Journal of Agricultural and Resource Economics 27(2):281–319

A Complete Characterization of the Linear, Log-Linear, and Semi-Log Incomplete Demand System Models

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This study extends LaFrance's (1985, 1986, 1990) previous research by deriving the necessary parameter restrictions for two additional classes of incomplete demand system models to be integrable. In contrast to LaFrance's earlier work, this analysis considers models that treat expenditures and expenditure shares as the dependent variables in the specified incomplete demand systems. With environmental economists increasingly turning to demand system approaches to value changes in environmental quality, these new results significantly expand the menu of empirical specifications which can be used to fit a given data set. Moreover, the alternative specifications considered in this study, in combination with LaFrance's original work, represent a complete characterization of the linear, log-linear, and semi-log incomplete demand system models.

Key words: incomplete demand systems, integrability

Introduction

With increasing regularity, environmental economists are turning to demand system models to value changes in environmental quality with revealed preference data (e.g., Phaneuf; Phaneuf, Kling, and Herriges; Shonkwiler; Englin, Boxall, and Watson; and von Haefen and Phaneuf). Relative to discrete-choice random utility maximization (RUM) approaches (e.g., Train), demand system approaches are appealing because they fully integrate the extensive commodity selection and intensive derived demand choices within a coherent and consistent model of consumer behavior.

Within the demand system framework, the *incomplete* demand system structure, originally proposed by Epstein and authoritatively analyzed by LaFrance and Hanemann, is an appealing framework for modeling consumer choice in environmental applications that focus on a subset of goods entering consumer preferences. Without resorting to restrictive aggregation and/or separability assumptions, the incomplete demand system structure represents a consistent strategy for modeling the demand for n goods as a function of $n+m$ prices ($m>1$).

Linear, log-linear, and semi-log incomplete demand structures are frequently used in applied demand analysis. In a series of papers, LaFrance (1985, 1986, 1990) considers the necessary parameter restrictions for these specifications to be integrable, i.e.,

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Review coordinated by Gary D. Thompson.

consistent with a rational preference ordering. A necessary restriction for integrability is that the $n \times n$ Slutsky matrix must be symmetric in a local neighborhood of observed prices and income. As LaFrance demonstrates, Slutsky symmetry implies relatively strong restrictions on price and income effects for the eight empirical specifications he examines.

At present, the stock of incomplete demand system structures that can be used in applied work has been limited to the eight structures considered by LaFrance. The current study attempts to expand this relative paucity by examining the integrability of sixteen additional incomplete demand system models. In contrast to the specifications considered by LaFrance, these structures treat the individual's expenditures and expenditure shares on the goods of interest as the dependent variables. The necessary parameter restrictions are derived for the linear, log-linear, and six variations of the semi-log expenditure and expenditure share models to have symmetric Slutsky matrices in a local neighborhood of observed prices and income. When closed-form solutions exist, the quasi-indirect utility functions for the restricted demand models are also derived. In combination with LaFrance's original work, the results presented here represent a complete characterization of the linear, log-linear, and semi-log incomplete demand system models.

Incomplete Demand Systems

Applied researchers are often interested in modeling the demand for a subset of goods (e.g., recreation sites) entering an individual's preference ordering. To consistently model consumption for these goods within the demand system framework, the analyst may employ one of three sets of assumptions. One approach assumes the goods of interest enter consumer preferences through a weakly separable subfunction. In this case, the analyst models consumption for the goods of interest conditional on total expenditures allocated to them. Alternatively, the analyst may assume the other goods' prices vary proportionately across individuals and/or time. In this situation, the other goods can be aggregated into a single Hicksian composite good, and the analyst models the demand for the goods of interest as functions of their prices, total income, and the composite good's price index. A third, and in many ways less restrictive, approach involves the specification of a demand system for the goods of interest as functions of their own prices, total income, and the other goods' prices which are assumed quasi-fixed. This final strategy falls under the rubric of incomplete demand system approaches and has been systematically investigated by Epstein, and by LaFrance and Hanemann.

The incomplete demand system framework assumes that consumer demand for a set of n goods can be represented by the following system of Marshallian demand functions:

$$x_i = x_i(\mathbf{p}, \mathbf{q}, y, \beta), \quad i = 1, \ldots, n, \tag{1}$$

where x_i is the Marshallian consumer's demand for good (site) i, $\mathbf{p}$ is a vector of prices for the n goods in (1), $\mathbf{q}$ is a vector of prices for m other goods (perhaps other sites) whose demands are not explicitly specified, y is the consumer's income, and β is a vector of structural parameters.

To avoid confusion and unnecessary notation, (1) does not explicitly depend on the quality attributes of the $n + m$ goods. However, using either the simple repackaging (e.g., Griliches) or the cross-product repackaging (Willig) frameworks, it is straightforward to introduce quality in a parsimonious manner consistent with the intuitively appealing notion of weak complementarity (Mäler). Following LaFrance (1985, 1986, 1990), $\mathbf{p}$, $\mathbf{q}$, and y are all normalized by $\pi(\mathbf{q})$, a homogeneous-of-degree-one price index for the m other goods, to ensure the demand equations are homogeneous of degree zero in prices and income. Because the analyst models the demand for the n goods in $\mathbf{x}$ as functions of all $n+m$ prices and income, the demand specification in (1) is incomplete.

In principle, the analyst can generate (1) by either: (*a*) specifying an indirect utility function and using Roy's Identity, or (*b*) specifying a system of incomplete demand equations directly. With either approach, a significant question for the analyst attempting to use (1) to generate consistent Hicksian welfare measures for a set of price changes is whether the system is consistent with a rational individual maximizing her utility subject to a linear budget constraint. This is the classic integrability problem.

As noted by LaFrance and Hanemann, there are at least three distinct concepts of integrability in the incomplete demand system framework. This analysis employs LaFrance and Hanemann's concept of weak integrability. This concept implies that within a local neighborhood of price and income values, there exists a continuous and increasing preference ordering which both gives rise to and is quasi-concave in $\mathbf{x}$ and z, where z is defined as total expenditures on the m other goods, i.e., $z = y - \Sigma_{i=1}^{n} p_i x_i$. Compared to other concepts of integrability in the incomplete demand system framework, weak integrability represents the minimal set of assumptions allowing the analyst to construct exact welfare measures for changes in $\mathbf{p}$ conditional on quasi-fixed values of $\mathbf{q}$.

Theorem 2 in LaFrance and Hanemann states that an incomplete demand system is weakly integrable if the following four conditions are satisfied: (*a*) $\mathbf{x}$ is homogeneous of degree zero in all prices and income; (*b*) $\mathbf{x}$ is nonnegative, i.e., $\mathbf{x} \geq 0$; (*c*) expenditures on the n goods included in the incomplete demand system are strictly less than income, i.e., $\Sigma_{i=1}^{n} p_i x_i < y$; and (*d*) the Slutsky substitution matrix—i.e., the $n \times n$ matrix whose elements consist of

$$\tilde{S}_{ij} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial y} x_j, \quad i, j \in 1, \ldots, n,$$

where $\partial x_i / \partial p_j$ and $\partial x_i / \partial y$ are partial derivatives of the Marshallian demand functions with respect to price and income, respectively—is symmetric and negative semidefinite. Symmetry implies that for each pair of goods $i, j \in 1, \ldots, n$, $i \neq j$, $\tilde{S}_{ij} = \tilde{S}_{ji}$, whereas negative semidefiniteness requires that the eigenvalues of the Slutsky matrix are nonpositive. The normalization of prices and income by the price index, $\pi(\mathbf{q})$, implies the first condition is satisfied, and the second and third conditions are innocuous in many applied situations and assumed to hold in an open neighborhood of prices and income. Thus, the necessary conditions for weak integrability which imply added structure for (1) are the symmetry and negative semidefiniteness of the Slutsky matrix.

In a series of papers, LaFrance derives the necessary parameter restrictions for the Slutsky matrix to be symmetric for eight incomplete demand system specifications—the linear model (1985), the log-linear or constant elasticity model (1986), and six alternative semi-log models (1990). These models or their logarithmic transformations share a common linear-in-parameters structure and are additive in their arguments.

Table 1. Incomplete Demand System Models

Model	Demand Specification	Model	Demand Specification
(x1)	$x_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i y, \ \forall i$	(x5)	$x_i = \alpha_i(\mathbf{q}) \exp\left\{\sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i y\right\}, \ \forall i$
(x2)	$x_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i \ln(y), \ \forall i$	(x6)	$x_i = \alpha_i(\mathbf{q}) \exp\left\{\sum_{k=1}^{n} \beta_{ik} p_k\right\} y^{\gamma_i}, \ \forall i$
(x3)	$x_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} \ln(p_k) + \gamma_i y, \ \forall i$	(x7)	$x_i = \alpha_i(\mathbf{q}) \left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\} \exp(\gamma_i y), \ \forall i$
(x4)	$x_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} \ln(p_k) + \gamma_i \ln(y), \ \forall i$	(x8)	$x_i = \alpha_i(\mathbf{q}) \left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\} y^{\gamma_i}, \ \forall i$

Notes: The (x1) model is considered by LaFrance (1985); the (x2)–(x7) models are considered by LaFrance (1990) and correspond with his models (m3), (m1), (m2), (m4), (m6), and (m5), respectively; and the (x8) model is examined in LaFrance (1986).

Table 1 lists the eight demand specifications examined by LaFrance. His results are extended here by deriving the implications of Slutsky symmetry for two additional classes of incomplete demand system models. Sixteen additional specifications are considered that treat either expenditures ($e_i = p_i x_i$, $e_i \geq 0$), expenditure shares ($s_i = p_i x_i / y$, $0 \leq s_i \leq 1$), or their logarithmic transformations as the dependent variables.

Since Stone's (1954a, b) pioneering work, it has been common in applied demand analysis for expenditures, expenditure shares, or their transformations to be specified as the dependent variables in the estimated system of equations.[1] For specification of incomplete demand systems, however, analysts have eschewed these possibilities to date. Tables 2 and 3 list the expenditure and expenditure share specifications considered in this analysis.

In addition to expanding the menu of specifications from which analysts can choose, these models may be of particular interest to environmental economists for at least two reasons. Demand system models estimated within the count data framework are becoming increasingly popular for the analysis of disaggregate consumption data for commodity groups such as recreation sites. These models require that all consumer demands be strictly positive values, and thus the (e5)–(e8) (table 2) and (s5)–(s8) (table 3) specifications are potentially appealing alternatives to the (x5)–(x8) specifications (table 1) which have been used in the literature.

The results presented here also can be used by analysts wishing to estimate micro-level dual representations of continuous demand system models (Bockstael, Hanemann, and Strand). At present, empirical specification of these models proposed by Lee and Pitt, and recently implemented by Phaneuf, only consider complete or weakly separable demand systems, but these models can also be estimated within the incomplete demand system framework. The empirical implementation of the dual models proposed by Lee and Pitt depends critically on the existence of closed-form solutions for the implied virtual price functions, i.e., the prices that would drive the consumer's demand for the non-consumed goods to zero (Neary and Roberts). Because LaFrance's (x1)–(x4) specifications

[1] Three of the most widely used empirical specifications, the linear expenditure system (Klein and Rubin), the indirect translog (Christensen, Jorgenson, and Lau), and the almost ideal demand system (Deaton and Muellbauer), treat expenditures or expenditure shares as the system's dependent variables.

Table 2. Incomplete Expenditure System Models

Model	Expenditure Specification	Model	Expenditure Specification
(e1)	$e_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i y, \ \forall i$	(e5)	$e_i = \alpha_i(\mathbf{q}) \exp\left\{\sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i y\right\}, \ \forall i$
(e2)	$e_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i \ln(y), \ \forall i$	(e6)	$e_i = \alpha_i(\mathbf{q}) \exp\left\{\sum_{k=1}^{n} \beta_{ik} p_k\right\} y^{\gamma_i}, \ \forall i$
(e3)	$e_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} \ln(p_k) + \gamma_i y, \ \forall i$	(e7)[a]	$e_i = \alpha_i(\mathbf{q}) \left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\} \exp(\gamma_i y), \ \forall i$
(e4)	$e_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} \ln(p_k) + \gamma_i \ln(y), \ \forall i$	(e8)[b]	$e_i = \alpha_i(\mathbf{q}) \left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\} y^{\gamma_i}, \ \forall i$

[a] Note the equivalence between this specification and (x7) (table 1) if the following parametric transformations are made: $\beta_{ii}^{(e7)} = \beta_{ii}^{(x7)} + 1, \forall i$.
[b] Note the equivalence between this specification and (x8) (table 1) if the following parametric transformations are made: $\beta_{ii}^{(e8)} = \beta_{ii}^{(x8)} + 1, \forall i$.

Table 3. Incomplete Expenditure Share System Models

Model	Expenditure Share Specification	Model	Expenditure Share Specification
(s1)	$s_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i y, \ \forall i$	(s5)	$s_i = \alpha_i(\mathbf{q}) \exp\left\{\sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i y\right\}, \ \forall i$
(s2)	$s_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i \ln(y), \ \forall i$	(s6)[a]	$s_i = \alpha_i(\mathbf{q}) \exp\left\{\sum_{k=1}^{n} \beta_{ik} p_k\right\} y^{\gamma_i}, \ \forall i$
(s3)	$s_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} \ln(p_k) + \gamma_i y, \ \forall i$	(s7)	$s_i = \alpha_i(\mathbf{q}) \left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\} \exp(\gamma_i y), \ \forall i$
(s4)	$s_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} \ln(p_k) + \gamma_i \ln(y), \ \forall i$	(s8)[b]	$s_i = \alpha_i(\mathbf{q}) \left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\} y^{\gamma_i}, \ \forall i$

[a] Note the equivalence between this specification and (e6) (table 2) if the following parametric transformations are made: $\gamma_i^{(s6)} = \gamma_i^{(e6)} - 1, \forall i$.
[b] Note the equivalence between this specification and (e8) (table 2) if the following parametric transformations are made: $\gamma_i^{(s8)} = \gamma_i^{(e8)} - 1, \forall i$. Note also the equivalence between this specification and (x8) (table 1) if the following parametric transformations are made: $\gamma_i^{(s8)} = \gamma_i^{(x8)} - 1$; $\beta_{ii}^{(s8)} = \beta_{ii}^{(x8)} + 1, \forall i$.

and the (e1)–(e4) and (s1)–(s4) structures allow for corner solutions and have $\mathbf{p}$ entering linearly or log-linearly, they can in principle be inverted to solve for the implied virtual price functions.

