Determining Firm-Specific Values for Risky Investments

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This article demonstrates that the usefulness of time-state contingent investment evaluation models need not be constrained by limited time-state contingent markets. Dual solutions to stochastic programs can be used to obtain firm-specific values for risky investments while allowing linear dependence between initial values and later time-state contingent income/technical coefficients. The model could be useful when the exogenous a priori determination of appropriate (and project-specific) risk-adjusted discount rates and/or certainty equivalents is difficult or when the cash equivalents of noncash investment effects are difficult to estimate.

Key words: asset, values, contingent prices, stochastic programming.

Agricultural producers frequently must evaluate assets which generate a series of risky returns across time. Market prices of the assets (if known) may differ from the value of the assets in the firm's portfolio. Firm-specific asset values will be affected by the operator's objectives and the firm's organization, technical constraints, resource availability, and financial limitations. Operator risk perceptions and attitudes also will affect a risky asset's value to the firm.

Several procedures have been used to evaluate an asset generating a series of risky returns. Under a restrictive set of assumptions (Fama), the single-period capital asset pricing model (CAPM) of Sharp and of Lintner can be extended to the multiperiod case (Merton). However, in addition to the normal criticisms of the CAPM (Hakansson), the multiperiod CAPM requires that economic agents know all future risk-free interest rates and market prices of risk with certainty. As indicated by Fama, these assumptions are quite strong and will often restrict the applicability of the multiperiod CAPM.

An alternative and more widely used method of asset valuation involves the use of risk-adjusted discounting or, equivalently (Booth), discounted certainty equivalents. Although each method is widely recommended (Weston and Brigham; Barry, Hopkin, and Baker), the required risk-adjusted discount rates and certainty equivalents usually are obtained in a somewhat ad hoc manner. The problems in choosing appropriate discount rates and/or certainty equivalents are not trivial. Annual risk premiums and risk-adjusted discount rates will be identical across time only if project risk increases appropriately with time (Weston and Brigham, pp. 378–81; Robichek and Myers). In general, risk-adjusted discount rates will be project and time specific as will appropriate certainty equivalents (Booth).

A third method of evaluating risky assets uses the time-state contingent model of Arrow, of Debreu, and of Hirshleifer (ADH). The time-state contingent model is more general than the multiperiod CAPM but has been applied less frequently due to empirical difficulties and the assumption of time-state contingent security markets (Merton).[1] ADH assume that complete markets exist for time-state contingent securities. As indicated by Merton, the absence of such markets has limited the practical applicability of the ADH model.

This article demonstrates that a discrete stochastic programming (DSP) model can have

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[1] The meaning of "time-state contingent" is not always clear in the literature. In this article, the term "state θ in time t" denotes a particular sequence of events from time 0 through time t. The time-state contingent claims of ADH are thus claims which must be redeemed if and only if a given sequence of events occurs.

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practical use in estimating the value of risky assets while incorporating firm-specific constraints and objectives. The DSP model can be used to estimate asset values when the use of risk-adjusted discount rates or certainty equivalents is difficult or inappropriate.[2] Assets can be valued while allowing linear dependence between an asset's initial acquisition cost and later period cash flows and/or technical coefficients. Examples of such dependence are common and include book values, property taxes, credit limits, and tax depreciation allowances which are directly determined by acquisition costs. The procedures presented are similar to those used by Hirshleifer but do not require a complete set of time-state contingent prices.

To use the ADH model in valuing assets, the analyst must have time-state contingent prices.[3] A number of researchers have used variants of Cocks' discrete stochastic programming model in investigating economic behavior in a dynamic and risky setting (Leatham and Baker; Rae 1971a, b; Yaron and Horowitz). If the DSP model incorporates possible future states of nature, as perceived by the decision maker, the solution's dual values can be used as time-state contingent prices. These prices then can be used to evaluate assets as is demonstrated below.

A Firm-Specific Time-State Contingent Model

A DSP model can be written as follows:

$$(1)\qquad \begin{aligned} &\text{Maximize } f(\boldsymbol{z}),\\ &\text{Subject to } \boldsymbol{Cx} - \boldsymbol{z} \geq \boldsymbol{0},\\ &\boldsymbol{Ax} \leq \boldsymbol{b}, \text{ and}\\ &\boldsymbol{x}, \boldsymbol{z} \geq \boldsymbol{0}, \end{aligned}$$

where $f(\boldsymbol{z})$ is a utility or value function defined over potential states of "money" or wealth; $\boldsymbol{z}$ is a vector of potential "money" states; $\boldsymbol{C}$ is a matrix of monetary coefficients; $\boldsymbol{x}$ is a vector of activities; $\boldsymbol{A}$ is a matrix of "nonmoney" coefficients, and $\boldsymbol{b}$ is a vector of available resources. All parameters and choice variables in (1) are time and state contingent.[4] Choice variables are assumed continuous.

