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Mending the Family Tree: A Reconciliation of the Linearization and Levels Schools of CGE Modelling

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ABSTRACT

This paper offers a critical comparison between the North American *levels* school of applied general equilibrium modelling and the Norwegian/Australian school of linearizers. The paper develops both the levels and linearized representations of a neoclassical, multiregion trade model. This development is used to focus attention on similarities and differences between the two schools. The main conclusions are as follows.

(i) The method used to solve applied general equilibrium models is not really the issue — the solution method used has become short-hand for a host of cultural differences reflecting the orientation of the two groups.

(ii) Levels or linearized versions of models are equally valid representations. Either representation is a natural starting point for obtaining accurate solutions of the model.

(iii) Linearized versions often aid transparency in explaining the mechanisms at work in a model.

(iv) In view of recent developments with the *GEMPACK* software suite, it is no longer necessary for linearizers to settle for solutions containing linearization errors.

(v) The two schools have a great deal in common and both would benefit from greater cooperation.

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I. *Introduction*

Applied general equilibrium analysts have often seemed obsessed with computational issues and solution algorithms. This usually strikes the uninitiated as rather peculiar, since economists normally leave these issues to specialists in algorithm development. However, the intellectual origins of research in applied general equilibrium are deeply rooted in computational concerns. The first problem, and one which preoccupied many welfare economists during the first half of this century, was whether or not it was actually feasible to perform a centralized computation of the Pareto optimal allocation of an economy's resources (Whalley (1986), pp. 30-34). Gradually attention shifted from the centralized "planner's problem" to the decentralized equilibrium problem. This transition culminated in Johansen's implementation, in the late 1950s, of the first applied general equilibrium (AGE) model. His innovation was to write the equations down in their linearized form, thus permitting the model to be solved by inverting a single matrix. The equilibrium problem which Johansen solved is most appropriately viewed as a local perturbation to an initial equilibrium position. This solution was, of course, only exact for an infinitesimal change in the exogenous variables. However, given the uncertainty associated with many model parameters, this method of solution was deemed appropriate for small changes, since linearization errors were likely to be less significant than the error bounds associated with parametric uncertainty.

Several long-lived research programs have evolved, based on what is often termed the "Johansen approach" to AGE modelling. Most notable are the efforts which followed up on Johansen's initial contribution in Norway (see Schreiner and Larson for an overview), and the work initiated by the Impact Project in Australia (Dixon *et al.* (1982); see also Powell and Lawson (1986)). Apart from the fact that these models are written down as systems of linearized equations, it cannot generally be said that they share any attributes which distinguish them uniquely from other types of AGE analysis. Indeed, the range of model types and the variety of applications of linearized AGE analysis is probably representative of the breadth of the field as a whole.

The obvious drawback of the linearized approach to AGE analysis is the lingering question: How small is a small perturbation? Alternatively: How rapidly does the local approximation deteriorate as one moves away from the initial equilibrium? Periodically, researchers have attempted to address this question by solving a *given* perturbation to a *particular* model with a *particular* set of parameters via both the linearized the non-linear routes. (The numerous qualifiers in the previous sentence indicate the difficulty of generalizing from such experience.) Examples include Dixon *et al.*. (1982), and Bovenberg and Keller (1981). Judging from the subsequent work of these authors, they seem to have concluded that the approximation errors were of an acceptable magnitude for many applications.

An alternative "school" of AGE modelling emerged in the late 1960s motivated by the algorithmic work of Scarf (1973) which provided a robust method for solving non-linear general equilibrium problems "in the levels", as distinct from solving a linearized represention for proportional changes in endogenous variables. Due to the specificity of these early non-linear solution algorithms, applications tended to be limited to those working with Scarf, or with his students (most notably Shoven and Whalley). Eventually, however, general-purpose software evolved, based on advances in the operations research field, and it is now relatively routine to solve nonlinear AGE models.[1]

Given the fact that most large-scale modelling work tends to become somewhat competitive, and also that, until recently, obtaining a non-linear general equilibrium solution was a non-trivial exercise, it is perhaps not surprising that something of a split developed between the linearization and levels schools of modelling. Typical arguments run as follows. Adherents of the former school will often argue that information about behavioural elasticities is only locally valid. Also, since most policy changes are only marginal, a local perturbution is adequate. Besides, the "linearizers" argue, the model itself is much simpler to interpret when expressed in elasticity form. The non-linear school quickly counters: while it may be true that most *actual* policy reforms are marginal, the linearizers often look at non-marginal hypothetical changes. Furthermore, despite parametric uncertainty, if the model's equations (a production function, for example) do not hold at the new equilibrium, how can one possibly do proper welfare analysis?

High-minded arguments aside, purely practical considerations have meant that most AGE modellers perpetuate the method which they themselves first learned. Since the form in which an AGE model is written is its most visible manifestation, the linear *versus* levels labels quickly became a proxy for other issues, such as preferred closure of the model, etc. To many outsiders, this schism between the two schools of AGE modelling appeared quite irrelevant, being dwarfed by concerns about the vast number of parameters required, and the validity of assuming a simultaneous equilibrium in all markets.

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1 GAMS (Brooks, Kendrick and Meeraus, 1988) and MPS/GE (Rutherford, 1989) are two popular alternatives. The latter is specifically designed for AGE models, while the former is not.

We believe that the linear/non-linear split is a red herring which has hindered progress in the field of AGE-based research. This paper represents one attempt to help bridge the illusory gap between these two schools of modelling. We do so by taking a neoclassical, multiregion trade model and developing both its levels and linearized forms. Along the way a number of interesting insights about the two alternative ways of expressing an AGE model crop up. We then demonstrate that by treating the linearized form as a system of differential equations to be numerically integrated, *the two models, when solved, will produce the same new equilibrium.* We conclude with a discussion of the pros and cons of each approach.

II. An Overview of the Model

The model employed in this paper is a relatively simple, multiregion trade model based on perfectly competitive behaviour. A further simplification involves the use of a single, representative household which absorbs all income in each region, and which generates all final demands. Primary factors of production are in fixed supply at both the global and regional levels. Thus new investment does not come "on line" over the course of the simulation. Each household purchases a portfolio of capital goods in order to satisfy its demand for savings. If these are foreign capital goods, then the transaction represents a capital outflow and conversely. All international transactions other than trade in merchandise, non-factor service trade, and international investment, are treated as transfer payments among households. Thus we abstract from the fact that each household's initial factor income may depend on returns to foreign assets. The latter possibility could be easily accommodated, given the availability of data on the pattern of international cross-ownership of assets.

Figure 1 provides an overview of the basic structure of each regional economy, as represented in the model. Starting at the bottom of this figure, we see that domestic industries compete for a common primary factor endowment in order to supply domestic and foreign markets with tradeable producer goods. Producer goods are absorbed by the so-called "margins industries", which assemble products from different origins and subsequently provide domestic consumers with a nontradeable consumer good. Note that this specification permits us to identify gross trade flows, since products leaving the country differ from similar products entering the country. In other words this is an "Armington" type of trade model.

Note also that all goods are treated in a symmetric fashion in Figure 1. Thus, in the case of capital goods, households are viewed as purchasing a portfolio of assets from the appropriate margins industry (e.g., banking and financial services). The composition of this portfolio depends on the relative attractiveness of domestic and foreign capital goods. Finally, transfers are handled by endowing the recipient region with a primary factor: "goodwill", and creating a demand for this factor in the donor region. In this way, these transfers are also translated into market transactions.[2]

Table 1 lays out the relevant notation for this model. Note that the price of each commodity may vary across all uses, so household prices are distinct from firm prices, which in turn may differ from domestic market prices. In the case of tradeable commodities, the domestic market price of i in region

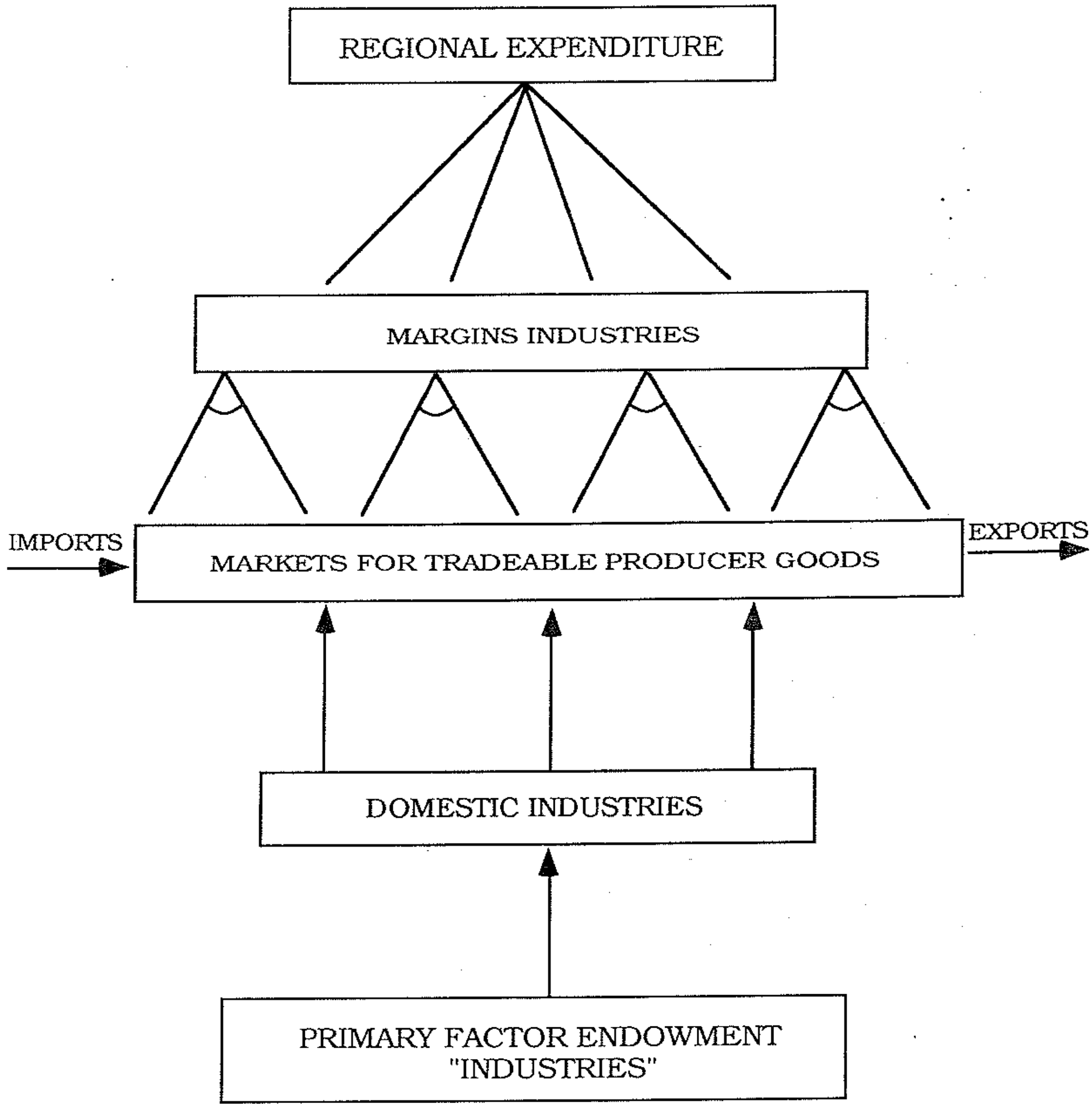

Figure 1: An overview of the model

2 This approach to handling transfers follows Rutherford (1989). Its main virtue is that it simplifies the model structure, as the pattern of transfers is provided by the data-base, rather than as modifications to the model's equations.

Table 1

*Notation Employed in the Model Exposition**

Variables	Description
Prices: P_{Wi}	World market price of commodity i (i∈ T)
P^r_{Mi}	Domestic price of i in region r
P^r_{Hi}	Household price of i in r
P^r_{Sij}	Supply price of i received by industry j in r
P^r_{Dij}	Demand (purchase) price of i paid by j in r
Taxes: $T^r_{Ti} = P^r_{Mi}/P_{Wi}$	*Ad valorem* border intervention on i in r (i∈ T)
$T^r_{Hi} = P^r_{Hi}/P^r_{Mi}$	Power of tax on i consumed by households in region r
$T^r_{Sij} = P^r_{Sij}/P^r_{Mi}$	Power of tax on sales of i by industry j in r
$T^r_{Dij} = P^r_{Dij}/P^r_{Mi}$	Power of tax on purchases of i by j in r
Quantities: Q^r_{Hi}	Demand for commodity i by households in region r
Q^r_{Sij}	Supply of i by industry j in r
Q^r_{Dij}	Demand for i by j in r
Values: $V^{rA}_{Hi} = P^r_{Hi}\, Q^r_{Hi}$	Household expenditures on commodity i in region r (valued at agent's prices)
$V^{rA}_{Sij} = P^r_{Sij}\, Q^r_{Sij}$	Revenues associated with sale of i by industry j in r (valued at agent's prices)
$V^{rA}_{Dij} = P^r_{Dij}\, Q^r_{Dij}$	Producer expenditures on i by j in r (valued at agent's prices)
V^{rA}	Value of a transaction in region r at agent's prices
V^{rM}	Value of a transaction in region r at market prices at domestic market prices ($P^r_M\, Q$)
V^{rW}	Value of a transaction in region r at world market prices ($P_W\, Q$)
Other Variables: Z^r_j	Activity level of industry j in region r
Y^r	Households' disposable income in region r
U^r_i	Subutility associated with purchases of i by region r
U^r	Aggregate utility in region r
Sets: T	Set of all tradeable commodities
NT	Nontradeable commodities
C	Consumption commodities
NC	Nonconsumption commodities
EI	Endowment industries
EC	Endowment commodities

* We adopt certain conventions to simplify notation used for quantification in this table and for quantification and summations in the equations of this paper. Unless otherwise indicated, index r ranges over all regions, index j ranges over all industries and indices i and k range over all commodities; this applies to quantifiers (where "∀" means "for all") and sums.

r (P^r_{Mi}) will also diverge from the world market price of i (P_{Wi}) in the presence of border interventions. All interventions are characterized in *ad valorem* equivalent form, and there is a complete vector of such taxes for each agent in each region. Finally, input-output separability in production is assumed, so that supplies (Q_S) may be uniquely distinguished from demands (Q_D).