For any pair of goods $i, j \in i, \ldots, n$; $i \neq j$, the Slutsky symmetry restrictions require that in an open neighborhood of prices and income, the following conditions must hold for the demand, expenditure, and expenditure share equations, respectively:

$$\frac{\partial x_j}{\partial p_i} + \frac{\partial x_j}{\partial y} x_i = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial y} x_j, \tag{2}$$

$$\frac{1}{p_i p_j}\left[\frac{\partial e_j}{\partial p_i} p_i + \frac{\partial e_j}{\partial y} e_i\right] = \frac{1}{p_i p_j}\left[\frac{\partial e_i}{\partial p_j} p_j + \frac{\partial e_i}{\partial y} e_j\right], \tag{3}$$

$$(4)\qquad \frac{y}{p_i p_j}\left[\frac{\partial s_j}{\partial p_i}p_i + \left\{\frac{s_j}{y} + \frac{\partial s_j}{\partial y}\right\}ys_i\right] = \frac{y}{p_i p_j}\left[\frac{\partial s_i}{\partial p_j}p_j + \left\{\frac{s_i}{y} + \frac{\partial s_i}{\partial y}\right\}ys_j\right],$$

where all derivatives are with respect to the Marshallian demands, expenditures, and expenditure shares, respectively. The specific structure of these equalities for each of the specifications included in tables 1, 2, and 3 can be found in the technical appendix at the end of this article.

In addition to ascertaining the necessary parameter restrictions implied by Slutsky symmetry, determining whether the restricted demand systems can be linked to closed-form representations of preferences may be of interest to applied researchers. For example, virtually every recently proposed method for linking multiple intensive and extensive margins of consumer choice in a behaviorally consistent framework (e.g., Cameron; Eom and Smith) assumes consumer preferences can be represented by a utility function with a closed-form solution. Without the closed form, these strategies would not be econometrically viable.

As noted by LaFrance and Hanemann, a difficulty with the incomplete demand system framework is that one cannot recover the complete structure of preferences with respect to all $n+m$ goods from an n-good demand system which satisfies the conditions for weak integrability. However, what Hausman has called the *quasi*-indirect utility function can be recovered by solving a series of partial differential equations. For the demand, expenditure, and expenditure share models, this can be accomplished sequentially by first solving one of the following partial differential equations:

$$\frac{\partial E(\cdot)}{\partial p_1} = x_1(\mathbf{p}, \mathbf{q}, E(\cdot), \beta),$$

$$\frac{\partial E(\cdot)}{\partial \ln(p_1)} = e_1(\mathbf{p}, \mathbf{q}, E(\cdot), \beta),$$

$$\frac{\partial \ln E(\cdot)}{\partial \ln(p_1)} = s_1(\mathbf{p}, \mathbf{q}, E(\cdot), \beta),$$

where $E(\cdot)$is the expenditure function evaluated at the baseline utility, $\bar{U}$, and good 1 is chosen arbitrarily with no loss in generality.

In some cases, the techniques of differential calculus can be used to derive closed-form solutions for $E(\cdot)$ [or $\ln E(\cdot)$] up to a constant of integration, $K_1(\bar{U}, \bar{\mathbf{p}}^{-1}, \mathbf{q})$, where $\bar{\mathbf{p}}^{-1}$ is the price vector for the $n-1$ remaining goods in the specified incomplete demand system. Because the constant of integration depends on the $n-1$ other prices, additional information can be recovered about the structure of the expenditure function by sequentially solving the following differential equations for $i = 2, \ldots, n$:

$$\frac{\partial \tilde{E}(\cdot)}{\partial p_i} + \frac{\partial K_{i-1}(\cdot)}{\partial p_i} = x_i(\mathbf{p}, \mathbf{q}, E(\cdot), \beta),$$

$$\frac{\partial \tilde{E}(\cdot)}{\partial \ln(p_i)} + \frac{\partial K_{i-1}(\cdot)}{\partial \ln(p_i)} = e_i(\mathbf{p}, \mathbf{q}, E(\cdot), \beta),$$

$$\frac{\partial \ln \tilde{E}(\cdot)}{\partial \ln(p_i)} + \frac{\partial K_{i-1}(\cdot)}{\partial \ln(p_i)} = s_i(\mathbf{p}, \mathbf{q}, E(\cdot), \beta),$$

where $K_{i-1}(\cdot)$ is the constant of integration arising from the evaluation of the first $i-1$ partial differential equations, and $\tilde{E}(\cdot)$ is the identified component of the individual's expenditure function (i.e., that portion of the expenditure function excluding the constant of integration).

When the analyst has solved all n differential equations, the individual's expenditure function is identified up to the constant of integration, $K_n(\bar{U}, \mathbf{q})$, which is independent of $\mathbf{p}$. The constant of integration is a function of the baseline utility as well as the other m goods' prices, suggesting the analyst cannot identify the full structure of the expenditure function with respect to all $n+m$ goods from an incomplete demand system. However, the quasi-indirect utility function can be obtained by treating $K_n(\bar{U}, \mathbf{q})$ as the quasi-baseline utility and inverting, i.e.:

$$\tilde{U} = K_n(\bar{U}, \mathbf{q}) = \phi(\mathbf{p}, \mathbf{q}, y, \beta).$$

LaFrance and Hanemann formally prove that $\phi(\mathbf{p}, \mathbf{q}, y, \beta)$ can be used to consistently evaluate the welfare implications of one or several price changes for the n goods.

Necessary Parameter Restrictions, the Structure of the Restricted Demand Systems, and the Quasi-Indirect Utility Functions

Tables 4, 5, and 6 report all possible combinations of parameter restrictions that satisfy Slutsky symmetry for the demand, expenditure, and share specifications reported in tables 1, 2, and 3, respectively.[2] The results in table 4 were reported originally in LaFrance (1985, 1986, 1990) and are presented here mainly for completeness.[3] For expositional purposes, these tables employ simplifying notation developed by LaFrance. Let J, K, and N denote index sets satisfying $\emptyset \subset J \subset K \subset N \equiv \{1, 2, ..., n\}$, and let ~ denote set differences, e.g., $N \sim J \equiv \{i \in N;\, i \notin J\}$. Further assume that if $J \neq \emptyset$, $1 \in J$, or if $K \neq \emptyset$, $1 \in K$.

The derivation of these results follows the logic laid out in LaFrance (1985, 1986). For each specification, three mutually exclusive and exhaustive types of income effects for goods i and j are considered: (a) no income effects, i.e., $\gamma_i = \gamma_j = 0$; (b) both goods having income effects, i.e., $\gamma_i \neq 0$, $\gamma_j \neq 0$; and (c) only one good having income effects, i.e., $\gamma_i \neq 0$, $\gamma_j = 0$. For each of these possibilities, the necessary parameter restrictions for Slutsky symmetry to hold in an open neighborhood of relevant prices and income were derived. The derivative properties of the Slutsky symmetry conditions were used extensively to identify these parameter restrictions. Because equations (2), (3), and (4) are assumed to hold over a range of price and income values, they are identities which can be differentiated to generate additional restrictions that may clarify the structure of the parameter restrictions.

[2] A technical appendix with the derivations for the expenditure and expenditure share parameter restrictions is provided at the end of this article. The appendix also contains derivations for LaFrance's (x2)–(x7) models, and the interested reader can consult LaFrance (1985, 1986) for the derivations of the (x1) and (x8) specifications.

[3] A review of the results reported in LaFrance (1990) uncovered minor extensions for the (x5) and (x6) specifications as well as a few typographical errors for the remaining specifications. The results reported in text table 4 and appendix tables A1 and A4 incorporate these extensions and correct for the errors.

Table 4. Slutsky Symmetry Restrictions for Incomplete Demand System Models

(x1)[a]	1a. $\beta_{ij} = \beta_{ji},\ i,j \in N$	2c. $\text{sgn}(\gamma_i) = \text{sgn}(\gamma_1) \neq 0,\ i \in J$
	1b. $\gamma_i = 0,\ i \in N$	2d. $\beta_{ij} = 0,\ i \in N \sim J, j \in N$
	2a. $\alpha_i(\mathbf{q}) = \frac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q}) + \frac{\beta_{11}}{\gamma_1} - \frac{\beta_{1i}}{\gamma_i}\right\},\ i \in J$	2e. $\gamma_i = 0,\ i \in N \sim J$
		2f. $\alpha_i(\mathbf{q}) = -\beta_{1i}/\gamma_1 > 0,\ i \in N \sim J$
	2b. $\beta_{ij} = (\gamma_i/\gamma_1)\beta_{1j},\ i \in J, j \in N$	
(x2)	1a. $\beta_{ij} = \beta_{ji},\ i,j \in N$	2a. $\alpha_i(\mathbf{q}) = (\gamma_i/\gamma_1)\alpha_1(\mathbf{q}),\ i \in N$
	1b. $\gamma_i = 0,\ i \in N$	2b. $\beta_{ij} = (\gamma_i\gamma_j/\gamma_1^2)\beta_{11},\ i,j \in N$
		2c. $\text{sgn}(\gamma_i) = \text{sgn}(\gamma_1) \neq 0,\ i \in N$
(x3)	1a. $\beta_{ij} = 0,\ i,j \in N, i \neq j$	2a. $\beta_{ij} = 0,\ i,j \in N$
&	1b. $\gamma_i = 0,\ i \in N$	2b. $\alpha_i(\mathbf{q}) = (\gamma_i/\gamma_1)\alpha_1(\mathbf{q}),\ i \in N$
(x4)		2c. $\text{sgn}(\gamma_i) = \text{sgn}(\gamma_1) \neq 0,\ i \in N$
(x5)[b,c]	1a. $\alpha_i(\mathbf{q}) = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q}) > 0,\ i \in J$	1d. $\beta_{ij} = 0,\ i \in J, j \in K \sim J;\ i \in K \sim J, j \in K, i \neq j;\ i \in N \sim K, j \in N$
	1b. $\gamma_i = \gamma_1,\ i \in K$	1e. $\gamma_i = 0,\ i \in N \sim K$
	1c. $\beta_{ij} = \beta_{jj},\ i,j \in J$	1f. $\alpha_i(\mathbf{q}) = -\beta_{1i}/\gamma_1 > 0,\ i \in N \sim K$
(x6)[c]	1a. $\alpha_i(\mathbf{q}) = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q}) > 0,\ i \in J$	2b. $\gamma_i = \gamma_1,\ i \in K$
	1b. $\gamma_i = \gamma_1,\ i \in N$	2c. $\beta_{ij} = \beta_{jj},\ i,j \in J$
	1c. $\beta_{ij} = \beta_{jj},\ i,j \in J$	2d. $\beta_{ij} = 0,\ i \in J, j \in K \sim J;\ i \in K \sim J, j \in K, i \neq j;\ i \in N \sim K, j \in N$
	1d. $\beta_{ij} = 0,\ i \in J, j \in N \sim J;\ i \in N \sim J, j \in N, i \neq j$	
	2a. $\alpha_i(\mathbf{q}) = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q}) > 0,\ i \in J$	2e. $\gamma_i = 1,\ i \in N \sim K$
		2f. $\alpha_i(\mathbf{q}) = \beta_{1i} > 0,\ i \in N \sim K$
(x7)[b,d]	1a. $\alpha_i(\mathbf{q}) = \alpha_1(\mathbf{q})\left\{\frac{1 + \beta_{ii}}{1 + \beta_{11}}\right\} > 0,\ i \in J$	1e. $\beta_{ij} = 0,\ i \in J, j \in K \sim J;\ i \in K \sim J, j \in K, i \neq j;\ i \in N \sim K, j \in N, i \neq j$
	1b. $\beta_{ij} = 1 + \beta_{jj},\ i,j \in J, i \neq j$	1f. $\gamma_i = 0,\ i \in N \sim K$
	1c. $\beta_{ij} = \beta_{1j},\ i \in K, j \in N \sim K$	1g. $\beta_{ii} = -1,\ i \in N \sim K$
	1d. $\gamma_i = \gamma_1,\ i \in K$	1h. $\alpha_i(\mathbf{q}) = -\beta_{1i}/\gamma_1 > 0,\ i \in N \sim K$
(x8)[d]	1a. $\alpha_i(\mathbf{q}) = \alpha_1(\mathbf{q})\left\{\frac{1 + \beta_{ii}}{1 + \beta_{11}}\right\} > 0,\ i \in J$	2c. $\alpha_i(\mathbf{q}) = \beta_{1i} > 0,\ i \in N \sim K$
		2d. $\gamma_i = 0,\ i \in K$
	1b. $\beta_{ij} = 1 + \beta_{jj},\ i,j \in J, i \neq j$	2e. $\gamma_i = 1,\ i \in N \sim K$
	1c. $\gamma_i = \gamma_1,\ i \in N$	2f. $\beta_{ij} = \beta_{1j},\ i \in K, j \in N \sim K$
	1d. $\beta_{ij} = 0,\ i \in J, j \in N \sim J;\ i \in N \sim J, j \in N, i \neq j$	2g. $\beta_{ii} = -1,\ i \in N \sim K$
	2a. $\alpha_i(\mathbf{q}) = \alpha_1(\mathbf{q})\left\{\frac{1 + \beta_{ii}}{1 + \beta_{11}}\right\} > 0,\ i \in J$	2h. $\beta_{ij} = 0,\ i \in J, j \in K \sim J;\ i \in K \sim J, j \in K, i \neq j;\ i \in N \sim K, j \in N, i \neq j$
	2b. $\beta_{ij} = 1 + \beta_{jj},\ i,j \in J, i \neq j$	

[a] LaFrance (1985) notes that an additional restriction arising from the negative semidefiniteness of the Slutsky matrix assumption is $\beta_{11} + \gamma_1 x_1 \leq 0$ for the J subset.
[b] Note that the $N \sim K$ subset is empty if $\gamma_1 = 0$.
[c] For the (x5-1) and (x6-2) restricted models, LaFrance (1990) further partitions the $K \sim J$ subset into one set with $\beta_{kk} = 0$ and another with $\beta_{kk} \neq 0$. Similarly, for the (x6-1) restricted model, LaFrance decomposes the $N \sim J$ subset into one set with $\beta_{kk} = 0$ and another with $\beta_{kk} \neq 0$.
[d] For the (x7-1) and (x8-2) restricted models, LaFrance (1990) further partitions the $K \sim J$ subset into one set with $\beta_{kk} = -1$ and another with $\beta_{kk} \neq -1$. Similarly, for the (x8-1) restricted model, LaFrance decomposes the $N \sim J$ subset into one set with $\beta_{kk} = -1$ and another with $\beta_{kk} \neq -1$.