Kuhn-Tucker conditions require that the several conditions must hold if a solution $\hat{\boldsymbol{x}}$ is to be optimal for system (1). Included in these requirements is that:

$$(2)\qquad \hat{\boldsymbol{d}}'\boldsymbol{c}_j - \hat{\boldsymbol{s}}'\boldsymbol{a}_j \leq 0 \text{ with all } \hat{\boldsymbol{x}}_j \geq 0.$$

In expression (2), $\hat{\boldsymbol{d}}'$ and $\hat{\boldsymbol{s}}'$ are nonnegative row vectors of dual values associated with monetary and nonmonetary constraints, respectively; $\boldsymbol{c}_j$ is the jth column of $\boldsymbol{C}$; and $\boldsymbol{a}_j$ is the jth column of $\boldsymbol{A}$. As with system (1), the dual values in (2) are both time and state specific.

Assume that the decision maker knows the optimal solution $\hat{\boldsymbol{x}}$ and the corresponding dual values $\hat{\boldsymbol{d}}$ and $\hat{\boldsymbol{s}}$. A new activity, y, not previously in the firm's investment portfolio, is to be evaluated. The activity can be either an investment or a disinvestment and is continuous. The associated activity level cannot be negative. Let v_y be the potential sale or purchase price of one unit of asset y. The new activity's constraint coefficient vectors are decomposed as:

$$(3a)\qquad \boldsymbol{c}_y = v_y\boldsymbol{e} + \boldsymbol{f}$$

and

$$(3b)\qquad \boldsymbol{a}_y = v_y\boldsymbol{g} + \boldsymbol{h},$$

where $\boldsymbol{c}_y$ are "monetary" coefficients associated with activity y, and $\boldsymbol{a}_y$ are "nonmonetary" constraint coefficients. Expression (2) allows any or all coefficients in $\boldsymbol{a}_y$ and $\boldsymbol{c}_y$ to be linear functions of the initial price of the activity.

If the activity is brought into the firm's portfolio, the decision maker's objective will be enhanced only if the following Kuhn-Tucker condition is satisfied:

$$(4)\qquad \hat{\boldsymbol{d}}'\boldsymbol{c}_y - \hat{\boldsymbol{s}}'\boldsymbol{a}_y > 0.$$

Note that expression (4) evaluates the new investment's effects with the dual values of the original optimal solution. Substituting (3) into

[2] The appendix demonstrates when cash equivalents for noncash investment effects can be obtained using time-state contingent prices. Methods to obtain risk-adjusted discount rates and certainty equivalents also are presented.

[3] Although complete time-state contingent markets do not exist, limited time-state contingent securities do. An example is an insurance instrument which commits the insurer to paying the insured a specified amount should an event occur.

[4] The author recognizes that other authors may use more complex notation to describe a discrete stochastic program. However, primal and dual solutions are obtained with conventional optimization algorithms. The assumption is made that the reader can properly construct a sequential stochastic program. The simpler notation in (1) is useful in the following derivations. More complex notation is introduced in the appendix as it is needed.

Table 1. The Two-Period, Three-State Model for the Example Problem

Row and Unit	Activities									Constraint
	Corn	Grain Sorghum	Wheat	Alfalfa	TTARG	*D1*	*D2*	*D3*	Transfer Expected Income	
Objective						*	*	*	1	Maximize
Land 1 (Acres)	1.0	1.0	1.0	1.0						≦ 320
Land 2 (Acres)				1.0						≦ 80
Labor (Hours)	2.8	1.8	2.0	2.6						≦ 2,000
Bad (Dollars)	31.5	42.5	63.0	−9.0	−1	+1				≧ 0
Average (Dollars)	89.7	29.3	77.2	101.5	−1		+1			≧ 0
Good (Dollars)	151.0	132.0	108.0	57.5	−1			+1		≧ 0
Target (Dollars)					1					= 20,000
EXPINC (Dollars)	90.3	52.5	80.5	70.6					−1	≧ 0

* The objective is to maximize $EXPINC - .001[.2\,(D1)^2 + .6(D2)^2 + .2(D3)^2]$.

