Journal of Agricultural and Resource Economics, 20(1):195–214

The Rhetoric of Duality

Quirino Paris and Michael R. Caputo

Agricultural economists' view of duality has often assumed the characteristics of an ambivalent relation. During the eighties, several authors published papers which put in doubt this or that aspect of duality. This study emphasizes the notion that duality is a time-honored approach suitable for solving problems that can be expressed mathematically. Contrary to many assertions that appeared in the agricultural economics literature, duality does not seem to suffer from any theoretical limitations any more than does the formulation of the primal problem. The article presents two problems that can be solved with dual methods. The authors are incapable of deriving the same results using a primal approach.

Key words: comparative statics, duality, testable hypotheses, symmetry

Introduction

This article is entirely about duality for two main reasons. First, primality represents the direct and natural way to formulate and analyze a problem: it needs no further elaboration as a methodological approach. Second, only recently has duality achieved a mature stage that allows us to dispel residual doubts regarding its applicability in analyzing any economic problem.

A dual specification does not exist independently of the primal formulation, but is in fact fully dependent upon it and conversely so. This assertion is neither obvious nor generally accepted, at least that is the impression one would get from reading some literature that has appeared in agricultural economics journals during the eighties. A primal formulation has the advantage of an immediate and intuitive interpretation, but for uncovering the qualitative structure of complex problems, duality may represent a more analytically convenient vista for some problems. We shall document this assertion by discussing economic problems for which their intrinsic qualitative properties can only be obtained via a dual approach. We have no idea how to achieve the same results using a primal framework.

It appears that agricultural economists need periodic reassurance about the validity and scope of duality in the analysis of economic problems. The vacillating mood of the profession with respect to duality surfaced in a 1982 study by Pope bearing the maverick title "To Dual or Not to Dual?" The opening paragraph of that article defined the tone for many years of doubt about the essential nature of duality (Pope, p. 337):

> Is duality theory a breakthrough of momentous proportions? Does it affect the applied researcher with the same magnitude as the theorist? Is it only a novel approach and hence enhances the chance of academic promotion for the user, or does it exhibit more simplicity and more power than other approaches—say those of Heady and Dillon? Does the dual approach only contribute when one is examining a production or demand system?

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An earlier version of this study was presented as an invited paper at the joint AAEA-WAEA meetings, San Diego, California, 7–10 August 1994. Giannini Foundation Paper No. 1120 (for identification purpose only).

These questions and the tentative answers that followed have imbued the attitude of many agricultural economists and graduate students ever since. To this day, it is not rare to hear questions such as "If a problem can be analyzed from a primal perspective, what is the advantage of using a dual approach? Isn't it true that the primal formulation is more intuitive than the dual one? Is duality 'one of those popular catchwords that often invades an academic discipline'?" (Pope, p. 337)

The theme of our article goes along the following lines: Duality is a logical framework as old as Leonhard Euler and Adrien Marie Legendre who lived about 250 years ago. It deals with the analysis in the parameter space of any problem which can be expressed mathematically. In other words, the parameters of the given problem are treated as if they were decision variables for some economic agent. Why did it take so long for economists to rediscover the elegance and the power of a dual analysis? Up to 1950 (200 years after Euler and Legendre), very few economists used duality explicitly (Antonelli; Hotelling; Court; Roy 1942). The majority of the profession learned how to use this powerful approach confidently only in the last two decades.

The introduction of duality to the general audience of economists (initiated with the works of Samuelson; Shephard; Houthakker 1960; Nerlove; Uzawa; McFadden; Diewert 1973; Jorgenson and Lau) took on, to a large extent, the pedagogical aspect of showing that the same results obtained from the primal side could be obtained also from the dual apparatus. Many of these analyses dealt with relatively simple economic problems, such as the static theory of the consumer and the competitive producer. During this phase, duality uncovered few strikingly new theoretical results; for the most part, it achieved only the reproduction of well-known conclusions albeit in a more elegant form.

If the role of duality were limited to provide more elegant derivations of results that could be obtained from a primal approach, the standoff between supporters of either primality or duality would continue for the foreseeable future. Fortunately, duality has a wider and more relevant scope.

The Scope of Duality

In economic applications, duality refers to the approach that regards the problem under consideration as a function of the parameters rather than of the decision variables. Deaton and Muellbauer (p. 47) assert that "the essential feature of the duality approach is a *change of variables*." This idea was made abundantly clear by René Roy in 1942 when he entitled a crucial section of his work as "The equilibrium equations in tangential coordinates." The change of variables referred to by Deaton and Muellbauer restates the utility problem from point coordinates (quantities) to tangential or plane coordinates (normalized prices). Roy recognizes explicitly that this change of variables constitutes the essence of the "*principe de dualité*"(p. 19). In a similar vein, Silberberg (1990, preface) asserts that duality results are "all derivations or applications of the envelope theorem...," which itself is a modern generalization of the Euler-Legendre transformation.

To give a sense of the general scope of duality, consider figure 1. The set S is not convex nor is its boundary, ∂S, differentiable everywhere. It is still possible, however, to define the smallest convex set that contains S, namely its convex hull *CoS*. Given the set S and a point, say P_1, that does not belong to S (i.e., $P_1 \notin S$), the primal relation between P_1 and S can be specified as the minimum of all possible distances between P_1 and S. In figure 1, the dashed

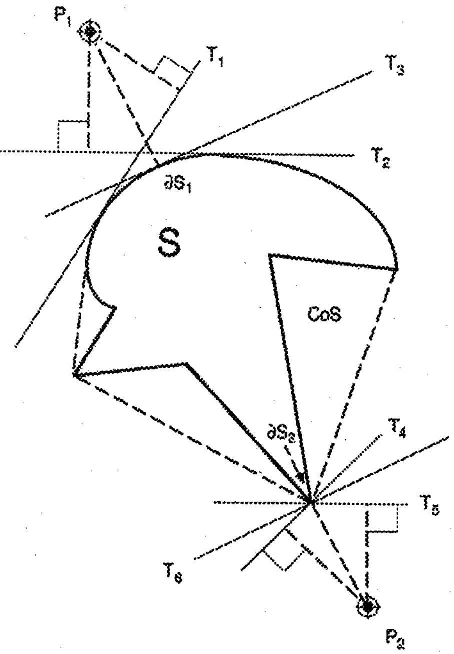

Figure 1. A general representation of duality

line from P_1 to ∂S_1 represents such a minimum distance as it is orthogonal to the tangent hyperplane T_3 at ∂S_1.

Alternatively, the dual relation between P_1 and S is the maximum distance between P_1 and all separating hyperplanes tangent to S. The dual relation between P_1 and S is particularly simple since, in the region facing P_1, the set S is convex and its boundary is differentiable in a neighborhood of ∂S_1. This scenario is not true for ∂S_2. Despite this fact, it is still possible to define a dual relationship between the convex hull of S, *CoS*, and a new point P_2, as figure 1 illustrates. The operations of minimum and maximum distance are the same as those described for point P_1.

It is important to emphasize that the notion of duality is predicated upon separating and supporting hyperplanes to a set and that no mention or use of convexity or differentiability is needed.