III. A Detailed Exposition of the Model in Both Nonlinear and Linearized Forms

This section lays out the detailed structure of the trade model, both in its levels and linearized[3] forms. Along the way we will discuss economically relevant differences between the two formulations and consider the ease with which each may be implemented and interpreted. We begin with the accounting equations. These are what define the model as *general equilibrium* in nature. We then turn to the behavioural equations which describe how each agent's market transactions are determined.

Accounting Conditions

Market clearing: There are two types of market clearing conditions in the model. The first set applies to commodities i which are not traded across regional borders (i ∈ NT). In the model described above, this set includes the primary factor endowments, and consumer goods which are supplied by the domestic margins industries. These may be written as follows:

$$\sum_j Q^r_{Sij} = \sum_j Q^r_{Dij} + Q^r_{Hi} \qquad \forall r;\ \forall i \in NT \qquad (1)$$

This equation is very direct, intuitive and easy to implement. Indeed all of the accounting conditions are most naturally expressed in terms of the levels of variables.

In order to "linearize" equation (1), we first totally differentiate it as follows:

$$\sum_j dQ^r_{Sij} = \sum_j dQ^r_{Dij} + dQ^r_{Hi}$$

Now multiply each element by Q/Q such that the differential pertubations are converted into quantity-weighted proportional changes, denoted here by the use of lower case letters (i.e., $q = dQ/Q$)

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3 Various linearizations are possible. In this paper we use the version in which variables are interpreted as percentage (or proportional) changes. A linearized representation based in actual changes is also possible (but less useful, in our view).

$$\sum_j Q^r_{Sij}\, q^r_{Sij} = \sum_j Q^r_{Dij}\, q^r_{Dij} + Q^r_{Hi}\, q^r_{Hi} \; .$$

Multiplying both sides of this equation by the market price of commodity i in r, we can now write the market clearing condition in terms of values, as opposed to quantities. This will be the form of the nontradeable market clearing conditions in the linearized model; hence the asterisk above the equation number.

$$\sum_j V^{rM}_{Sij}\, q^r_{Sij} = \sum_j V^{rM}_{Dij}\, q^r_{Dij} + V^{rM}_{Hi}\, q^r_{Hi} \qquad \forall\, r;\ \forall\, i \in NT \qquad (1^*)$$

The fact that market clearing conditions may be expressed in terms of appropriately normalized values (which are used to weight proportional changes in quantities) has important implications for implementations based on a linearized representations. In particular, *it will never be necessary to compute the level of quantities or prices in the process of solving the nonlinear general equilibrium problem.*[4] The benchmark equilibrium data set is most naturally supplied in value terms anyway. Furthermore, this has certain advantages when it comes to solving the model via its linearized representation, as will be discussed below. Of course, in order to implement the process of iteration and extrapolation needed to obtain accurate solutions of the nonlinear problem we will require the initial data-base to be systematically updated. But this may also be done without reference to the *level* of quantities or prices, since:

$$dV/V = d(PQ)/(PQ) = p + q. \qquad (F1)$$

In sum, *even though the linearized model is expressed in terms of proportional changes in prices and quantities, the underlying levels of those variables will never be computed.* This has implications for model calibration as well, which will also be discussed below.

The market clearing condition for the tradeable commodities, expressed in levels, is given by equation (2). This is identical to (1), only now it is *global* supplies which must equal *global* demands. Note that this condition is only imposed for (N–1) of the N tradeable commodities. Equilibrium in the N^{th} market follows by virtue of Walras' Law.[5]

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4 This is an advantage which is peculiar to proportional or percentage change linearized representations. A linearization based on actual changes would need to compute changes in levels of prices and quantities.

5 By including a "dummy" equation which computes excess demand in the omitted market one obtains a valuable consistency check on the model. However, in the linearized representation (2*) the form of this omitted equation becomes a relevant consideration. At first sight, it seems natural to add a variable set equal to the difference between the two sides of (2*) for the Nth commodity. Unfortunately, since in some simulations the two values whose difference is being calculated are moderately large, this can lead to substantial loss of

$$\sum_r \sum_j Q^r_{Sij} = \sum_r \sum_j Q^r_{Dij} + \sum_r Q^r_{Hi} \qquad i = 1, \ldots, N-1 \in T \qquad (2)$$

The linearized form of (2) is likewise analogous to (1*), with the important exception that now one must also sum over regions. In order to place all quantity changes on a common basis, the value weights must be evaluated at the common *world price* for commodity i.

$$\sum_r \sum_j V^{rW}_{Sij}\, q^r_{Sij} = \sum_r \sum_j V^{rW}_{Dij}\, q^r_{Dij} + \sum_r V^{rW}_{Hi}\, q^r_{Hi} \qquad i = 1, \ldots, N-1 \in T \qquad (2^*)$$

Price Linkage: The price linkage equations in this model are very straightforward, as all interventions are expressed in *ad valorem* equivalent form. They are given in (3) and (3*):

$$P^r_{Mi} = T^r_{Ti}\, P_{Wi} \qquad \forall\, r;\ \forall\, i \in T$$

$$P^r_{Hi} = T^r_{Hi}\, P^r_{Mi} \qquad \forall\ r, i$$

$$P^r_{Sij} = T^r_{Sij}\, P^r_{Mi} \qquad \forall\ r, i, j \qquad (3)$$

$$P^r_{Dij} = T^r_{Dij}\, P^r_{Mi} \qquad \forall\ r, j, i$$

$$p^r_{Mi} = t^r_{Ti} + p_{Wi} \qquad \forall\, r;\ \forall\, i \in T$$

$$p^r_{Hi} = t^r_{Hi} + p^r_{Mi} \qquad \forall\ r, i$$

$$p^r_{Sij} = t^r_{Sij} + p^r_{Mi} \qquad \forall\ r, i, j \qquad (3^*)$$

$$p^r_{Dij} = t^r_{Dij} + p^r_{Mi} \qquad \forall\ r, i, j$$

Zero Profits: Since we assume input-output separability, industries never alter the set of commodities for which they are net suppliers. Thus the zero profits condition (which does not apply to the endowment industries) becomes:

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accuracy (so-called "subtractive cancellation"), which can cause the extrapolated results to be unreliable. This problem is overcome by adding two variables, one for each side of (2*), and comparing them. In the application below, we found excellent agreement between the two values obtained.

$$\sum_i P^r_{Sij}\, Q^r_{Sij} = \sum_i P^r_{Dij}\, Q^r_{Dij} \qquad \forall\, r;\ \forall\, j \notin EI \qquad (4)$$

The linearized form of (4) is most commonly expressed in terms of revenue and cost shares: $S^r_{Sij} = V^{rA}_{Sij} / \sum_k V^{rA}_{Skj}$, and $S^{rA}_{Dij} = V^r_{Dij} / \sum_k V^{rA}_{Dkj}$. This yields the following restriction on price changes (quantity changes drop out due to the envelope theorem):

$$\sum_i S^r_{Sij}\, p^r_{Sij} = \sum_i S^r_{Dij}\, p^r_{Dij} \quad .$$

Note that since $\sum_i V^{rA}_{Sij} = \sum_i V^{rA}_{Dij}$ [by equation (4)], it is also possible to express the linearized zero profit condition directly in terms of values at agent's prices:

$$\sum_i V^{rA}_{Sij}\, p^r_{Sij} = \sum_i V^{rA}_{Dij}\, p^r_{Dij} \qquad \forall\, r;\ \forall\, j \notin EI \qquad (4^*)$$

Regional Income: The final accounting condition in the model permits us to calculate regional (household) income as the sum of payments to primary factor endowments ($i \in EC$), supplied by the endowment "industries" ($j \in EI$), and taxes, net of subsidies. Note that the definition of the tax rates in (3) means that tax revenue for a given transaction will be positive if $T^r_{Dij} > 1$, $T^r_{Hi} > 1$, $T^r_{Sij} < 1$, $T^r_{Ti} > 1$ for imports and $T^r_{Ti} < 1$ in the case of exports. Thus we have:

$$Y^r = \sum_{j \in EI} \sum_{i \in EC} P^r_{Sij}\, Q^r_{Sij} + \sum_i (T^r_{Hi} - 1)\, P^r_{Mi}\, Q^r_{Mi} + \sum_j \sum_i (1 - T^r_{Sij})\, P^r_{Mi}\, Q^r_{Sij}$$

$$+ \sum_j \sum_i (T^r_{Dij} - 1)\, P^r_{Mi}\, Q^r_{Dij} + \sum_{i \in T} (T^r_{Ti} - 1)\, P_{Wi}\, [Q^r_{Hi} + \sum_j (Q^r_{Dij} - Q^r_{Sij})] \qquad \forall\ r \quad (5)$$

From the final term, it is clear that when $T^r_{Ti} > 1$, the border intervention in question will raise revenue when the term in brackets [•] is positive, i.e., when the region is a net importer of commodity i. Conversely a domestic price in excess of the world price implies a net subsidy when domestic supply exceeds demand.[6]

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6 Use of the term "net importer" does not preclude the possibility of a "gross trade" model in which products are differentiated by origin. In such a framework Q^r_S

Linearization of (5) is facilitated by first expressing this equation in terms of a sum of differences in the value variables, evaluated at appropriate prices. This gives the following expression, which will be denoted (5')

$$Y^r = \sum_{j\in EI}\sum_{i\in EC} V^{rA}_{Sij} + \sum_i (V^{rA}_{Hi} - V^{rM}_{Hi}) + \sum_j\sum_i (V^{rM}_{Sij} - V^{rA}_{Sij}) + \sum_j\sum_i (V^{rA}_{Dij} - V^{rM}_{Dij})$$

$$+ \sum_{i\in T}\left\{(V^{rM}_{Hi} - V^{rW}_{Hi}) + \sum_j\left[(V^{rM}_{Dij} - V^{rW}_{Dij}) + (V^{rW}_{Sij} - V^{rM}_{Sij})\right]\right\} \qquad (5')$$

Given the complexity of this expression for regional income, the notation has been carefully selected in order to facilitate easy checking. For example, all subscripts within a given summation must be the same. This is because net tax revenue is always found by examining differences in the value of a *given agent's transaction* at prices inclusive and exclusive of the tax in question. In the case of domestic taxes, this will be the difference between the value at agent's prices (superscript A) and that at domestic market prices (M). For trade taxes we will look at the difference between V^{rM} and V^{rW}.

Having reexpressed (5) as (5'), linearization proceeds in a straightforward manner, analogous to the earlier accounting conditions. The only difference is that now we must take into account both changes in prices and quantities. Since the value differences in (5') are evaluated at *different* prices, their rate of change will also be distinguished in this manner. The rates of change in quantities behind each of these value pairs must be the same, since both refer to the same transaction — simply viewed from different sides of the policy wedge. Total differentiation of (5') and reexpression in terms of proportional changes yields (5*):

$$Y^r y^r = \sum_{j\in EI}\sum_{i\in EC} V^{rA}_{Sij}(p^r_{Sij} + q^r_{Sij}) + \sum_i [V^{rA}_{Hi}(p^r_{Hi} + q^r_{Hi}) - V^{rM}_{Hi}(p^r_{Mi} + q^r_{HI})]$$

$$+ \sum_j\sum_i [V^{rM}_{Sij}(p^r_{Mij} + q^r_{Sij}) - V^{rA}_{Sij}(p^r_{Sij} + q^r_{Sij})]$$

$$+ \sum_j\sum_i [V^{rA}_{Dij}(p^r_{Dij} + q^r_{Dij}) - V^{rM}_{Dij}(p^r_{Mij} + q^r_{Dij})]$$

$$+ \sum_{i\in T}\Big([V^{rM}_{Hi}(p^r_{Mi} + q^r_{Hi}) - V^{rW}_{Hi}(p_{Wi} + q^r_{Hi})]$$

(equation continues next page)

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will simply be a very sparse matrix, such that domestic producers never supply imported commodities.

$$+\sum_{j} \left\{ [V^{rM}_{Dij}\,(p^{r}_{Mi} + q^{r}_{Dij}) - V^{rW}_{Dij}\,(p_{Wi} + q^{r}_{Dij})] \right.$$

$$\left. + [V^{rW}_{Sij}\,(p_{Wi} + q^{r}_{Sij}) - V^{rM}_{Sij}(p^{r}_{Mi} + q^{r}_{Sij})] \right\} \Big) \qquad \forall\, r. \qquad (5^*)$$

Behavioural Equations

Whereas the accounting conditions are more naturally expressed in terms of levels, the opposite is true for the behavioural equations. They are more naturally expressed in terms of *elasticities*, which in turn call for a representation in terms of proportional changes. Consequently, in this section we will sometimes proceed by first considering the linearized equation, thereupon turning to the levels form of the behavioural relationship in question. The latter relationships, as well as the formulae defining the elasticities in the linearized equations, will depend critically on the form of the function chosen to represent technology or preferences. As such it is not possible to proceed in full generality. However, we have chosen what we believe to be an interesting mix of functional forms for different parts of the model. In particular, we consider both *explicit* and *implicit* functional forms to highlight some important differences between the linearized and levels representations of the model.

Demands: The proportional change in industry demands in the model are expressed as *conditional* derived demand equations:

$$q^{r}_{Dij} = \sum_{k} \eta^{r}_{Dikj}\; p^{r}_{Dkj} + z^{r}_{j} \qquad \forall\, r, j, i \quad , \qquad (6^*)$$

where z\S\up4(r,j) is the proportional change in the activity level of industry j, and η^{r}_{Dik} is the derived demand elasticity, conditional on a given activity level. This equation follows directly as a consequence of constant returns to scale. It does not vary with the choice of functional form, although the formula for calculating η^{r}_{ikj} as a function of prices and technology parameters obviously does vary with the form of the cost function chosen. This invariance, along with the simplicity and ease of interpreting (6*) is one of the great attractions of the linearized form of this model.

In this model, all cost functions $C^{r}_{j}(.)$ are of the constant elasticity of substitution (CES) variety, so that:

$$C^{r}_{j}\,(\mathbf{P}^{r}_{Dj}\,,\, Z^{r}_{j}) = \left[\sum_{j} B^{r}_{Dij}\,(P^{r}_{Dij})^{-\rho^{r}_{j}} \right]^{1/\rho^{r}_{j}} Z^{r}_{j} \quad , \qquad \forall\, j\, ,r$$

where $\mathbf{P}^{r}_{Dj}$ is the vector of input prices facing j in r. The B^{r}_{Dij}'s are scale parameters and $\sigma^{r}_{j} = (\rho^{r}_{j} + 1)$ is the elasticity of substitution in production.