Table 5. Slutsky Symmetry Restrictions for Incomplete Expenditure System Models

Model			
(e1)	1a. $\beta_{ij}=0,\ i,j\in N,\ i\neq j$	2h.	$\gamma_i=0,\ i\in N\sim K$
	1b. $\gamma_i=0,\ i\in N$	2i.	$\alpha_i(\mathbf{q})=0,\ i\in N\sim K$
	2a. $\gamma_i=1,\ i\in J$	2j.	$\beta_{ii}=-\beta_{1i}/\gamma_1>0,\ i\in N\sim K$
	2b. $\beta_{ii}=0,\ i\in J$	3a.	$\gamma_1=1$
	2c. $\alpha_i(\mathbf{q})=(\gamma_i/\gamma_1)\alpha_1(\mathbf{q}),\ i\in K$	3b.	$\gamma_i=0,\ i\in N,\ i\neq 1$
	2d. $\gamma_i\neq 0,1,\ i\in K\sim J$	3c.	$\beta_{ij}=0,\ i,j\in N,\ i\neq 1,\ j\neq i,1$
	2e. $\beta_{ij}=\gamma_i\beta_{jj}/(\gamma_j-1),\ i\in K,\ j\in K\sim J$	3d.	$\beta_{ii}=-\beta_{1i},\ i\in N,\ i\neq 1$
	2f. $\beta_{ik}=(\gamma_i/\gamma_j)\beta_{jk},\ i,j\in K,\ k\in N,\ k\neq i,j$	3e.	$\alpha_i(\mathbf{q})=0,\ i\in N,\ i\neq 1$
	2g. $\beta_{ij}=0,\ i\in N\sim K,\ j\in N,\ i\neq j$		
(e2)	1a. $\beta_{ij}=0,\ i,j\in N,\ i\neq j$	2a.	$\beta_{ij}=0,\ i,j\in N$
	1b. $\gamma_i=0,\ i\in N$	2b.	$\alpha_i(\mathbf{q})=(\gamma_i/\gamma_1)\alpha_1(\mathbf{q}),\ i\in N$
		2c.	$\mathrm{sgn}(\gamma_i)=\mathrm{sgn}(\gamma_1)\neq 0,\ i\in N$
(e3)	1a. $\beta_{ij}=\beta_{ji},\ i,j\in N$	2d.	$\gamma_i=0,\ i\in N\sim J$
	1b. $\gamma_i=0,\ i\in N$	2e.	$\alpha_i(\mathbf{q})=\frac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q})-\frac{\beta_{1i}}{\gamma_i}+\frac{\beta_{i1}}{\gamma_i}\right\},\ i\in J$
	2a. $\beta_{ij}=(\gamma_i/\gamma_1)\beta_{1j},\ i,j\in J$		
	2b. $\beta_{ij}=0,\ i\in N\sim J,\ j\in N$	2f.	$\alpha_i(\mathbf{q})=-\beta_{1i}/\gamma_1>0,\ i\in N\sim J$
	2c. $\mathrm{sgn}(\gamma_i)\neq 0,\ i\in J$		
(e4)	1a. $\beta_{ij}=\beta_{ji},\ i,j\in N$	2a.	$\alpha_i(\mathbf{q})=(\gamma_i/\gamma_1)\alpha_1(\mathbf{q}),\ i\in N$
	1b. $\gamma_i=0,\ i\in N$	2b.	$\beta_{ij}=(\gamma_i\gamma_j/\gamma_1^2)\beta_{11},\ i,j\in N$
		2c.	$\mathrm{sgn}(\gamma_i)=\mathrm{sgn}(\gamma_1)\neq 0,\ i\in N$
(e5) & (e6)	1a. $\beta_{ij}=0,\ i,j\in N,\ i\neq j$		
	1b. $\gamma_i=\gamma_1,\ i\in N$		
(e7)[a]	1a. $\alpha_i(\mathbf{q})=(\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q})>0,\ i\in J$	1e.	$\beta_{ij}=0,\ i\in J,\ j\in K\sim J;\ i\in K\sim J,\ j\in K,\ i\neq j;\ i\in N\sim K,\ j\in N$
	1b. $\beta_{ij}=\beta_{jj},\ i,j\in J$		
	1c. $\beta_{ij}=\beta_{1j},\ i\in K,\ j\in N\sim K$	1f.	$\gamma_i=0,\ i\in N\sim K$
	1d. $\gamma_i=\gamma_1,\ i\in K$	1g.	$\alpha_i(\mathbf{q})=-\beta_{1i}/\gamma_1>0,\ i\in N\sim K$
(e8)	1a. $\alpha_i(\mathbf{q})=(\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q})>0,\ i\in J$	2c.	$\alpha_i(\mathbf{q})=\beta_{1i}>0,\ i\in N\sim K$
	1b. $\beta_{ij}=\beta_{jj},\ i,j\in J$	2d.	$\gamma_i=0,\ i\in K$
	1c. $\gamma_i=\gamma_1,\ i\in N$	2e.	$\gamma_i=1,\ i\in N\sim K$
	1d. $\beta_{ij}=0,\ i\in J,\ j\in N\sim J;\ i\in N\sim J,\ j\in N,\ i\neq j$	2f.	$\beta_{ij}=\beta_{1j},\ i\in K,\ j\in N\sim K$
	2a. $\alpha_i(\mathbf{q})=(\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q})>0,\ i\in J$	2g.	$\beta_{ij}=0,\ i\in J,\ j\in K\sim J;\ i\in K\sim J,\ j\in K,\ i\neq j;\ i\in N\sim K,\ j\in N$
	2b. $\beta_{ij}=\beta_{jj},\ i,j\in J$		

[a]Note that the $N\sim K$ subset is empty if $\gamma_1=0$.

Table 6. Slutsky Symmetry Restrictions for Incomplete Expenditure Share System Models

(s1)	1a. $\gamma_i = 0,\ i \in N$	2a. $\alpha_i(\mathbf{q}) = (\gamma_i/\gamma_1)\alpha_1(\mathbf{q}),\ i \in N$
	1b. $\beta_{ij} = 0,\ i,j \in N,\ i \neq j$	2b. $\beta_{ij} = 0,\ i,j \in N$
		2c. $\mathrm{sgn}(\gamma_i) = \mathrm{sgn}(\gamma_1) \neq 0,\ i \in N$
(s2)	1a. $\gamma_i = 0,\ i \in N$	2g. $\beta_{ik} = (\gamma_i/\gamma_j)\beta_{jk},\ i,j \in K,\ k \in N,\ k \neq i,j$
	1b. $\beta_{ij} = 0,\ i,j \in N,\ i \neq j$	2h. $\beta_{ij} = 0,\ i \in N\sim K,\ j \in N,\ i \neq j$
	2a. $\gamma_i = 1,\ i \in J$	2i. $\alpha_i(\mathbf{q}) = 0,\ i \in N\sim K$
	2b. $\gamma_i \neq 0, 1,\ i \in K\sim J$	2j. $\beta_{ii} = -\beta_{1i}/\gamma_1 > 0,\ i \in N\sim K$
	2c. $\gamma_i = 0,\ i \in N\sim K$	3a. $\gamma_1 = 1$
	2d. $\alpha_i(\mathbf{q}) = (\gamma_i/\gamma_1)\alpha_1(\mathbf{q}),\ i \in K$	3b. $\gamma_i = 0,\ i \in N,\ i \neq 1$
	2e. $\beta_{ii} = 0,\ i \in J$	3c. $\beta_{ij} = 0,\ i,j \in N,\ i \neq 1,\ j \neq i,1$
	2f. $\beta_{ij} = \dfrac{\gamma_i \beta_{jj}}{\gamma_j - 1},\ i \in K,\ j \in K\sim J,\ i \neq j$	3d. $\beta_{ii} = -\beta_{1i},\ i \in N,\ i \neq 1$
		3e. $\alpha_i(\mathbf{q}) = 0,\ i \in N,\ i \neq 1$
(s3)	1a. $\beta_{ij} = \beta_{ji},\ i,j \in N$	2a. $\alpha_i(\mathbf{q}) = (\gamma_i/\gamma_1)\alpha_1(\mathbf{q}),\ i \in N$
	1b. $\gamma_i = 0,\ i \in N$	2b. $\beta_{ij} = (\gamma_i\gamma_j/\gamma_1^2)\beta_{11},\ i,j \in N$
		2c. $\mathrm{sgn}(\gamma_i) = \mathrm{sgn}(\gamma_1) \neq 0,\ i \in N$
(s4)	1a. $\beta_{ij} = \beta_{ji},\ i,j \in N$	2c. $\alpha_i(\mathbf{q}) = -\beta_{1i}/\gamma_1 > 0,\ i \in N\sim J$
	1b. $\gamma_i = 0,\ i \in N$	2d. $\beta_{ij} = 0,\ i, \in N\sim J,\ j \in N$
	2a. $\alpha_i(\mathbf{q}) = \dfrac{\gamma_i}{\gamma_1}\left\{\dfrac{\beta_{i1}}{\gamma_i} - \dfrac{\beta_{1i}}{\gamma_i} + \alpha_1(\mathbf{q})\right\},\ i \in J$	2e. $\gamma_i = 0,\ i \in N\sim J$
		2f. $\mathrm{sgn}(\gamma_i) \neq 0,\ i \in J$
	2b. $\beta_{ij} = (\gamma_i/\gamma_1)\beta_{1j},\ i \in J,\ j \in N$	
(s5) & (s6)	1a. $\beta_{ij} = 0,\ i,j \in N,\ i \neq j$	
	1b. $\gamma_i = \gamma_1,\ i \in N$	
(s7)	1a. $\alpha_i(\mathbf{q}) = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q}) > 0,\ i \in J$	1c. $\gamma_i = \gamma_1,\ i \in N$
	1b. $\beta_{ij} = \beta_{jj},\ i,j \in J$	1d. $\beta_{ij} = 0,\ i \in J,\ j \in N\sim J;\ i \in N\sim J,\ j \in N,\ i \neq j$
(s8)	1a. $\alpha_i(\mathbf{q}) = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q}) > 0,\ i \in J$	2c. $\alpha_i(\mathbf{q}) = \beta_{1i} > 0,\ i \in N\sim K$
	1b. $\beta_{ij} = \beta_{jj},\ i,j \in J$	2d. $\gamma_i = -1,\ i \in K$
	1c. $\gamma_i = \gamma_1,\ i \in N$	2e. $\gamma_i = 0,\ i \in N\sim K$
	1d. $\beta_{ij} = 0,\ i \in J, j \in N\sim J;\ i \in N\sim J, j \in N,\ i \neq j$	2f. $\beta_{ij} = \beta_{1j},\ i \in K, j \in N\sim K$
	2a. $\alpha_i(\mathbf{q}) = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q}) > 0,\ i \in J$	2g. $\beta_{ij} = 0,\ i \in J, j \in K\sim J;\ i \in K\sim J, j \in K,\ i \neq j;\ i \in N\sim K,\ j \in N$
	2b. $\beta_{ij} = \beta_{jj},\ i,j \in J$	

Theorem 2 in LaFrance and Hanemann identifies the following two equalities:

$$\frac{\partial \tilde{S}_{ji}}{\partial p_k} = \frac{\partial \tilde{S}_{ij}}{\partial p_k}, \quad i,j,k \in 1, \ldots, n; \; i \neq j,$$

$$\frac{\partial \tilde{S}_{ji}}{\partial y} = \frac{\partial \tilde{S}_{ij}}{\partial y}, \quad i,j \in 1, \ldots, n; \; i \neq j.$$

It should be noted, however, that these equalities are only a subset of the restrictions which can be generated by differentiating the Slutsky symmetry identities. In principle, one can multiply and/or add the same functions of market prices and income to both sides of the Slutsky symmetry conditions and still preserve the identity relationship. These modified Slutsky identities can then be differentiated to generate additional equalities that may help to identify the necessary parameter restrictions. Once the parameter restrictions were identified for the three distinct income relationships, consistent combinations of the three sets of parameter restrictions were then determined, and the results are reported in tables 4, 5, and 6.

To help clarify the implications of the parameter restrictions reported in tables 4, 5, and 6, the structure of the restricted incomplete demand systems is presented in appendix tables A1, A2, and A3. Not all cross-equation restrictions within sets of goods can be represented in the restricted demand specifications, so these tables should only be interpreted as suggestive of the general structure. Appendix tables A4, A5, and A6 also present the structure of the quasi-indirect utility functions for all restricted models with closed-form solutions. These tables suggest roughly one-half of the restricted models can be linked to closed-form representation of consumer preferences.

Collectively, the results reported in text tables 4–6 and appendix tables A1–A6 imply that none of the 24 structures considered in this study allow for both flexible income and Marshallian cross-price effects, and some do not allow for either. Perhaps the most general specifications are the (s3-1) and (s4-1) models which allow for general cross-price effects but restrictively assume all consumer demand equations are homothetic in income. The overall findings of this analysis suggest that strong, and in many cases implausible, assumptions about the structure of consumer preferences are required for analysts employing linear, semi-log, and log-linear incomplete demand system models.

Discussion

This study has extended LaFrance's earlier research by identifying the necessary parameter restrictions for systems of linear-in-parameters incomplete expenditure and expenditure share equations to satisfy the integrability condition of Slutsky symmetry. Although Slutsky symmetry is a necessary condition for the existence of a rational underlying preference ordering, it is not sufficient. As noted above, integrability also requires that the Slutsky matrix must be negative semidefinite, i.e., the matrix's eigenvalues must be nonpositive over the full range of relevant price and income values for the welfare scenarios under consideration. Imposing this latter condition is difficult in practice because the elements of the Slutsky matrix are in general nonlinear functions of prices, income, and the demand system's structural parameters. As a result, the Slutsky matrix may only be negative semidefinite over a subregion of the relevant range.

Existing approaches to imposing curvature restrictions on systems of equations can be grouped into two broad categories: (*a*) those that impose negative semidefiniteness of the Slutsky matrix at a single point (such as each individual's observed prices and income, or the sample average of these values); and (*b*) those that impose negative semidefiniteness globally over the full range of relevant price and income values through parameter restrictions (see Pitt and Millimet, and Diewert and Wales for discussions of existing approaches). Although the latter approach is similar in spirit to the strategy for insuring Slutsky symmetry described in this study, the former suggests a conceptually different strategy. In principle, the analyst could treat the Slutsky symmetry conditions as binding nonlinear constraints evaluated at the observed market price and income values when estimating the structural parameters of the demand equations.

Although estimation of a system of equations subject to side constraints can be computationally burdensome, the approach has some precedent in the existing literature (LaFrance 1991) and has both advantages and drawbacks. On the one hand, the results presented in the previous sections strongly suggest that imposing Slutsky symmetry on linear-in-parameters demand, expenditure, and expenditure share systems greatly limits the analyst's ability to allow for flexible income and Marshallian cross-price effects. Imposing symmetry on the Slutsky matrix at a single point, however, allows the analyst to incorporate these effects while preserving some degree of theoretical consistency.

On the other hand, economists interested in using the estimated system of equations to evaluate the welfare implications of nonmarginal price changes may find it troubling that the model is capable of generating only approximate Hicksian values. Moreover, because symmetry of the Slutsky matrix is not preserved over the entire range of the relevant price changes, the approximate welfare measures are not independent of the ordering of the price changes. (For a possible resolution to this problem, see LaFrance 1991.) Although these factors suggest that imposing Slutsky symmetry at a single point does not strictly dominate the approach pursued in this analysis, it may be preferable in some applications.

[Received May 2002; final revision received September 2002.]

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Technical Appendix

Auxiliary Tables A1–A6

The structure of the restricted incomplete demand systems is presented in auxiliary appendix tables A1, A2, and A3 below. Tables A4, A5, and A6 then present the structure of the quasi-indirect utility functions for all restricted models with closed-form solutions.