(4) and rearranging implies that the objective will be enhanced only if:

$$\text{(5a)}\qquad v_y < -\frac{[\hat{d}'f - \hat{s}'h]}{[\hat{d}'e - \hat{s}'g]} \text{ if } \hat{d}'e - \hat{s}'g < 0$$

or

$$\text{(5b)}\qquad v_y > -\frac{[\hat{d}'f - \hat{s}'h]}{[\hat{d}'e - \hat{s}'g]} \text{ if } \hat{d}'e - \hat{s}'g > 0.$$

Expression (5) thus gives an upper or lower bound on an activity's value to the firm. Should the market price of the activity fail to satisfy (5), bringing the activity into the firm's portfolio will not enhance the decision maker's objective and may diminish firm performance. Expression (5) thus gives a "break-even price" or value, such as those examined by Robison and Burghardt, at which the decision maker would be indifferent concerning the acquisition. The "break-even price" so generated can be shown to be equivalent to prices obtained with risk-adjusted discounting or certainty equivalents when the latter can be properly used. However, as the following examples demonstrate, the results of expression (5) also can be used to obtain values when DSP models do not contain complete time-state contingent monetary constraints, nonmonetary constraints, and dual values. The interested reader is referred to the appendix for a more complete discussion.

The results given in (5) can be extended if the objective function of system (1) is linear. In such cases procedures presented by Gal can be modified to enable the identification of firm-specific "demand" schedules for new activities while allowing linear dependence among initial acquisition costs and other constraint coefficients. A detailed derivation of these schedules is beyond the scope of this article but is available from the author.

To demonstrate the usefulness of the above procedures, two numerical examples are presented. The first example is a nonlinear target semivariance model. The second examines multiyear lease rates for an irrigated farm in south-central Nebraska.

A Three-State Target Semivariance Example

Table 1 presents the tableau of a three-state target semivariance model. The target semivariance model, using squared deviations, was chosen because of its ability to generate third-degree stochastically efficient portfolios (Fishburn). The tableau is different than the linear Target MOTAD model as presented by Held, Watts, and Helmers; Tauer; and Watts, Held, and Helmers in that mean income and weighted deviations below the income target directly enter the objective function.

The first four columns represent four crops (corn, grain sorghum, continuous wheat, and alfalfa) grown on irrigated land in south-central Nebraska. A yield and nominal price series (1975–85) was obtained for each crop from annual Bureau of Reclamation reports on the Missouri River Basin, Frenchman-Cambridge Division, H&RW Irrigation District Project. Prices were normalized to 1982 dollars using the GNP implicit inflation index. A real per-acre gross revenue series was constructed and detrended. To reduce the dimensions of the discrete stochastic program and to simplify the

Table 2. Dual Values and "Investment" Vectors for the Example Problem

					Decomposition of Vectors			
					c_5 a_5		c_6 a_6	
Constraint	State Probability	Dual Value ($)	"Investment" Coefficients $(c_5' a_5')'$	$(c_6' a_6')'$	$(e_5' g_5')'$	$(f_5' h_5')'$	$(e_6' g_6')'$	$(f_6' h_6')'$
EXPINC	—	1.0	$-v_5$	$-v_6 + .2$	−1	0	−1	.2
*C*1—'Bad' State	.2	.311	$-v_5$	$-v_6 + 1$	−1	0	−1	1
*C*2—'Average' State	.6	0	$-v_5$	$-v_6$	−1	0	−1	0
*C*3—'Good' State	.2	0	$-v_5$	$-v_6$	−1	0	−1	0
Land 1	—	100.1	−1	0	0	−1	0	0
Land 2	—	67.8	0	0	0	0	0	0
Labor	—	0	0	0	0	0	0	0
$\hat{d}'f - \hat{s}'h$			100.1	.511				
$\hat{d}'e - \hat{s}'g$			−1.311	−1.311				
Maximal Price			$76.35	.390				

example, the time series was modified to obtain an estimate of good, average, and bad states of nature for the project. The real revenues per acre were multiplied by the acres grown to obtain a series of real gross revenues of the project. All observations of gross project revenue more than one standard deviation below and above the mean were assumed associated with bad (*C*1) and good (*C*3) outcomes, respectively. The remaining observations were used to approximate the average outcome (*C*2). The mean gross per-acre return from each of the three states was used to represent the per-acre gross return in the bad, average, and good states for each crop. The number of observations in each state was used to obtain probability estimates.