This functional form is standard "workhorse" of CGE models. It gives rise to the following conditional demand equation:

$$Q^r_{Dij} = \left\{ B^r_{Dij}\,(P^r_{Dij})^{-\sigma^r_j} / [\sum_k B^r_{Dkj}\,(P^r_{Dkj})^{-\sigma^r_j}]^{\sigma^r_j/(\sigma^r_j - 1)} \right\} Z^r_j \quad \forall\, r, j, i. \qquad (6)$$

These demand equations in (6) are obviously more complex than (6*). Despite this complexity, the formulae for the derived demand elasticities are quite straightforward. Indeed, it was via this route that the CES was initially invented (Arrow *et al.*. (1961)):

$$\eta^r_{Dikj} = (S^r_{Dkj} - \delta_{ik})\ \sigma^r_j \qquad \forall\, r, j, i, k \qquad (F2)$$

where S^r_{Dij} is the cost share of input i in industry j, region r, as defined previously, and $\delta_{ii} = 1$, $\delta_{ik} = 0$ for $i \neq k$.

Industry Supplies: While only some industries in the model are truly multiproduct, they will be universally treated as such. (When there is only one product produced, then the following supply equation degenerates to an identity.) Once again, we begin with the linearized form of this behavioural equation:

$$q^r_{Sij} = \sum_k \eta^r_{Sikj}\ p^r_{Skj} + z^r_j \qquad \forall\, r, j, i\ . \qquad (7a^*)$$

Here the η^r_{Sikj} are activity-level constant, supply elasticities. Thus they embody only a transformation effect.

These supply equations may be derived from a maximum revenue function: $R(\mathbf{P}_S, Z)$. Since we wish to have somewhat greater flexibility in specifying supply elasticities than is afforded by a Constant Elasticity of Transformation (CET) revenue function, we have chosen a more general functional form — the Constant Difference Elasticity (CDE) revenue function (Hanoch (1975); see also Hertel *et al.*. (1991)). This embodies the CET as a special case. Furthermore, since the CDE is an *implicit* function, it provides us with an opportunity to highlight some substantive differences between the linearized and levels formulations.

The CDE implicit maximum revenue function, with constant returns to scale imposed, is written as follows:

$$\sum_i B^r_{Sij}\ (Z^r_j)^{\alpha^r_{Sij}}\ [P^r_{Sij}/R(\mathbf{P}^r_{Sj}, Z^r_j)^{\alpha^r_{Sij}}] \ \equiv\ 1 \qquad \forall r\ , j.$$

The $B^r_{Sij} > 0$ are scale parameters, while the parameters $\alpha^r_{Sij} > 1$ relate to the ease of transformation among products, thus determining the conditional supply elasticities. Unless $\alpha^r_{Sij} = \bar{\alpha}^r_j$, $\forall i$, maximum revenue, $R(\mathbf{P}^r_{Sj}, Z^r_j)$, cannot be isolated. This renders the revenue function implicit in R. When all of the transformation parameters *are* equal this reduces to the (*explicit*) CET revenue function. If we combine the fact that $\partial R(\bullet)/\partial Z = R(\bullet)/Z$, under constant returns to scale, with $\partial R(\bullet)/\partial Z = P_Z$ = the shadow price of activity in the industry, then the revenue function can be simplified to the following price frontier:

$$\sum_i B^r_{Sij} \, (P^r_{Sij}/P^r_{Zj})^{\alpha^r_{Sij}} \equiv 1 \quad \forall \, r, j \, . \tag{7b}$$

Application of the implicit function theorem and Hotelling's lemma yields the following conditional supply equations:

$$Q^r_{Sij} = \frac{B^r_{Sij} \, \alpha^r_{Sij} \, (P^r_{Sij}/P^r_{Zj})^{(\alpha^r_{Sij} - 1)}}{\sum_k B^r_{Skj} \, \alpha^r_{Skj} \, (P^r_{Skj}/P^r_{Zj})^{\alpha^r_{Skj}}} \, Z^r_j \quad \forall \, r, j \, . \tag{7a}$$

But there is a complication introduced by the form of (7a). Given a vector of supply prices P_S, we need to compute unit revenues (R/Z or P_z) before we can evaluate these supply equations. This problem may be handled by calling a special subroutine designed to solve (7a) for P_Z (Hertel *et al.* (1991)).

In contrast, the linearized representation treats both explicit and implicit functional forms with equanimity. All that is required in addition to (7a*) are the formula for computing η^r_{Sij} as a function of prices (or revenue shares) and the transformation parameters. This is provided by Hanoch (1975), and is given in (F3):

$$\eta^r_{Sikj} = S^r_{Skj} \left\{ (1 - \alpha^r_{Sij}) + (1 - \alpha^r_{Skj}) - \sum_m S^r_{Smj} \, (1 - \alpha^r_{Smj})] - \delta_{ik} \, (1 - \alpha^r_{Sij}) \right\}$$

$$\forall \; r, j, i, k, \tag{F3}$$

where S^r_{Sij} is revenue share of sales of commodity i by industry j in region r, as defined previously, $\delta_{ii} = 1$ and $\delta_{ik} = 0$ if $i \neq k$.

While the linearized version of (7b) is not required, it is useful to provide it here, for future reference:

$$\sum_i B^r_{Sij} \, \alpha^r_{Sij} \, (P^r_{Sij}/P^r_{Zj})^{\alpha^r_{Sij}} \, (p^r_{Sij} - p^r_{Zj}) = 0 \tag{7b*}$$

Note further, that if both sides of (7a) are multiplied by $(P^r_{Sij}/P^r_{Zj} \, Z^r_j)$ we obtain an expression for the share of commodity i supplies in total revenue (S^r_{Sij}) which may be substituted into (7b*) so that it simplifies to the following expression for p^r_{Zj}.

$$p^r_{Zj} = \sum_i S^r_{Sij}\, p^r_{Sij} .$$

Clearly this is optional and need not be included in the linearized representation of the model, as p^r_{Zj} is simply an intermediate variable.

Consumer Demand: The structure of preferences in this model is designed to hold the share of disposable income allocated to current consumption constant, while providing considerable scope for the incorporation of independent econometric estimates of expenditure and price elasticities of consumption demand. Thus aggregate utility is a Cobb Douglas function of individual subutilities derived from purchases of goodwill (transfer payments), capital goods purchases (savings) and current consumption:

$$U^r = (U^r_{CONS})^{\beta^r_{CONS}} (U^r_{SAV})^{\beta^r_{SAV}} (U^r_{TRANS})^{\beta^r_{TRANS}} \quad \forall r . \tag{8a}$$

In the case of savings and transfer payments (i.e., "non-consumption commodities", $i \in NC$), the level of U^r_i is equal to the quantity of these goods purchased:

$$U^r_i = Q^r_{Hi} = (\beta^r_i Y^r)/P^r_{Hi} \qquad \forall r ;\ \forall i \in NC . \tag{8b}$$

In the case of the subutility of composite consumption, U^r_{CONS} is determined by the following implicit CDE minimum expenditure function, $E(\mathbf{P}^r_H, U^r_{CONS})$ governing purchases of consumption commodities ($i \in C$):

$$\sum_{i \in C} B^r_{Hi} (U^r_{CONS})^{\alpha^r_{Hi} \gamma^r_{Hi}} [P^r_{Hi}/E(\mathbf{P}^r_H, U^r_{CONS})]^{\alpha^r_{Hi}} \equiv 1 \qquad \forall r. \tag{8c}$$

Once again the B's are scale parameters, this time for demands, with $B^r_{Hi} > 0$. The $\alpha^r_{Hi} < 1$ parameters govern substitution in consumption, and all of these values must lie on the same side of zero for global regularity to apply (Hanoch (1975); see also Hertel *et al.*. (1991), footnote 2). The $\gamma^r_{Hi} > 0$ parameters appear due to non-homotheticity in consumption ($\gamma^r_{Hi} \neq 1$). Equation (8c) is used to evaluate minimum expenditures, given prices and U^r_{CONS}. This in turn must equal the share of disposable income allocated to current consumption:

$$E(\mathbf{P}^r_H, U^r_{CONS}) = \beta^r_{CONS}\, Y^r \qquad \forall r . \tag{8d}$$

Finally, application of Shepard's lemma and the implicit function theroem to (8c) gives us the following *utility compensated* household demands

$$Q^r_{Hi} = \left\{ B^r_{Hi}\, \alpha^r_{Hi} (U^r_{CONS})^{\alpha^r_{Hi} \gamma^r_{Hi}} \,[(P^r_{Hi}/E(\mathbf{P}^r_{Hi}, U^r_{CONS})]^{(\alpha^r_{Hi}-1)}) \right\}$$

$$\left\{ \sum_{i \in C} B^r_{Hi} \alpha^r_{Hi} (U^r_{CONS})^{\alpha^r_{Hi} \gamma^r_{Hi}} [P^r_{Hi}/E(P^r_H, U^r_{CONS})]^{\alpha^r_{Hi}} \right\}^{-1} \qquad \forall r ;\ \forall\ i \in C. \tag{8e}$$

The sub-system of equations given by (8a) – (8e) are clearly the most complex part of the model, as written in the levels variables. In contrast, these equations are greatly simplified when the model is expressed in linearized form. Beginning with the differential of the aggregate utility function, and the associated non-consumption demands, we have:

$$u^r = \beta^r_{CONS}\, u^r_{CONS} + \beta^r_{SAV}\, u^r_{SAV} + \beta^r_{TRANS}\, u^r_{TRANS} \quad \forall\, r \qquad (8a^*)$$

$$u^r_i = q^r_{Hi} = y^r - p^r_{Hi} \qquad \forall\, r;\ \forall\, i \in NC. \qquad (8b^*)$$

As was the case with the implicit, CDE revenue function, since minimum expenditures will not appear in the linearized demand equations, it is no longer necessary to evaluate to evaluate $E(\mathbf{P}^r_H, U^r)$ as an implicit function of the other variables. Thus we do not need the analogue to (8c). However, we do need to know the value of u^r_{CONS} in order to evaluate (8a*). The relevant equation [(8d*) below] may be derived from (8d). Use of this budget constraint and total differentiation yields:

$$dE(\mathbf{P}^r_H, U^r_{CONS}) = dY^r \quad ,$$

or

$$\sum_{i \in C} (\partial E(\bullet)/\partial P^r_{Hi}) + (\partial E/\partial U^r_{CONS})\, d\, U^r_{CONS} = dY^r$$

Dividing through by $E(\bullet) = Y^r$, multiplying terms by (P^r_{Hi}/P^r_{Hi}), (U^r_{CONS}/U^r_{CONS}), and employing Shepard's lemma, gives:

$$\sum_{i \in C} S^r_{Hi}\, p^r_{Hi} + \left\{ \sum_i \gamma^r_{Hi}\, S^r_{Hi} \right\} u^r_{CONS} = y^r \quad \forall\, r \qquad (8d^*)$$

where $S^r_{Hi} = P^r_{Hi}\, Q^r_{Hi}/E(\mathbf{P}^r_H, U^r_{CONS})$, and we have used the fact that $(\partial E/\partial U^r)U^r/E(\bullet) = \sum_{i \in C} \gamma^r_{Hi}\, S^r_{Hi}$ = the expenditure elasticity with respect to utility (Hanoch).

The linearized demand equations associated with current consumption items are given in (8e*). Once again, they are invariant to the choice of functional form:

$$q^r_{Hi} = \sum_k \eta^r_{Hik}\, p^r_{Hi} + \eta^r_{Hi}\, y^r \quad \forall\, r, i \qquad (8e^*)$$

where η^r_{Hik} is the *uncompensated* cross-price elasticity of demand between i and k and η^r_{Hi} is the income elasticity of demand for i in region r. These demand elasticities are related to the preference parameters by the following formulae:

$$\eta^r_{Hik} = (\sigma^r_{Hik} - \eta^r_{Hi})\, S^r_{Hk} \qquad \forall\, r;\ \forall\, i, k \in C,$$

$$\sigma^r_{Hik} = (1 - \alpha^r_{Hi}) + (1 - \alpha^r_{Hk}) - \sum_m S^r_{Hm}(1 - \alpha^r_{Hm}) - \delta_{ik}\ [(1 - \alpha^r_{Hi})/S^r_{Hi}] \quad \forall\ r; \forall\ i, k \in C, \tag{F4}$$

$$\eta^r_{Hi} = \left[\sum_m S^r_{Hm}\gamma^r_{Hm}\right]^{-1}\left[(\gamma^r_{Hi}\,\alpha^r_{Hi}) + \sum_m S^r_{Hm}\gamma^r_{Hm}(1 - \alpha^r_{Hm})\right]$$

$$+\left[(1 - \alpha^r_{Hi}) - \sum_m S^r_{Hm}(1 - \alpha^r_{Hm})\right] \quad \forall\ r; \forall\ i \in C\ ,$$

where $\delta_{ii} = 1$ and $\delta_{ik} = 0$, $i \neq k$ and $\delta_{ii} = 1$.

It is interesting to note that the scale parameters (B's) for $C(\mathbf{P}^r_{Dj}, Z_j)$, $R(\mathbf{P}^r_{Sj}, Z_j)$ and $E(\mathbf{P}^r_H, U^r_{CONS})$ do not appear in the linearized representation of the model. This stems from the fact that we are never required to compute the level of the quantities $\mathbf{Q}^r_{Dj}$, $\mathbf{Q}^r_{Sj}$ or $\mathbf{Q}^r_H$.

IV. Model Simulation

Having laid out the equations for two equivalent representations of a general equilibrium trade model, the question arises: How shall we solve each of these models? What special properties do these alternative solution strategies possess?

Calibration and Benchmarking

The data-base underlying a typical AGE model consists of: (a) tables of flows showing transactions between agents, and (b) files of behavioural parameters (such as substitution elasicities). It is assumed that this data is consistent with a solution of the model. The raw data-base does not include, however, levels values of prices (which would allow us to infer quantities from flows) or the full set of parameters. For models formulated in the levels, these additional values are required. Calibration refers to the process of deducing the missing parameters from the data available and from the model's behavioural equations.