Metaphors of Duality

The working of duality may be understood more easily when it is guided by a metaphorical language. In particular, the conceptual view of parameters as decision variables would seem to beg the existence of an economic agent to which the choice of parameters should be ascribed. Observe, in fact, that even before the use of duality became widespread, economists had introduced into their jargon the notions of *invisible hand* and *social planner* which represent phantom economic agents whose domain of operation is the parameter (i.e., price) space.

Such a thought process is equally as relevant for describing economic models in a mathematical programming framework. In order to understand fully the structure of duality as expressed by the dual objective function and the dual constraints, it is convenient to exemplify the problem by assuming a bidding competition between two entrepreneurs: the "primal" owner of the firm's resources whose main objective is profit maximization, and the "dual" economic agent whose objective is that of buying the owner's firm. This framework is valid for behavior of both purely competitive and monopolistic markets.

A second metaphor regards duality as a process of *tunneling through* from the primal space to the dual space. The idea is that, after a period of acquaintance with the structure of duality, a researcher is capable of activating a thought process that (after the specification of the primal problem) leads her directly to think of the dual problem as a convenient framework (often the most convenient) for deriving the qualitative properties of the given model. Under these circumstances, duality becomes a way of thinking, and although each problem continues to present its specific challenges of discovery, the magic of reaching a solution unfolds almost effortlessy with the added confidence of being on the right path.

We will briefly present two examples of tunneling through which deal with rather complex problems which are rarely, if ever, discussed in the literature. The first problem presents empirically verifiable hypotheses for a general specification of price-induced technical progress, in which output and input prices enter into the firm's production function as shift parameters of the technology frontier. The second problem presents the qualitative properties of an adjustment-cost model of the firm with a vector of quasi-fixed factors. In each of these cases, we are incapable of deriving the same results using a primal approach.

Notes of Historical Interest

Apparently, the first mathematician to use the word *duality* was Boole (p. 376) who asserted: "There exists in partial differential equations a remarkable *duality*, in virtue of which each equation stands connected with some other equation of the same order by relations of a perfectly reciprocal character."

In 1886, Antonelli published a monograph that presents a rather comprehensive mathematical treatment of the consumer problem. He derives demand functions as well as price functions for n commodities. In particular, he obtains a utility function that depends on commodity prices and the budget. He, then, derives from it the precise expressions that much later will be called "Roy's identities." He did not mention duality explicitly but he made skillful use of it.

It appears that Hotelling (p. 594) was the first (part-time) economist to utter the word *dually* in the following passage:

> Just as we have a utility (or profit) function u of the quantities consumed whose derivatives are the prices, there is, *dually*, a function of the prices whose derivatives are the quantities consumed. The existence of such a function, which heretofore does not seem to have been noticed, is assured by [the symmetry of a certain matrix]. On the basis of physical analogies we may call this the "price potential."

Among economists (and only among them) this statement has become known as "Hotelling Lemma," a name assigned to it by Diewert (1994) in 1973.

Unaware of Antonelli's contribution, Hotelling simply mentioned the existence of a "price potential" (a dual function of prices), but did not elaborate its properties. The first rigorous and extensive discussion of duality appearing in the Anglo-Saxon economic literature is that of Court, who asserted to have "discovered the intimate relation subsisting between the parent price functions and the utility function" in 1938 (p. 283). This work is cast in a general mathematical context which is suitable also for the treatment of dynamic problems. Hence, the discussion may be taken to be the first rigorous elaboration of dynamic duality appearing in an economic journal. To the economists of the forties, Court's paper must have appeared impenetrable. Indeed, it constitutes a very challenging reading even for the mathematically trained economist of today. The mathematical sophistication used by Court in the treatment of duality explains why such an important paper was and still is ignored.

It is curious (or unfortunate) that Court elected to call the function that expresses utility in terms of prices and income the *inverse utility function* (p. 135) rather than the *dual* utility function. The use of such a name might have unwittingly initiated a period of improper and confusing terminology that has plagued duality ever since.

At about the same time (1942) and apparently in complete independence of Antonelli, Hotelling, and Court, the French economist René Roy rediscovered the duality of the consumer problem discussed originally by Antonelli 55 years before. The worldwide audience of economists became aware of his fundamental contribution through his 1947 paper published in *Econometrica*. Roy appears to have understood from the beginning the theoretical and empirical importance of duality, at least as much as his followers, for René Roy was also an accomplished econometrician.

To better understand the intellectual path that led to the notion of duality, it is instructive to retrace the consumer equilibrium as elaborated by Roy. A market planner, wishing to direct consumers' choices toward a predetermined basket of goods, x^*, wants to find prices and income levels that will induce a representative consumer to purchase the given basket. The main objective of the analysis, therefore, is to derive normalized price functions which are the analogous counterparts to the quantity choice functions in the primal approach. This problem can be stated as minimizing the dual utility function $\Phi(p, r)$ with respect to the price vector p and income r, subject to a linear budget constraint where, now, the vector of commodity quantities x^* is predetermined. More formally,

$$\min_{p,r} \Phi(p,r) \quad \text{subject to} \quad p'x^* - r = 0. \tag{1}$$

It is clear that this problem is formulated in the dual (or parameter) space, as the price vector p and income r are the choice variables in (1). From the associated Lagrangean function, $L(p,r) = \Phi(p,r) + \mu[p'x^* - r]$, the first-order necessary conditions are

$$\frac{\partial L}{\partial p} = \Phi_p + \mu x^* = 0, \text{ and} \tag{2}$$

$$\frac{\partial L}{\partial r} = \Phi_r - \mu = 0, \tag{3}$$

and the budget constraint. At this stage, two developments are possible. The first avenue is to continue pursuing the initial objective of deriving the optimal price functions of the market planner. This goal corresponds to the strategy followed by Roy in his 1942 pamphlet where he restated the first-order necessary conditions as:

$$\frac{\Phi_{p_1}}{x_1^*} = \frac{\Phi_{p_2}}{x_2^*} = \ldots = \frac{\Phi_{p_n}}{x_n^*} = -\Phi_r. \tag{4}$$

The n equilibrium relations stated in (4), in principle, allow the determination of the n prices $p_1, p_2, \ldots, p_n$ and the income r up to a factor of proportionality. This dual operation invokes the implicit function theorem and, in terms of complexity, is no different from the primal objective of determining optimal choice functions for the goods.

The second development corresponds to a rearrangement of the first-order necessary conditions (2) and (3). Under this second viewpoint (which appeared for the first time in Antonelli's work published in 1886), the initial goal of deriving optimal price functions is abandoned in favor of recognizing a convenient way of computing the primal problem's optimal choice functions:

$$x^* = -\frac{\Phi_p}{\Phi_r}. \tag{5}$$

Equations (5) received the name of "Roy's identities" from Houthakker (1994). When referring to *equilibrium* conditions (4) and (5), the term identity is improper. They become identities only when the *optimal* primal solution vector $x^*(p, r)$ is reinserted into (4) and (5). The name of "Roy's identities" stuck nevertheless, revealing the profession's degree of inattentiveness. As it happens oftentimes in history, the first discoverer (Antonelli) was deprived of due recognition. We propose to rename equations (5) as Antonelli-Roy Lemma.

