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1. INTRODUCTION

CONSIDERABLE PROGRESS was achieved recently in the development of urban travel demand functions, in particular the successful calibration of modal split function of the logit type [6]. If the travel demand function is specified and successfully calibrated so that it is responsive to changes in the service offered by the transportation system, then a proper estimate of the effect of changes in the transportation system on the resulting trip interchanges and link volumes can be achieved within the framework of an equilibration procedure.

The purpose of this paper is to present a method of combining a demand function with a network equilibrium procedure that results in an extended multimodal equilibrium type model; the modes are the private car and one or more public transit service. This undertaking is possible as the prediction of traffic flows on congested networks by using equilibrium type methods is a transportation planning tool of some demonstrated validity [13].

The flow of vehicles by all modes on the road network is modelled in the framework of a road traffic equilibrium model by considering public transit modes that are assigned fixed itineraries, such as bus lines. Then distinction is made between the flow of vehicles on the road network and the flow of passengers that use public transit vehicles. It is important to remark that in order to model the interaction on the road network, the public transit occupancy is irrelevant. However, the choice of passengers between the modes has a direct effect on the congestion of the road network since the switching from private cars to public transit diminishes the number of private cars that use the road network. As in all modelling efforts that are concerned with behavioural phenomena, considerable empiricism will be used, both in the description of congestion effects and in the use of various demand model structures for predicting the modal choice. All the empirical relations that will be assumed are concordant with the accepted state of the art in the description of vehicular traffic and transportation demand modelling. The problem that we will succeed to solve in part is that of providing a method for urban transportation system analysis that is consistent with the demand model and the network interaction effects. The problem and the deficiencies of the standard UTP computations are clearly described by Manheim [4].

2. ALTERNATIVE STRUCTURE FOR THE DEMAND MODEL

We adopt the following notation in order to refer to the person trips that are of interest. We let g be the total number of trips by all modes and between all origins and destinations. g_{pq} denotes the number of person trips between origin p and destination q. We denote the modes by index m, that is $g_{pq}{}^{m}$ is the number of person trips between p and q that occur by mode m. In particular we shall sometimes refer to $g_{pq}{}^{au}$ as the trips by private car and $g_{pq}{}^{tr}$ as the trips taken on a transit mode.

We consider two broad classes of demand models: models that determine $g_{pq}{}^{m}$ as a function of accessibility variables by all modes between the pair (p,q) alone and models that determine $g_{pq}{}^{m}$ as a function of accessibility variables by all modes between all pairs (p,q). As we shall show later the exact specification of the demand model is of secondary importance as we are concerned primarily with the variation of $g_{pq}{}^{m}$ with $u_{pq}{}^{m}$, the accessibility between pair (p,q) by mode m in time or generalized time units. Thus, for a model of the first class mentioned the general form would be

$$(1)\quad g_{pq}{}^{m} = f\,\{ u_{pq}{}^{m}\ ,\ m = 1,\ldots, M\ ;\ \text{OTHER} \}\qquad ,\ m = 1,\ldots, M$$

where M denotes the number of modes considered and OTHER denotes all the remaining explanatory variables such as fares, costs, car ownership, socio-economic characteristics, etc. In an eventual application, the OTHER variables contribute a constant to the functional form. We require only that these functions behave in a reasonable way, that is $g_{pq}{}^{m}$ decreases with increasing $u_{pq}{}^{m}$. A specific example of such a function is

$$(2)\quad g_{pq}{}^{*m^*} = \bar{g}_{pq} \cdot \frac{e^{\Theta\, u_{pq}{}^{*m^*}}}{\sum_{m} e^{\Theta\, u_{pq}{}^{m}}}\ ,\qquad m = 1,\ldots, m^*,\ldots, M$$

That is, we consider a fixed total number of trips between p and q and we adopt a logit model to describe the division of trips between modes. Other specific forms for the class of functions (1) may be considered.