Table A1. Restricted Incomplete Demand System Models

(x1)	1.	$x_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik} p_k, \quad i \in N$
	2.	$x_i = \frac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q}) + \frac{\beta_{11}}{\gamma_1} - \frac{\beta_{1i}}{\gamma_i} + \sum_{k\in N}\beta_{1k}p_k + \gamma_1 y\right\}, \quad i \in J$ $x_i = -\beta_{1i}/\gamma_1, \quad i \in N\sim J$
(x2)	1.	$x_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik} p_k, \quad i \in N$
	2.	$x_i = \frac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q}) + \frac{\beta_{11}}{\gamma_1}\sum_{k\in N}\gamma_k p_k + \gamma_1 \ln(y)\right\}, \quad i \in N$
(x3)	1.	$x_i = \alpha_i(\mathbf{q}) + \beta_{ii}\ln(p_i), \quad i \in N$ 2. $x_i = (\gamma_i/\gamma_1)(\alpha_1(\mathbf{q}) + \gamma_1 y), \quad i \in N$
(x4)	1.	$x_i = \alpha_i(\mathbf{q}) + \beta_{ii}\ln(p_i), \quad i \in N$ 2. $x_i = (\gamma_i/\gamma_1)(\alpha_1(\mathbf{q}) + \gamma_1 \ln(y)), \quad i \in N$
(x5)[a]	1.	$x_i = \frac{\beta_{ii}}{\beta_{11}}\alpha_1(\mathbf{q})\exp\left\{\sum_{k\in J}\beta_{kk}p_k + \sum_{k\in N\sim K}\beta_{1k}p_k + \gamma_1 y\right\}, \quad i \in J$ $x_i = \alpha_i(\mathbf{q})\exp\left\{\beta_{ii}p_i + \sum_{k\in N\sim K}\beta_{1k}p_k + \gamma_1 y\right\}, \quad i \in K\sim J$ $x_i = -\beta_{1i}/\gamma_1, \quad i \in N\sim K$
(x6)	1.	$x_i = \frac{\beta_{ii}}{\beta_{11}}\alpha_1(\mathbf{q})\exp\left\{\sum_{k\in J}\beta_{kk}p_k\right\}y^{\gamma_1}, \quad i \in J$ $x_i = \alpha_i(\mathbf{q})\exp(\beta_{ii}p_i)y^{\gamma_1}, \quad i \in N\sim J$
	2.	$x_i = \frac{\beta_{ii}}{\beta_{11}}\alpha_1(\mathbf{q})\exp\left\{\sum_{k\in J}\beta_{kk}p_k + \sum_{k\in N\sim K}\beta_{1k}p_k\right\}, \quad i \in J$ $x_i = \alpha_i(\mathbf{q})\exp\left\{\beta_{ii}p_i + \sum_{k\in N\sim K}\beta_{1k}p_k\right\}, \quad i \in K\sim J$ $x_i = \beta_{1i}y, \quad i \in N\sim K$
(x7)[a]	1.	$x_i = \alpha_1(\mathbf{q})\left\{\frac{1+\beta_{ii}}{1+\beta_{11}}\right\}p_i^{-1}\left\{\prod_{k\in J}p_k^{1+\beta_{kk}}\right\}\left\{\prod_{k\in N\sim K}p_k^{\beta_{1k}}\right\}\exp(\gamma_1 y), \quad i \in J$ $x_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}\left\{\prod_{k\in N\sim K}p_k^{\beta_{1k}}\right\}\exp(\gamma_1 y), \quad i \in K\sim J$ $x_i = -(\beta_{1i}/\gamma_1)p_i^{-1}, \quad i \in N\sim K$
(x8)	1.	$x_i = \alpha_1(\mathbf{q})\left\{\frac{1+\beta_{ii}}{1+\beta_{11}}\right\}p_i^{-1}\left\{\prod_{k\in J}p_k^{1+\beta_{kk}}\right\}y^{\gamma_1}, \quad i \in J$ $x_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}y^{\gamma_1}, \quad i \in N\sim J$
	2.	$x_i = \alpha_1(\mathbf{q})\left\{\frac{1+\beta_{ii}}{1+\beta_{11}}\right\}p_i^{-1}\left\{\prod_{k\in J}p_k^{1+\beta_{kk}}\right\}\left\{\prod_{k\in N\sim K}p_k^{\beta_{1k}}\right\}, \quad i \in J$ $x_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}\left\{\prod_{k\in N\sim K}p_k^{\beta_{1k}}\right\}, \quad i \in K\sim J$ $x_i = \beta_{1i}p_i^{-1}y, \quad i \in N\sim K$

[a] Note that the $N\sim K$ subset is empty if $\gamma_1 = 0$.

Table A2. Restricted Incomplete Expenditure System Models

(e1)	1.	$e_i = \alpha_i(\mathbf{q}) + \beta_{ii}p_i, \quad i \in N$
	2.	$e_i = \alpha_1(\mathbf{q}) + \sum_{k\in K, i\neq k} \beta_{ik}p_k + \sum_{k\in N\sim K} \beta_{1k}p_k + y, \quad i \in J$
		$e_i = \frac{\gamma_i}{\gamma_1}\alpha_1(\mathbf{q}) + \sum_{k\in K} \beta_{ik}p_k + \frac{\gamma_i}{\gamma_1}\sum_{k\in N\sim K} \beta_{1k}p_k + \gamma_i y, \quad i \in K\sim J$
		$e_i = -(\beta_{1i}/\gamma_1)p_i, \quad i \in N\sim K$
	3.	$e_1 = \alpha_1(\mathbf{q}) + \sum_{k\in N} \beta_{1k}p_k + y$
		$e_i = \beta_{i1}p_1 - \beta_{1i}p_i, \quad i \in N,\ i \neq 1$
(e2)	1.	$e_i = \alpha_i(\mathbf{q}) + \beta_{ii}p_i, \quad i \in N$
	2.	$e_i = (\gamma_i/\gamma_1)(\alpha_1(\mathbf{q}) + \gamma_1\ln(y)), \quad i \in N$
(e3)	1.	$e_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik}\ln(p_k), \quad i \in N$
	2.	$e_i = \frac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q}) - \frac{\beta_{1i}}{\gamma_i} + \frac{\beta_{i1}}{\gamma_i} + \sum_{k\in N}\beta_{1k}\ln(p_k) + \gamma_1 y\right\}, \quad i \in J$
		$e_i = -\beta_{1i}/\gamma_1, \quad i \in N\sim J$
(e4)	1.	$e_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik}\ln(p_k), \quad i \in N$
	2.	$e_i = \frac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q}) + \frac{\beta_{11}}{\gamma_1}\sum_{k\in N}\gamma_k\ln(p_k) + \gamma_1\ln(y)\right\}, \quad i \in N$
(e5)	1.	$e_i = \alpha_i(\mathbf{q})\exp(\beta_{ii}p_i + \gamma_1 y), \quad i \in N$
(e6)	1.	$e_i = \alpha_i(\mathbf{q})\exp(\beta_{ii}p_i)y^{\gamma_1}, \quad i \in N$
(e7)[a]	1.	$e_i = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q})\left\{\prod_{k\in J} p_k^{\beta_{kk}}\right\}\left\{\prod_{k\in N\sim K} p_k^{\beta_{1k}}\right\}\exp(\gamma_1 y), \quad i \in J$
		$e_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}\left\{\prod_{k\in N\sim K} p_k^{\beta_{1k}}\right\}\exp(\gamma_1 y), \quad i \in K\sim J$
		$e_i = -(\beta_{1i}/\gamma_1), \quad i \in N\sim K$
(e8)	1.	$e_i = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q})\left\{\prod_{k\in J} p_k^{\beta_{kk}}\right\}y^{\gamma_1}, \quad i \in J$
		$e_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}y^{\gamma_1}, \quad i \in N\sim J$
	2.	$e_i = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q})\left\{\prod_{k\in J} p_k^{\beta_{kk}}\right\}\left\{\prod_{k\in N\sim K} p_k^{\beta_{1k}}\right\}, \quad i \in J$
		$e_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}\left\{\prod_{k\in N\sim K} p_k^{\beta_{1k}}\right\}, \quad i \in K\sim J$
		$e_i = \beta_{1i}y, \quad i \in N\sim K$

[a] Note that the $N\sim K$ subset is empty if $\gamma_1 = 0$.

Table A3. Restricted Incomplete Expenditure Share System Models

(s1)	1.	$s_i = \alpha_i(\mathbf{q}) + \beta_{ii}p_i, \quad i \in N$
	2.	$s_i = (\gamma_i/\gamma_1)(\alpha_1(\mathbf{q}) + \gamma_1 y), \quad i \in N$
(s2)	1.	$s_i = \alpha_i(\mathbf{q}) + \beta_{ii}p_i, \quad i \in N$
	2.	$s_i = \alpha_1(\mathbf{q}) + \sum_{k\in K, i\neq k} \beta_{ik}p_k + \sum_{k\in N\sim K} \beta_{1k}p_k + \ln(y), \quad i \in J$
		$s_i = \frac{\gamma_i}{\gamma_1}\alpha_1(\mathbf{q}) + \sum_{k\in K} \beta_{ik}p_k + \frac{\gamma_i}{\gamma_1}\sum_{k\in N\sim K} \beta_{1k}p_k + \gamma_i\ln(y), \quad i \in K\sim J$
		$s_i = -(\beta_{1i}/\gamma_1)p_i, \quad i \in N\sim K$
	3.	$s_1 = \alpha_1(\mathbf{q}) + \sum_{k\in N} \beta_{1k}p_k + \ln(y)$
		$s_i = \beta_{i1}p_1 - \beta_{1i}p_i, \quad i \in N,\ i \neq 1$
(s3)	1.	$s_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik}\ln(p_k), \quad i \in N$
	2.	$s_i = \frac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q}) + \frac{\beta_{11}}{\gamma_1}\sum_{k\in N}\gamma_k\ln(p_k) + \gamma_1 y\right\}, \quad i \in N$
(s4)	1.	$s_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik}\ln(p_k), \quad i \in N$
	2.	$s_i = \frac{\gamma_i}{\gamma_1}\left\{\frac{\beta_{i1}}{\gamma_i} - \frac{\beta_{1i}}{\gamma_i} + \alpha_1(\mathbf{q}) + \sum_{k\in N}\beta_{1k}\ln(p_k) + \gamma_1\ln(y)\right\}, \quad i \in J$
		$s_i = -\beta_{1i}/\gamma_1, \quad i \in N\sim J$
(s5)	1.	$s_i = \alpha_i(\mathbf{q})\exp(\beta_{ii}p_i + \gamma_1 y), \quad i \in N$
(s6)	1.	$s_i = \alpha_i(\mathbf{q})\exp(\beta_{ii}p_i)y^{\gamma_1}, \quad i \in N$
(s7)	1.	$s_i = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q})\left\{\prod_{k\in J} p_k^{\beta_{kk}}\right\}\exp(\gamma_1 y), \quad i \in J$
		$s_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}\exp(\gamma_1 y), \quad i \in N\sim J$
(s8)	1.	$s_i = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q})\left\{\prod_{k\in J} p_k^{\beta_{kk}}\right\}y^{\gamma_1}, \quad i \in J$
		$s_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}y^{\gamma_1}, \quad i \in N\sim J$
	2.	$s_i = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q})\left\{\prod_{k\in J} p_k^{\beta_{kk}}\right\}\left\{\prod_{k\in N\sim K} p_k^{\beta_{1k}}\right\}y^{-1}, \quad i \in J$
		$s_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}\left\{\prod_{k\in N\sim K} p_k^{\beta_{1k}}\right\}y^{-1}, \quad i \in K\sim J$
		$s_i = \beta_{1i}, \quad i \in N\sim K$

Table A4. Quasi-Indirect Utility Functions for Restricted Incomplete Demand System Models

Model	Restrictions	Quasi-Indirect Utility Function
(x1-1) & (x2-1)		$\phi(\mathbf{p},\mathbf{q},y) = y - \sum_{k\in N}\alpha_k(\mathbf{q})p_k - \frac{1}{2}\sum_{k\in N}\sum_{j\in N}\beta_{kj}p_k p_j$
(x1-2)		$\phi(\mathbf{p},\mathbf{q},y) = \left\{y + \frac{1}{\gamma_1}\left[\sum_{k\in N}\beta_{1k}p_k + \alpha_1(\mathbf{q}) + \beta_{11}/\gamma_1\right]\right\}\exp\left\{-\sum_{k\in J}\gamma_k p_k\right\}$
(x3-1) & (x4-1)		$\phi(\mathbf{p},\mathbf{q},y) = y - \sum_{k\in N}\alpha_k(\mathbf{q})p_k - \sum_{k\in N}\beta_{kk}p_k(\ln(p_k) - 1)$
(x3-2)		$\phi(\mathbf{p},\mathbf{q},y) = \left\{y + \frac{\alpha_1(\mathbf{q})}{\gamma_1}\right\}\exp\left\{-\sum_{k\in N}\gamma_k p_k\right\}$
(x5-1)	$\gamma_1 = 0$, $N\sim K = \emptyset$	$\phi(\mathbf{p},\mathbf{q},y) = y - \frac{\alpha_1(\mathbf{q})}{\beta_{11}}\exp\left\{\sum_{k\in J}\beta_{kk}p_k\right\} - \sum_{\substack{k\in K\sim J\\ \beta_{kk}\neq 0}}\frac{\alpha_k(\mathbf{q})}{\beta_{kk}}\exp(\beta_{kk}p_k) - \sum_{\substack{k\in K\sim J\\ \beta_{kk}=0}}\alpha_k(\mathbf{q})p_k$
(x5-1)	$\gamma_1 \neq 0$	$\phi(\mathbf{p},\mathbf{q},y) = \frac{-\exp(-\gamma_1 y)}{\gamma_1}\exp\left\{-\sum_{k\in N\sim K}\beta_{1k}p_k\right\} - \frac{\alpha_1(\mathbf{q})}{\beta_{11}}\exp\left\{\sum_{k\in J}\beta_{kk}p_k\right\} - \sum_{\substack{k\in K\sim J\\ \beta_{kk}\neq 0}}\frac{\alpha_k(\mathbf{q})}{\beta_{kk}}\exp(\beta_{kk}p_k) - \sum_{\substack{k\in K\sim J\\ \beta_{kk}=0}}\alpha_k(\mathbf{q})p_k$
(x6-1)	$\gamma_1 = 1$	$\phi(\mathbf{p},\mathbf{q},y) = \ln(y) - \frac{\alpha_1(\mathbf{q})}{\beta_{11}}\exp\left\{\sum_{k\in J}\beta_{kk}p_k\right\} - \sum_{\substack{k\in N\sim J\\ \beta_{kk}\neq 0}}\frac{\alpha_k(\mathbf{q})}{\beta_{kk}}\exp(\beta_{kk}p_k) - \sum_{\substack{k\in N\sim J\\ \beta_{kk}=0}}\alpha_k(\mathbf{q})p_k$
(x6-1)	$\gamma_1 \neq 1$	$\phi(\mathbf{p},\mathbf{q},y) = \frac{y^{1-\gamma_1}}{1-\gamma_1} - \frac{\alpha_1(\mathbf{q})}{\beta_{11}}\exp\left\{\sum_{k\in J}\beta_{kk}p_k\right\} - \sum_{\substack{k\in N\sim J\\ \beta_{kk}\neq 0}}\frac{\alpha_k(\mathbf{q})}{\beta_{kk}}\exp(\beta_{kk}p_k) - \sum_{\substack{k\in N\sim J\\ \beta_{kk}=0}}\alpha_k(\mathbf{q})p_k$
(x6-2)		$\phi(\mathbf{p},\mathbf{q},y) = y\exp\left\{-\sum_{k\in N\sim K}\beta_{1k}p_k\right\} - \frac{\alpha_1(\mathbf{q})}{\beta_{11}}\exp\left\{\sum_{k\in J}\beta_{kk}p_k\right\} - \sum_{\substack{k\in K\sim J\\ \beta_{kk}\neq 0}}\frac{\alpha_k(\mathbf{q})}{\beta_{kk}}\exp(\beta_{kk}p_k) - \sum_{\substack{k\in K\sim J\\ \beta_{kk}=0}}\alpha_k(\mathbf{q})p_k$
(x7-1)	$\gamma_1 = 0$, $N\sim K = \emptyset$	$\phi(\mathbf{p},\mathbf{q},y) = y - \frac{\alpha_1(\mathbf{q})}{1+\beta_{11}}\prod_{k\in J}p_k^{1+\beta_{kk}} - \sum_{\substack{k\in K\sim J\\ \beta_{kk}\neq -1}}\frac{\alpha_k(\mathbf{q})}{1+\beta_{kk}}p_k^{1+\beta_{kk}} - \sum_{\substack{k\in K\sim J\\ \beta_{kk}=-1}}\alpha_k(\mathbf{q})\ln(p_k)$
(x7-1)	$\gamma_1 \neq 0$	$\phi(\mathbf{p},\mathbf{q},y) = \frac{-\exp(-\gamma_1 y)}{\gamma_1}\prod_{k\in N\sim K}p_k^{-\beta_{1k}} - \frac{\alpha_1(\mathbf{q})}{1+\beta_{11}}\prod_{k\in J}p_k^{1+\beta_{kk}} - \sum_{\substack{k\in K\sim J\\ \beta_{kk}\neq -1}}\frac{\alpha_k(\mathbf{q})}{1+\beta_{kk}}p_k^{1+\beta_{kk}} - \sum_{\substack{k\in K\sim J\\ \beta_{kk}=-1}}\alpha_k(\mathbf{q})\ln(p_k)$