Expenses and labor requirements were estimated using published crop budgets (Nebraska Cooperative Extension Service). Returns to land, labor, and management are reported in table 1. Three hundred twenty acres of cropland, 80 acres of alfalfa, and 2,000 hours of labor are available. In table 1 deviations are measured below $20,000—a level chosen arbitrarily for this example. (Actual target levels chosen by a decision maker could be those associated with family living expenses, debt payments, or other commitments of the firm.) The solution to the system in table 1 has an objective value of about $31,355 with a mean income of $31,476. Eighty acres of alfalfa are harvested. Wheat is planted on 313 acres with an additional seven acres of corn being grown. Additional information concerning the solution is presented in table 2.

Table 2 presents the dual values of the constraints as well as two example cash investments. The first investment's coefficients are denoted $[c_5' a_5']'$. The activity represents a one-year lease of additional cropland. A cursory examination of the dual values might lead one to conclude that the decision maker would be willing to pay $100.10 to lease an acre of cropland. However, this ignores the effect of leasing land upon the distribution of cash flows. In applying (5a), $(\hat{d}'f_5 - \hat{s}'h_5) = 100.10$; $(\hat{d}'e_5 - \hat{s}'g_5) = -1.311$ generating a maximal value of $76.35 for an acre of leased cropland.[5]

The second investment's coefficients are denoted as $[c_6' a_6']'$. The activity represents the acquisition of an insurance-like asset. One unit of the asset returns $1 if the low-income state *C*1 occurs and zero otherwise. Again, one might be tempted to conclude that an additional unit of insurance is worth $.311, the dual value of the *C*1 constraint. However, when (5a) is applied, the actual value of a unit of insurance is $.390.

The reader will note that for at least some quantity of insurance the decision maker is willing to incur expected losses to obtain income in the bad state of nature. Although the expected value of one unit of the asset is only $.20, the decision maker would incur up to $.390 in expected costs to obtain the insurance.

With both "investments" in table 2, a sim-

[5] With larger scale models, the values $(\hat{d}'e - \hat{s}'g)$ and $(\hat{d}'f - \hat{s}'h)$ can be obtained by appending $(e'g')'$ and $(f'h')'$ to the constraint matrix. Using a procedure similar to the Big-M method in linear programming will guarantee that the $(e'g')'$ and $(f'h')'$ columns are not in the solution. The corresponding "reduced cost" coefficient then can be used to obtain the values needed in (5).

Table 3. Solutions to Chance-Constrained Stochastic Program with P[(Debt/Assets) > .4] ≤ L*

	$L = 0$			$L = .05$		
	Year			Year		
Activities	1	2	3	1	2	3
Corn (Acres)	0	0	0	0	0	0
Grain Sorghum (Acres)	0	0	0	0	0	0
Wheat (Acres)	320	320	320	320	320	320
Alfalfa (Acres)	80	80	61	80	80	80
Work (Hours)	1,152	1,152	1,202	1,152	1,152	1,152
Expected Net Worth ($000)	313.4	335.6	358.9	313.4	335.6	359.7
$\hat{d}'e - \hat{s}'g$		−3.3823			−2.7530	
$\hat{d}'f - \hat{s}'h$		161.84			162.37	
Maximal Nominal Lease Price ($/Acre)	—	47.85	—	—	58.98	—

ple evaluation of dual values associated with a particular constraint would lead to erroneous evaluations of the investment's value to the firm. The acquisition cost of assets often will affect several of the firm's effective constraints simultaneously. This example demonstrates the ease with which such effects can be evaluated. The following lease evaluation example demonstrates the model's effectiveness in a more complex setting.

Determining Multiyear Contractual Lease Rates

In this example a maximum annual payment is derived for a three-year land lease. The model is a three-year stochastic program which selects the planned sequence of production decisions. Expected ending net worth is maximized subject to technical constraints, consumption constraints, and chance constraints upon possible debt-to-asset ratios (see Atwood, Watts, and Helmers). The consumption function is assumed to be