Notice, therefore, that the essence of duality is not to make the derivation of primal choice functions easy but, rather, to carry out the analysis of the given problem in the parameter space to the point of deriving empirically verifiable hypotheses. If the consumer problem were restated using a nonlinear budget constraint, the dual analysis would follow Roy's development in every step thus obtaining optimal price functions. It may not be analytically tractable, under a nonlinear budget, to obtain optimal choice functions by relying on the structure of equations (5). We should not and would not, however, speak of duality's failure because it does not deliver the choice functions by a simple derivative.

Rhetoric of Static Duality

In the preceding section, it was shown that the duality approach to economic analysis began in the thirties and was largely completed during the sixties and seventies. The general economist never questioned or doubted the scope and applicability of the dual approach whether in a theoretical or empirical setting. It is hard to explain, therefore, the flurry of papers that put in doubt this or that aspect of duality that appeared in the agricultural economics literature during the eighties.

Probably the best place to begin the review of this literature is the 1982 WAEA session entitled "Relevance of Duality Theory to the Practicing Agricultural Economist," in particular, the paper "To Dual or Not to Dual" by Pope. Initially we are drawn to the section "Dualities Failings."

Pope (p. 349) asserts that "it seems that duality works poorly when the objective function is nonlinear in the parameters" and points out that the expected utility framework "may not be thoroughly treated by dual methods." We believe that these assertions are misleading. Pope derives his equation system (32) for a profit maximizing firm with a fixed input using the primal-dual methodology of Silberberg (1974) and claims that the restrictions in his equation system (32) are not easily applied. It is important to emphasize that this conclusion does not indicate a failure of duality; in fact, it does not indicate the failure of anything, since the primal methodology leads, in principle, precisely to the same restrictions. It just points out that as economic models become more complex and realistic, the resulting qualitative properties are more elaborate. The fact that his equation system (32) yields restrictions involving technology (or the fact that the analysis of risk problems yields restrictions not independent of preferences) is not a failure of duality but, rather, an indication of the primal-dual methodology's ability to uncover the fundamental qualitative structure of any optimization problem. Moreover, Pope's use of a dual (rather than a primal) methodology to derive his equation system (32) and its qualitative properties attest to the relative ease with which problems that are nonlinear in the parameters are analyzed by a dual approach. Often, in order to obtain qualitative restrictions more readily applicable in empirical analyses, it is necessary to restate the problem in an equivalent but radically different form.We will provide an example of this strategy when developing a test for technical progress further on.

The relevant question then becomes: Can the qualitative restrictions produced by the primal-dual methodology be expressed solely in terms of observable variables and parameters? It is this question that Pope answered in the negative when referring to risk problems. Using a dual methodology, Paris, Caputo, and Holloway have shown, however, that in the context of output and input price uncertainty, the qualitative comparative statics properties of an expected utility maximizing firm in the long-run (the key specification) are contained in a symmetric positive semidefinite Slutsky-type matrix, expressable entirely in terms of observable variables and parameters. In addition, these empirically verifiable restrictions are independent of preferences, just like the archetype certainty theory.

In summary, qualitative results obtained by a primal methodology can also be obtained (in principle, at least) by a dual methodology, and conversely. The dual approach is simply another complementary way of looking at a given economic problem. Sometimes it is easier to use a primal methodology to extract qualitative information from a model, and other times it is easier to use a dual methodology. But a primal solution cannot exist if a dual solution does not and vice versa.

A study by Taylor (1989) is also perplexing because of its emphasis on the "pitfalls of duality." It turns out that all of the pitfalls mentioned in this article (e.g., invalid behavioral hypotheses, incorrect constraints or information sets, wrong economic model) are not pitfalls of duality per se. We believe that the insistence on duality limitations and pitfalls that appeared in the agricultural economics literature of the eighties has been misleading to students and newcomers to the duality fold. It is unfortunate that some researchers have cast blame on a methodology that simply offers an elegant and often unexpected vista of a given problem. These aspects of duality can certainly never be bad, since with duality we get two modes of analysis and two perspectives for any given problem.

Rhetoric of Dynamic Duality

Pope is also quite negative on the application of dual methods to dynamic (or intertemporal) economic problems. Epstein has proven that conditions on the third-order partial derivatives of the optimal value function of a discounted autonomous infinite horizon optimal control problem are required in order to obtain a complete qualitative characterization of dual relationships. Because of this result, Pope (p. 349) asserts that "duality under dynamics is much more cumbersome" than, presumably, under statics. If a more mathematically complex (e.g., dynamic) economic problem is analyzed, one would not expect that a dual view of the problem could be equally as simple as a dual view of a static problem. So, while agreeing with Pope on the additional complexity brought about by the dynamic framework, we remind the reader of a similar complexity which characterizes the primal analysis of the same problem. Hence, we do not see complexity as either a failing of duality or as an unexpected event. Rather, we view the more elaborate implications of dynamic models that are uncovered using a dual methodology as enriching the information set about a given class of problems. Moreover, given that few dynamic problems have explicit solutions, a dual view of them is a must. In fact, the pessimism of Pope conflicts directly with the praise of Epstein (p. 82):

> Explicit solutions for calculus of variations problems are even rarer and the implicit representation of solutions generally involves a second-order nonlinear differential equation (system) and nontrivial boundary conditions. The differential equation system can serve as a basis for estimation only if the generally unrealistic assumption is made that the firm does not revise its plans for several periods and continues along the same optimal path. Thus duality is indispensible for empirical work based on functional forms that are too complicated to be derived directly from the technology as explicit solutions of a problem of intertemporal optimization.

Finally, it is not clear to us how one could establish the curvature properties developed by Epstein from a primal point of view, while a dual perpective leads directly (almost trivially one might add) to the relevant qualitative features of the dynamic model.

In another study by Taylor (1984, p. 352), the point is made that for a competitive firm with Markovian expectations about prices, "a stochastic, dynamic analogue of Hotelling's Lemma does not exists." Taylor's view of Hotelling's Lemma is particularly narrow, as he sees it strictly as a differentiation of the optimal value function (or indirect objective function) with respect to, say, output price in order to recover directly the output supply function. This particular view of Hotelling's Lemma is true only for the static archetype model of the competitive profit maximizing firm. One would naturally expect that for more complex problems, as in dynamic models, the application of the envelope theorem to the problem's value function would not generate the standard static result. Epstein (theorem 2) has shown that in the context of the adjustment-cost model of the firm with static expectations, intertemporal analogues of Hotelling's Lemma do indeed exist and involve first and second partial derivatives of the optimal value function. Moreover, Caputo (1992) has shown that the dynamic envelope theorem recovers directly the cumulative discounted factor demand and output supply functionals (more on this in a later section). The expected additional complexity of the intertemporal Hotelling's Lemma over its static counterpart has not stopped numerous agricultural economists (e.g, Taylor and Monson; Vasada and Chambers; Howard and Shumway; Vasada and Ball) from using the dynamic dual approach for investigating dynamic adjustments in U.S. agriculture. In passing, we remark that the title of the 1989 article by Howard and Shumway, "Nonrobustness of Dynamic Dual Models of the U.S. Dairy Industry," is misleading because the *nonrobustness* referred to in the article has nothing to do with dynamic dual models, but is simply an empirical feature of the functional forms adopted in the analysis.