(continued . . .)

Table A4. Continued

Model	Restrictions	Quasi-Indirect Utility Function
(x8-1)	$\gamma_1 = 1$	$\phi(\mathbf{p}, \mathbf{q}, y) = \ln(y) - \frac{\alpha_1(\mathbf{q})}{1+\beta_{11}} \prod_{k\in J} p_k^{1+\beta_{kk}} - \sum_{\substack{k\in N\sim J \\ \beta_{kk}\neq -1}} \frac{\alpha_k(\mathbf{q})}{1+\beta_{kk}} p_k^{1+\beta_{kk}} - \sum_{\substack{k\in N\sim J \\ \beta_{kk}=-1}} \alpha_k(\mathbf{q})\ln(p_k)$
(x8-1)	$\gamma_1 \neq 1$	$\phi(\mathbf{p}, \mathbf{q}, y) = \frac{y^{1-\gamma_1}}{1-\gamma_1} - \frac{\alpha_1(\mathbf{q})}{1+\beta_{11}} \prod_{k\in J} p_k^{1+\beta_{kk}} - \sum_{\substack{k\in N\sim J \\ \beta_{kk}\neq -1}} \frac{\alpha_k(\mathbf{q})}{1+\beta_{kk}} p_k^{1+\beta_{kk}} - \sum_{\substack{k\in N\sim J \\ \beta_{kk}=-1}} \alpha_k(\mathbf{q})\ln(p_k)$
(x8-2)		$\phi(\mathbf{p}, \mathbf{q}, y) = y \prod_{k\in N\sim K} p_k^{-\beta_{1k}} - \frac{\alpha_1(\mathbf{q})}{1+\beta_{11}} \prod_{k\in J} p_k^{1+\beta_{kk}} - \sum_{\substack{k\in K\sim J \\ \beta_{kk}\neq -1}} \frac{\alpha_k(\mathbf{q})}{1+\beta_{kk}} p_k^{1+\beta_{kk}} - \sum_{\substack{k\in K\sim J \\ \beta_{kk}=-1}} \alpha_k(\mathbf{q})\ln(p_k)$

Note: The results reported here correct for typographical errors found in LaFrance (1990).

Table A5. Quasi-Indirect Utility Functions for Restricted Incomplete Expenditure System Models

Model	Restrictions	Quasi-Indirect Utility Function
(e1-1) & (e2-1)		$\phi(\mathbf{p}, \mathbf{q}, y) = y - \sum_{k\in N} (\alpha_k(\mathbf{q})\ln(p_k) + \beta_{kk} p_k)$
(e3-1) & (e4-1)		$\phi(\mathbf{p}, \mathbf{q}, y) = y - \sum_{k\in N} \alpha_k(\mathbf{q})\ln(p_k) - \frac{1}{2}\sum_{k\in N}\sum_{j\in N} \beta_{kj}\ln(p_k)\ln(p_j)$
(e7-1)	$\gamma_1 = 0$, $N\sim K = \emptyset$	$\phi(\mathbf{p}, \mathbf{q}, y) = y - \frac{\alpha_1(\mathbf{q})}{\beta_{11}} \prod_{k\in J} p_k^{\beta_{kk}} - \sum_{\substack{k\in K\sim J \\ \beta_{kk}\neq 0}} \frac{\alpha_k(\mathbf{q})}{\beta_{kk}} p_k^{\beta_{kk}} - \sum_{\substack{k\in K\sim J \\ \beta_{kk}=0}} \alpha_k(\mathbf{q})\ln(p_k)$
(e7-1)	$\gamma_1 \neq 0$	$\phi(\mathbf{p}, \mathbf{q}, y) = \frac{-\exp(-\gamma_1 y)}{\gamma_1} \prod_{k\in N\sim K} p_k^{-\beta_{1k}} - \frac{\alpha_1(\mathbf{q})}{\beta_{11}} \prod_{k\in J} p_k^{\beta_{kk}} - \sum_{\substack{k\in K\sim J \\ \beta_{kk}\neq 0}} \frac{\alpha_k(\mathbf{q})}{\beta_{kk}} p_k^{\beta_{kk}}$ $- \sum_{\substack{k\in K\sim J \\ \beta_{kk}=0}} \alpha_k(\mathbf{q})\ln(p_k)$
(e8-1)	$\gamma_1 = 1$	$\phi(\mathbf{p}, \mathbf{q}, y) = \ln(y) - \frac{\alpha_1(\mathbf{q})}{\beta_{11}} \prod_{k\in J} p_k^{\beta_{kk}} - \sum_{\substack{k\in N\sim J \\ \beta_{kk}\neq 0}} \frac{\alpha_k(\mathbf{q})}{\beta_{kk}} p_k^{\beta_{kk}} - \sum_{\substack{k\in N\sim J \\ \beta_{kk}=0}} \alpha_k(\mathbf{q})\ln(p_k)$
(e8-1)	$\gamma_1 \neq 1$	$\phi(\mathbf{p}, \mathbf{q}, y) = \frac{y^{1-\gamma_1}}{1-\gamma_1} - \frac{\alpha_1(\mathbf{q})}{\beta_{11}} \prod_{k\in J} p_k^{\beta_{kk}} - \sum_{\substack{k\in N\sim J \\ \beta_{kk}\neq 0}} \frac{\alpha_k(\mathbf{q})}{\beta_{kk}} p_k^{\beta_{kk}} - \sum_{\substack{k\in N\sim J \\ \beta_{kk}=0}} \alpha_k(\mathbf{q})\ln(p_k)$
(e8-2)		$\phi(\mathbf{p}, \mathbf{q}, y) = y \prod_{k\in N\sim K} p_k^{-\beta_{1k}} - \frac{\alpha_1(\mathbf{q})}{\beta_{11}} \prod_{k\in J} p_k^{\beta_{kk}} - \sum_{\substack{k\in K\sim J \\ \beta_{kk}\neq 0}} \frac{\alpha_k(\mathbf{q})}{\beta_{kk}} p_k^{\beta_{kk}} - \sum_{\substack{k\in K\sim J \\ \beta_{kk}=0}} \alpha_k(\mathbf{q})\ln(p_k)$

Table A6. Quasi-Indirect Utility Functions for Restricted Incomplete Expenditure Share System Models

Model	Restrictions	Quasi-Indirect Utility Function
(s1-1) & (s2-1)		$\phi(\mathbf{p},\mathbf{q},y) = y\prod_{k\in N} p_k^{-\alpha_k(\mathbf{q})}\exp\left\{-\sum_{k\in N}\beta_{kk}p_k\right\}$
(s3-1) & (s4-1)		$\phi(\mathbf{p},\mathbf{q},y) = \ln(y) - \sum_{k\in N}\alpha_k(\mathbf{q})\ln(p_k) - \frac{1}{2}\sum_{k\in N}\sum_{j\in N}\beta_{kj}\ln(p_k)\ln(p_j)$
(s8-1)	$\gamma_1 = 0$	$\phi(\mathbf{p},\mathbf{q},y) = \ln(y) - \frac{\alpha_1(\mathbf{q})}{\beta_{11}}\prod_{k\in J} p_k^{\beta_{kk}} - \sum_{\substack{k\in N\sim J\\ \beta_{kk}\neq 0}}\frac{\alpha_k(\mathbf{q})}{\beta_{kk}}p_k^{\beta_{kk}} - \sum_{\substack{k\in N\sim J\\ \beta_{kk}=0}}\alpha_k(\mathbf{q})\ln(p_k)$
(s8-1)	$\gamma_1 \neq 0$	$\phi(\mathbf{p},\mathbf{q},y) = \frac{y^{-\gamma_1}}{-\gamma_1} - \frac{\alpha_1(\mathbf{q})}{\beta_{11}}\prod_{k\in J} p_k^{\beta_{kk}} - \sum_{\substack{k\in N\sim J\\ \beta_{kk}\neq 0}}\frac{\alpha_k(\mathbf{q})}{\beta_{kk}}p_k^{\beta_{kk}} - \sum_{\substack{k\in N\sim J\\ \beta_{kk}=0}}\alpha_k(\mathbf{q})\ln(p_k)$
(s8-2)		$\phi(\mathbf{p},\mathbf{q},y) = y\prod_{k\in N\sim K} p_k^{-\beta_{1k}} - \frac{\alpha_1(\mathbf{q})}{\beta_{11}}\prod_{k\in J} p_k^{\beta_{kk}} - \sum_{\substack{k\in K\sim J\\ \beta_{kk}\neq 0}}\frac{\alpha_k(\mathbf{q})}{\beta_{kk}}p_k^{\beta_{kk}} - \sum_{\substack{k\in K\sim J\\ \beta_{kk}=0}}\alpha_k(\mathbf{q})\ln(p_k)$

Derivation of the Parameter Restrictions for the Twenty-Four Models

This appendix section derives the necessary parameter restrictions for Slutsky symmetry to hold in an open neighborhood around observed prices and income. The approach employed is similar to LaFrance (1985, 1986). For each of the 24 models, three mutually exclusive and exhaustive cases with alternative income effects for goods i and j ($i \neq j$) are considered: (a) no income effects, i.e., $\gamma_i = \gamma_j = 0$; (b) both goods having income effects, i.e., $\gamma_i \neq 0$, $\gamma_j \neq 0$; and (c) only one good having income effects, i.e., $\gamma_i \neq 0$, $\gamma_j = 0$.

For each of these possibilities, the necessary parameter restrictions for Slutsky symmetry to hold regardless of prices and income were derived. Restrictions implied by the derivative properties of the Slutsky summetry conditions were used extensively for this task. Once the parameter restrictions were identified for the three distinct income relationships, consistent combinations of the three sets of parameter restrictions were then determined. Tables A7, A8, and A9 summarize the conditions for Slutsky symmetry to hold.

Table A7. Slutsky Symmetry Conditions for Incomplete Demand System Models

(x1)	$\beta_{ji} + \gamma_j x_i = \beta_{ij} + \gamma_i x_j,\ \forall i,j$	(x5)	$\beta_{ji}x_j + \gamma_j x_i x_j = \beta_{ij}x_i + \gamma_i x_i x_j,\ \forall i,j$
(x2)	$\beta_{ji} + \frac{\gamma_j}{y}x_i = \beta_{ij} + \frac{\gamma_i}{y}x_j,\ \forall i,j$	(x6)	$\beta_{ji}x_j + \frac{\gamma_j}{y}x_i x_j = \beta_{ij}x_i + \frac{\gamma_i}{y}x_i x_j,\ \forall i,j$
(x3)	$\frac{\beta_{ji}}{p_i} + \gamma_j x_i = \frac{\beta_{ij}}{p_j} + \gamma_i x_j,\ \forall i,j$	(x7)	$\frac{\beta_{ji}}{p_i}x_j + \gamma_j x_i x_j = \frac{\beta_{ij}}{p_j}x_i + \gamma_i x_i x_j,\ \forall i,j$
(x4)	$\frac{\beta_{ji}}{p_i} + \frac{\gamma_j}{y}x_i = \frac{\beta_{ij}}{p_j} + \frac{\gamma_i}{y}x_j,\ \forall i,j$	(x8)	$\frac{\beta_{ji}}{p_i}x_j + \frac{\gamma_j}{y}x_i x_j = \frac{\beta_{ij}}{p_j}x_i + \frac{\gamma_i}{y}x_i x_j,\ \forall i,j$