The second class of demand models considered is of general form

$$(3)\quad g_{pq}{}^{m} = f\,\{ u_{pq}{}^{m}\ ,\ m = 1,\ldots, M \text{ and } q = 1,\ldots, Q;\ \text{OTHER} \}$$

Urban Travel Demand Models and Multi-Modal Traffic Equilibrium

*by Michael Florian**

Implicit in this structure is that choices of modes and choices of destinations are determined simultaneously and possibly, the total number of trips as well. A specific example of such a function may be derived from Dial's [8] extended logit model as

$$(4)\quad g_{pq}{}^{*m^*} = \bar{g}_p \cdot \frac{r_q \cdot e^{\Theta\, u_{pq}{}^{*m^*}}}{\sum_q \sum_m r_q \cdot e^{\Theta\, u_{pq}{}^{m}}},$$

$$m = 1,\ldots,m^*,\ldots,M$$
$$q = 1,\ldots,q^*,\ldots,Q$$

where $\bar{g}_p$ is the total number of trips originating at p and r_q is an index of the attraction of zone q that must be calibrated from data. It is worthwhile to remark that in the classical UTP sequence, an origin destination matrix is forecast by a gravity type model and the usual post distribution modal split model belongs to the class (1). Also the (intercity) direct demand models such as the Quandt and Baumol [26] mode abstract model and McLynn's composite analytic model [21] belong to the class (1).

In addition we shall assume that the number of person trips by private car is converted to a number of vehicle trips by using a car occupancy factor γ_{pq} which may vary by origin, by destination or both by origin, and destination. In practice, car occupancy factors are widely used in transportation studies although the assumption that it does not vary with mode characteristics may pose problems if one of the modes considered is travel by a car pool.

3. SPECIFICATION OF THE ROAD NETWORK MODEL

We consider a network model of the street and road network of an urban area. We identify nodes N and oriented links A. A node represents an origin, a destination or an intersection of streets. A two way street is represented by directed links of opposite orientation. On each link of the network, the congestion effects are represented by link volume delay functions $s_a\ (v_a)$ for links $a \in A$. These are convex increasing functions of the link flows v_a. A good discussion of volume delay functions is given by Branston [4]. On the road network, the private car travellers from p to q use a set of routes that carry flows $h_{k,pq}$. The public transit lines, that share the use of the street network, are represented by lines ℓ that are incident to the links on their itinerary. We define variables $\triangle_{a\ell}$ for all such lines.

$$(5)\quad \triangle_{a\ell} = \begin{cases} 1 & \text{if line } \ell \text{ uses link } a \\ 0 & \text{otherwise} \end{cases}$$

All public transit services that have exclusive rights of way, such as subways or commuter trains, do not interfere with the road traffic. We shall consider these services explicity when specifying the public transit network.

On a link which carries vehicles of both modes, the public transit services consume a fixed amount of the capacity of the link. We recall that equilibrium models are static models that consider the traffic flow in a preselected time slice, which may be of different duration. Thus, for a given time period, the number of public transit vehicles which use a link are given by the line frequency f_ℓ for the time period chosen. In order to determine the effect of these vehicles on the private car traffic we shall assume that a transit vehicle is equivalent to a multiple of private cars. This multiple, α, may be determined empirically. In traffic engineering studies [14] a bus is equivalent to 3 or 4 private cars. Thus, the total flow on a link is

$$(6)\quad v_a = \sum_{\ell} \triangle_{a\ell} \cdot f_\ell \cdot \alpha + \sum_{(p,q)} \sum_{k} \delta_{ak,pq} \cdot h_{k,pq}\,,\ \text{all } a$$

where

$$(7)\quad \delta_{ak,pq} = \begin{cases} 1 & \text{of route } k \text{ for pair } (p,q) \text{ uses link } a \\ 0 & \text{otherwise} \end{cases}$$

The first part of the summation in (6) is fixed, while the route flows $h_{k,pq}$ and hence v_a are variable. It is worthwhile

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to note that implicit to this formulation is the assumption that transit vehicles are equally likely to use any of the lanes in a multi lane street, although practical experience suggests that buses are more likely to use the inside lane due to their stopping requirements. We judge that the bias introduced in this way is not more serious than the similar assumption that private cars are equally likely to use any of the lanes of a street, which is implicit in the definition of the volume-delay functions.