$$C_t = 12{,}000 + .4 \cdot (W_t - W_{t-1}), \tag{6}$$

where C_t is the consumption in period t and W_t is the expected wealth at the beginning of year t. C_1 is set at $18,000. The firm initially is assumed to have $10,000 in cash, $400,000 in other assets, and $120,000 in liabilities. Real long-term borrowing, short-term borrowing, and short-term saving rates are assumed to be 6%, 4%, and 3%, respectively. Inflation is assumed to be 5% per year giving nominal interest rates (using the Fisher effect) of 11.3%, 9.2%, and 8.15%. Annual costs and returns as well as resource use and availability are the same as presented in table 1. Additionally, it is assumed that labor can be hired at a rate of $5 per hour. Excess labor can be employed off farm at a rate of $4.90 per hour. States of nature are assumed independent across years. A three-year lease of land is available at the beginning of the first period. The lessee is required to pay a constant nominal price at the beginning of each of the three years. This problem has 13 coefficients in the lease vector which are linearly related to the bid price. The linearly related coefficients are those associated with opening cash, each of the three possible cash states at the beginning of the second year, and each of the nine possible cash states at the beginning of the third year. The activity also generates one acre of land for each year. The decision maker must determine the value of the lease to the firm.

Table 3 presents information from selected solutions to the above problem. The solutions were obtained by varying the upper limit on the probability of debt-to-asset ratios exceeding 40%. The reader will note that as the probability limit, L*, is changed, the maximal annual lease price obtained from (5) increases from $47.85 per acre to $58.98 per acre. The solution indicates that irrigated wheat is grown on the cropland. However, the opportunity costs associated with corn production are not large ($10–$15/acre) when no probability of debt/asset ratios exceeding .4 is accepted. At higher probability levels, the opportunity costs are lower ($3–$5/acre) indicating that corn and wheat appear to be competitive in the analysis area. Indeed, while more acres of corn than wheat were actually grown in the project during the analysis period, corn acreage decreased significantly while wheat acreage increased steadily.

Summary and Conclusions

This article has presented methods to evaluate investments while incorporating operator-specific objectives, the operator's subjective expectations concerning future events, and firm-specific technical constraints. Dependence between acquisition prices and later-period constraint coefficients can be easily incorporated into such models. The methods presented do not require (*a*) that all noncash investment effects be exogenously converted to cash equivalents, (*b*) that the appropriate discount rates be identified a priori, or (*c*) that complete time-state contingent markets and prices exist. The absence of such requirements should make the model useful when it is difficult to accurately use discounting procedures due to problems in identifying risk-adjusted discount rates and certainty-equivalent values. The fact that dual evaluation techniques do not require complete time-state markets, as contrasted to the traditional ADH model, also should increase the usefulness of the proposed methodology. However, the reader should note that dual evaluation methods suffer from the "curse of dimensionality" which affects all discrete stochastic programming applications. In addition, the decision maker must subjectively assess future states including possible crop yields, prices, and market interest rates. Subjective probabilities also must be determined for such states of nature. The elicitation of subjective probabilities at times may be as difficult as obtaining risk-adjusted discount rates or certainty equivalents.

Finally, the presented dual evaluation methods are not intended to supplant the use of discounting procedures in evaluating investments. However, the ability to examine the effects of various operator objectives and firm-level constraints upon firm-specific investment values should prove useful to both decision makers and applied researchers.

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Appendix

A Discussion of Cash Equivalents, Risk-Adjusted Discount Rates, and Certainty Equivalents

The use of risk-adjusted discount rates (RADR), certainty equivalents (CE), or the traditional ADH time-state contingent model requires that the monetary equivalents of nonmonetary investment effects be known before the evaluation process can proceed. The a priori identification of cash equivalents and their distribution is often difficult when an investment is used in a multiple input-output firm. The model presented in the text does not require that the cash equivalents of investment effects be known *ex ante.* The model thus could be valuable when the exogenous identification of cash equivalents is difficult. Additionally, the results of the text's model can be used to generate cash equivalents, risk-adjusted discount rates, and certainty equivalents. However, if such results are to be obtained, the model must be constructed with a complete set of time-state constraints on monetary and nonmonetary investment effects. This requirement often may be quite restricting and is not satisfied with either example presented in the text. Both models would need to be significantly restructured if they were to be analyzed with RADR or CE methods.