A Problem Where Duality Really Matters

For many years, duality has been regarded as a second methodology for achieving results that could be achieved easily from a primal perspective. Under this point of view, it is simply a matter of choice for the researcher to adopt a primal or a dual framework. These were the years that allowed some agricultural economists to ask the question "What is there to be gained by adopting a dual approach?"

In this section, therefore, we present a problem solved with a dual approach which we are incapable of solving using a traditional primal formulation. The goal is to discover empirically verifiable hypotheses of a general nature for the price-induced-technical-progress conjecture (Paris). One of the more interesting propositions about technical progress states that relative prices induce shifts in the production frontier because the cheapening of some factors forces the discovery and development of new production techniques. To implement this conjecture in its most general formulation, we discard the assumption of factor augmenting technical progress and incorporate output and input prices directly into the production function where they act as shift parameters of the technological frontier. As indicated below, it becomes evident that a primal approach is not suitable for achieving the desired objective of discovering refutable and observable hypotheses.

Let $y = f(x, p, r)$ be a twice-differentiable production function for a single output y, where x is an n-vector of inputs, r is an n-vector of input prices, and p is output price. The production function is strongly concave in inputs, x, while nothing is assumed with respect to prices. Short-run profit (π) maximization requires that the competitive firm solves

(6) $$\max_{x} \pi = pf(x, p, r) - r'x,$$

where ($'$) denotes transposition. First-order necessary conditions are

(7) $$\frac{\partial \pi}{\partial x} = pf_x(x, p, r) - r = 0,$$

while second-order sufficiency conditions require

(8) $$\frac{\partial^2 \pi}{\partial x \partial x'} = pf_{xx'},$$

a symmetric negative definite matrix.

Under the postulated assumptions, relations (7) can be solved for the vector of input demand functions $x(p, r)$, while the short-run supply function is $y(p, r) = f[x(p, r), p, r]$. These functions are not homogeneous of degree zero in prices because the production function is nonlinear in p and r.

According to the primal approach, comparative statics relations are obtained by inserting $x(p,r)$ into (7) and differentiating with respect to r and p to obtain

(9) $$\frac{\partial x}{\partial r'} = \frac{f_{xx'}^{-1}}{p} - f_{xx'}^{-1} f_{xr'}, \text{ and}$$

(10) $$\frac{\partial x}{\partial p} = -\frac{f_{xx'}^{-1} f_x}{p} - f_{xx'}^{-1} f_{xp}.$$

Similarly, by differentiating the supply function with respect to p and r we find

(11)
$$\frac{\partial y}{\partial p} = f_x' \frac{\partial x}{\partial p} + f_p = -\frac{f_x' f_{xx'}^{-1} f_x}{p} - f_x' f_{xx'}^{-1} f_{xp} + f_p, \text{ and}$$

(12)
$$\frac{\partial y}{\partial r'} = f_x' \frac{\partial x}{\partial r} + f_{r'} = \frac{f_x' f_{xx'}^{-1}}{p} - f_x' f_{xx'}^{-1} f_{xr'} + f_{r'}.$$

None of the derivatives (9), (10), (11), nor (12) are signable because of the presence of cross-derivatives of the production function involving the quantity and the prices of all commodities, a point noted also by Fulginiti. By following a primal perspective, therefore, no testable hypothesis seems readily available for verifying the profit-maximizing behavior of entrepreneurs operating under the stated scenario. It appears, therefore, that because of her adherance to the primal perspective, Fulginiti (p. 165) was lead to the erroneous conclusion that "without placing qualitative restrictions on the latter term [of equation (9) of her paper], we would be unable to deduce qualitative properties of the 'observable relation' on the left side of equation (6)." We show below that no additional restrictions need be imposed on the price-induced technological model to derive the symmetric negative semidefinite matrix which characterizes the fundamental testable implications of the model. This remarkable result is achieved by a dual view of the optimization problem describing price-induced technical progress.

In order to obtain testable hypotheses consistent with the specification of a price-dependent production function it is convenient to change strategy and to adopt a dual perspective. The conclusion of the preceding analysis found that the direct derivatives of the choice functions $y(p, r)$ and $x(p, r)$ (or combinations of them) cannot be signed. The obvious direction, therefore, is the formulation of a problem amenable to a compensation scheme so that the compensated derivatives of the corresponding choice functions can provide the scaffolding for empirically verifiable hypotheses.

One avenue toward the solution of this problem is offered by the Lagrangean tranpose theorem. This theorem is in general overlooked, but on occasion, it can provide a powerful approach for economic problems that otherwise would be intractable.

The Lagrangean transpose theorem is proven by Panik (pp. 207–11).

Lagrange an transposition principle: Maximizing (minimizing) $L(x,\lambda) = f(x) + \lambda\hat{g}(x)$ is equivalent to minimizing (maximizing) $M(x,\mu) = g(x) + \mu\hat{f}(x)$.

In the above principle, if we assume that the function $y = f(x)$ is concave, and $\hat{g}(x) \equiv b - g(x)$, where g is convex, then the maximization and minimization operations are consistent with the specification of the problem's components. Similarly, $\hat{f}(x) \equiv y - f(x)$.

Consider now the following model of the competitive firm under price-induced technical progress:

(13)
$$\max_{y,x} \pi = p'y - r'x$$
$$\text{subject to} \quad F(y,x,p,r) = 0,$$

where y and p are m-vectors of output quantities and output prices, $x^*(p, r)$ is the optimal vector of factor demand functions, and $y^*(p, r)$ is the optimal supply vector. In other words, the above specification generalizes the original problem to handle multiple outputs. Furthermore, the implicit production function F is twice differentiable and convex in (y, x) (Chambers, p. 260). By the Lagrangean transposition principle, the following problem is equivalent to (13):

$$\text{(14)} \qquad \begin{aligned} &\min_{y,x} F(y,x,p,r) \\ &\text{subject to} \quad \pi - p'y + r'x = 0, \end{aligned}$$

where $\hat{x}(p,r,\pi)$ and $\hat{y}(p,r,\pi)$ are the solution to (14). The preference for specification (14) over (13) is due to the presence of a constraint (such as the profit constraint) that is linear in the problem's parameters. The profit constraint constitues the compensating scheme required in order to achieve testable hypotheses, as demonstrated in the ensuing analysis. A natural selection for the benchmark level of profit is the zero profit which characterizes the long-run framework of the competitive industry. Within this context, specification (14) constitutes the problem of the representative entrepreneur.

The dual function of problem (14) is defined as:

$$\text{(15)} \qquad V(p,r,\pi) \equiv \min_{y,x}\{F(y,x,p,r) | \pi - p'y + r'x = 0.$$

The function $V(p, r, \pi)$ must be interpreted as the implicit profit function of the firm operating under a price-induced technical progress as described by problem (14).