Table A8. Slutsky Symmetry Conditions for Incomplete Expenditure System Models

(e1)	$\frac{1}{p_i p_j}\{\beta_{ji}p_i + \gamma_j e_i\} = \frac{1}{p_i p_j}\{\beta_{ij}p_j + \gamma_i e_j\}, \quad \forall i,j$
(e2)	$\frac{1}{p_i p_j}\left\{\beta_{ji}p_i + \frac{\gamma_j}{y}e_i\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij}p_j + \frac{\gamma_i}{y}e_j\right\}, \quad \forall i,j$
(e3)	$\frac{1}{p_i p_j}\{\beta_{ji} + \gamma_j e_i\} = \frac{1}{p_i p_j}\{\beta_{ij} + \gamma_i e_j\}, \quad \forall i,j$
(e4)	$\frac{1}{p_i p_j}\left\{\beta_{ji} + \frac{\gamma_j}{y}e_i\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij} + \frac{\gamma_i}{y}e_j\right\}, \quad \forall i,j$
(e5)	$\frac{1}{p_i p_j}\{\beta_{ji}p_i e_j + \gamma_j e_i e_j\} = \frac{1}{p_i p_j}\{\beta_{ij}p_j e_i + \gamma_i e_i e_j\}, \quad \forall i,j$
(e6)	$\frac{1}{p_i p_j}\left\{\beta_{ji}p_i e_j + \frac{\gamma_j}{y}e_i e_j\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij}p_j e_i + \frac{\gamma_i}{y}e_i e_j\right\}, \quad \forall i,j$
(e7)	$\frac{1}{p_i p_j}\{\beta_{ji}e_j + \gamma_j e_i e_j\} = \frac{1}{p_i p_j}\{\beta_{ij}e_i + \gamma_i e_i e_j\}, \quad \forall i,j$
(e8)	$\frac{1}{p_i p_j}\left\{\beta_{ji}e_j + \frac{\gamma_j}{y}e_i e_j\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij}e_i + \frac{\gamma_i}{y}e_i e_j\right\}, \quad \forall i,j$

Table A9. Slutsky Symmetry Conditions for Incomplete Expenditure Share System Models

(s1)	$\frac{y}{p_i p_j}\{\beta_{ji}p_i + (s_j + \gamma_j y)s_i\} = \frac{y}{p_i p_j}\{\beta_{ij}p_j + (s_i + \gamma_i y)s_j\}, \quad \forall i,j$
(s2)	$\frac{y}{p_i p_j}\{\beta_{ji}p_i + (s_j + \gamma_j)s_i\} = \frac{y}{p_i p_j}\{\beta_{ij}p_j + (s_i + \gamma_i)s_j\}, \quad \forall i,j$
(s3)	$\frac{y}{p_i p_j}\{\beta_{ji} + (s_j + \gamma_j y)s_i\} = \frac{y}{p_i p_j}\{\beta_{ij} + (s_i + \gamma_i y)s_j\}, \quad \forall i,j$
(s4)	$\frac{y}{p_i p_j}\{\beta_{ji} + (s_j + \gamma_j)s_i\} = \frac{y}{p_i p_j}\{\beta_{ij} + (s_i + \gamma_i)s_j\}, \quad \forall i,j$
(s5)	$\frac{y}{p_i p_j}\{\beta_{ji}p_i s_j + (1 + \gamma_j y)s_i s_j\} = \frac{y}{p_i p_j}\{\beta_{ij}p_j s_i + (1 + \gamma_i y)s_i s_j\}, \quad \forall i,j$
(s6)	$\frac{y}{p_i p_j}\{\beta_{ji}p_i s_j + (1 + \gamma_j)s_i s_j\} = \frac{y}{p_i p_j}\{\beta_{ij}p_j s_i + (1 + \gamma_i)s_i s_j\}, \quad \forall i,j$
(s7)	$\frac{y}{p_i p_j}\{\beta_{ji}s_j + (1 + \gamma_j y)s_i s_j\} = \frac{y}{p_i p_j}\{\beta_{ij}s_i + (1 + \gamma_i y)s_i s_j\}, \quad \forall i,j$
(s8)	$\frac{y}{p_i p_j}\{\beta_{ji}s_j + (1 + \gamma_j)s_i s_j\} = \frac{y}{p_i p_j}\{\beta_{ij}s_i + (1 + \gamma_i)s_i s_j\}, \quad \forall i,j$

The Twenty-Four Models

1. The (x1) Model

Consider the (x1) unrestricted model specification:

(x1) $$x_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i y, \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(x1-1) $$\beta_{ji} + \gamma_j x_i = \beta_{ij} + \gamma_i x_j.$$

Refer to LaFrance (1985) for the derivation of the necessary parameter restrictions.

2. The (x2) Model

Consider the (x2) unrestricted model specification:

(x2) $$x_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i \ln(y), \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(x2-1) $$\beta_{ji} + \frac{\gamma_j}{y} x_i = \beta_{ij} + \frac{\gamma_i}{y} x_j.$$

The derivative of (x2-1) with respect to p_k, $k = 1,\ldots,N$, implies the following restriction:

(x2-2) $$\gamma_j \beta_{ik} = \gamma_i \beta_{jk}.$$

The derivative of (x2-1) with respect to y implies the following restriction:

(x2-3) $$\gamma_j x_i = \gamma_i x_j.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (x2-1) implies:

(x2-4) $$\beta_{ji} = \beta_{ij}.$$

CASE II. $\gamma_i \neq 0$; $\gamma_j \neq 0$

- Expression (x2-2) implies:

(x2-5) $$\beta_{jk} = (\gamma_j/\gamma_i)\beta_{ik}, \quad \forall k.$$

- Expressions (x2-3) and (x2-5) together imply:

(x2-6) $$\alpha_j(\mathbf{q}) = (\gamma_j/\gamma_i)\alpha_i(\mathbf{q}).$$

- Plugging (x2-5) and (x2-6) into (x2-1) and simplifying implies:

(x2-7) $$\beta_{ij} = \beta_{ji}.$$

- One can combine (x2-5) and (x2-7) as:

(x2-8) $$\beta_{ij} = (\gamma_i\gamma_j/\gamma_k^2)\beta_{kk}, \quad \forall k.$$

- Expressions (x2-6) and (x2-8) jointly imply that:

(x2-9) $$\operatorname{sgn}(\gamma_i) = \operatorname{sgn}(\gamma_j) \neq 0.$$

- Thus, (x2-6), (x2-8), and (x2-9) are the necessary parameter restrictions.

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (x2-3) implies this case is only possible if $\gamma_i = 0$, a contradiction.

The restricted model specification takes the following form:
1. $x_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik}p_k,\ i \in N$
2. $x_i = \dfrac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q}) + \dfrac{\beta_{11}}{\gamma_1}\sum_{k\in N}\gamma_k p_k + \gamma_1\ln(y)\right\},\ i \in N$

3. The (x3) Model

Consider the (x3) unrestricted model specification:

(x3) $$x_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n}\beta_{ik}\ln(p_k) + \gamma_i y, \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j $(i \neq j)$ are:

(x3-1) $$\frac{\beta_{ji}}{p_i} + \gamma_j x_i = \frac{\beta_{ij}}{p_j} + \gamma_i x_j.$$

The derivative of (x3-1) with respect to p_j implies:

(x3-2) $$\beta_{ij}(\gamma_j + 1/p_j) = \gamma_i\beta_{jj}.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expressions (x3-1) and (x3-2) are only satisfied if:

(x3-3) $$\beta_{ij} = \beta_{ji} = 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- Expression (x3-2) holds in general only if:

(x3-4) $$\beta_{ik} = \beta_{jk} = 0, \quad \forall k.$$

- Expressions (x3-4) and (x3-1) imply:

(x3-5) $$\alpha_j(\mathbf{q}) = (\gamma_j/\gamma_i)\alpha_i(\mathbf{q}),$$

which further implies:

(x3-6) $$\operatorname{sgn}(\gamma_i) = \operatorname{sgn}(\gamma_j) \neq 0.$$

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (x3-2) implies the restriction in (x3-4), which with (x3-1) implies $\gamma_i = 0$, a contradiction.

The restricted model specification takes the following form:
1. $x_i = \alpha_i(\mathbf{q}) + \beta_{ii}\ln(p_i),\ i \in N$ 2. $x_i = (\gamma_i/\gamma_1)(\alpha_1(\mathbf{q}) + \gamma_1 y),\ i \in N$

4. The (x4) Model

Consider the (x4) unrestricted model specification:

(x4) $$x_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n}\beta_{ik}\ln(p_k) + \gamma_i\ln(y), \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(x4-1) $$\frac{\beta_{ji}}{p_i} + \frac{\gamma_j}{y}x_i = \frac{\beta_{ij}}{p_j} + \frac{\gamma_i}{y}x_j.$$

The derivative of (x4-1) with respect to y implies:

(x4-2) $$\gamma_j x_i = \gamma_i x_j.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (x4-1) is satisfied only if:

(x4-3) $$\beta_{ij} = \beta_{ji} = 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- Plugging $x_j = (\gamma_j/\gamma_i)x_i$ from (x4-2) into (x4-1) implies (x4-3). Expressions (x4-3) and (x4-2), along with the structure of (x4), imply the following three restrictions:

(x4-4) $$\beta_{ik} = \beta_{jk} = 0, \quad \forall k,$$

(x4-5) $$\alpha_j(\mathbf{q}) = (\gamma_j/\gamma_i)\alpha_i(\mathbf{q}),$$

(x4-6) $$\mathrm{sgn}(\gamma_i) = \mathrm{sgn}(\gamma_j) \neq 0.$$

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (x4-2) implies $\gamma_i = 0$, a contradiction.

The restricted model specification takes the following form:
1. $x_i = \alpha_i(\mathbf{q}) + \beta_{ii}\ln(p_i),\ i \in N$ 2. $x_i = (\gamma_i/\gamma_1)(\alpha_1(\mathbf{q}) + \gamma_1\ln(y)),\ i \in N$

5. The (x5) Model

Consider the (x5) unrestricted model specification:

(x5) $$x_i = \alpha_i(\mathbf{q})\exp\left\{\sum_{k=1}^{n}\beta_{ik}p_k + \gamma_i y\right\}, \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(x5-1) $$\beta_{ji}x_j + \gamma_j x_i x_j = \beta_{ij}x_i + \gamma_i x_i x_j.$$

The derivative of (x5-1) with respect to y implies:

(x5-2) $$\gamma_i \tilde{S}_{ij} = \gamma_j \tilde{S}_{ji}.$$

The derivative of (x5-1) with respect to p_k, $k = 1, ..., N$, implies:

(x5-3) $$\beta_{ik}(\tilde{S}_{ij} - \gamma_j x_i x_j) = \beta_{jk}(\tilde{S}_{ji} - \gamma_i x_i x_j).$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (x5-1) implies $\beta_{ji}x_j = \beta_{ij}x_i$, which is satisfied only if:

(x5-4) $$\beta_{ij} = \beta_{ji} = 0,$$

or

(x5-5) $$\beta_{ik} = \beta_{jk} = \beta_{kk}, \quad \forall k$$

(x5-6) $$\alpha_i(\mathbf{q}) = (\beta_{ii}/\beta_{jj})\alpha_j(\mathbf{q}) > 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- Expression (x5-2) implies $\gamma_i = \gamma_j$, and this case collapses into Case I above.

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (x5-2) implies $\tilde{S}_{ij} = 0$; (x5-1) implies that:

(x5-7) $$x_j = -\beta_{ij}/\gamma_i.$$

The restricted model specification takes the following form:
1. $x_i = \dfrac{\beta_{ii}}{\beta_{11}}\alpha_1(\mathbf{q})\exp\left\{\sum_{k\in J}\beta_{kk}p_k + \sum_{k\in N\sim K}\beta_{1k}p_k + \gamma_1 y\right\},\ i \in J$ $x_i = \alpha_i(\mathbf{q})\exp\left\{\beta_{ii}p_i + \sum_{k\in N\sim K}\beta_{1k}p_k + \gamma_1 y\right\},\ i \in K\sim J$ $x_i = -\beta_{1i}/\gamma_1,\ i \in N\sim K$ (Note that the subset $N\sim K$ is empty if $\gamma_1 = 0$.)

6. The (x6) Model

Consider the (x6) unrestricted model specification:

(x6) $$x_i = \alpha_i(\mathbf{q})\exp\left\{\sum_{k=1}^{n} \beta_{ik} p_k\right\} y^{\gamma_i}, \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(x6-1) $$\beta_{ji} x_j + \frac{\gamma_j}{y} x_i x_j = \beta_{ij} x_i + \frac{\gamma_i}{y} x_i x_j.$$

The derivative of (x6-1) with respect to y implies:

(x6-2) $$\gamma_i(\tilde{S}_{ij} - x_i x_j / y) = \gamma_j(\tilde{S}_{ji} - x_i x_j / y).$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (x6-1) simplifies to $\beta_{ji} x_j = \beta_{ij} x_i$. As with the (x5) model, this condition is satisfied only if:

(x6-3) $$\beta_{ij} = \beta_{ji} = 0,$$

or

(x6-4) $$\beta_{ik} = \beta_{jk} = \beta_{kk}, \quad \forall k,$$

(x6-5) $$\alpha_i(\mathbf{q}) = (\beta_{ii}/\beta_{jj})\alpha_j(\mathbf{q}) > 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- Expression (x6-2) implies $\gamma_i = \gamma_j$ or $\tilde{S}_{ij} = \tilde{S}_{ji} = x_i x_j / y$; however, the latter condition is only satisfied if $\gamma_i = \gamma_j = 1$, and $\beta_{ij} = \beta_{ji} = 0$. Thus, the following condition must hold:

(x6-6) $$\gamma_i = \gamma_j.$$

- Expression (x6-6) implies $\beta_{ji} x_j = \beta_{ij} x_i$, and thus either the conditions in (x6-3) or (x6-4) and (x6-5) must be satisfied.

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (x6-2) implies that $\tilde{S}_{ij} = x_i x_j / y$, which when plugged back into (x6-1) implies:

(x6-7) $$x_i = \beta_{ji} y.$$

The restricted model specification takes the following form:
1. $x_i = \frac{\beta_{ii}}{\beta_{11}} \alpha_1(\mathbf{q})\exp\left\{\sum_{k\in J} \beta_{kk} p_k\right\} y^{\gamma_1},\ i \in J$ $x_i = \alpha_i(\mathbf{q})\exp(\beta_{ii} p_i) y^{\gamma_1},\ i \in N\sim J$ 2. $x_i = \frac{\beta_{ii}}{\beta_{11}} \alpha_1(\mathbf{q})\exp\left\{\sum_{k\in J} \beta_{kk} p_k + \sum_{k\in N\sim K} \beta_{1k} p_k\right\},\ i \in J$ $x_i = \alpha_i(\mathbf{q})\exp\left\{\beta_{ii} p_i + \sum_{k\in N\sim K} \beta_{1k} p_k\right\},\ i \in K\sim J$ $x_i = \beta_{1i} y,\ i \in N\sim K$

7. The (x7) Model

Consider the (x7) unrestricted model specification:

(x7) $$x_i = \alpha_i(\mathbf{q})\left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\}\exp(\gamma_i y), \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(x7-1) $$\frac{\beta_{ji}}{p_i}x_j + \gamma_j x_i x_j = \frac{\beta_{ij}}{p_j}x_i + \gamma_i x_i x_j.$$

The derivative of (x7-1) with respect to y implies:

(x7-2) $$\gamma_j \tilde{S}_{ij} = \gamma_i \tilde{S}_{ji}.$$

The derivative of (x7-1) with respect to p_k, $k \neq i, j$, implies:

(x7-3) $$\beta_{jk}(\tilde{S}_{ji} - \gamma_i x_i x_j) = \beta_{ik}(\tilde{S}_{ij} - \gamma_j x_i x_j).$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (x7-1) simplifies to $\beta_{ji} p_j x_j = \beta_{ij} p_i x_i$, which is satisfied if:

(x7-4) $$\beta_{ij} = \beta_{ji} = 0.$$

- It can also be shown that (x7-1) is satisfied if:

(x7-5) $$\beta_{kl} = 1 + \beta_{ll}, \quad k = i,j; \; l = i,j; \; k \neq l,$$

(x7-6) $$\beta_{jk} = \beta_{ik}, \quad \forall k; \; k \neq i,j,$$

(x7-7) $$\alpha_i(\mathbf{q}) = \left\{\frac{1 + \beta_{ii}}{1 + \beta_{jj}}\right\}\alpha_j(\mathbf{q}) > 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- Expression (x7-2) implies that $\gamma_i = \gamma_j$, and with this restriction the case collapses into Case I above.

