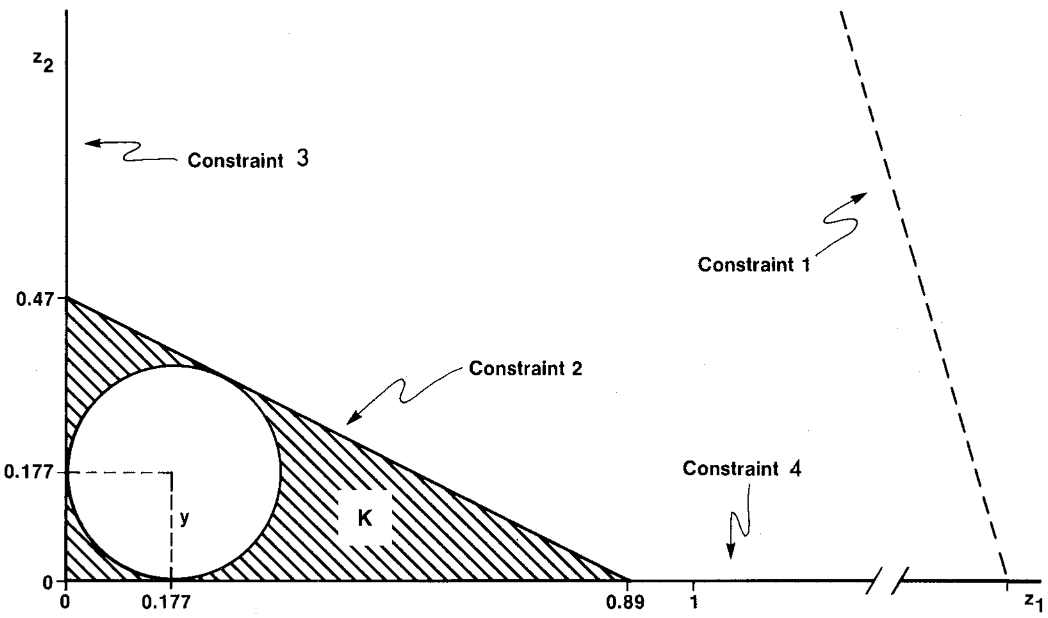

Figure 1. The Set of Solutions, K, to System (8).

ically, and proceeding with Mattheiss' algorithm, it is convenient to graph it. Figure 1 indicates that the convex polytope K of feasible solutions to (8), whose vertices are sought, possesses three extreme points (0, 0), (0.0, 0.89) and (0.47, 0.0) and that constraint 1 is redundant. It also shows that the largest sphere inscribable in the convex set of feasible solutions, K, has a radius y = 0.177.

The initial and the final Tableaux of Mattheiss' set up are presented in Tableaux 11 and 12, respectively. The primal simplex algorithm is used for solving this part of the problem.

Tableau 12 shows that, at this stage the basic variables are z_1, z_2, y and s_1. The nonbasic variables are s_2, s_3 and s_4 which have been starred to indicate that the corresponding constraints are binding. The values of $\bar{z}_1$ and $\bar{z}_2$ (as well as y) are all equal to .1769. They are to be interpreted as the coordinates of the center of the maximum circumference (sphere, in higher dimensions) inscribed in the K-polytope, as illustrated in Figure 1.

Mattheiss' algorithm requires a thorough analysis of Tableau 12. First of all $H_1 = \{2, 3, 4\}$ defines the set of binding constraints for this Tableau. A record R_1 is defined by the value of the linear objective function (the radius of the largest sphere) and by the set of binding constraints, that is, $R_1 = \{.1769, (2, 3, 4)\}$. In the process of analyzing a record, either a new record or a set of vertices of K are

TABLEAU 11. Initial Primal Tableau, Example 3, System (8).

Z	z_1	z_2	y	s_1	s_2	s_3	s_4	B	Basic Variables
1	0	0	−1	0	0	0	0	0	Z
0	7/22	1/22	.3214	1				8/11	s_1
0	9/22	17/22	.8743		1			4/11	s_2
0	−1	0	1.0			1		0	s_3
0	0	−1	1.0				1	0	s_4

TABLEAU 12. Final Tableau, Example 3, system (8) (Reordered).