To do this, initial values of prices must be assumed. This amounts to an arbitrary choice of units. With prices and quantities given, we can deduce the scale parameters in production and utility functions. For example, in the case of the CES demand equations (6), the B^r_{Dij}'s may be inferred from σ^r_j once expenditures have been partitioned into prices (P^r_{Dij}) and quantities (Q^r_{Dij}) (see, for example, Mansur and Whalley). Similarly in the case of CDE supplies (7a) the scale parameters (B^r_{Sij}) may be inferred from the transformation parameters (α^r_{Sij}), once prices and quantities are specified (Hertel *et al.* (1991)).

In contrast, linearization in proportional or percentage changes takes advantage of the invariance to units implicit in rational economic behaviour. Those parameters deduced in the levels calibration process are not required, since their values merely reflect arbitrary price assumptions. Thus no such calibration step is needed.

Benchmarking in a levels model is the process of reproducing the initial equilibrium by using the model solution procedure. Since calibration depends on the initial truth of all behavioural equations, the benchmarking process amounts to: (a) checking that the calibration program correctly implemented the behavioural equations, and (b) checking that the model data-base satisfies simple accounting identities. For a linearized model, step (a) does not apply.

Condensing a Model

The size of a model (that is, number of variables and number of equations to be solved) can be reduced by making algebraic substitutions. Consider, for example, the demands Q^r_{Dij} in the model described in Section III. Equation (6) gives an expression for these in terms of other variables (P_D's and Z's in this case) of the model. We can use this expression to replace all occurences of Q^r_{Dij} in the other equations (equations (1), (2), (4) and (5)) by the appropriate expression in terms of the P_D's and Z's. Then, provided we are not interested in shocking or reporting values of these Q^r_{Dij} variables, we can ignore the equations given by (6) and solve just the remaining equations (as modified). This one algebraic substitution removes $I \times J \times R$ actual equations and variables from the original model, (where I is the total number of commodities in the model, J is the common number of industries in each region and R is the number of regions). Similarly, in the linearized representation, equation (6*) can be used to eliminate q^r_{Dij} from all other equations, thus producing an equivalent equation and variable reduction. For large models several such substitutions may be essential to reduce the system to be solved to a size that can be handled on a particular computer.

This method of *condensing* a model by substituting out one or more variables can only be applied when it is possible to isolate an explicit algebraic formula for the variables in question. If the variable in question is only given by an implicit function (for example P^r_{Zj} in equation (7b)) it may not be possible to obtain such an explicit formula. This limits the amount of condensation possible using a levels representation. However, with a linearized representation, implicit functions are never a problem since each variable in an equation can always be isolated.[7] For example, equation (7b*) can be rewritten as

7 This is because the equations are linear; for example, from the equation Tb + Uc + Vd = 0 it is easy to isolate b as b = -(Uc + Vd)/T. The saving involved is usually only worthwhile when a single algebraic substitution covers all possible indices

$$p^r_{Zj} = \sum_i \left[B^r_{Sij}\, \alpha^r_{Sij}\, (P^r_{Sij} / P^r_{Zj})^{\alpha^r_{Sij}} \Big/ \sum_k B^r_{Skj}\, \alpha^r_{Skj} (P^r_{Skj} / P^r_{Zj})^{\alpha^r_{Skj}} \right] p^r_{Sij} \quad \forall\ j$$

to isolate each p^r_{Zj} (in terms of the p^r_{Sij}'s). Thus it is easy to eliminate (7b*) and the p^r_{Zj}'s from the linearized representation,[8] although this is not possible with the levels representation.

Solving the Equilibrium Problem

It is important to distinguish between the way a model's equations are written down and the algorithm used to solve the model. Applied general equilibrium models are inherently nonlinear, as illustrated by equations (1)–(8e) above. However, most models[9] can also be expressed in linearized form, as in (1*) – (8e*). The important thing is that either of these representations of the model can be used as a starting point for obtaining true solutions of the model; indeed several different algorithms can be used to obtain these true solutions (as we indicate below). Some algorithms rely on the levels representation and others on the linearized representation.

Considerable confusion has occurred in the past due to the tendency to classify models according to the solution method employed, as opposed to the economic content of the model. Thus models whose equations are usually represented and discussed in linearized form have been referred to as models of the "Johansen class". The confusion has occurred because these models have often been solved by *Johansen's method*[10]. This method is not capable of producing arbitrarily accurate simulation results; the numerical results obtained are only approximations to the true results and Johansen's method provides no way of increasing the accuracy of these results. In our view the confusion would be avoided by referring to such models as models which are

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associated with the variable in question (as for the p^r_{Zj} example in the next sentence of the main text). Covering all indices is only possible when there are as many actual equations specified by the single symbolic equation in question as there are instances of the variable in question. For example, while the single actual equation (5*) could be used to isolate p_{Wi} for any <u>one</u> commodity i, the resulting expression would involve p_{Wt} for all other commodities t; such a substitution is algebraically messy and would not be helpful in practice.

8 Carrying out substitutions like this with pencil and paper is time consuming and error prone. For this reason, GEMPACK (Codsi and Pearson, 1988) automates such substitutions. The user simply nominates which variable to eliminate and which equation to use. Then the software carries out all of the algebra (including isolating the relevant variables, if necessary, as above).

9 Some models require inequalities as well as equations. See the paragraph below headed "Limitations of Linearized Representations" for relevant comments.

10 This method is described in the subsection "Obtaining Accurate Results from a Linearized Representation" below.

implemented via a linearized representation.[11] This representation can be used as the starting point for (approximate) solution by Johansen's method. But, more importantly, it can also be the starting point for accurate solution by other methods, as we describe in the subsection "Obtaining Accurate Results from a Linearized Representation" below.

The most important feature of any model is its basic economic content. Of secondary importance is the way its equations are written down. Of only tertiary importance is the algorithm usually used to solve it since, in general, several different algorithms can be used to obtain the same results. The only qualifications that should be made to the previous sentence is to add "provided the algorithm used is capable of producing numerical results of any desired accuracy".[12] In particular, we believe that models represented in linearized form should be solved as a matter of routine by an algorithm (such as those described below) that is able to produce such arbitrarily accurate results.[13]

Naturally software specialists will want to compare the efficiency of different algorithms (that is, how much computing time and other resources are required to obtain a given level of accuracy). But, most economic modellers will choose the type of model representation (levels or linearized) on the basis of which they find more helpful for formulating their model, modifying it and understanding and communicating its results. Also of great importance to modellers who are not computer experts is the range of general-purpose software which they can access.[14]

Obtaining Accurate Solutions from a Linearized Representation: It may be of interest to practitioners used to either levels or linearized representations of their models to see how a linearized representation can be used to obtain accurate solutions. Firstly we illustrate this in the special case where there is just one exogenous variable x and one endogenous variable y, and where the only equation of the model $g(x, y) = 0$ is as shown in Figure 2. Suppose we

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11 The term "Johansen model" should either be dropped or, perhaps more fittingly, be used to describe models whose economic foundations are similar to those of Johansen's 1960 model.

12 Of course, the accuracy of any actual calculation is limited by machine accuracy and the fact that rounding errors inevitably occur in any large calculation.

13 For this reason, Euler's method with extrapolation (see below) is currently the default in GEMPACK (Pearson, 1991).

14 In this regard, important questions include the following.

(i) Does any programming have to be done to communicate the model's theoretical structure and data to the software? (In some cases the user must provide subroutines which can calculate the values or derivatives of certain functions in the model. Other software such as GAMS, GEMPACK, HERCULES and MPS/GE do not require any code to be written; they can read and interpret symbolic (or numeric, in the case of MPS/GE) representations of the model prepared by users.)

(ii) Does the software give access to algorithms which are robust in the sense that they converge to the solutions of well-posed problems with little or no user intervention?

know an initial solution of the model, say $y = y_0$ when $x = x_0$ (see the point A in Figure 2) and we wish to calculate the value of y when x increases to x_1. The true solution is $y = y_1$ (see point B in Figure 2). In this case the linearized representation of the equation is used to calculate the gradient $\frac{dy}{dx}$ (in terms of x and y) at any point on the curve $g(x, y) = 0$ (and at points just off it).

Johansen's method is to calculate $\frac{dy}{dx}$ at A and then, in passing from x_0 to x_1, to move along the tangent to the curve at A. This brings us to the point B_J in the figure and so this produces the estimate y_J. The idea behind other more accurate methods is to follow the curve $g(x, y) = 0$ more closely. For example, start out in the direction of the tangent at A but, after a small distance, stop, recompute the direction to move in (that is, recompute $\frac{dy}{dx}$ in terms of the new values of x and y reached), move in the new direction for a short distance, then recompute the direction, and so on until x reaches x_1.

Euler's method is to divide the interval $[x_0, x_1]$ into N equal subintervals and then proceed as indicated above, recomputing the direction to move in at the end of such subinterval. This is shown in Figure 2 for $N = 2$, when the path is from A to C_2 to B_2 which produces the approximate solution y_{E2}. The important point to note is that, by taking N sufficiently large, the approximate solution y_{EN} obtained can be made arbitrarily close to the true solution y_1.[15] Note also that Johansen's method produces the Euler result y_{E1} when $N = 1$.

When there are more exogenous and endogenous variables, it is less easy to visualize what happens when Euler's method is used, but the method does generalize. If there are n exogenous variables $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ with initial values $\underset{\sim}{a} = (a_1, a_2, ..., a_n)$ and final values $\underset{\sim}{b} = (b_1, b_2, ..., b_n)$, one way of visualizing this is to imagine a straight line in n dimensions joining $\underset{\sim}{a}$ to $\underset{\sim}{b}$ and a scalar variable t equal to 0 at $\underset{\sim}{a}$ and to 1 at $\underset{\sim}{b}$. The linearized representation of the model makes it possible to calculate $\frac{dz_i}{dt}$ for all of the endogenous variables $z_1, ..., z_m$ of the model. (This is done by solving a system of linear equations in m unknowns derived directly from the linearized representation of the model.) The levels equations of the model define a surface (the set of all solutions of the model) and there will be a path on this surface from the initial value $\underset{\sim}{z}_0$ of $(z_1, ..., z_m)$ to its final value $\underset{\sim}{z}^*$ which $\underset{\sim}{z}$ moves along as t goes from 0 to 1 (or $\underset{\sim}{x}$ goes from $\underset{\sim}{a}$ to $\underset{\sim}{b}$ along a straight line). The idea is to approximate this path along the surface in $(m + n)$-dimensional space by a sequence of short line segments, much as in Figure 2 in 2 dimensions.

Although Euler's method is guaranteed to converge to the true solution for large classes of models,[16] its convergence is not quick. Indeed the errors are only approximately halved when N is doubled. However, this basic method can be refined to give much faster convergence. Firstly, there is a procedure known as *extrapolation* which, given two Euler solutions (say for

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15 The relevant theorem places certain continuity and differentiability restrictions on the function $g(x, y)$; see Pearson (1991) for details.

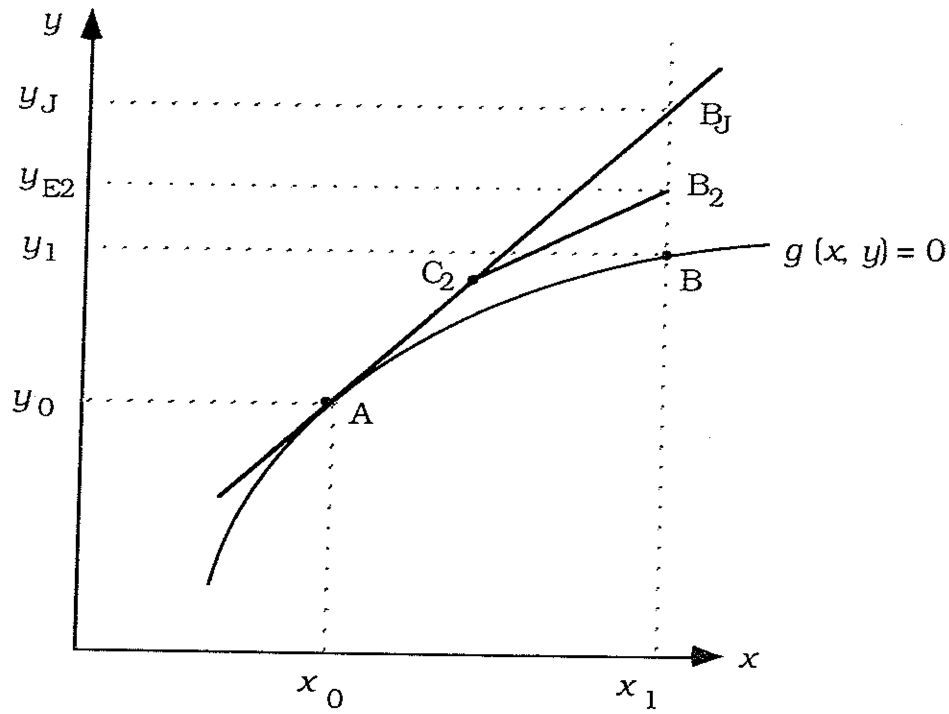

Figure 2: Illustration of Euler's method

$N = 10$ and $N = 20$), calculates an approximation which is significantly more accurate than either of them; extrapolations based on 3 different Euler solutions are even more accurate. Secondly, Euler's method is just the first and simplest of a whole class of methods which can be applied once the derivatives are known via a linear representation of the model.[17] Details of alternative methods, extrapolation, rates of convergence and underlying theory which guarantee convergence are given in Pearson (1991).

Updating Instructions: The equations of a linearized representation (such as (1*) – (8e*) above) are usually solved once for each step of a multi-step solution procedure such as Euler's method described above. The solutions obtained are percentage changes (or, occassionally, actual changes) in the relevant (levels) variables for this step. The levels of the variables at the end of each straight line segment must be estimated on the basis of these percentage changes. This is done via update formulae of the form[18]

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16 See Pearson (1991) for details.

17 In Pearson (1991), it is shown how the simulation problem can be converted to a so-called Initial Value problem. Many different algorithms are known for such problems; indeed most textbooks on numerical computation or numerical analysis contain at least a chapter on such problems. (See, for example, Chapter 15 of Press *et al.* (1986) or Chapter 8 of Dahlquist and Björck (1974)).