An application of the primal-dual methodology (Silberberg 1974) to problem (14) begins with the statement of the primal-dual problem:

$$\text{(16)} \qquad \max L = V(p,r,\pi) - F(y,x,p,r) - \lambda(\pi - p'y + r'x).$$

The relevant first-order necessary conditions (or envelope relations) are

$$\text{(17)} \qquad L_p = V_p - F_p + \lambda y = 0, \text{ and}$$

$$\text{(18)} \qquad L_r = V_r - F_r - \lambda x = 0, \text{ and}$$

$$\text{(19)} \qquad L_\pi = V_\pi - \lambda = 0.$$

Second-order necessary conditions of the above problem can be stated as:

$$\text{(20)} \qquad u'L_{\alpha\alpha'}u \leq 0 \quad \text{for all } u \in \Re^{(m+n+1)},$$

such that $g'_\alpha u = 0$, where $\alpha \equiv (p,r,\pi)$ and $g \equiv \pi - p'y + r'x$.

Finally, all comparative statics relations of problem (14) are contained in the following symmetric negative semidefinite matrix, which is simply a restatement of (20):

$$S = \frac{A'L_{\alpha\alpha'}A}{V_\pi}$$

$$(21)\quad = \frac{1}{V_\pi}\begin{bmatrix} I & 0 & y \\ 0 & I & -x \end{bmatrix}\begin{bmatrix} (V_{pp'} - F_{pp'}) & (V_{pr'} - F_{pr'}) & V_{p\pi} \\ (V_{rp'} - F_{rp'}) & (V_{rr'} - F_{rr'}) & V_{r\pi} \\ V_{\pi p'} & V_{\pi r'} & V_{\pi\pi} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & I \\ y' & -x' \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

$$= \frac{1}{V_\pi}\begin{bmatrix} (V_{pp'} - F_{pp'} + V_{p\pi}y' + yV_{\pi p'} + yy'V_{\pi\pi}) & (V_{pr'} - F_{pr'} - V_{p\pi}x' + yV_{\pi r'} - yx'V_{\pi\pi}) \\ (V_{rp'} - F_{rp'} + V_{r\pi}y' - xV_{\pi p'} - xy'V_{\pi\pi}) & (V_{rr'} - F_{rr'} - V_{r\pi}x' - xV_{\pi r'} + xx'V_{\pi\pi}) \end{bmatrix},$$

where $A' \equiv \begin{bmatrix} I & 0 & y \\ 0 & I & -x \end{bmatrix}$ is a matrix that satisfies the constraint $g'_\alpha u = 0$ of the above second-order necessary conditions. The symmetry and negative semidefinitenes of the matrix S follows from the symmetry and negative semidefiniteness of $L_{\alpha\alpha'}$ and the fact that $A'A$ is symmetric positive semidefinite for any matrix A. Furthermore, $V_\pi > 0$ as it represents the marginal cost of the constraint in problem (14).

In order to attribute economic significance to each element of the matrix S, it is necessary to consider the envelope relations obtained from (17), (18), and (19). Dividing (17) by (19) we obtain the envelope relations for the output supplies in the form of

$$(22)\qquad y = -\frac{(V_p - F_p)}{V_\pi} = \hat{y}(p, r, \pi).$$

These output supply functions are relations specific of a price-induced technical progress as stated in (14). In general, they are not homogeneous of any degree in prices (p, r), and furthermore, they are functions of the profit level π. The lack of homogeneity is attributable directly to the presence of prices in the production function. In general, therefore, relative prices are no longer meaningful.

The compensated derivatives of (22) with respect to p and r produce the desired comparative statics relations associated with the output supply functions:

$$(23)\qquad \left(\frac{\partial\hat{y}}{\partial p'} + \frac{\partial\hat{y}}{\partial\pi}y'\right) - \frac{F_{py'}}{V_\pi}\left(\frac{\partial\hat{y}}{\partial p'} + \frac{\partial\hat{y}}{\partial\pi}y'\right) - \frac{F_{px'}}{V_\pi}\left(\frac{\partial\hat{x}}{\partial p'} + \frac{\partial\hat{x}}{\partial\pi}y'\right) = -s_{11},$$

and

$$(24)\qquad \left(\frac{\partial\hat{y}}{\partial r'} - \frac{\partial\hat{y}}{\partial\pi}x'\right) - \frac{F_{py'}}{V_\pi}\left(\frac{\partial\hat{y}}{\partial r'} - \frac{\partial\hat{y}}{\partial\pi}x'\right) - \frac{F_{px'}}{V_\pi}\left(\frac{\partial\hat{x}}{\partial r'} - \frac{\partial\hat{x}}{\partial\pi}x'\right) = s_{12}.$$

The terms on the left-hand-side of equation (23) constitute a symmetric positive semidefinite matrix since s_{11} is a symmetric negative semidefinite matrix according to (21).

Similarly, dividing (18) by (19) we obtain the envelope relations for the input demands:

(25) $$x = \frac{(V_r - F_r)}{V_\pi} = \hat{x}(p,r,\pi).$$

These input demand functions possess properties similar to those of the output supply functions, and in particular, they are not homogeneous of zero degree in input and output prices (p, r).

Again, the compensated derivatives of (25) produce the comparative statics relations associated with the input demand functions:

(26) $$\left(\frac{\partial \hat{x}}{\partial p'} + \frac{\partial \hat{x}}{\partial \pi} y'\right) + \frac{F_{ry'}}{V_\pi}\left(\frac{\partial \hat{y}}{\partial p'} + \frac{\partial \hat{y}}{\partial \pi} y'\right) + \frac{F_{rx'}}{V_\pi}\left(\frac{\partial \hat{x}}{\partial p'} + \frac{\partial \hat{x}}{\partial \pi} y'\right) = s_{21},$$

and

(27) $$\left(\frac{\partial \hat{x}}{\partial r'} - \frac{\partial \hat{x}}{\partial \pi} x'\right) + \frac{F_{ry'}}{V_\pi}\left(\frac{\partial \hat{y}}{\partial r'} - \frac{\partial \hat{y}}{\partial \pi} x'\right) + \frac{F_{rx'}}{V_\pi}\left(\frac{\partial \hat{x}}{\partial r'} - \frac{\partial \hat{x}}{\partial \pi} x'\right) = s_{22}.$$

The terms on the left-hand-side of equation (27) constitute a symmetric negative semidefinite matrix. Furthermore, by the symmetry of s_{12} and s_{21}, relation (24) is equal to the negative transpose of (26). Relations (23), (24), (26), and (27) constitute the comparative statics of problem (14) and are, in principle, empirically testable. Notice, however, that they contain elements of both the dual (e.g., V_π) and the primal (e.g., $F_{ry'}$) problem. This is a novel result in production economics, although it is not in mathematical programming. The additional complexity of the problem describing a price-induced technical progress, however, does not prevent the derivation of empirically verifiable hypotheses. In this case, the presence of cross partial derivatives of the production functions in all the four relations (23), (24), (26), and (27) requires that the implicit profit function $V(p, r, \pi)$ be estimated concomitantly with the production function $F(y, x, p, r)$.