Z	$\bar{z}_2$	$\bar{z}_3$	y	s_1	s_2^*	s_3^*	s_4^*	B	Basic Variables
1					.4863	.1990	.3758	.1769	Z
	1				.4863	−.8010	.3758	.1769	$\bar{z}_1$
		1			.4863	.1990	−.6242	.1769	$\bar{z}_2$
			1		(.4863)	(.1990)	(.3758)	.1769	y
				1	−.3332	.1819	.2120	.6061	s_1

obtained. A list is a set of records. When all the records have been analyzed and eliminated from the list, the algorithm terminates.

The analysis of a record is performed through a set of pivot operations. Recall that it is admissible to pivot only in the rows corresponding to either y or slack variables. Choose a pivot in each column of the nonbasic variables s* such that it maintains the feasibility of the solution. A pivot executed in a slack row generates a new record. A pivot executed in the y row generates a vertex of K.

Let us proceed to the analysis of Tableau 12, (R_1).

Step 1. $H_1 = \{2, 3, 4\}$.

Step 2. The pivot in the first nonbasic column, s_2^*, is a pivot in the y row, UY, (pivot is enclosed in parentheses) which generates the vertex of K, $Z_1 = (0, 0)$. In fact, the solution column corresponding to this pivot execution is:

$$\begin{array}{ll} \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.3636 \\ 0.7273 \end{bmatrix} & \begin{array}{l} Z \\ \bar{z}_1 \\ \bar{z}_2 \\ s_2^* \\ s_1 \end{array} \end{array}$$

Step 3. The pivot in column s_3^* is, again, a UY pivot corresponding to the vertex of K, $Z_2 = (0.8889, 0.0)$. The solution column corresponding to this pivot execution is:

$$\begin{array}{ll} \begin{bmatrix} 0.0 \\ 0.8889 \\ 0.0 \\ 0.8889 \\ 0.4444 \end{bmatrix} & \begin{array}{l} Z \\ \bar{z}_1 \\ \bar{z}_2 \\ s_3^* \\ s_1 \end{array} \end{array}$$

Step 4. The pivot in column s_4^* is a UY pivot corresponding to the vertex of K, $Z_3 = (0.0, 0.4706)$. The solution column corresponding to this pivot execution is:

$$\begin{array}{ll} \begin{bmatrix} 0.0 \\ 0.0 \\ 0.4706 \\ 0.4706 \\ 0.7059 \end{bmatrix} & \begin{array}{l} Z \\ \bar{z}_1 \\ \bar{z}_2 \\ s_4^* \\ s_1 \end{array} \end{array}$$

The analysis of record R_1 is completed. R_1 is removed from the list. No other record is in the list and the algorithm is terminated. All vertices of K have been identified together with the redundant constraint corresponding to the slack variable s_1 which, for this reason, was not starred.

Notice that in terms of the original linear programming problem of Example 3, the slack variables s_1 and s_2 of Mattheiss' problem correspond to the variables x_3 and x_4. To summarize the enumeration of all the basic optimal solutions of Example 3, we have:

		Optimal Solutions		
	Variables	Vertex 1	Vertex 2	Vertex 3
PRIMAL	x_1	0.0	0.8889	0.0
	x_2	0.0	0.0	0.4706
	x_3	0.7273	0.4444	0.7059
	x_4	0.3636	0.0	0.0
DUAL	y_1	12/11	12/11	12/11
	y_2	5/22	5/22	5/22

It can easily be verified that all three primal basic solutions generate the same op-