The interaction of the transit vehicles and private cars on the network affects the speed of the transit vehicles. Over a link, the time implied from the volume delay function, $s_a(v_a)$ applies to private cars.

$$(8)\quad s_{a,\ell}^{tr}(v_a) = s_a(v_a) \cdot \beta_{a,\ell}$$

where tr denotes transit and $\beta_{a,\ell}$ is a constant that may vary with line ℓ and link a. If link a has some bus priority rule implemented, then $\beta_{a,\ell}$ may vary accordingly. In general it is plausible that $\beta_{a,\ell} > 1$ and we assume that it can be determined empirically.

4. SPECIFICATION OF THE TRANSIT NETWORK MODEL

We assume that the itineraries for the transit lines and their frequencies f_ℓ are fixed for the purpose of computing the equilibrium flows. The simulation of the flows of transit users relies almost entirely on the behavioural assumption that passengers select and use shortest routes in the transit network. Although the "all-or-nothing" assumption is recognized to be too restrictive, most operational models rely entirely on it. Typical examples are the work of Dial and Bunyan [10], L'Abbe and Scherer [16] and more recently the development of TRANSCOM by Robillard et al [1].

The capacity of a line is not considered explicitly. This is justifiable if there always is sufficient capacity to transport all passengers who wish to travel. When this condition is not satisfied, the all-or-nothing method is deficient, however its continued use reflects on the lack of a method which considers capacities explicitly.

We consider a network model of the transit network that identifies lines $\ell \in L$ and nodes N. The nodes N may be centroids or intersections of lines. We assume that the access to the transit line and the transfer between lines are modeled implicitly or explicitly. The origin to destination access time by transit is

$$(9)\quad u_{pq}^{tr} = u_{pq}^{tr}\,(\text{access}) + u_{pq}^{tr}\,(\text{in vehicle}) + u_{pq}^{tr}\,(\text{transfer}) + u_{pq}^{tr}\,(\text{waiting})$$

As mentioned before the in vehicle time varies with the link congestion levels. If the frequencies f_ℓ are maintained all other time components may be assumed constant, except the waiting time, which increases when the public transit service operates near capacity. In most operational transit simulation models, waiting time is nevertheless assumed to be constant. For transit services that have reserved rights of way, the in vehicle time is independent of the road network.

The accessibility between p and q by transit is obtained then by computing shortest routes on the transit network, by considering all the time components measured in generalized time units as in (9).

Although we have considered the frequencies and itineraries fixed, we shall show later that this equilibrium model may serve to test the effect of supply changes on the road and transit networks. These changes may be considered as discrete, parametric changes that are exogeneous to the model.

5. THE PRIVATE CAR AND TRANSIT EQUILIBRIUM MODEL

For simplicity of the exposition we develop the model for private car and one public transit mode. We suppose that a "user-optimized" equilibrium occurs on the road network. That is

$$(10)\quad \sum_a \delta_{ak,pq} \cdot s_a(v_a) = u_{pq}^{au} \quad \text{if } h_{k,pq} > 0$$

and

$$(11)\quad \sum_a \delta_{ak,pq} \cdot s_a(v_a) \geqslant u_{pq}^{au} \quad \text{if } h_{k,pq} = 0$$

Also, the flows are conserved and thus

$$(12)\quad \sum_k h_{k,pq} = g_{pq}^{au}(u_{pq}^{au}; u_{pq}^{tr}) \quad \text{all } (p,q)$$

$$(13)\quad h_{k,pq} \geqslant 0 \text{ for all } k \text{ and all } (p,q)$$

and v_a is defined as in (6).