18 Here "old-V" refers to the value of V at the left-hand end of the line segment and new-V at the right-hand end in Figure 2.

$$\text{new V} = \text{old V} [1 + (\% \text{ change in V}) / 100] .$$

In practice (as in our model), percentage changes in the flows (values in dollar terms) are not explicitly in the linearized equations solved, just the associated prices and quantities. If V = PQ in levels (Flow = price times quantity) the percentage change linearization is v = p + q and so the update formula is

$$\text{new V} = \text{old V} [1 + (p + q) / 100] .$$

The discipline of specifying these update formulae helps to ensure that there is a well defined system of nonlinear equations (that is, a full behavioural specification) underlying the linearized representation.

How does this compare to "nonlinear" algorithms? Despite the fact that it is based on a linearized representation, Euler's method and its extensions are true nonlinear methods in that, like other "nonlinear" methods, they can be used to obtain solutions of any desired accuracy.[19] Indeed, the way Euler's method relies on calculations of derivatives and partial derivatives has a great deal in common with many other nonlinear algorithms. Most of those iterate towards the final value by moving along short straight line segments (whose direction is based on calculations of derivatives). Some algorithms calculate values of derivatives numerically from calculation of function values[20], while others require users to supply routines[21] which calculate these derivatives.

In summary, either a levels or a linearized representation of a model can be taken as the starting point for algorithms which calculate accurate solutions to the *nonlinear* equilibrium problem. Many models can be represented in either form and so solved by algorithms suited to either representation.

Limitations of Linearized Representations: Since explicit differentiation is carried out in forming a linearized representation of a model, such a representation is only valid for models which can be expressed as a system of equations

$$g_i (z_1, \ldots, z_m, x_1, \ldots, x_n) = 0 \qquad (i = 1, \ldots, m)$$

whose underlying functions g_i are differentiable in some suitable domain containing the initial and final equilibria. However these limitations are not unique to the linearized approach since many of the traditional nonlinear algorithms only apply to equations and are based on calculations of derivatives.

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19 Modellers used to tradional nonlinear software may think it produces "exact" solutions. However, because of limited precision in storing real numbers in computers and because of rounding errors that accumulate whenever a complicated numerical calculation is carried out on a computer, the results obtained by any software are only approximations (hopefully, quite accurate ones) to the true solution.

20 To estimate the derivative of a function f at a point a, take two values b, c close to a and calculate $(f(b) - f(c))/(b - c)$.

21 Such routines may be based on explicit algebraic formulae for the derivatives, as is the case in any linearized representation of the model.

Note also that inequalities ($a \leq b$ etc) cannot be directly represented in linearized form (although in some cases it may be possible to convert these to equalities). Thus linearized representations may not be suitable for models in which inequalities play a vital role or in which corner solutions are sought. Once again this is also a limitation of many of the traditional nonlinear algorithms.[22]

Systematic Sensitivity Analysis

As the solution of nonlinear equilibrium problems has become more routine, attention has turned to the problem of conducting systematic sensitivity analysis with respect to key parameters (Harrison *et al.*. (1991), Wigle (1991)). Complete, *unconditional* systematic sensitivity analysis is extremely costly, requiring m^n model solutions, where m is the number of different levels at which n parameters of interest are to be set.[23] Thus it is more common to vary one parameter at a time, thereby conducting a *conditional* systematic sensitivity analysis. Even this more modest approach is costly and is thus often reserved for special applications of well-established models. Since any sensitivity analysis is *also* specific to the experiment in question, most general equilibrium analysis proceeds in the absence of any information about the robustness of model results to changes in parameter values.

One virtue of expressing the general equilibrium problem in its linearized form is that it is very easy (and cheap) to obtain a complete local sensitivity analysis for a local perturbation to the model. The methodology for doing so is developed in Hanslow and Pearson (1991). Their approach can be readily understood by considering the industry demands as characterized by (6*) and (F2). Denote the one-step (i.e., Johansen) general equilibrium solution to a particular shock (say a subsidy on food production) by appending an asterisk to the variable in question. Thus, in the linearized equations, the perturbation in the food subsidy results in the following set of industry demands:

$$q_{Dij}^{r*} = \sum_k \eta_{Dikj}^{r} \, p_{Dkj}^{r*} + z_j^{r*} \qquad \forall\, r, j, i \,. \qquad (6^{**})$$

Now, assume that we would like to know how sensitive the general equilibrium solution is to the elasticities of substitution in all of the industry cost functions in all regions (i.e., the σ_j^r's). A perturbation in σ_j^r affects (6**) through (F2), so we totally differentiate (F2), recognizing that the cost shares (S_{Dkj}^r) will be left unchanged:

$$d\eta_{Dikj}^{r} = (S_{Dkj}^{r} - \delta_{ik})\, d\sigma_j^{r} \qquad (\Delta F2)$$

---

22 At present, the nonlinear complementarity methods used by Rutherford (1989) and Preckel (1988) are most suitable for solving models in which inequalities play a vital role.

23 Clearly, since m^n grows so quickly, an unconditional systematic sensitivity analysis becomes impractical for even moderately large values of m and n.

Similarly totally differentiate (6**) around the Johansen solution, recognizing that $d\sigma_j^r$ will affect both the elasticity of demand and the general equilibrium price change:

$$dq_{Dij}^{r*} = \sum_k (d\eta_{Dikj}^{r} \, p_{Dkj}^{r*} + \eta_{Dikj}^{r} \, dp_{Dkj}^{r*}) + dz_j^{r*} \qquad (\Delta 6^{**})$$

Now if $d\sigma_j^r$ is the only parametic perturbation of interest, then differentiation of the remainder of the linearized model, (i.e., (1*) – (5*) and (7*) – (8*)) is easy. Indeed simply append the differential operator and asterisk to all of the variables, i.e., p becomes dp*, and we are left with a system of equations which describes the *changes* in the one-step general equilibrium solution (itself expressed in terms of percentage changes), as a result of perturbing the σ_j^r's. The model is solved for these changes as a function of the original solution (as embodied in p_{Dkj}^{r*}) and the parameteric perturbations, $d\sigma_j^r$, and any combination of these perturbations may be examined from this one solution of the model. While this approach is not a substitute for global systematic sensitivity analysis, it is a useful method which can be easily automated and routinely employed in order to determine the potential relevance of parametric uncertainty in the context of any particular simulation of interest.[24] (An example of this will be provided in the next section.)

V. *Application of the Model*

In order to illustrate the points made previously in this paper and in particular to verify that the two methods of model implementation give the same solution, we have selected a simple application which is already available in published form (Hertel *et al.*, 1991). In this empirical model there are two regions: the United States (U.S.) and the rest of the world (R.O.W). Each region produces, consumes and trades three basic commodities: food, manufactures and services, as well as a composite capital good used to satisfy the demand for domestic and foreign savings. The data refers to 1982, and is based on an aggregation of that reported in Peterson (1989). The experiment which will be used to compare the two models is the introduction of a 20 per cent subsidy on the output of food in the U.S. (No distortions are built into the initial equilibrium data set.) Documentation of the computer code used to implement the levels version of the model using NCPLU (Preckel. 1988) is discussed in Hertel *et al.*. (1991). The linearized representation used to solve this via GEMPACK (Codsi and Pearson, 1988) is provided in Appendix B.

From Table 2 it can be seen that, in the initial equilibrium, both regions are on their individual budget constraints (once the transfer from ROW to US

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24 Although we have only described here how the method in Hanslow and Pearson (1991) can be used to obtain the local sensitivity of the Johansen solution, their method can also be used to obtain the local sensitivity of true nonlinear solutions. The basic idea is to take into account possible changes in the parameters as well as in the variables when forming the linearized representation to be used for sensitivity calculations.

Table 2

Benchmark Equilibrium Data ($US 1982 Trillion)

Commodity	U.S. Net Output	U.S. Investment Demand	U.S. Household Demand	Trade U.S. Exports	Trade R.O.W. Exports	R.O.W. Houshold Demand	R.O.W. Investment Demand	R.O.W. Net Output
Food	24204	—	0.22552	0.03101	0.01449	1.30000	—	1.28348
Manufactures	0.54618	0.16865	0.43705	0.16515	0.22467	1.456009	0.827677	2.343206
Services	1.68051	0.28114	1.38995	0.06879	0.05937	4.541441	1.379737	5.911758
Capital goods	—	—	0.44876	0.094078	0.093048	2.208444	—	—
Transfer	—	—	—	0.03255	—	—	—	—
Totals	2.46873	0.44979	2.50128	0.391578	0.391578	9.505894	2.207414	9.538444

is taken into account) and they are thus (equivalently) in balance of payments equilibrium with one another. There are really only two agents in eacheconomy: a representative consumer and an aggregate, multiproduct industry. However, implementation of this model via equations (1) – (8e) or (1*) – (8e*) is facilitated by defining seven additional "dummy industries" in each region. These include: (a) two endowment industries (one to distribute the primary factor endowment to the aggregate domestic firm which in turn supplies the three net outputs, and one to supply the "goodwill"), (b) a capital goods industry which assembles investment goods, and (c) four "margins" industries, assembling foreign and domestic food, manufactures, services, or capital goods (savings), thereupon making it available to domestic consumers. In practice, such margins industries will themselves absorb real resources. However, we abstract from that possibility here.

As noted in the context of the behavioural equations developed above, multiproduct technology is represented by a CDE implicit revenue function, with constant returns to scale imposed. This applies to the production possibilities frontier which determines each economy's net output, based on its exogenous primary factor endowments. All other industries are single product enterprises represented by a CES cost function which also exhibits constant returns to scale. Finally, consumption behaviour is represented via a non-homothetic, CDE implicit minimum expenditure function. Details on the model's parameters are provided in an Appendix A.

Comparison of Empirical Results

Table 3 presents the predicted outcomes for key variables derived from implementation of the 20 per cent U.S. food production subsidy using both the linearized and the levels systems of equations. In the case of the linearized representation, two alternative solutions are presented. The right hand column presents estimates based on a one-step solution. This is the

Table 3

A Comparison of Results from the Levels and Linearized Implementations[a] of the Two Region Model: 20 Per Cent Subsidy on U.S. Food Production— Predicted Levels of Selected Variables in New Equilibrium (Percentage changes in parentheses)

Levels Implementation[b]		Linearized Implementation	
Variable		Multistep[c]	One-step
World prices:			
U.S. food	0.9261922	0.9261927	0.9144747
R.O.W. food	0.9919922	0.9919928	0.9915167
U.S. manufactures	1.0088813	1.0088818	1.0094076
R.O.W. manufactures	1.0019112	1.0019119	1.0020536
U.S. services	1.0195380	1.0195385	1.0204131
R.O.W. services	1.0009723	1.0009728	1.0010278
U.S. capital goods	1.0155291	1.0155296	1.0162866
R.O.W. capital goods	1.0013243	1.0013248	1.0014124
Disposable Income:			
U.S.	(0.66566%)	(0.66570%)	(0.74930%)
R.O.W.[d]	Zero	Zero	Zero
Relative Factor Returns:	(2.6635%)	(2.6635%)	(2.7200%)
Utility of Consumption:			
U.S.	1.0008319	1.0008319	1.0025373
R.O.W.	1.0004729	1.0004729	1.0004654
Aggregate Utility:			
U.S.	(e)	(-0.03621%)	(0.08280%)
R.O.W.	(e)	(-0.008187%)	(-0.01177%)
Walras Law:	0.000000003	0.00000000	-0.00000006

(a) The "Levels Implementation" results here were obtained by NCPLU (Preckel, 1988) running in double prevision. The "Linearized Implementation" results were obtained by GEMPACK (see Codsi and Pearson (1988) running in single precision.

(b) These results differ from those results reported in Hertel *et al.* (1991), Table 4 due to the fact that those results are based on a simulation whereby the subsidy was deducted from current consumption instead of disposable income.

(c) The multistep solution was obtained by extrapolating on the basis of three steps, where the first is 10 steps, followed by 20 and 40 step solutions.

(d) The price of the R.O.W. endowment has been chosen as the numeraire.

(e) Aggregate utility indices are not strictly comparable due to the differential treatment of transfer payments in the utility function. In the linearized implementation, "good will" enters the aggregate utility function explicitly. However, in both cases, the value of the transfer remains constant in terms of the numeraire (ROW endowment price). Thus the two equilibrium problems are otherwise identical.

standard "Johansen" approach, by which the initial equilibrium value weights and elasticities in (1*) – (8*e) are treated as parameters. Consequently they are not adjusted to reflect the effect of changing relative prices, and are thus inconsistent with the new equilibrium position. In contrast, the multi-step implementation of the linearized model uses formulae (F1) – (F4) to update values and shares as discussed in the previous section. Thus the new equilibrium values and elasticities satisfy all of the accounting identities and behavioural restrictions implied by neoclassical theory.

Comparison of the entries in the first two columns of Table 3 reveals remarkable agreement between the levels, and linearized-multistep solutions of the model. Differences in predicted price *levels* appear only in the seventh decimal place, which is at the limit of the single precision arithmetic used to produce the "Linearized Implementation" results in Table 3. In contrast, the one-step implementation occassionally exhibits discrepancies in the second digit. More importantly, the first two implementations agree exactly on their evaluation of utility of consumption, based on the implicit CDE minimum expenditure function. Unfortunately, the same cannot be said for the one-step implementation. Since the model's initial equilibrium is undistorted, the output subsidy introduces no measurable excess burden into a one-step perturbation of the model. This means that the one step solution *overstates* disposable income, and hence also overstates utility. Indeed, this error is large enough to *reverse* the direction of change in U.S. aggregate utility. Furthermore, when converted to real income this is a nontrivial error amounting to roughly $3 billion, or about six percent of the subsidy payments.