The interpretation of relations (23), (24), (26), and (27) reveals the double role of prices as scarcity signals and as shift parameters. In the absence of price-induced technical progress, prices function only as a source of scarcity information and the first term of each relation is sufficient for guaranteeing the proper curvature of the envelope relations as well as an unambiguous hypothesis testing framework. When prices enter the production function as shift parameters of the technology frontier, the curvature of the supply and demand functions depends on a contribution of the second cross derivatives of the production function ($F_{py'}$, $F_{ry'}$, $F_{px'}$, $F_{rx'}$) which act as "weights" for a rather complex combination of compensated price slopes of the envelope relations. The second and third terms in (23), (24), (26), and (27) can, therefore, be interpreted as the effects of technical progress induced by price variations which stimulate technological innovations through the relative cheapening of some commodity.

By casting the problem into a primal-*dual* mold, the second-order necessary conditions with respect to prices lead directly to the refutable implications of the price-induced technology model. Given the complexity of the symmetric negative semidefinite matrix composed of submatrices (23), (24), (26), and (27), one must be thankful for the relative ease with which the primal-dual methodology uncovered the fundamental qualitative properties of the model. The triumph of duality in this case lies in the following observation: using the primal methodology it is not clear (*a*) *why* anyone would examine relations such as (23), (24), (26), and (27), as they are nonintuitive generalizations of the prototype Hicksian results; and (*b*) *how* one could prove that S is a symmetric negative semidefinite matrix. A dual view of the same problem has almost reduced these concerns to a triviality.

The Beauty of Duality in Dynamic Problems

Epstein (p. 81) has argued that for dynamic models of firm behavior, a dual approach is superior to a primal approach whenever there are more than two quasi-fixed stocks of resources. This is becasuse the primal approach becomes rapidly intractable in such cases, while the dual approach readily accommodates any number of quasi-fixed stocks. Moreover, the dual approach is capable of encompassing a wider class of adjustment mechanisms than is the primal approach. We will continue this line of argument in the current section by focusing the discussion on the theoretical advantages of duality in continuous-time intertemporal problems.

In static optimization problems there are three methods by which one can determine the qualitative or comparative statics properties of the model's choice functions: (*a*) a primal approach which involves differentiating the first-order necessary conditions of the optimization problem, using the second-order sufficient conditions of the problem, the implicit function theorem, and some solution method for linear equations such as Cramer's rule [the approach was introduced systematically into economics by Samuelson (1947)]; (*b*) a dual approach, whereby one first determines the monotonicity, curvature, and homogeneity properties of the indirect (dual) objective function of the optimization problem, and then uses the envelope theorem to determine the comparative statics properties of the choice functions; and (*c*) a dual approach using the primal-dual methodology of Silberberg (1974) as outlined in the last section, or the gain function methodology of Hatta. All three approaches, in principle, yield the same information, but for any given optimization problem, one method may yield the information more easily than the others.

In continuous-time dynamic optimization problems there are also three methodologies available to the analyst for exacting the qualitative or comparative dynamics properties of the optimal solution paths: (*a*) a primal approach useful for autonomous infinite-horizon optimal control problems that involves a local stability analysis of the steady state, a comparative statics analysis of the steady state, and a local comparative dynamics analysis of the solution to the Taylor series approximation of the canonical differential equations (see, e.g., Caputo 1989); (*b*) a primal approach valid for any class of optimal control problems that involves differentiating the identity form of the canonical differential equations with respect to the parameters of interest, resulting in the so-called variational differential equations (see, e.g., Caputo 1990a); and (*c*) a dual approach using the dynamic primal-dual methodology of Caputo (1990b, c). The two primal approaches, in principle, can be applied to control problems with more than one state variable, but as a practical matter, they are tractable only when a single state variable is present. This is because in order to do a comparative dynamics analysis of the problem, use of a phase diagram becomes important, and with $n \geq 2$ state variables one ends up with a $2n \geq 4$ dimensional phase-diagram, a situation which does not permit graphing. Primal approach (*a*) also suffers in scope because it can be applied only to infinite-horizon control problems where a steady state exists, and in addition, only to control problems that are autonomous in current value form. Its scope, therefore, is much narrower than primal approach (*b*) or the dynamic primal-dual methodology.

The dynamic primal-dual methodology does not suffer from any of the limitations mentioned above and, in fact, is just as easily applicable to optimal control problems with one or $n \geq 2$ state variables. The dynamic primal-dual methodology is *not* a substitute for the two primal approaches, however, as it provides qualitative information about any optimal control model that *complements* information obtained (if any) from a primal investigation.

To solidify the discussion, consider the now renowned adjustment-cost model of the firm:

$$J^*(\beta) \equiv \max_{x(t)} \int_0^{+\infty} [pf(x(t),\dot{x}(t)) - w'x(t) - g'\dot{x}(t)]e^{-rt}dt \quad (28)$$
$$\text{subject to } x(0) = x_0,$$

where f is a generalized production function; $f_x > 0$ signifies the vector of positive marginal products of capital stocks $x(t) \in \Re^n_{++}$; the vector $f_{\dot{x}} < 0$ reflects the adjustment and installation costs (i.e., foregone output) from investing in the capital stock $x(t)$ at the rate $\dot{x}(t) \in \Re^n$; $p > 0$ is the market determined price of output; $w \in \Re^n_{++}$ is the vector of exogenous holding or maintenance costs per unit of capital; $g \in \Re^n_{++}$ is the vector of purchase prices of the investment goods; $r > 0$ is the firm's discounting rate; x_0 is the firm's initial stock of capital goods; and $\beta' \equiv (g', p, w')$ is the time independent vector of prices. The assumption that β is constant over time conforms to the literature on the adjustment-cost model and can be interpreted as saying that the firm operator has perfect foresight and static expectations. The function $J^*(\beta)$ is the maximum (discounted) present value of profits that the firm can earn over the indefinite future, when it begins operating with capital stock x_0, faces prices β, and discounts instantaneous profits at the rate r. Readers are referred to Caputo (1990c, 1992) for the mathematical details of the ensuing analysis.

Because problem (28) has n capital stocks and an unspecified functional form of the generalized production function, it is impossible to solve the system of Euler equations (necessary conditions) for an explicit solution. This obstacle, however, has not prevented researchers such as Treadway, Mortensen, and Brock from investigating the comparative statics properties of the steady state from a primal perspective via the implicit function theorem. Treadway has shown, however, that unless strong assumptions are placed on the production function, no refutable comparative statics properties are forthcoming for the steady state capital stock and investment demand functions.