Note that we have assumed that the demand function is of the first class described (1) and we have left out the OTHER variables for convenience.

The computation of flows that satisfy the equilibrium conditions is a very difficult problem unless it is possible to derive an equivalent minimization problem. Beckman [2] showed that an equivalent problem exists when g_{pq} is a function of u_{pq} alone. In our model, the in vehicle transit time and hence u_{pq}^{tr} varies in a non linear way with changes in the link travel times s_a (v_a). The shortest route in the transit network which yields u_{pq}^{tr} may vary in length only or it may change altogether, with resulting changes in all time components of (9). We were unable to derive an equivalent minimization problem without the following simplification. We consider u_{pq}^{tr} be constant for the purpose of computing the equilibrium flows and reconsider its variations parametrically, in an ulterior stage of the computations. We obtain then

$$(12') \quad \sum_k h_{k,pq} = g_{pq}^{au} \cdot (u_{pq}^{au}, u_{pq}^{-tr}) \qquad \text{all } (p,q)$$

If the inverse of the demand function exists, that is

$$(14) \quad u_{pq}^{au} = f^{-1} (g_{pq}^{au}, u_{pq}^{-tr})$$

then the equivalent minimization problem is

$$(15) \quad \text{Min} \sum_a \int_0^{v_a} s_a (x)\, dx - \sum_{(p,q)} \int_0^{g_{pq}^{au}} f^{-1} (y, u_{pq}^{-tr})\, dy$$

subject to constraints (6), (13) and

$$(12'') \quad \sum_k h_{k,pq} = g_{pq}^{au}$$

The equivalence of this minimization problem may be verified easily by deriving the Kuhn-Tucker necessary conditions. These are precisely equations (10) and they are sufficient as well since objective function (15) is convex. The convexity of (15) is due to the convexity of the volume functions s_a (v_a) and the fact that if the inverse function (14) exists, it is decreasing with increasing g_{pq}^{au}.

We recognize this problem as a form of a traffic equilibrium problem with elastic demand. Algorithms for this problem were developed by Bruynooghe et al [5], Florian and Nguyen [12] and Nguyen [24]. The computational experience reported in [12] indicates that problems of realistic size may be solved numerically without the requirement of excessive computer time or cost.

The overall computational scheme that we suggest is as following:

Step 0: Select initial estimates of transit vehicle travel times s_a^{tr} for links such that $\triangle_{a\ell} = 1$.

Step 1: Compute shortest routes in the transit network and obtain initial estimates of u_{pq}^{-tr} for all (p,q)

Step 2: Solve the traffic equilibrium problem with elastic demand

$$\text{Min} \sum_a \int_0^{v_a} s_a (x)\, dx - \sum_{(p,q)} \int_0^{g_{pq}^{au}} f^{-1} (y, u_{pq}^{-tr})\, dy$$

subject to

$$\sum_k h_{k,pq} = g_{pq}^{au}$$

$$h_{k,pq} \geqslant 0, \text{ all } k \text{ and all } (p,q)$$

and

$$v_a = \sum_\ell \triangle_{a\ell} \cdot f_\ell \cdot \alpha + \sum_{(p,q)} \sum_k \delta_{ak,pq} \cdot h_{k,pq}, \text{ all } a$$

and obtain equilibrium flows v_a^*, $h_{k,pq}^*$

Step 3: Compute derived transit times s_a^{tr} (v_a^*) from s_a (v_a^*) for all links such as $\triangle_{a\ell} = 1$ and recompute shortest routes to obtain u_{pq}^{tr*}. If $| u_{pq}^{tr*} - u_{pq}^{tr} | > \epsilon$, where ϵ is a suitably chosen convergence parameter, the set $u_{pq}^{tr} = u_{pq}^{tr*}$ and return to Step 2; otherwise continue to Step 4.