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (x7-2) implies $s_{ji} = 0$, which is satisfied only if:

(x7-8) $$x_j = -(\beta_{ij}/\gamma_i)/p_j.$$

The restricted model specification takes the following form:
1. $x_i = \alpha_1(\mathbf{q})\left\{\frac{1 + \beta_{ii}}{1 + \beta_{11}}\right\}p_i^{-1}\left\{\prod_{k\in J} p_k^{1+\beta_{kk}}\right\}\left\{\prod_{k\in N\sim K} p_k^{\beta_{1k}}\right\}\exp(\gamma_1 y),\ i \in J$
$x_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}\left\{\prod_{k\in N\sim K} p_k^{\beta_{1k}}\right\}\exp(\gamma_1 y),\ i \in K\sim J$
$x_i = -(\beta_{1i}/\gamma_1)p_i^{-1},\ i \in N\sim K$
(Note that the $N\sim K$ set must be empty if $\gamma_1 = 0$.)

8. The (x8) Model

Consider the (x8) unrestricted model specification:

$$x_i = \alpha_i(\mathbf{q})\left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\} y^{\gamma_i}, \quad \forall i. \tag{x8}$$

The implied Slutsky symmetry conditions for goods i and j $(i \neq j)$ are:

$$\frac{\beta_{ji}}{p_i}x_j + \frac{\gamma_j}{y}x_i x_j = \frac{\beta_{ij}}{p_j}x_i + \frac{\gamma_i}{y}x_i x_j. \tag{x8-1}$$

See LaFrance (1986) for the derivation of the necessary parameter restrictions.

9. The (e1) Model

Consider the (e1) unrestricted model specification:

$$e_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik}p_k + \gamma_i y, \quad \forall i. \tag{e1}$$

The implied Slutsky symmetry conditions for goods i and j $(i \neq j)$ are:

$$\frac{1}{p_i p_j}\left\{\beta_{ji}p_i + \gamma_j e_i\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij}p_j + \gamma_i e_j\right\}. \tag{e1-1}$$

The derivative of (e1-1) with respect to p_i implies:

$$\beta_{ji}(\gamma_i - 1) = \gamma_j \beta_{ii}. \tag{e1-2}$$

The derivative of (e1-1) with respect to p_k, $k \neq i, j$, implies:

$$\gamma_j \beta_{ik} = \gamma_i \beta_{jk}. \tag{e1-3}$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expressions (e1-1) and (e1-2) imply:

$$\beta_{ij} = \beta_{ji} = 0. \tag{e1-4}$$

CASE II. $\gamma_i \neq 0$; $\gamma_j \neq 0$

- Expression (e1-3) implies:

$$\beta_{ik} = (\gamma_i/\gamma_j)\beta_{jk}, \quad \forall k;\ k \neq i, j. \tag{e1-5}$$

- For (e1-2) to hold, it must be the case that:

$$\beta_{ii} = 0 \ \text{ if } \ \gamma_i = 1, \tag{e1-6}$$

$$\beta_{ji} = \frac{\gamma_j \beta_{ii}}{\gamma_i - 1} \ \text{ if } \ \gamma_i \neq 1. \tag{e1-7}$$

Thus, (e1-5), (e1-6), and (e1-7) are the necessary parameter restrictions.

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (e1-1) simplifies to $e_j = -(\beta_{ij}/\gamma_i)p_j + (\beta_{ji}/\gamma_i)p_i$. Two possibilities are implied by this structure:

$$\text{(e1-8)} \qquad \gamma_i = 1 \text{ and } \beta_{jj} = -\beta_{ij} \Rightarrow e_j = -\beta_{ij}p_j + \beta_{ji}p_i,$$

$$\text{(e1-9)} \qquad \gamma_i \neq 1 \text{ and } \beta_{ji} = 0 \Rightarrow e_j = -(\beta_{ij}/\gamma_i)p_j.$$

The restricted model specification takes the following form:
1. $e_i = \alpha_i(\mathbf{q}) + \beta_{ii}p_i,\ i \in N$
2. $e_i = \alpha_1(\mathbf{q}) + \sum_{k \in K, i \neq k} \beta_{ik}p_k + \sum_{k \in N\sim K} \beta_{1k}p_k + y,\ i \in J$
$e_i = \frac{\gamma_i}{\gamma_1}\alpha_1(\mathbf{q}) + \sum_{k \in K} \beta_{ik}p_k + \frac{\gamma_i}{\gamma_1}\sum_{k \in N\sim K} \beta_{1k}p_k + \gamma_i y,\ i \in K\sim J$
$e_i = -(\beta_{1i}/\gamma_1)p_i,\ i \in N\sim K$
3. $e_1 = \alpha_1(\mathbf{q}) + \sum_{k \in N} \beta_{1k}p_k + y$
$e_i = \beta_{i1}p_1 - \beta_{1i}p_i,\ i \in N,\ i \neq 1$

10. The (e2) Model

Consider the (e2) unrestricted model specification:

$$\text{(e2)} \qquad e_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik}p_k + \gamma_i \ln(y), \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

$$\text{(e2-1)} \qquad \frac{1}{p_i p_j}\left\{\beta_{ji}p_i + \frac{\gamma_j}{y}e_i\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij}p_j + \frac{\gamma_i}{y}e_j\right\}.$$

The derivative of (e2-1) with respect to p_j implies:

$$\text{(e2-2)} \qquad \gamma_j\beta_{ij}/y = \beta_{ij} + \gamma_i\beta_{jj}/y.$$

The derivative of (e2-1) with respect to p_k, $k \neq i, j$, implies:

$$\text{(e2-3)} \qquad \gamma_j\beta_{ik} = \gamma_i\beta_{jk}.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (e2-2) implies:

$$\text{(e2-4)} \qquad \beta_{ij} = \beta_{ji} = 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- Expression (e2-2) implies (e2-4) must hold. Expressions (e2-1), (e2-2), and (e2-4) imply:

(e2-5) $$\alpha_i(\mathbf{q}) = (\gamma_i/\gamma_j)\alpha_j(\mathbf{q}),$$

(e2-6) $$\beta_{ik} = \beta_{jk} = 0, \ \forall k,$$

(e2-7) $$\mathrm{sgn}(\gamma_i) = \mathrm{sgn}(\gamma_j) \neq 0.$$

CASE III. $\gamma_i \neq 0; \ \gamma_j = 0$

- Expression (e2-1) implies $e_i = (y/\gamma_j)(\beta_{ij}p_j - \beta_{ji}p_i)$, which is inconsistent with the structure of the (e2) model.

The restricted model specification takes the following form:
1. $e_i = \alpha_i(\mathbf{q}) + \beta_{ii}p_i, \ i \in N$
2. $e_i = (\gamma_i/\gamma_1)(\alpha_1(\mathbf{q}) + \gamma_1 \ln(y)), \ i \in N$

11. The (e3) Model

Consider the (e3) unrestricted model specification:

(e3) $$e_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} \ln(p_k) + \gamma_i y, \ \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(e3-1) $$\frac{1}{p_i p_j}\left\{\beta_{ji} + \gamma_j e_i\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij} + \gamma_i e_j\right\}.$$

The derivative of (e3-1) with respect to p_k, $k = 1, \ldots, N$, implies:

(e3-2) $$\gamma_j \beta_{ik} = \gamma_i \beta_{jk}, \ \forall i, j, k.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (e3-1) implies:

(e3-3) $$\beta_{ji} = \beta_{ij}.$$

CASE II. $\gamma_i \neq 0; \ \gamma_j \neq 0$

- Expression (e3-2) implies:

(e3-4) $$\beta_{ik} = \frac{\gamma_i}{\gamma_j}\beta_{jk}, \ \forall k.$$

- Plugging (e3-4) back into (e3-1) implies the following restriction:

(e3-5) $$\beta_{ji} - \beta_{ij} + \gamma_j \alpha_i(\mathbf{q}) - \gamma_i \alpha_j(\mathbf{q}) = 0.$$

CASE III. $\gamma_i \neq 0$; $\gamma_j = 0$

- Expression (e3-1) simplifies to $e_j = \beta_{ji}/\gamma_i - \beta_{ij}/\gamma_i$. To be consistent with (e3), it must be the case that:

(e3-6) $$\beta_{jk} = 0, \ \forall k,$$

(e3-7) $$\alpha_j(\mathbf{q}) = -\beta_{ij}/\gamma_i > 0.$$

The restricted model specification takes the following form:
1. $e_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik} \ln(p_k), \ i \in N$
2. $e_i = \frac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q}) - \frac{\beta_{1i}}{\gamma_i} + \frac{\beta_{i1}}{\gamma_i} + \sum_{k\in N} \beta_{1k}\ln(p_k) + \gamma_1 y\right\}, \ i \in J$ $e_i = -\beta_{1i}/\gamma_1, \ i \in N\sim J$

12. The (e4) Model

Consider the (e4) unrestricted model specification:

(e4) $$e_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik}\ln(p_k) + \gamma_i \ln(y), \ \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(e4-1) $$\frac{1}{p_i p_j}\left\{\beta_{ji} + \frac{\gamma_j}{y}e_i\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij} + \frac{\gamma_i}{y}e_j\right\}.$$

The derivative of (e4-1) with respect to y implies:

(e4-2) $$\gamma_i e_j = \gamma_j e_i.$$

The derivative of (e4-1) with respect to p_k, $k = 1,\ldots,N$, implies:

(e4-3) $$\gamma_j \beta_{ik} = \gamma_i \beta_{jk}, \ \forall k.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (e4-1) implies:

(e4-4) $$\beta_{ji} = \beta_{ij}.$$

CASE II. $\gamma_i \neq 0$; $\gamma_j \neq 0$

- Plugging (e4-2) into (e4-1) and simplifying implies (e4-4). Expressions (e4-4) and (e4-3) together imply:

(e4-5) $$\beta_{ij} = \frac{\gamma_i\gamma_j}{\gamma_k^2}\beta_{kk}, \ \forall k,$$

(e4-6) $$\text{sgn}(\gamma_i) = \text{sgn}(\gamma_j) \neq 0.$$

■ Plugging (e4-5) back into (e4-2) then implies:

(e4-7)
$$\alpha_i(\mathbf{q}) = \frac{\gamma_i}{\gamma_j}\alpha_j(\mathbf{q}).$$

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

■ Expression (e4-2) implies this case is not possible.

The restricted model specification takes the following form:
1. $e_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik}\ln(p_k),\ i\in N$
2. $e_i = \frac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q}) + \frac{\beta_{11}}{\gamma_1}\sum_{k\in N}\gamma_k\ln(p_k) + \gamma_1\ln(y)\right\},\ i\in N$

13. The (e5) Model

Consider the (e5) unrestricted model specification:

(e5)
$$e_i = \alpha_i(\mathbf{q})\exp\left\{\sum_{k=1}^{n}\beta_{ik}p_k + \gamma_i y\right\},\quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(e5-1)
$$\frac{1}{p_i p_j}\left\{\beta_{ji}p_i e_j + \gamma_j e_i e_j\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij}p_j e_i + \gamma_i e_i e_j\right\}.$$

The derivative of (e5-1) with respect to y implies:

(e5-2)
$$\gamma_j\tilde{S}_{ji} = \gamma_i\tilde{S}_{ij}.$$

CASE I. $\gamma_i = \gamma_j = 0$

■ Expression (e5-1) simplifies to $\beta_{ji}p_i e_j = \beta_{ij}p_j e_i$, which is not in general satisfied unless:

(e5-3)
$$\beta_{ij} = \beta_{ji} = 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

■ Expression (e5-2) implies:

(e5-4)
$$\gamma_i = \gamma_j.$$

■ Expression (e5-4) implies that (e5-1) simplifies to $\beta_{ji}p_i e_j = \beta_{ij}p_j e_i$, and thus (e5-3) must also be satisfied.

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

■ Expression (e5-2) implies this case is not possible.

The restricted model specification takes the following form:
1. $e_i = \alpha_i(\mathbf{q})\exp(\beta_{ii}p_i + \gamma_1 y),\ i\in N$

14. The (e6) Model

Consider the (e6) unrestricted model specification:

(e6) $$e_i = \alpha_i(\mathbf{q})\exp\left\{\sum_{k=1}^{n} \beta_{ik} p_k\right\} y^{\gamma_i}, \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(e6-1) $$\frac{1}{p_i p_j}\left\{\beta_{ji} p_i e_j + \frac{\gamma_j}{y} e_i e_j\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij} p_j e_i + \frac{\gamma_i}{y} e_i e_j\right\}.$$

The derivative of (e6-1) with respect to y implies:

(e6-2) $$\gamma_j\left\{\tilde{S}_{ji} - \frac{e_j e_i}{p_j p_i y}\right\} = \gamma_i\left\{\tilde{S}_{ij} - \frac{e_j e_i}{p_j p_i y}\right\}.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (e6-1) simplifies to $\beta_{ji} p_i e_j = \beta_{ij} p_j e_i$, which is not in general satisfied unless:

(e6-3) $$\beta_{ji} = \beta_{ij}.$$

CASE II. $\gamma_i \neq 0$; $\gamma_j \neq 0$

- Expression (e6-1) is not in general satisfied unless (e6-3) and the following condition are satisfied:

(e6-4) $$\gamma_i = \gamma_j.$$

CASE III. $\gamma_i \neq 0$; $\gamma_j = 0$

- Expression (e6-2) implies $\tilde{S}_{ij} = e_j e_i/(p_j p_i y)$. This restriction, along with (e6-1), implies $e_i = \beta_{ji} p_i y$, which is inconsistent with the structure of the (e6) model.