In summary, the evidence in Table 3 may be viewed as lending support to both the "Johansen" and levels schools of modelling. The advocates of simple one-step matrix inversion of a locally linearized AGE representation can look at the price discrepancies, and given the degree of uncertainty about model specification and parameters, they might pronounce the answers acceptably close. "What we really care about is the direction and relative magnitude of the changes." Meanwhile, advocates of the levels school of modelling need only look at the utility predictions to confirm their worst fears about the so-called "Johansen" modellers. The point of this paper is that such altercations are no longer relevant. Convenient software permitting algebraic implementation of either (1) – (8e) or (1*) – (8e*) is now readily available. The "linearizers" have no defense for not employing the multistep procedures in order to obtain the true solution to the nonlinear problem as posed by (1) – (8e). Similarly, the "levels" school of modelling can no longer discount a model on the grounds that it is implemented via a linearized representation. *Provided the linearized representation is used as the starting point for an algorithm which produces accurate (nonlinear) results, it provides a solution to the same nonlinear equilibrium problem as would be obtained via a levels formulation.*

Systematic Sensitivity Analysis

It is well known that general equilibrium trade models in which products are differentiated by origin are quite sensitive to specified values for the

Armington parameters which determine the ease of substitutability between foreign and domestic products, and among different sources of foreign products (Brown, 1987). Indeed this is the focus of the sensitivity analysis paper by Harrison *et al.* (1991). The model presented above is no exception to this rule. Since there is only one "foreign" source for each of the region's imports, there is only one Armington elasticity of substitution for each product grouping in each region, i.e., σ_j^r, $j \in$ margins industries. In this section we will use the methodology outlined at the end of Section IV in order to examine the sensitivity of model results to perturbations in each of these parameters individually, as well as simultaneous shocks to various groupings of them.

The variable which we focus on is the rate of return to the U.S. primary factor endowment. Since the foreign primary factor return is the numeraire in this model, we are effectively focussing on the *relative* rate of return to primary factors in the two regions. This variable is the key to achieving balance of payments equilibrium in the model. In effect it is the "real exahange rate". If a given policy intervention reduces U.S. excess demand for R.O.W. commodities, then an *appreciation* of this variable $p_{PF}^{US} > 0$ is required to restore equilibrium. This raises the relative price of U.S. goods and simultaneously increases total U.S. spending, thus eliminating the excess demand and bringing the two regions back into balance. The magnitude of p_{PF}^{US} required to reequilibrate the system following such a shock obviously depends on the ease of substitution between U.S. and R.O.W. in both regions. The larger these Armington elasticities, the smaller the required adjustment in relative primary factor returns.

Table 4 reports the systematic sensitivity results derived by examining the local sensitivity of the one-step results to perturbations in each of the Armington parameters. Before discussing these numbers, several qualifications are in order. First recall from Table 3 that the one-step estimate of p_{PF}^{US} *overstates* the true adjustment required (2.7200 per cent *versus* 2.6635 per cent) as reported in the first two colums of that table. This is because the linearized model does not permit the R.O.W. share to increase as U.S. prices rise. Thus the linearized model *understates* the own-price elasticity of demand for U.S. products in the new equilibrium [(see F(2)]. A second point to be made is that, while the perturbation represented by (ΔF2) and (Δ6**) is only strictly correct for infinitesimal changes, the shocks considered in Table 4 are rather more substantial. Thus this is a somewhat crude, but extremely cheap method of assessing the model's sensitivity to the Armington parameters. All the numbers in this table were generated by inverting a single matrix of slightly greater dimension than the basic model. Thus the computational cost is far less than solving the levels representation even once. Furthermore, implementation is sufficiently straightforward to permit routine application of this technique.

The first column of numbers in Table 4 reports the equilibrium solution to the local perturbation of the one-step outcome: dp_{PF}^{US*}. The first row in the table reports the case where no parameters are perturbed, so that $dp_{PF}^{US*} = 0$,

Table 4
Sensitivity of One-Step Solution to Perturbations in the Armington Parameters

Nature of Perturbation ($d\sigma_j^r$)	Difference in Relative Primary Factor Returns (dp_{PF}^{US*})(b)	Revised Percentage Change in Primary Factor Return ($p_{PF}^{US**} = p_{PF}^{US*} + dp_{PF}^{US*}$)
None	0	2.7200 per cent
Doubling $\sigma(r, j)$(a):		
(U.S., food)	0.3615	3.0815 per cent
(U.S., mnfc.)	–0.1407	2.5793 per cent
(U.S., svces)	–0.1920	2.5280 per cent
(U.S., c.goods)	–0.2514	2.4686 per cent
(U.S., all)	**–0.2226**	**2.4974 per cent**
(R.O.W., food)	0.8070	3.5270 per cent
(R.O.W., mnfc)	–0.1887	2.5313 per cent
(R.O.W., svces)	–0.2288	2.4912 per cent
(R.O.W., cgds)	–0.3069	2.4131 per cent
(R.O.W., all)	**0.0826**	**2.8026 per cent**
(Global, all)	**–0.1400**	**2.5800 per cent**

(a) The value of the Armington elasticities of substitution in the original model are as follows: food = 5, manufactures and capital goods = 3, and services = 2.

(b) The results in this column are the estimated actual changes in the original simulation result of 2.7200 per cent. Thus the value 0.3615 in this column indicates that the one-step simulation result would change from 2.7200 per cent to approximately 2.7200 = 0.3615 = 3.0815 per cent.

and $p_{PF}^{US**} = p_{PF}^{US*} = 2.7200$ per cent, as reported in the third column of Table 3. The remaining entries in Table 4 involve a doubling of the initial Armington elasticity of subsitutiton. Because the policy shock involves a subsidy on U.S. food output, the first-round effects are determined in large part by the elasticity of substitution between U.S. and R.O.W. food in two regions. If this is increased, then the initial disequilibrium in transactions between the two regions is exacerbated as U.S. food displaces R.O.W. food in both regions. Given unchanged Armington parameters for other commodities, it takes a larger value of p_{PF}^{US} to erase this increased excess demand for U.S. products. Thus it is not surprising that increasing either σ^{US}_{FOOD} or σ^{RW}_{FOOD} results in a value of $dp^{US*}_{PF} > 0$. Indeed, doubling both of these parameters, while leaving the remaining σ^{r}_{j}'s unchanged results in an approximate value of p^{US**}_{PF} which is almost half again as large as the initial estimate.

Of course an increase in any of the nonfood Armington parameters dampens p^{US*}_{PF}, since it is now easier to adjust to the regional imbalance induced by the food subsidy. Indeed, the two changes are roughly offsetting. Thus a doubling of *all* Armington parameters, either in the U.S. or in R.O.W., or in both regions, causes a relatively small change in p^{US*}_{PF}. This is useful information. Furthermore, in a very large trade model, where there are many Armington parameters, results such as those in Table 4 would prove very helpful in establishing an economical design for further, more accurate sensitivity calculations.

VI. Summary and Conclusions

As reflected in its title, the objective of this paper has been to "mend the family tree" of applied general equilibrium economics. More specifically, we have sought to reconcile the linearization and levels schools of modelling. Historically interesting prototypes aside, there is no such thing as a *fully* linear general equilibrium model. Rather there are linearized and levels representations of GE models, and either is a natural starting point for solving the model; the numerical solutions obtained are, of course, independent of the choice of starting point. There is no cogent reason for distinguishing between classes of models by the manner in which their members are represented. Of far greater importance is the economic content of the model(s) in question. While the force of these observations is immediately obvious, the historical lack of communication between the two schools of modellers provides ample evidence of the need for such a reconciliation.

Towards this end we have provided, side-by-side, both linearized and levels representations of a fairly general trade model, together with a comparison of the two based on a number of criteria of interest. We then discussed issues of model implementation and solution. Of particular note is the approach to solving the true nonlinear equilibrium problem using the linearized representation of the model and a set of formulae for updating model coefficients. A simple 2 region example served to illustrate the equivalence of the two approaches.

In summary, this paper carries a message to both schools of AGE modelling. To the "linearizers" we emphasize the point that there is no longer any excuse for providing only "approximate" solutions to AGE models, based on "Johansen method" matrix inversion. Most properly specified models can be readily implemented in linearized form, yet solved for the true nonlinear solution. Of course this calls for a *complete* specification of preferences and technology so that it is clear how the model's elasticities vary as a function of prices. The disciplines of having to specify how the levels data is updated (between steps of the multi-step solution process) and how elasticities depend on this data ensure that modellers using a linearized representation have a complete, nonlinear behavioural specification underpinning their model.

We also carry a message to the levels school of modellers: from both a theoretical and a computational point of view, the linearized approach to AGE modelling is equally valid, provided the aforementioned disciplines are observed. Furthermore, the linearized representation has some appealing properties which make it an attractive alternative in many instances. These include: a more straightforward representation of behavioral relationships, indifference to implicit *versus* explicit representations of preferences and technology, as well as the availability of inexpensive, local sensitivity analysis.

Most importantly we wish to point out that the two schools of AGE modelling have a great deal in common. Both would benefit from greater interaction.

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Appendix A Behavioural Parameters

Commodity	Elasticity of Substitution (σ_j)[a]		Transformation Parameter (α_S)		Substitution Parameter (α_H)		Expansion Parameter (γ_H)	
	U.S.	R.O.W.	U.S.	R.O.W.	U.S.	R.O.W.	U.S.	R.O.W.
Food	5.0	5.0	2.030633	1.794214	0.627078	0.543540	0.328013	0.699897
Manufacturing	3.0	3.0	2.345713	2.490010	0.415685	0.355754	0.572383	0.995484
Services	2.0	2.0	3.732460	3.468794	0.664946	0.843377	1.0	1.0
Capial Goods	3.0[c]	3.0[c]	(b)	(b)	(b)	(b)	(b)	(b)

(a) These are the "Armington" elasticities of substitution used to combine foreign and domestic products.

(b) These are not applicable.

(c) The elasticity of substitution among inputs used to create capital goods is unity.