The dual analysis of problem (28) begins by formulating a dynamic primal-dual problem *viewing the vector of prices* β *as the vector of decision variables*. It then follows from the second-order necessary conditions of the dynamic primal-dual problem that J^* is convex in β (see Caputo 1990c, corollary 1; or Caputo 1992, curvature lemma; or La France and Barney, theorem 2). Moreover, the key to determining what the curvature property of J^* implies for qualitative restrictions on economic behavior comes about rather easily by using the dynamic envelope theorem (see Caputo 1990c, dynamic envelope theorem; or Caputo 1992, dynamic envelope theorem; or La France and Barney, theorem 1), which asserts that, for variational calculus problems, the partial derivative of the optimal value function with respect to a parameter can be found by (*a*) differentiating the integrand of the variational problem with respect to the *explicit* appearance of the parameter prior to optimization (i.e., holding x and $\dot{x}$ constant); (*b*) evaluating the derivative in (*a*) along the optimal solution path for the problem; and (*c*) integrating the result in (*b*) over the relevant planning horizon. The application of the dynamic envelope theorem to J^* yields

(29a) $$\frac{\partial J^*(\beta)}{\partial g} = -\int_0^{+\infty} \dot{z}(t;\beta)e^{-rt}dt(<,>)0,$$

(29b) $$\frac{\partial J^*(\beta)}{\partial p} = \int_0^{+\infty} y(t;\beta)e^{-rt}dt \equiv \int_0^{+\infty} f(z(t;\beta),\dot{z}(t;\beta))e^{-rt}dt > 0, \text{ and}$$

(29c) $$\frac{\partial J^*(\beta)}{\partial w} = -\int_0^{+\infty} z(t;\beta)e^{-rt}dt < 0,$$

where $(z(t;\beta),\dot{z}(t;\beta))$ are the optimal paths of the capital stocks and investment rates, and $y(t;\beta)$ is the optimal path of output supply. The first important feature revealed by the dynamic envelope theorem is that it recovers the cumulative discounted open-loop solution of a decision variable. For example, relation (29b) asserts that the partial derivative of the present value profit function (i.e., optimal value function) with respect to the output price is the cumulative discounted supply function. Relation (29a) asserts that the partial derivative of the present value profit function with respect to the price of the *i*th investment good is the negative of the *i*th cumulative discounted investment demand function. In contrast to the archetypal static profit maximizing firm, where the static envelope theorem recovers the supply and factor demand functions directly, the dynamic envelope theorem applied to the adjustment-cost model of the firm shows that the qualitative properties of the model fall on the shoulders of the cumulative discounted open-loop demand and supply functions.

The envelope results derived by Epstein (theorem 2) differ markedly from (29) and for a good reason: Epstein uses the Hamilton-Jacobi-Bellman partial differential equation for current-value autonomous infinite-horizon control problems as the basis for his duality results, and thus works with the closed-loop form of the decision variables, whereas the primal-dual methodology of Caputo (1990c) employed here works directly off the optimal value function as defined in (28), which implies the open-loop form of the decision variables. The remark by Taylor (1984, p. 351) that "dynamic product supply or factor demands cannot be obtained directly by partial differentiation of an indirect function as can be done in the static, deterministic case," while true, misses the reason for examining a dynamic problem from a dual perspective, which is to uncover the qualitative properties of the model, and not exclusively to recover the primal choice functions by a simple differentiation of the optimal value function.

Given the dynamic envelope results in (29), one may now differentiate them again and recall the convexity of J^* in β to assert that the Hessian matrix:

$$J^*_{\beta\beta'} = \begin{bmatrix} -\frac{\partial}{\partial g}\int_0^{+\infty} \dot{z}(t;\beta)e^{-rt}dt & -\frac{\partial}{\partial p}\int_0^{+\infty} \dot{z}(t;\beta)e^{-rt}dt & -\frac{\partial}{\partial w}\int_0^{+\infty} \dot{z}(t;\beta)e^{-rt}dt \\ \frac{\partial}{\partial g}\int_0^{+\infty} y(t;\beta)e^{-rt}dt & \frac{\partial}{\partial p}\int_0^{+\infty} y(t;\beta)e^{-rt}dt & \frac{\partial}{\partial w}\int_0^{+\infty} y(t;\beta)e^{-rt}dt \\ -\frac{\partial}{\partial g}\int_0^{+\infty} z(t;\beta)e^{-rt}dt & -\frac{\partial}{\partial p}\int_0^{+\infty} z(t;\beta)e^{-rt}dt & -\frac{\partial}{\partial w}\int_0^{+\infty} z(t;\beta)e^{-rt}dt \end{bmatrix}$$

(30)

$$= \begin{bmatrix} -\int_0^{+\infty} \frac{\partial \dot{z}}{\partial g}(t;\beta)e^{-rt}dt & -\int_0^{+\infty} \frac{\partial \dot{z}}{\partial p}(t;\beta)e^{-rt}dt & -\int_0^{+\infty} \frac{\partial \dot{z}}{\partial w}(t;\beta)e^{-rt}dt \\ \int_0^{+\infty} \frac{\partial y}{\partial g}(t;\beta)e^{-rt}dt & \int_0^{+\infty} \frac{\partial y}{\partial p}(t;\beta)e^{-rt}dt & \int_0^{+\infty} \frac{\partial y}{\partial w}(t;\beta)e^{-rt}dt \\ -\int_0^{+\infty} \frac{\partial z}{\partial g}(t;\beta)e^{-rt}dt & -\int_0^{+\infty} \frac{\partial z}{\partial p}(t;\beta)e^{-rt}dt & -\int_0^{+\infty} \frac{\partial z}{\partial w}(t;\beta)e^{-rt}dt \end{bmatrix}$$

is symmetric positive semidefinite. This result is the intertemporal analogue to the convexity of the static dual profit function in output and input prices and its corresponding envelope results. For example, the diagonal elements of $J^*_{\beta\beta'}$ are all nonnegative. Hence, the primal-dual methodology yields the comparative dynamics result:

(31)
$$J^*_{pp}(\beta) = \frac{\partial}{\partial p}\int_0^{+\infty} y(t;\beta)e^{-rt}dt = \int_0^{+\infty} \frac{\partial y}{\partial p}(t;\beta)e^{-rt}dt \geq 0,$$

which asserts that the cumulative discounted supply of the firm's output will not fall when output price rises, or equivalently, that the discounted slope of the firm's supply function is nonnegative when viewed (i.e., integrated) over the firm's planning horizon. Similar results and interpretations apply to the remaining diagonal elements of $J^*_{\beta\beta'}$, the slopes of cumulative discounted capital stock and investment demand functions. Thus, rational dynamic behavior restricts the cumulative discounted demand and supply functions to adhere to the kind of qualitative properties one would expect in static theory when the proper adjustment for the horizon is taken into account. It does not restrict the instantaneous response of dynamic supply and demand functions. Hence observing that $\partial y(t;\beta)/\partial p < 0$ holds at various points in the firm's planning horizon, or even over some finite time period, is perfectly consistent with rational dynamic behavior but refutes the static theory of profit maximization.

Conditions (30) also contain symmetry and reciprocity relations that generate important implications for dynamic behavior. Consider, for example, the reciprocity relation:

(32)
$$J^*_{pg_i}(\beta) = \frac{\partial}{\partial g_i}\int_0^{+\infty} y(t;\beta)e^{-rt}dt = \int_0^{+\infty} \frac{\partial y}{\partial g_i}(t;\beta)e^{-rt}dt$$
$$= -\int_0^{+\infty} \frac{\partial \dot{z}_i}{\partial p}(t;\beta)e^{-rt}dt = -\frac{\partial}{\partial p}\int_0^{+\infty} \dot{z}_i(t;\beta)e^{-rt}dt = J^*_{g_i p}(\beta),$$

which is the dynamic counterpart to the prototype Hicksian reciprocity relation between factor demand and output supply functions in the static profit maximization model. The important aspect revealed by (32) is that the symmetry properties of the model are embodied in the cumulative discounted demand and supply functions, not in the demand and supply functions at each point in time of the planning horizon. Moreover, since $J^*(\beta)$ is positively homogeneous of degree one in the prices β, by the derivative theorem of homogeneous functions, the cumulative discounted demand and supply functions in (29) are homogeneous of degree zero in prices. This point further solidifies our assertion that for the adjustment cost model of the firm, it is the cumulative discounted demand and supply functions that possess the qualitative properties that are analogous to those of the static profit maximizing model of the firm.