Step 4: Determine the (fixed) origin-destination demands

$$g_{pq}^{tr*} = g_{pq}^{tr} (u_{pq}^{au*}, u_{pq}^{tr*})$$

and assign these along the current shortest routes in the transit network.

The above algorithm results in a nontrivial decomposition of the problem into an elastic demand traffic equilibrium problem that may require the computation of several shortest routes on the transit network and into an "all-or-nothing" assignment of a fixed origin-destination matrix on the transit network. This is possible mainly due to the assumption that the transit travellers all use generalized cost shortest routes. Should more refined methods become available for simulating transit flows, then the above model would have to be reformulated accordingly.

The problem in Step 2 may be solved to ϵ equilibrium by using any of the methods in [24] or that in [12]. The convergence of the entire algorithm is not guaranteed due to the relaxing assumption that we have made about u_{pq}^{tr}. However we have good empirical reasons to postulate that the procedure converges nevertheless.

Empiric modal split calibrations studies indicate that the choice among the car and transit mode is not very sensitive to changes in u_{pq}^{tr} (in vehicle). Thus implied difference in the modal demands as a function of this variable may be very small and should not induce oscillatory, non-convergent behaviour of computations. For instance, in the modal split logit type mode estimated by Charles River Associates [6] for the city of Pittsburgh, waiting time is much more important in determining modal split than the difference of "in vehicle" travel time.

It is imperative to point out that the equilibrium model imposes rather severe restrictions on the function g_{pq}^{au} (u_{pq}^{au}, u_{pq}^{-tr}). These are that the inverse function (14) exist, and that it be integrable. We will show however that several typical functional forms of demand models satisfy these restrictions.

We consider next demand functions of the second class, such as (3). The conservation of flow equations are

$$(16) \sum_k h_{k,pq} = g_{pq}^{au} (u_{pq}^{au} ; u_{pq}^{tr} , q = 1,\ldots,Q) \text{ , all } (p,q)$$

It is evident that the inverse function g_{pq}^{au-1} as a function of g_{pq}^{au} alone does not exist, since g_{pq}^{au} depends on the accessibility by car (and transit) to all feasible destinations q. Thus, we are unable to derive an equivalent minimization problem for this class of demand function.

The computation of the equilibrium flows $h_{k,pq}$ for this type of model becomes a remote possibility, since the numerical solution of the simultaneous equations (10), (11), (13) and (16) is a most difficult task. In order to highlight this difficulty simply consider that for an urban area of medium size there may be of the order of 10,000 equations (16), that is there may be of the order of 10,000 origin destination pairs that have a nonzero trip interchange.

In any particular application, it will become evident which form of demand function must be used. A priori, it does not seem evident that the specification of demand functions should be restricted to those functional forms which are invertible. However, if the aims of the transportation study include the explicit estimation of network equilibrium in a consistent manner, then the structural characteristics of the postulated demand functions must be evaluated in the way described above.

6. A COMBINED MODAL SPLIT, ROAD AND TRANSIT ASSIGNMENT MODEL

We consider now the very popular logit model in the form (2) that is, a fixed origin-destination matrix $\bar{g}_{pq}$, p = 1,...,P; q = 1,...,Q is known and the logit model is used for modal split alone. Thus, the incorporation of such a modal split function in the general computational framework outlined earlier would result in the equivalent, in UTPS terms, of a combined modal split, road assignment and transit assignment model. All we have to show is that the logit model is invertible and integrable. The demand function is

$$(17)\ g_{pq}^{au} = \bar{g}_{pq} (1 + e^{-\Theta u_{pq}^{au} + CST})^{-1}$$

Its inverse is given by

$$(18)\ u_{pq}^{au} = \frac{CST}{\Theta} - \frac{1}{\Theta} \ln \left(\frac{\bar{g}_{pq}}{g_{pq}^{au}} - 1\right)$$

which is equivalent to

$$(19)\ u_{pq}^{au} = \frac{CST}{\Theta} - \frac{1}{\Theta} \{\ln (\bar{g}_{pq} - g_{pq}^{au}) - \ln g_{pq}^{au}\}$$

$$\text{for } 0 < g_{pq}^{au} < \bar{g}_{pq}$$

This function is easily integrable since $\int \ln x = x\ln x - x$.