Appendix B TABLO File for the Trade Model

```
! Multiregion Trade Model With Updates. For Documentation see:
    Hertel, T.W., J.M. Horridge, and K.R. Pearson,
     "Mending the Family Tree: A Reconcilation of the
      Linearization and Levels Schools of Applied General
      Equilibrium Modelling", Impact Project Preliminary
      Working Paper, University of Melbourne.
      This is referred to below as HHP. !

! Text between exclamation marks is a comment !
! Text between hashes # is labelling inforamtion !

! SETS !

! Sets define relevant groupings of entities over which we will
  be performing operations in the model. Subsets are defined
  in order to facilitate summation over only a portion of a given
  group, e.g. tradeable commodities. They will also be used later
  on when we wish to condense the model in various ways, eg. to
  generate an aggregate regional revenue function. By assigning
  each element a name, we can later identify them/extract them
  easily. !


SET REG # regions # (USA, ROW) ;

SET IND # industries # (prfactor, gdwill, ppf, cgds, mfood, mmnfc, msvces,
  msavings) ;

SET MARG_IND # margins industries #  (mfood, mmnfc, msvces, msavings) ;
```

```
SET ENDW_IND # endowment industries # (prfactor, gdwill ) ;

SET NEND_IND # nonendowment industries # (ppf, cgds, mfood, mmnfc, msvces,
  msavings) ;

SET COMM # commodities # (prfactor, gdwill, foodus, foodrw, mnfcus,
  mnfcrw, svcesus, svcesrw, cgdsus, cgdsrw, cfood, cmnfc,
  csvces, csavings) ;

! note that only the traded goods which are differentiated by origin
  (food, mnfc, and svces), need to be separately identified. Gdwill
  is homogeneous (one region supplies it and the other purchases it.
  The nontraded goods are identified by the market in which they are
  produced or consumed. !

SET TRAD_COMM # traded commodities # (gdwill, foodus, foodrw, mnfcus,
  mnfcrw, svcesus, svcesrw, cgdsus, cgdsrw);

SET WALR_COMM # excludes one good - mnfcus # (gdwill, foodus, foodrw,
  mnfcrw, svcesus, svcesrw, cgdsus, cgdsrw) ;

SET NTRD_COMM # the set of all nontraded commodities # (prfactor,
  cfood, cmnfc, csvces, csavings) ;

SET HHLD_COMM # consumer commodities # (cfood, cmnfc,
  csvces, csavings, gdwill) ;

SET CONS_COMM # consumption goods # (cfood, cmnfc, csvces );

SET NONC_COMM # hhld savings and goodwill and all other commodities #
  (prfactor, gdwill, foodus, foodrw, mnfcus,
  mnfcrw, svcesus, svcesrw, cgdsus, cgdsrw, csavings) ;

SET ENDW_COMM # endowment commodities # (prfactor, gdwill ) ;

SUBSET MARG_IND IS SUBSET OF IND ;
SUBSET ENDW_IND IS SUBSET OF IND ;
SUBSET NEND_IND IS SUBSET OF IND ;
SUBSET TRAD_COMM IS SUBSET OF COMM ;
SUBSET WALR_COMM IS SUBSET OF TRAD_COMM ;
SUBSET NTRD_COMM IS SUBSET OF COMM ;
SUBSET HHLD_COMM IS SUBSET OF COMM ;
SUBSET CONS_COMM IS SUBSET OF HHLD_COMM ;
SUBSET NONC_COMM IS SUBSET OF COMM ;
SUBSET ENDW_COMM IS SUBSET OF COMM ;


! FILES
  ----- !

FILE BASEDATA # The file containing all base data for the economy. # ;
```

```
! VARIABLES: These are divided into four groups -- percentage
changes in quantities, prices, taxes, and finally income and
utility. In GEMPACK, variables refer to those items which will be
changing endogenously with each Johansen solution. They are assigned
lower case labels to denote the fact that they are percentage changes. !


! quantity variables !

VARIABLE (all,j,IND)(all,r,REG) z(j,r)
        # activity level of industry j in region r # ;

VARIABLE (all,i,COMM)(all,j,IND)(all,r,REG) qs(i,j,r)
        # supply of commodity i from industry j in region r # ;

VARIABLE (all,i,COMM)(all,j,IND)(all,r,REG) qd(i,j,r)
        # demand for commodity i by industry j in region r # ;

VARIABLE (all,i,COMM)(all,r,REG) qh(i,r)
        # household demand for commodity i in region r # ;

VARIABLE walras_dem
        # demand in the omitted market - should equal "walras_sup" # ;

VARIABLE walras_sup
        # supply in the omitted market - should equal "walras_dem" # ;


! price variables !

VARIABLE (all,i,COMM)(all,j,IND)(all,r,REG) ps(i,j,r)
        # supply price of commodity i from industry j in region r # ;

VARIABLE (all,i,COMM)(all,j,IND)(all,r,REG) pd(i,j,r)
        # demand price for commodity i by industry j in region r # ;

VARIABLE (all,i,COMM)(all,r,REG) ph(i,r)
        # household price for commodity i in region r # ;

VARIABLE (all,i,COMM)(all,r,REG) pm(i,r)
        # domestic price for good i in region r # ;

VARIABLE (all,i,TRAD_COMM) pw(i)
        # world price of tradeable good i # ;


! tax variables !

VARIABLE (all,i,COMM)(all,j,IND)(all,r,REG) ts(i,j,r)
        # tax on commodity i supplied by industry j in region r # ;

VARIABLE (all,i,COMM)(all,j,IND)(all,r,REG) td(i,j,r)
```

```
        # tax on commodity i demanded by industry j in region r # ;

VARIABLE  (all,i,COMM)(all,r,REG)  th(i,r)
        # tax on commodity i purchased by housedholds in region r # ;

VARIABLE  (all,i,TRAD_COMM)(all,r,REG)  tt(i,r)
        # border tax on good i in region r # ;

! income and utility !

VARIABLE  (all,r,REG)  y(r)
        # household income in region r # ;

VARIABLE  (all,r,REG)  u(r)
        # aggregate utility of household in region r #  ;

VARIABLE  (all,r,REG)  uc(r)
        # utility of consumption for household in region r #  ;


!  BASE DATA
   ---------  !


! The base data are divided into four sections:  base revenues/expendi-
  tures at agent's prices, base revenues/expenditures at market prices,
  and base revs/exps at world prices for all tradeable commodities,
  followed by the technology and preference parameters.   Since these are
  invariant for each solution of the model, they are termed coefficients.
  Coefficients are assigned upper case labels to distinguish them from
  variables.  The updating command indicates how the new level of the coef-
  ficient will be computed based on the previous Johansen solution.  Note that
  the notation used in the update commands is a shorthand for total differen-
  tials of these coefficient values.  Thus, w * v indicates that we want
  to take the total differential of W * V, plug in the calculated
  values of w and v, and add this to the base level in order to
  obtain a revised value for this product. Of course the technology
  preference parameters do not change at all and so require no update
  statement.  !

! base revenues and expenditures at agent's prices !

COEFFICIENT  (all,i,COMM)(all,j,IND)(all,r,REG)   VSA(i,j,r)
        ! producer revenue from commodity i in industry j in region r ! ;
UPDATE       (all,i,COMM)(all,j,IND)(all,r,REG)
     VSA(i,j,r)= ps(i,j,r) * qs(i,j,r) ;

COEFFICIENT  (all,i,COMM)(all,j,IND)(all,r,REG)   VDA(i,j,r)
        ! producer expenditure on commodity i in industry j, region r ! ;
UPDATE       (all,i,COMM)(all,j,IND)(all,r,REG)
     VDA(i,j,r)= pd(i,j,r) * qd(i,j,r) ;
```

```
COEFFICIENT (all,i,COMM)(all,r,REG) VHA(i,r)
        ! household expenditure on commodity i in region r ! ;
UPDATE      (all,i,COMM)(all,r,REG)
     VHA(i,r)= ph(i,r) * qh(i,r) ;

! base revenues and expenditures at market prices !

COEFFICIENT (all,i,COMM)(all,j,IND)(all,r,REG) VSM(i,j,r)
        ! producer revenue from commodity i in industry j in region r
         valued at domestic market prices ! ;
UPDATE      (all,i,COMM)(all,j,IND)(all,r,REG)
     VSM(i,j,r)= pm(i,r) * qs(i,j,r) ;

COEFFICIENT (all,i,COMM)(all,j,IND)(all,r,REG) VDM(i,j,r)
        ! producer expenditure on commodity i in industry j, region r
         valued at domestic market prices ! ;
UPDATE      (all,i,COMM)(all,j,IND)(all,r,REG)
     VDM(i,j,r)= pm(i,r) * qd(i,j,r) ;

COEFFICIENT (all,i,COMM)(all,r,REG) VHM(i,r)
        ! household expenditure on commodity i in region r
         valued at domestic market prices ! ;
UPDATE      (all,i,COMM)(all,r,REG)
     VHM(i,r)= pm(i,r) * qh(i,r) ;


! base revenues and expenditures at world prices !

COEFFICIENT (all,i,TRAD_COMM)(all,j,IND)(all,r,REG) VSW(i,j,r)
        ! producer revenue from commodity i in industry j in region r
         valued at world prices (tradeables only) ! ;
UPDATE      (all,i,TRAD_COMM)(all,j,IND)(all,r,REG)
     VSW(i,j,r) = pw(i) * qs(i,j,r) ;

COEFFICIENT (all,i,TRAD_COMM)(all,j,IND)(all,r,REG) VDW(i,j,r)
        ! producer expenditure on commodity i in industry j, region r
         valued at world prices (tradeables only) ! ;
UPDATE      (all,i,TRAD_COMM)(all,j,IND)(all,r,REG)
     VDW(i,j,r) = pw(i) * qd(i,j,r) ;

COEFFICIENT (all,i,TRAD_COMM)(all,r,REG) VHW(i,r)
        ! household expenditure on commodity i in region r
         valued at world prices (tradeables only) ! ;
UPDATE      (all,i,TRAD_COMM)(all,r,REG)
     VHW(i,r)= pw(i) * qh(i,r) ;

! technology and preference parameters !

! consumer demand parameters !

COEFFICIENT (all,i,CONS_COMM)(all,r,REG)    SUBPAR(i,r)
        ! the substitution parameter in the CDE
         minimum expenditure function ! ;
```

```
COEFFICIENT  (all,i,CONS_COMM)(all,r,REG)    INCPAR(i,r)
         ! expansion parameter in the CDE minimum expenditure
         function ! ;

! technology parameters !

COEFFICIENT  (all,i,COMM)(all,j,IND)(all,r,REG)    TRNPAR(i,j,r)
         ! the transformation parameter in the CDE
          maximum revenue function ! ;

COEFFICIENT  (all,j,IND)(all,r,REG)    ESUB(j,r)
         ! elasticity of substitution among inputs in
         production ! ;


! reading in the base data !

! data for the United States !

READ (all,i,COMM)(all,j,IND) VSA(i,j,"USA") FROM FILE BASEDATA HEADER "PRUA" ;
READ (all,i,COMM)(all,j,IND) VDA(i,j,"USA") FROM FILE BASEDATA HEADER "PEUA" ;
READ (all,i,COMM) VHA(i,"USA") FROM FILE BASEDATA HEADER "HEUA" ;
READ (all,i,COMM)(all,j,IND) VSM(i,j,"USA") FROM FILE BASEDATA HEADER "PRUM"
;
READ (all,i,COMM)(all,j,IND) VDM(i,j,"USA") FROM FILE BASEDATA HEADER "PEUM"
;
READ (all,i,COMM) VHM(i,"USA") FROM FILE BASEDATA HEADER "HEUM" ;
READ (all,i,TRAD_COMM)(all,j,IND) VSW(i,j,"USA")
              FROM FILE BASEDATA HEADER "PRUW" ;
READ (all,i,TRAD_COMM)(all,j,IND) VDW(i,j,"USA")
              FROM FILE BASEDATA HEADER "PEUW" ;
READ (all,i,TRAD_COMM) VHW(i,"USA")
              FROM FILE BASEDATA HEADER "HEUW" ;

READ (all,i,CONS_COMM) INCPAR(i,"USA") FROM FILE BASEDATA HEADER "INUS" ;
READ (all,i,CONS_COMM) SUBPAR(i,"USA") FROM FILE BASEDATA HEADER "SUUS" ;
READ (all,i,COMM)(all,j,IND) TRNPAR(i,j,"USA") FROM FILE BASEDATA HEADER
"TRUS";
READ (all,j,IND) ESUB(j,"USA") FROM FILE BASEDATA HEADER "ESUS" ;

! data for the rest of the world !

READ (all,i,COMM)(all,j,IND) VSA(i,j,"ROW") FROM FILE BASEDATA HEADER "PRRA" ;
READ (all,i,COMM)(all,j,IND) VDA(i,j,"ROW") FROM FILE BASEDATA HEADER "PERA" ;
READ (all,i,COMM) VHA(i,"ROW") FROM FILE BASEDATA HEADER "HERA" ;
READ (all,i,COMM)(all,j,IND) VSM(i,j,"ROW") FROM FILE BASEDATA HEADER "PRRM"
;
READ (all,i,COMM)(all,j,IND) VDM(i,j,"ROW") FROM FILE BASEDATA HEADER "PERM"
;
READ (all,i,COMM) VHM(i,"ROW") FROM FILE BASEDATA HEADER "HERM" ;
READ (all,i,TRAD_COMM)(all,j,IND) VSW(i,j,"ROW")
              FROM FILE BASEDATA HEADER "PRRW" ;
```

```
READ (all,i,TRAD_COMM)(all,j,IND) VDW(i,j,"ROW")
                FROM FILE BASEDATA HEADER "PERW" ;
READ (all,i,TRAD_COMM) VHW(i,"ROW")
                FROM FILE BASEDATA HEADER "HERW" ;

READ (all,i,CONS_COMM) INCPAR(i,"ROW") FROM FILE BASEDATA HEADER "INRW" ;
READ (all,i,CONS_COMM) SUBPAR(i,"ROW") FROM FILE BASEDATA HEADER "SURW" ;
READ (all,i,COMM)(all,j,IND) TRNPAR(i,j,"ROW") FROM FILE BASEDATA HEADER
"TRRW";
READ (all,j,IND) ESUB(j,"ROW") FROM FILE BASEDATA HEADER "ESRW" ;

! DERIVATIVES OF THE BASE DATA
 ---------------------------  !

! derivatives of the base data include computations of
household income, budget shares, and elasticities.
(The parameters needed to calculate the elasticities will be
read in at the point when they are required.)
Again, since these are constant for each Johansen solution, they
are termed coefficients. !


! household income !

COEFFICIENT (all,r,REG) INCOME(r)
      ! level of income in region r ! ;
FORMULA     (all,r,REG)
     INCOME(r) = sum(i,HHLD_COMM, VHA(i,r)) ;


! budget shares !

COEFFICIENT (all,i,COMM)(all,j,IND)(all,r,REG)   SI(i,j,r)
      ! revenue share of commodity i in industry j, region r ! ;
FORMULA     (all,i,COMM)(all,j,IND)(all,r,REG)
 SI(i,j,r) = VSA(i,j,r) / sum(m,COMM,VSA(m,j,r)) ;

COEFFICIENT (all,i,COMM)(all,j,IND)(all,r,REG)   CI(i,j,r)
      ! cost share of commodity i in industry j, region r ! ;
FORMULA     (all,i,COMM)(all,j,IND)(all,r,REG)
 CI(i,j,r) = VDA(i,j,r) / sum(m,COMM,VDA(m,j,r)) ;

COEFFICIENT (all,i,HHLD_COMM)(all,r,REG)   BI(i,r)
      ! budget share of commodity i in region r ! ;
FORMULA     (all,i,HHLD_COMM)(all,r,REG)
 BI(i,r) = VHA(i,r) / sum(m,HHLD_COMM,VHA(m,r)) ;


! checking the base data !

COEFFICIENT     (all,j,IND)(all,r,REG)   PROFITS(j,r)
         ! This variable computes the sum of revenues minus
           expenditures in each sector. This should be zero. !;
```

```