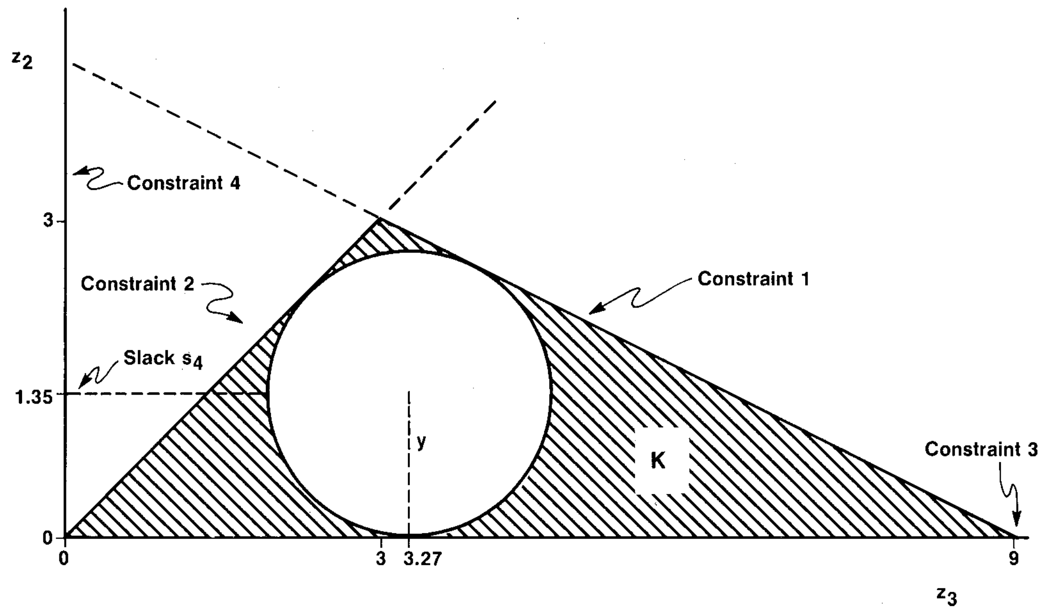

Figure 2. The Set of Solutions to system (9).

timal value of the linear objective function in Example 3, that is $48/11 \cong 4.3636$.

Example 4

To complete the description of the procedure to generate all optimal solutions of a QP problem, Example 2 of the previous section will be fully analyzed. Consider Tableau 6.

The $\bar{M}_{\rho\rho}$ matrix corresponding to $\rho = \{2, 3, 5\}$ is such that $(\bar{M}_{\rho\rho} + \bar{M}_{\rho\rho}')$ is a null matrix. Therefore, also in this example, constraints (7) are inoperative. The $\bar{M}_{\cdot\rho}$ matrix establishes the following relevant inequalities corresponding to (6):

$$-\begin{bmatrix} 2/3 & 1/3 & 0 \\ 0 & 0 & -1/3 \\ 0 & 0 & 1/3 \\ 0 & 0 & 2/3 \\ 1/3 & -1/3 & 0 \end{bmatrix}\begin{bmatrix} z_2 \\ z_3 \\ z_5 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} z_2 \\ z_3 \\ z_5 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Notice that, by inspection, one can immediately conclude that $z_5 = 0$. Thus, it is possible to reduce the problem to two inequalities:

$$\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}\begin{bmatrix} z_2 \\ z_3 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} z_2 \\ z_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (9)$$

The initial and optimal tableaux of Mattheiss' algorithm are presented in Tableaux 13 and 14, respectively. From Tableau 14, record R_1 is $R_1 = \{1.35, (1, 2, 3)\}$. Figure 2 illustrates this record. It shows that the three vertices are (0, 0), (0, 9), (3, 3), while the radius of the largest sphere is 1.35. The distance of the circumference from constraint 4 is the slack $s_4 = 1.92$.

TABLEAU 13. Mattheiss' Initial Primal Tableau, Example 4.

Z	z_2	z_3	y	s_1	s_2	s_3	s_4	B	Basic Variables
1	0	0	−1					0	Z
	2/3	1/3	.7454	1				3	s_1
	1/3	−1/3	.4714		1			0	s_2
	−1	0	1			1		0	s_3
	0	−1	1				1	0	s_4

TABLEAU 14. Mattheiss' Optimal Tableau, Record R_1, Example 4 (Reordered). Pivots in parentheses.

Z	$\bar{z}_2$	$\bar{z}_3$	y	s_4	s_1^*	s_2^*	s_3^*	B	Basic Variables
1					.4511	.4511	.4511	1.3533	Z
	1				.4511	.4511	−.5492	1.3533	$\bar{z}_2$
		1			1.0891	−1.9111	.0903	3.2673	$\bar{z}_3$
			1		(.4511)	(.4511)	(.4511)	1.3533	y
				1	(.6380)	−2.3623	−.3606	1.9140	s_4

Analysis of Tableau 14 starts with the starring of s_1*, s_2*, s_3* because the corresponding constraints are binding. Pivots are in parentheses.

Step 1. The selection of pivot in column s_1* indicates a tie with pivots in both the UY and US block. The pivot executed in the UY row gives the vertex of K, $Z_1 = \{\bar{z}_2 = 0, \bar{z}_3 = 0\}$. The pivot executed in the US block creates a new record, $R_2 = \{0, (2, 3, 4)\}$ corresponding to Tableau 15. The list of records comprises R_1 and R_2.

Step 2. The pivot executed in column s_2* is a UY pivot and gives a vertex of K, $Z_2 = (0, 9)$.

Step 3. The pivot executed in column s_3* is a UY pivot and gives a vertex of K, $Z_3 = (3, 3)$.