Before deriving RADR or CE values, cash equivalents of investment effects must be derived. A new investment's value to the firm can be written as

$$\text{(A1)} \qquad \hat{\boldsymbol{d}}'\boldsymbol{c}_y - \hat{\boldsymbol{s}}'\boldsymbol{a}_y = \sum_t \sum_\theta \hat{d}_{t\theta} c_{t\theta} - \sum_t \sum_\theta \hat{\boldsymbol{s}}_{t\theta}'\boldsymbol{a}_{t\theta}$$

where the time (t) and state (θ) components of $\hat{\boldsymbol{d}}$, c_y, $\hat{\boldsymbol{s}}$, and $\boldsymbol{a}_y$ have been delineated. The vector $\hat{\boldsymbol{s}}_{t\theta}$ contains the dual values of the nonmonetary constraints associated with time t and state θ. To obtain time-state monetary equivalents, $c_{t\theta}^*$, the following condition is required for all t and θ:

$$\text{(A2)} \qquad \begin{aligned} \hat{d}_{t\theta} c_{t\theta}^* &\equiv \hat{d}_{t\theta} c_{t\theta} - \hat{\boldsymbol{s}}_{t\theta}'\boldsymbol{a}_{t\theta} <=> c_{t\theta}^* \\ &\equiv c_{t\theta} - (1/d_{t\theta})\, \hat{\boldsymbol{s}}_{t\theta}'\boldsymbol{a}_{t\theta}, \end{aligned}$$

assuming that cash in time t and state θ is valued. With this definition, the investment's value to the firm can be seen to be:

$$\text{(A3)} \qquad \hat{\boldsymbol{d}}'\boldsymbol{c}_y - \hat{\boldsymbol{s}}'\boldsymbol{a}_y = \sum_t \sum_\theta \hat{d}_{t\theta} c_{t\theta}^*.$$

Once cash equivalents are obtained, the procedures given in Hirshleifer can be used to identify risk-adjusted discount rates, risk-free discount rates, and certainty equivalents. The derivations are briefly reviewed here.

Assume that at time 0 the decision maker knows his or her present cash position with certainty. The value of the investment then can be written as:

$$\text{(A4)} \qquad W_0 = \hat{d}_0 c_0^* + \sum_{t>0} \sum_\theta \hat{d}_{t\theta} c_{t\theta}^* = c_0^* + \sum_{t>0} \sum_\theta P_{t\theta} c_{t\theta}^*,$$

where W_0 is Hirshleifer's Present Value Certainty Equivalent (p. 261), $P_{t\theta} = \hat{d}_{t\theta}/\hat{d}_0$ is a price or discount factor which converts further time-state money units to time 0 values, and $\hat{d}_0$ is assumed positive.

Risk-Adjusted Discount Rates

Let $\pi_{t\theta}$ denote the probability of being in state θ at time t. The risk-adjusted discount factor, $\tilde{r}_t$, is defined as (Booth):

$$\text{(A5)} \qquad (1 + \tilde{r}_t)^{-1} = \left[\sum_\theta P_{t\theta} c_{t\theta}^*\right] \Big/ \left[\sum_\theta \pi_{t\theta} c_{t\theta}^*\right].$$

This allows (A4) to be rewritten as:

$$\text{(A6)} \qquad W_0 = c_0^* + \sum_{t>0} (1 + \tilde{r}_t)^{-1} \bar{c}_t^*,$$

where $\bar{c}_t^* = \sum_\theta \pi_{t\theta} c_{t\theta}^*$ is the expected value of the cash equivalent in time t.

Certainty Equivalents

To obtain certainty equivalents, let the risk-free discount rate, r_t, be defined as (Hirshleifer):

$$\text{(A7)} \qquad (1 + r_t)^{-1} = \sum_\theta P_{t\theta}.$$

The certainty equivalent then can be obtained as:

$$\text{(A8)} \qquad CE_t^* = (1 + r_t) \sum_\theta P_{t\theta} c_{t\theta}^* = \frac{(1 + r_t)}{(1 + \tilde{r}_t)} \bar{c}_t^*.$$

The reader can verify that:

$$\text{(A9)} \qquad W_0 = c_0^* + \sum_{t>0} (1 + r_t)^{-1}\, CE_t^*.$$

The maximum that would be spent to acquire the investment is $\sum_{t>0} (1 + \tilde{r}_t)^{-1}\, \bar{c}_t^* = \sum_{t>0} (1 + r_t)^{-1}\, CE_t^*$. If later period monetary or nonmonetary investment effects are directly influenced by the initial acquisition price, expressions (A4) through (A9) can be modified so as to give results similar to those in expression (5) of the text.

The reader will note that the derivation of (A1)–(A9) requires strict modeling assumptions concerning the availability of complete time-state contingent dual values. Under these assumptions the methods presented in the text generate equivalent values and can be determined in a more straightforward manner.