FORMULA        (all,j,IND)(all,r,REG)
  PROFITS(j,r) = sum(i,COMM, VSA(i,j,r) - VDA(i,j,r)) ;

COEFFICIENT      (all,r,REG)   SURPLUS(r)
          ! This variable computes the excess of income (including
            transfers and net borrowing) over expenditures for
           each region in the data base.  It should be zero. ! ;
FORMULA          (all,r,REG)
  SURPLUS(r) = sum(j,ENDW_IND, sum(i,ENDW_COMM, VSA(i,j,r)))
      + sum(i,HHLD_COMM, VHA(i,r) - VHM(i,r))
      + sum(j,IND, sum(i,COMM, VSM(i,j,r) - VSA(i,j,r)))
      + sum(j,IND, sum(i,COMM, VDA(i,j,r) - VDM(i,j,r)))
      + sum(i,TRAD_COMM, (VHM(i,r) - VHW(i,r))
          + sum(j,IND, (VDM(i,j,r) - VDW(i,j,r))
                - (VSM(i,j,r) - VSW(i,j,r))))
      - sum(i,HHLD_COMM, VHA(i,r));

DISPLAY PROFITS ; DISPLAY SURPLUS ; DISPLAY SI ; DISPLAY CI ; DISPLAY BI ;

! elasticities  !

! computing the elasticities  !

COEFFICIENT   (all,i,CONS_COMM)(all,r,REG)      DALPHA(i,r)
          ! one minus the substitution parameter in the CDE
           minimum expenditure function ! ;
FORMULA        (all,i,CONS_COMM)(all,r,REG)
  DALPHA(i,r) = (1 - SUBPAR(i,r)) ;

COEFFICIENT   (all,i,COMM)(all,j,IND)(all,r,REG)      SALPHA(i,j,r)
          ! one minus the transformation parameter in the CDE
           maximum revenue function ! ;
FORMULA        (all,i,COMM)(all,j,IND)(all,r,REG)
  SALPHA(i,j,r) = (1 - TRNPAR(i,j,r)) ;

COEFFICIENT  (all,i,CONS_COMM)(all,r,REG)  CONSHR(i,r)
         ! the share in total consumption expenditure of
          good i in region r ! ;
FORMULA       (all,i,CONS_COMM)(all,r,REG)
  CONSHR(i,r) = BI(i,r)/sum(k,CONS_COMM, BI(k,r)) ;

COEFFICIENT  (all,i,CONS_COMM)(all,k,CONS_COMM)(all,r,REG)  AEP(i,k,r)
         ! the allen partial elasticity of substitution between
          goods i and k in region r ! ;
FORMULA       (all,i,CONS_COMM)(all,k,CONS_COMM)(all,r,REG)
  AEP(i,k,r) =  DALPHA(i,r) + DALPHA(k,r) -
           sum(m,CONS_COMM, CONSHR(m,r) * DALPHA(m,r)) ;
FORMULA       (all,i,CONS_COMM)(all,r,REG)
  AEP(i,i,r) =  2.0 * DALPHA(i,r) -
           sum(m,CONS_COMM, CONSHR(m,r) * DALPHA(m,r)) -
          DALPHA(i,r) / CONSHR(i,r)  ;
```

```
COEFFICIENT  (all,i,CONS_COMM)(all,r,REG)  COMPDEM(i,r)
        ! the own-price compensated elasticity of household demand ! ;
FORMULA      (all,i,CONS_COMM)(all,r,REG)
  COMPDEM(i,r) = AEP(i,i,r) * CONSHR(i,r) ;

COEFFICIENT  (all,i,CONS_COMM)(all,r,REG)  EY(i,r)
        ! the income elasticity of household demand for
          good i region r ! ;
FORMULA      (all,i,CONS_COMM)(all,r,REG)
  EY(i,r) =  (sum(m,CONS_COMM, CONSHR(m,r) * INCPAR(m,r)) ^ (-1.0))
        * (INCPAR(i,r) * (1.0 - DALPHA(i,r))
          + sum(m,CONS_COMM, CONSHR(m,r) * INCPAR(m,r) * DALPHA(m,r)))
        + (DALPHA(i,r) - sum(m,CONS_COMM, CONSHR(m,r) * DALPHA(m,r))) ;

COEFFICIENT  (all,i,CONS_COMM)(all,k,CONS_COMM)(all,r,REG)  EP(i,k,r)
        ! the uncompensated cross-price elasticity of hhld
          demand for good i wrt the kth price in region r ! ;
FORMULA      (all,i,CONS_COMM)(all,k,CONS_COMM)(all,r,REG)
  EP(i,k,r) =  (AEP(i,k,r) - EY(i,r)) * CONSHR(k,r) ;

COEFFICIENT  (all,i,COMM)(all,k,COMM)(all,j,IND)(all,r,REG)  ES(i,k,j,r)
        ! the compensated cross-price elasticity of
          supply of good i wrt the kth price in industry
        j of region r ! ;
FORMULA      (all,i,COMM)(all,k,COMM)(all,j,IND)(all,r,REG)
  ES(i,k,j,r) = SI(k,j,r) *
          (SALPHA(i,j,r) + SALPHA(k,j,r)
          - sum(m,COMM, SI(m,j,r) * SALPHA(m,j,r)));
FORMULA      (all,i,COMM)(all,j,IND)(all,r,REG)
  ES(i,i,j,r) = SI(i,j,r) *
        (2.0 * SALPHA(i,j,r)
          - sum(m,COMM, SI(m,j,r) * SALPHA(m,j,r))
          - SALPHA(i,j,r) / SI(i,j,r)) ;

COEFFICIENT  (all,i,COMM)(all,j,IND)(all,r,REG)  COMPSUP(i,j,r)
        ! the compensated own-price elasticity of
          supply of good i in industry
        j of region r ! ;
FORMULA      (all,i,COMM)(all,j,IND)(all,r,REG)
  COMPSUP(i,j,r) = ES(i,i,j,r) ;

COEFFICIENT  (all,i,COMM)(all,k,COMM)(all,j,IND)(all,r,REG)  ED(i,k,j,r)
        ! the compensated cross-price elasticity of
          demand for good i wrt the kth price in industry
        j of region r ! ;
FORMULA      (all,i,COMM)(all,k,COMM)(all,j,IND)(all,r,REG)
  ED(i,k,j,r) =  CI(k,j,r) * ESUB(j,r) ;
FORMULA      (all,i,COMM)(all,j,IND)(all,r,REG)
  ED(i,i,j,r) =  (CI(i,j,r) - 1.0) * ESUB(j,r) ;


! checking the elasticities !
```

```
COEFFICIENT (all,r,REG) ENGELAGG(r)
        ! a check on the Engel aggregation condition for the
          aggregate household in region r ! ;
FORMULA    (all,r,REG)
  ENGELAGG(r) = sum(i,CONS_COMM, (BI(i,r)/sum(m,CONS_COMM, BI(m,r)))
               *EY(i,r)) ;

COEFFICIENT (all,i,CONS_COMM)(all,r,REG) COURNOT(i,r)
        ! the rowsum for the ith row of the matrix of
          uncompensated cross-price elasticities of hhld
          demand for good i wrt the kth price in region r
          plus the income elasticity of demand.
          This must equal zero if demand is to be
          homogeneous of degree zero in prices and income. ! ;
FORMULA    (all,i,CONS_COMM)(all,r,REG)
  COURNOT(i,r) = sum(k,CONS_COMM, EP(i,k,r)) + EY(i,r) ;

COEFFICIENT (all,i,COMM)(all,j,IND)(all,r,REG) ROWSUMED(i,j,r)
        ! rowsum for the ith row of the matrix of compensated
          demand elasticities for the jth industry in the
        rth region ! ;
FORMULA    (all,i,COMM)(all,j,IND)(all,r,REG)
  ROWSUMED(i,j,r) = sum(k,COMM, ED(i,k,j,r));

COEFFICIENT (all,k,COMM)(all,j,IND)(all,r,REG) COLSUMED(k,j,r)
        ! cost share-weighted
          colsum for the kth column of the matrix of compensated
          demand elasticities for the jth industry in the
        rth region ! ;
FORMULA    (all,k,COMM)(all,j,IND)(all,r,REG)
  COLSUMED(k,j,r) = sum(i,COMM, CI(i,j,r) * ED(i,k,j,r));

COEFFICIENT (all,i,COMM)(all,j,IND)(all,r,REG) ROWSUMES(i,j,r)
        ! rowsum for the ith row of the matrix of compensated
          supply elasticities for the jth industry in the
        rth region ! ;
FORMULA    (all,i,COMM)(all,j,IND)(all,r,REG)
  ROWSUMES(i,j,r) = sum(k,COMM, ES(i,k,j,r));

COEFFICIENT (all,k,COMM)(all,j,IND)(all,r,REG) COLSUMES(k,j,r)
        ! share-weighted column sum for the
          kth column of the matrix of compensated
          supply elasticities for the jth industry in the
        rth region ! ;
FORMULA    (all,k,COMM)(all,j,IND)(all,r,REG)
  COLSUMES(k,j,r) = sum(i,COMM, SI(i,j,r) * ES(i,k,j,r));

DISPLAY EY ; DISPLAY COMPDEM ; DISPLAY COMPSUP ; DISPLAY ENGELAGG ;

DISPLAY COURNOT ; DISPLAY ROWSUMED ; DISPLAY COLSUMED ;
```

```
DISPLAY ROWSUMES ; DISPLAY COLSUMES ;

! EQUATIONS  !

! The equations in this model pertain to three levels of indexation.
The first set involves behavioral relationships which apply at the
sectoral level. These determine sectoral supplies and demands,
both conditional on activity levels, as well as the zero profits
condition which determines overall activity level in a sector.
The second group of equations are region-specific. Because there
is only a single household in each region, there is a one-to-one
mapping between households and regions. These equations include
the household demands in a given region, the regional/hhld budget
constraint -- in the form of the CDE expenditure function evaluated
at the level of income available to the domestic household for consumption,
and market clearing conditions for the nontradeables.
The final group of equations pertain to the world at large. The first
of these is the market clearing condition for the tradeable goods. Due
to Walras' Law, we may exclude one of the model's equations. We
drop the market clearing condition for one of the tradeable goods
as specified above. This group of equations also contains the
price linkage equations which relate prices for a common commodity
across uses. In order to facilitate cross-referencing between with
the paper, comments also note the corresponding equation number in HHP. !

! the following equations must hold for all sectors !

EQUATION INDSUPPLIES
 ! Industry supply equations. When the industry in question
   is not multiproduct, this is redundant, since supply and
   activity level are identical. Equation (7a*) in HHP. !
   (all,i,COMM)(all,j,IND)(all,r,REG)
qs(i,j,r) = sum(k,COMM, ES(i,k,j,r) * ps(k,j,r)) + z(j,r) ;

EQUATION INDDEMANDS
 ! Industry demand equations, HHP # (6*). !
   (all,i,COMM)(all,j,IND)(all,r,REG)
qd(i,j,r) = sum(k,COMM, ED(i,k,j,r) * pd(k,j,r)) + z(j,r) ;

EQUATION ZEROPROFITS
 ! Industry zero profits condition. This condition permits us to
   determine the endogenous activity level for each of the non-
   endowment sectors. The level of activity in the endowment
   sectors is determined exogenously. Equation (4*) in HHP. !
   (all,j,NEND_IND)(all,r,REG)
sum(i,COMM, SI(i,j,r) * ps(i,j,r)) = sum(i,COMM, CI(i,j,r) * pd(i,j,r)) ;


! the following equations must hold for all regions !

EQUATION REGIONALINCOME
 ! This equation computes regional income as the sum of
   primary factor payments and tax receipts, HHP #(5*). !
```

```
   (all,r,REG)
 INCOME(r) * y(r)
  = sum(i,ENDW_COMM, sum(j,ENDW_IND, VSA(i,j,r) * (ps(i,j,r) + qs(i,j,r))))
  + sum(i,HHLD_COMM, (VHA(i,r) * (ph(i,r) + qh(i,r))) -
             (VHM(i,r) * (pm(i,r) + qh(i,r))))
  + sum(i,COMM, sum(j,IND, (VSM(i,j,r) * (pm(i,r) + qs(i,j,r))) -
                  (VSA(i,j,r) * (ps(i,j,r) + qs(i,j,r)))))
  + sum(i,COMM, sum(j,IND, (VDA(i,j,r) * (pd(i,j,r) + qd(i,j,r))) -
                  (VDM(i,j,r) * (pm(i,r) + qd(i,j,r)))))
  + sum(i,TRAD_COMM, (VHM(i,r) * (pm(i,r) + qh(i,r))) -
             (VHW(i,r) * (pw(i) + qh(i,r))))
  + sum(i,TRAD_COMM, sum(j,IND, (VDM(i,j,r) * (pm(i,r) +  qd(i,j,r))) -
                    (VDW(i,j,r) * (pw(i) + qd(i,j,r)))))
  + sum(i,TRAD_COMM, sum(j,IND, (VSW(i,j,r) * (pw(i) + qs(i,j,r))) -
                    (VSM(i,j,r) * (pm(i,r) + qs(i,j,r))))) ;

EQUATION  BUDGETCONSTRAINT
 ! This equation forces the minium expenditure necessary to attain
   a given utility level to equal income.  It's only function here is
   to permit utility to be computed endogenously rather than being
   forced to evaluate the implicit function after the fact. HHP #(8d*) !
   (all,r,REG)
 sum(i,CONS_COMM, (CONSHR(i,r) * ph(i,r))) +
 (sum(i,CONS_COMM, (CONSHR(i,r) * INCPAR(i,r))) * uc(r)) = y(r);

EQUATION  OTHERDEMANDS
 ! Household demands for other goods, i.e. savings and goodwill. HHP#(8b*) !
   (all,i,NONC_COMM)(all,r,REG)
 qh(i,r) = y(r) - ph(i,r) ;

EQUATION  UTILITY
 ! computation of aggregate utility for each region.  HHP #(8a*). !
   (all,r,REG)
 u(r) = [BI("gdwill",r) * qh("gdwill",r)]
      + [BI("csavings",r) * qh("csavings",r)]
      + [sum(i,CONS_COMM, BI(i,r)) * uc(r)]  ;

EQUATION  CONSDEMANDS
 ! Household demands for consumption goods in region r.  HHP#(8e*).  !
   (all,i,CONS_COMM)(all,r,REG)
 qh(i,r) = sum(k,CONS_COMM, EP(i,k,r) * ph(k,r)) + EY(i,r) * y(r) ;

EQUATION  MKTCLNONTRD
 ! This equation assures market clearing in the nontraded goods
   markets in each of the regions.  HHP#(1*). !
   (all,i,NTRD_COMM)(all,r,REG)
 sum(j,IND, VSM(i,j,r) * qs(i,j,r))
  = sum(j,IND, VDM(i,j,r) * qd(i,j,r))
   + VHM(i,r) * qh(i,r) ;

!  the following equations hold for the world as a whole !
```

```
EQUATION MKTPRICES
  ! This equation links domestic and world prices.  It holds only
    for tradeable commodities.  HHP#(3*).  !
     (all,i,TRAD_COMM)(all,r,REG)
 pm(i,r) = tt(i,r) + pw(i)  ;

EQUATION HHPRICES
  ! This equation links domestic and household prices.
    It holds only for household goods. HHP#(3*). !
     (all,i,COMM)(all,r,REG)
 ph(i,r) = th(i,r) + pm(i,r)  ;


EQUATION  SUPPLYPRICES
  ! This equation links domestic and firm supply prices.
    It holds for all goods.  HHP#(3*). !
     (all,i,COMM)(all,j,IND)(all,r,REG)
 ps(i,j,r) =  ts(i,j,r) + pm(i,r)  ;

EQUATION  DEMANDPRICES
  ! This equation links domestic and firm demand prices.
    It holds for all goods.  HHP#(3*). !
     (all,i,COMM)(all,j,IND)(all,r,REG)
 pd(i,j,r) = td(i,j,r) + pm(i,r)  ;

EQUATION MKTCLTRD
  ! This equation assures market clearing in the traded goods markets.
    Due to Walras Law we omitt the last of these which is the US mnfcs
    producer good. HHP#(2*). !
     (all,i,WALR_COMM)
 sum(r,REG, sum(j,IND, VSW(i,j,r) * qs(i,j,r))) =
 sum(r,REG, sum(j,IND, VDW(i,j,r) * qd(i,j,r))) +
 sum(r,REG, VHW(i,r) * qh(i,r)) ;

EQUATION  WALRAS_D
  ! This is an extra equation which simply computes change in demand
    in the omitted market.  !
sum(r,REG, sum(j,IND, VSW("mnfcus",j,r) )) * walras_dem =
 sum(r,REG, sum(j,IND, VDW("mnfcus",j,r) * qd("mnfcus",j,r)))
 + sum(r,REG, VHW("mnfcus",r) * qh("mnfcus",r)) ;

EQUATION  WALRAS_S
  ! This is an extra equation which simply computes change in supply
    in the omitted market.  !
sum(r,REG, sum(j,IND, VSW("mnfcus",j,r) )) * walras_sup =
 sum(r,REG, sum(j,IND, VSW("mnfcus",j,r) * qs("mnfcus",j,r))) ;
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