Thus in the framework of the traffic equilibrium problem with elastic demand, the objective function to minimize is

$$(15')\ \text{Min} \sum_a \int_0^{v_a} sc(x)dx - \frac{1}{\Theta} \sum_{(p,q)} \int_0^{g_{pq}^{au}} \{\ln (\bar{g}_{pq} - y) - \ln y\}\, dy + CST1 \text{ for } 0 < g_{pq}^{au} < \bar{g}_{pq}$$

An additional transformation would be required for the first log term in the curly brackets. It is interesting to note that algorithms for the equilibrium problem with elastic demands require explicit knowledge of g_{pq}, an upper bound for the demand for trips between p and q. See for instance [13] and [24].

7. DISCUSSION OF MODEL SENSITIVITY, DATA REQUIREMENTS AND COMPUTATIONAL ASPECTS

All the models that belong to the general class of the model described in section 5, depend on the calibration of an appropriate demand function. In the development of models that serve to plan future transportation systems, the demand function seems at present to be the weakest link. The UTP sequence of generation, distribution and modal split for the prediction of demand for travel is cumbersome and has been widely criticized. However, no clear alternative has emerged and each important study seems to produce a different form for the demand function.

The requirement that is often asked of transportation planning models is that they be policy responsive, in the sense that some of the variables are explicit policy variables. These are variables that may be used as political parameters to change the distribution of traffic flows over specified links, modes or between origins and destinations. Viewed in this light the model that we propose is able to predict, within the behavioural assumption of traffic equilibrium, the effects of changes in policy variables, provided that the demand model has the proper sensitivity to all exogenous policy variables, (these are precisely the OTHER variables that we have conveniently left out the demand model specification) as well as to accessibility variables which may change as a result of the interaction on the network. We emphasize that a valid demand model is a prerequisite for use of such a multimodal model for short range or long range policy evaluation.

If we suppose that such a demand model is available, then a variety of policy changes may be evaluated via such a model. Some of these are:

- the effect of changes in public transit services such as line routing, line frequency, comfort, fares, introduction of express lines, etc. . .
- the effect of road pricing and increases in parking rates
- the effect of changes in the road network such as changes in one way orientation of streets, closing or opening of streets, etc. . .

The algorithms that are available for solving the problem in Step 2 of the algorithm have been tested partially. The computational experience obtained by Florian and Nguyen [12] indicates that if the computations are initiated with a "good" solution (which is sometimes available as a byproduct of an origin-destination survey), the total computational effort spent is 1/4 to 1/3 more than that required to solve a fixed demand traffic assignment problem. The current equilibrium codes available for solving such fixed demand problems, such as TRAFFIC [25], can obtain equilibrium flows for networks of the order of 5000 links and 150 zones in about 10 min. of CDC CYBER 74 time or about $300 in cost. Thus, the computational aspects do not pose a problem. The computation of shortest routes in the transit network should not cause any difficulty either, as super efficient algorithms such as that developed by Dial [9] have been implemented in most existing transit assignment codes.

8. CONCLUSION

In the preceding sections we have specified a traffic equilibrium model of travel by private car and one or more public transit modes. Efficient computational methods are available to obtain numerically the equilibrium flows for demand functions of the first class (1), however the validity of the model outputs depends on the validity of the demand function used. The model is sufficiently general to accommodate a wide variety of such demand functions, if they may be suitably calibrated.

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