Record R_1 is completely analyzed and is discarded from the list. The analysis of record R_2 indicates that by pivoting in columns s_2* and s_3* vertices already identified are generated. The pivot of column s_4* is in the US block and its execution creates a new record $R_3 = R_1$, already analyzed. Hence, the algorithm terminates successfully, having identified all vertices of K.

Notice that, in this example, slack s_1 corresponds to x_1 of the original QP problem. To summarize, the three optimal solutions of the QP problem in Examples 2 and 4 are:

	Variables	Complementary Solutions		
		Vertex 1	Vertex 2	Vertex 3
P R	x_1	3	0	0
I M	x_2	0	0	3
A L	x_3	0	9	3

It can easily be verified that each of these solutions corresponds to a value of the QP objective function of 22.5. Furthermore, any convex combination of these three solutions is another optimal solution. Hence, all the three activities can be operated efficiently at positive levels.

TABLEAU 15. Record R_2 of Example 4.

Z	$\bar{z}_2$	$\bar{z}_3$	y	s_1^*	s_2^*	s_3^*	s_4^*	B	Basic Variables
1					2.1213	.7058	−.7070	0	Z
	1				2.1213	−.2942	−.7070	0	$\bar{z}_2$
		1			2.1213	.7058	−1.7070	0	$\bar{z}_3$
			1		2.1213	.7058	−.7070	0	y
				1	−3.7026	−.5652	1.5674	3	s_1^*

Conclusions

In the 1980s, the determination of the number and the value of multiple optimal solutions in QP is a feasible problem. All basic optimal solutions can be obtained in a rather efficient way if the computational scheme illustrated in this paper is adopted. This applies also to LP problems.

There remains the problem of choosing the solution to recommend or to implement among all the multiple optimal solutions. Depending on the goals of the empirical study, different criteria may be adopted for this task. A particularly appealing one is to choose that optimal solution which minimizes the squared distance from present practices, as suggested by Paris. This procedure requires the identification of all basic optimal solutions first and, secondly, the computation of the optimal weights for combining these basic solutions into an optimal convex combination. Another possibility is to compute first any optimal solution and its corresponding value of the objective function, say Z*. Then, by extending a suggestion by McCarl and Nelson, an optimal solution having the property of minimizing the distance from present practices can be computed by solving the following nonlinear problem

$$\begin{aligned} &\text{minimize } (x_a - x)'(x_a - x)/2 \\ &\text{subject to } c'x - k_x x'Dx/2 \geq Z^* \\ &\qquad Ax \leq b, \quad x \geq 0 \end{aligned}$$

where x_a is the vector of activity levels actually operated. This problem is quadratic both in the objective function and in one crucial constraint. Suitable algorithms already exist for solving such a problem. Its main advantage lies with the fact that it does not require the enumeration of all the basic optimal solutions. Its disadvantage consists in the nonlinear constraint. Furthermore, this procedure does not yield any information on how different various optimal basic solutions might be. The computation of all basic optimal solutions is more informative because it provides a complete analysis of the given QP (LC) problem.

References

Adler, I. and D. Gale. "On the Solutions of the Positive Semi-Definite Complementarity Problem." Report ORC 75-12, Department of Industrial Engineering and Operations Research, University of California, August 1975.

Kaneko, I. "The Number of Solutions of a Class of Linear Complementarity Problems." *Mathematical Programming*, 17(1979): 104–5.

Lemke, C. E. "On Complementary Pivot Theory," in *Mathematics for the Decision Sciences*, Part I, G. B. Dantzig and A. F. Veinott, Jr., editors, Providence, Rhode Island: American Mathematical Society, 1968, pp. 95–114.

McCarl, B. and C. H. Nelson. "Multiple Optimal Solutions in Linear Programming Models: Comment." *American Journal of Agricultural Economics*, 65(1983): 181–83.

Mattheiss, T. H. "An Algorithm for Determining Irrelevant Constraints and All Vertices in Systems of Linear Inequalities." *Operations Research*, 21(1972): 247–60.

Paris, Q. "Multiple Optimal Solutions in Linear Programming Models." *American Journal of Agricultural Economics*, 63(1981): 724–27.

Takayama, T. and G. G. Judge. *Spatial Temporal and Price Allocation Models*. Amsterdam: North Holland Pub. Co., 1971.

von Oppen, M. and J. Scott. "A Spatial Equilibrium Model For Plant Location and Interregional Trade." *American Journal of Agricultural Economics*, 58(1978): 437–45.