The restricted model specification takes the following form:
1. $e_i = \alpha_i(\mathbf{q})\exp(\beta_{ii} p_i) y^{\gamma_1}, \; i \in N$

15. The (e7) Model

Consider the (e7) unrestricted model specification:

(e7) $$e_i = \alpha_i(\mathbf{q})\left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\}\exp(\gamma_i y), \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(e7-1) $$\frac{1}{p_i p_j}\left\{\beta_{ji} e_j + \gamma_j e_i e_j\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij} e_i + \gamma_i e_i e_j\right\}.$$

This model is observationally equivalent to the (x7) model up to a parametric transformation. See the (x7) model section for the derivation of the necessary parameter restrictions.

16. The (e8) Model

Consider the (e8) unrestricted model specification:

(e8) $$e_i = \alpha_i(\mathbf{q})\left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\} y^{\gamma_i}, \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(e8-1) $$\frac{1}{p_i p_j}\left\{\beta_{ji} e_j + \frac{\gamma_j}{y} e_i e_j\right\} = \frac{1}{p_i p_j}\left\{\beta_{ij} e_i + \frac{\gamma_i}{y} e_i e_j\right\}.$$

This model is observationally equivalent to the (x8) model up to a parametric transformation. See LaFrance (1986) for the derivation of the necessary parameter restrictions.

17. The (s1) Model

Consider the (s1) unrestricted model specification:

(s1) $$s_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i y, \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(s1-1) $$\frac{y}{p_i p_j}\left\{\beta_{ji} p_i + (s_j + \gamma_j y) s_i\right\} = \frac{y}{p_i p_j}\left\{\beta_{ij} p_j + (s_i + \gamma_i y) s_j\right\}.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (s1-1) simplifies in this case to $\beta_{ji} p_i = \beta_{ij} p_j$, which is satisfied only if:

(s1-2) $$\beta_{ij} = \beta_{ji} = 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- Expression (s1-1) simplifies to $\beta_{ji} p_i + \gamma_j y s_i = \beta_{ij} p_j + \gamma_i y s_j$ which, when differentiated with respect to y, implies $\gamma_j s_i = \gamma_i s_j$, and when differentiated with respect to p_i implies $\beta_{ji}(1 - \gamma_i y) = -\gamma_j y \beta_{ii}$. These two conditions hold in general only if:

(s1-3) $$\beta_{ik} = \beta_{jk} = 0, \quad \forall k,$$

(s1-4) $$\alpha_i(\mathbf{q}) = (\gamma_i/\gamma_j)\alpha_j(\mathbf{q}),$$

(s1-5) $$\mathrm{sgn}(\gamma_i) = \mathrm{sgn}(\gamma_j) \neq 0.$$

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (s1-1) in this case simplifies to $s_j = (\beta_{ji} p_i - \beta_{ij} p_j)/(\gamma_i y)$, which is inconsistent with the structure of (s1).

The restricted model specification takes the following form:
1. $s_i = \alpha_i(\mathbf{q}) + \beta_{ii} p_i,\ i \in N$
2. $s_i = (\gamma_i/\gamma_1)(\alpha_1(\mathbf{q}) + \gamma_1 y),\ i \in N$

18. The (s2) Model

Consider the (s2) unrestricted model specification:

(s2) $$s_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} p_k + \gamma_i \ln(y), \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(s2-1) $$\frac{y}{p_i p_j}\left\{\beta_{ji} p_i + (s_j + \gamma_j) s_i\right\} = \frac{y}{p_i p_j}\left\{\beta_{ij} p_j + (s_i + \gamma_i) s_j\right\}.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (s2-1) simplifies in this case to $\beta_{ji} p_i = \beta_{ij} p_j$, which is satisfied only if:

(s2-2) $$\beta_{ij} = \beta_{ji} = 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- Expression (s2-1) simplifies to $\beta_{ji} p_i + \gamma_j s_i = \beta_{ij} p_j + \gamma_i s_j$, whose derivative with respect to p_i is $\beta_{ji}(\gamma_i - 1) = \gamma_j \beta_{ii}$, and whose derivative with respect to p_k, $k \neq i, j$, is $\gamma_j \beta_{ik} = \gamma_i \beta_{jk}$. For these conditions to hold in general, either:

(s2-3) $$\gamma_i = 1,$$

(s2-4) $$\beta_{ii} = 0,$$

(s2-5) $$\beta_{ik} = \beta_{jk}/\gamma_j, \quad \forall k;\ k \neq i, j,$$

or

(s2-6) $$\gamma_i \neq 1,$$

(s2-7) $$\beta_{ji} = \frac{\gamma_j \beta_{ii}}{\gamma_i - 1}.$$

(s2-8) $$\beta_{ik} = (\gamma_i/\gamma_j)\beta_{jk}, \quad \forall k,\ k \neq i, j.$$

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (s2-1) in this case simplifies to $s_j = (\beta_{ji} p_i - \beta_{ij} p_j)/\gamma_i$. To be consistent with (s2), this condition requires that either:

(s2-9) $$\gamma_i = 1,$$

or

(s2-10) $$\beta_{ji} = 0,$$

(s2-11) $$\beta_{jj} = -\beta_{ij}/\gamma_i > 0.$$

The restricted model specification takes the following form:
1. $s_i = \alpha_i(\mathbf{q}) + \beta_{ii} p_i,\ i \in N$
2. $s_i = \alpha_1(\mathbf{q}) + \sum_{k \in K, i \neq k} \beta_{ik} p_k + \sum_{k \in N \sim K} \beta_{1k} p_k + \ln(y),\ i \in J$ $s_i = \frac{\gamma_i}{\gamma_1} \alpha_1(\mathbf{q}) + \sum_{k \in K} \beta_{ik} p_k + \frac{\gamma_i}{\gamma_1} \sum_{k \in N \sim K} \beta_{1k} p_k + \gamma_i \ln(y),\ i \in K \sim J$ $s_i = -(\beta_{1i}/\gamma_1) p_i,\ i \in N \sim K$
3. $s_1 = \alpha_1(\mathbf{q}) + \sum_{k \in N} \beta_{1k} p_k + \ln(y)$ $s_i = \beta_{i1} p_1 - \beta_{1i} p_i,\ i \in N,\ i \neq 1$

19. The (s3) Model

Consider the (s3) unrestricted model specification:

(s3)
$$s_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik} \ln(p_k) + \gamma_i y, \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(s3-1)
$$\frac{y}{p_i p_j} \left\{ \beta_{ji} + (s_j + \gamma_j y) s_i \right\} = \frac{y}{p_i p_j} \left\{ \beta_{ij} + (s_i + \gamma_i y) s_j \right\}.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (s3-1) implies:

(s3-2)
$$\beta_{ij} = \beta_{ji}.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- The derivative of (s3-1) with respect to p_k, $k = 1, \ldots, N$, implies:

(s3-3)
$$\beta_{ik} = \frac{\gamma_i}{\gamma_j} \beta_{jk}.$$

- Plugging (s3-3) into (s3-1) implies:

(s3-4)
$$\alpha_i(\mathbf{q}) = \frac{\gamma_i}{\gamma_j} \alpha_j(\mathbf{q}),$$

(s3-5)
$$\beta_{ij} = \beta_{ji}.$$

- Expressions (s3-3) and (s3-5) can be combined as follows:

(s3-6)
$$\beta_{ij} = \frac{\gamma_i \gamma_j}{\gamma_k^2} \beta_{kk}, \quad \forall k.$$

- Thus, (s3-4) and (s3-6) are the necessary restrictions for this case.

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (s3-1) simplifies to $s_j = (\beta_{ji} - \beta_{ij})/(\gamma_i y)$, which is inconsistent with the structure of the (s3) model.

The restricted model specification takes the following form:
1. $s_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik}\ln(p_k),\ \ i \in N$ 2. $s_i = \frac{\gamma_i}{\gamma_1}\left\{\alpha_1(\mathbf{q}) + \frac{\beta_{11}}{\gamma_1}\sum_{k\in N}\gamma_k \ln(p_k) + \gamma_1 y\right\},\ \ i \in N$

20. The (s4) Model

Consider the (s4) unrestricted model specification:

(s4) $$s_i = \alpha_i(\mathbf{q}) + \sum_{k=1}^{n} \beta_{ik}\ln(p_k) + \gamma_i \ln(y),\quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(s4-1) $$\frac{y}{p_i p_j}\left\{\beta_{ji} + (s_j + \gamma_j)s_i\right\} = \frac{y}{p_i p_j}\left\{\beta_{ij} + (s_i + \gamma_i)s_j\right\}.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (s4-1) implies:

(s4-2) $$\beta_{ij} = \beta_{ji}.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- The derivative of (s4-1) with respect to p_k, $k = 1,\ldots,N$, implies:

(s4-3) $$\beta_{ik} = \frac{\gamma_i}{\gamma_j}\beta_{jk},\quad \forall k.$$

- Plugging (s4-3) into (s4-1) implies:

(s4-4) $$\alpha_i(\mathbf{q}) = \frac{\gamma_i}{\gamma_j}\left\{\alpha_j(\mathbf{q}) - \frac{\beta_{ji}}{\gamma_i} + \frac{\beta_{ij}}{\gamma_i}\right\}.$$

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (s4-1) simplifies to $s_j = (\beta_{ji} - \beta_{ij})/\gamma_i$, but the structure of (s4) requires that:

(s4-5) $$\beta_{ji} = 0.$$

The restricted model specification takes the following form:
1. $s_i = \alpha_i(\mathbf{q}) + \sum_{k\in N} \beta_{ik}\ln(p_k),\ \ i \in N$ 2. $s_i = \frac{\gamma_i}{\gamma_1}\left\{\frac{\beta_{i1}}{\gamma_i} - \frac{\beta_{1i}}{\gamma_i} + \alpha_1(\mathbf{q}) + \sum_{k\in N}\beta_{1k}\ln(p_k) + \gamma_1\ln(y)\right\},\ \ i \in J$ $s_i = -\beta_{1i}/\gamma_1,\ \ i \in N\sim J$

21. The (s5) Model

Consider the (s5) unrestricted model specification:

$$\text{(s5)} \qquad s_i = \alpha_i(\mathbf{q})\exp\left\{\sum_{k=1}^{n}\beta_{ik}p_k + \gamma_i y\right\}, \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

$$\text{(s5-1)} \qquad \frac{y}{p_i p_j}\left\{\beta_{ji}p_i s_j + (1 + \gamma_j y)s_i s_j\right\} = \frac{y}{p_i p_j}\left\{\beta_{ij}p_j s_i + (1 + \gamma_i y)s_i s_j\right\}.$$

The derivative of (s5-1) with respect to y implies:

$$\text{(s5-2)} \qquad (1/y + \gamma_j)\tilde{S}_{ji} = (1/y + \gamma_i)\tilde{S}_{ij}.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (s5-1) simplifies to $\beta_{ji}p_i s_j = \beta_{ij}p_j s_i$, which holds in general only if:

$$\text{(s5-3)} \qquad \beta_{ij} = \beta_{ji} = 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- Expressions (s5-1) and (s5-2) imply that:

$$\text{(s5-4)} \qquad \gamma_i = \gamma_j.$$

- Given (s5-4), expression (s5-1) simplifies to $\beta_{ji}p_i s_j = \beta_{ij}p_j s_i$. As a result, (s5-3) must also hold.

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (s5-2) requires that $\gamma_i = 0$, a contradiction.

The restricted model specification takes the following form:
1. $s_i = \alpha_i(\mathbf{q})\exp(\beta_{ii}p_i + \gamma_1 y),\ i \in N$

22. The (s6) Model

Consider the (s6) unrestricted model specification:

$$\text{(s6)} \qquad s_i = \alpha_i(\mathbf{q})\exp\left\{\sum_{k=1}^{n}\beta_{ik}p_k\right\}y^{\gamma_i}, \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

$$\text{(s6-1)} \qquad \frac{y}{p_i p_j}\left\{\beta_{ji}p_i s_j + (1 + \gamma_j)s_i s_j\right\} = \frac{y}{p_i p_j}\left\{\beta_{ij}p_j s_i + (1 + \gamma_i)s_i s_j\right\}.$$

This model is observationally equivalent to the (e6) model up to a parametric transformation. See the (e6) model section for the derivation of the necessary parameter restrictions.

23. The (s7) Model

Consider the (s7) unrestricted model specification:

(s7) $$s_i = \alpha_i(\mathbf{q})\left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\}\exp(\gamma_i y), \quad \forall i.$$

The implied Slutsky symmetry conditions for goods i and j ($i \neq j$) are:

(s7-1) $$\frac{y}{p_i p_j}\left\{\beta_{ji}s_j + (1 + \gamma_j y)s_i s_j\right\} = \frac{y}{p_i p_j}\left\{\beta_{ij}s_i + (1 + \gamma_i y)s_i s_j\right\}.$$

The derivative of (s7-1) with respect to y implies:

(s7-2) $$(1/y + \gamma_j)\tilde{S}_{ji} = (1/y + \gamma_i)\tilde{S}_{ij}.$$

CASE I. $\gamma_i = \gamma_j = 0$

- Expression (s7-1) simplifies to $\beta_{ji}s_j = \beta_{ij}s_i$, which in general holds either if:

(s7-3) $$\beta_{ij} = \beta_{ji} = 0,$$

or

(s7-4) $$\beta_{jk} = \beta_{ik}, \quad \forall k,$$

(s7-5) $$\alpha_i(\mathbf{q}) = \frac{\beta_{ii}}{\beta_{jj}}\alpha_j(\mathbf{q}) > 0.$$

CASE II. $\gamma_i \neq 0;\ \gamma_j \neq 0$

- Expression (s7-2) implies:

(s7-6) $$\gamma_i = \gamma_j.$$

- With (s7-6), expression (s7-1) simplifies to $\beta_{ji}s_j = \beta_{ij}s_i$, which implies either (s7-3) or (s7-4) and (s7-5) must also hold.

CASE III. $\gamma_i \neq 0;\ \gamma_j = 0$

- Expression (s7-2) implies $\gamma_i = 0$, a contradiction.

The restricted model specification takes the following form:
1. $s_i = (\beta_{ii}/\beta_{11})\alpha_1(\mathbf{q})\left\{\prod_{k\in J} p_k^{\beta_{kk}}\right\}\exp(\gamma_1 y), \quad i \in J$ $s_i = \alpha_i(\mathbf{q})p_i^{\beta_{ii}}\exp(\gamma_1 y), \quad i \in N\sim J$

24. The (s8) Model

Consider the (s8) unrestricted model specification:

$$s_i = \alpha_i(\mathbf{q})\left\{\prod_{k=1}^{n} p_k^{\beta_{ik}}\right\}y^{\gamma_i}, \quad \forall i. \tag{s8}$$

The implied Slutsky symmetry conditions for goods i and j $(i \neq j)$ are:

$$\frac{y}{p_i p_j}\left\{\beta_{ji}s_j + (1+\gamma_j)s_i s_j\right\} = \frac{y}{p_i p_j}\left\{\beta_{ij}s_i + (1+\gamma_i)s_i s_j\right\}. \tag{s8-1}$$

This model is observationally equivalent to either the (x8) or (e8) model up to a parametric transformation. See LaFrance (1986) for the derivation of the necessary parameter restrictions.