In closing out this section, we are brought back to our earlier assertion that for more complicated economic models a dual view of the problem is a necessary strategy if one desires to obtain qualitative characterizations as a basis for deriving empirically verifiable hypotheses. For the adjustment-cost model a dual view of the problem has led to a quite unexpected and novel set of qualitative properties. It is not clear to us how one could reach the same conclusions by relying exclusively on a primal view of the problem. In fact, given that the adjustment-cost model has been in existence for over 30 years and no researcher has derived the integral form of the qualitative restrictions presented above using a primal methodology, we can safely conclude that it is extremely difficult to do so from a primal perspective. The results of Treadway, Mortensen, and Brock, which were obtained by primal methods and focused almost exclusively on the steady state, rely on strong assumptions imposed on the production function and, moreover, are valid only for the infinite-horizon variant of the adjustment-cost model, as are the duality results of Epstein. In contrast, all the results given in Caputo (1992) and summarized here hold for infinite- and finite-horizon versions of the adjustment-cost model.

We must emphasize that a dual view of any dynamic problem complements any qualitative information extracted from a primal analysis and cannot be regarded as a substitute for it. Nonetheless, it is true that optimal control problems with two or more state variables become cumbersome to analyze from a primal perspective, while a dual view of the problem is just as easy with one or many state variables. Finally, the empirical relevance of multiple stock control problems requires a dual view of such problems, as Epstein noted in 1981.

Conclusion

The advent of duality in economic analysis has brought a better awareness of the properties which characterize economic problems and of the strategy to obtain them. In particular, the duality approach has allowed a fruitful analysis of economic problems of increasing complexity which researchers could never dream to tackle using exclusively a primal perspective. In support of this assertion we have presented two economic models, a static model of a profit maximizing firm operating under the influence of price-induced technical progress and a dynamic model of a wealth maximizing firm facing adjustment costs of its capital stocks. In both cases, a primal view of the problem fell far short of what we would consider a thorough qualitative analysis of the model. After shifting the perspective from the quantity to the price space, however, we were able to achieve a rather thorough qualitative analysis of both models with relative ease. Moreover, the primal-dual methodology, whether static or dynamic, was the vehicle by which the models' properties were revealed. It has been our experience that the primal-dual methodology is unrelenting in its ability to uncover the fundamental qualitative structure of any *optimization* model, and because it contains both the primal *and* dual optimization problems as a special case (hence its name), we cannot imagine investigating any economic model for its qualitative properties without employing this methodology.

The scope, power, and insight of duality are fully revealed when the objective function and constraints of static and dynamic optimization problems are nonlinear in the parameters. Moreover, duality has no known theoretical limitations. Almost invariably, its confident use reveals unexpected but elegant results that serve as a scaffolding for empirically verifiable hypotheses of great generality.

[*Received August 1994; final version received May 1995.*]

References

Antonelli, G. B. *Sulla Teoria Matematica della Economia Politica.* Monograph published in "Nella Tipografia del Folchetto," Pisa, Italy, 1886. Translated by J. S. Chipman in *Preferences, Utility, and Demand*, eds. J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, pp. 333–64. New York: Harcourt Brace Jovanovich, Inc. 1971.

Boole, G. *A Treatise on Differential Equations*, 5th ed., New York: Chelsea Publishing Co., 1859.

Brock, W. A. "A Revised Version of Samuelson's Correspondence Principle: Applications of Recent Results on the Asymtopic Stability of Optimal Control to the Problem of Comparing Long-Run Equilibria." In *Models of Economic Dynamics*, ed., Hugo Sonnenschein. Berlin: Springer Verlag, 1986.

Caputo, M. R. "The Qualitative Content of Renewable Resource Models." *Natural Resource Modeling* 3(1989):241–59.

———. "A Qualitative Characterization of the Competitive Nonrenewable Resource Extracting Firm." *J. Environ. Econ. and Manage.* 18(1990a):206–26.

———. "Comparative Dynamics via Envelope Methods in Variational Calculus." *Rev. Econ. Stud.* 57(1990b):689-97.

———. "How to Do Comparative Dynamics on the Back of an Envelope in Optimal Control Theory." *J. Econ. Dynamics and Control* 14(1990c):655–83.

———. "Fundamental Symmetries and Qualitative Properties in the Adjustment Cost Model of the Firm." *J. Mathematical Economics* 21(1992):99–112.

Chambers, R. G. *Applied Production Analysis.* Cambridge: Cambridge University Press, 1988.

Court, L. M. "Entrepreneurial and Consumer Demand Theories for Commodity Spectra." *Econometrica* Part I 9(1941):135–62; Part II 9(1941):241–97.

Deaton, A., and J. Muellbauer. *Economics and Consumer Behavior.* New York: Cambridge University Press, 1980.

Diewert, W.E. "Functional Forms for Profit and Transformation Functions." *J. Econ. Theory* 6(1973):284–316

———. Personal communication, 1994.

Epstein, L. G. "Duality Theory and Functional Forms for Dynamic Factor Demands." *Rev. Econ. Stud.* 48(1981):81–95.

Fulginiti, L. E. "Price-Conditional Technology." *J. Agr. Resour. Econ.* 19(1994):161–72.

Hatta, T. "Structure of the Correspondence Principle at an Extremum Point." *Rev. Econ. Stud.* 47(1980):987–97.

Hotelling, H. "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions." *J. Polit. Econ.* 40(1932):557–616.

Howard, W. H., and C. R. Shumway. "Dynamic Adjustment in the U.S. Dairy Industry." *Amer. J. Agr. Econ.* 70(1988):837–47.

———. "Nonrobustness of Dynamic Dual Models of the U.S. Dairy Industry." *Northeast. J. Agr. and Resour. Econ.* 18(1989):18–25.

Houthakker, H. S. "Additive Preferences." *Econometrica* 28(1960):244–57.

———. Personal communication, 1994.

Jorgenson, D. W. and L. J. Lau. "Duality and Differentiablity in Production." *J. Econ. Theory* 9(1974):23–42.

La France, J. T., and L. D. Barney, "The Envelope Theorem in Dynamic Optimization." *J. Econ. Dynamics and Control* 15(1991):355–85.
McFadden, D. L. "Cost, Revenue, and Profit Functions: A Cursory Review." Work. Pap. No. 86, Institute of Business and Economic Research, University of California, Berkeley, 1966.
Mortensen, D. T. "Generalized Costs of Adjustment and Dynamic Factor Demand Theory." *Econometrica* 41(1974):657–65.
Nerlove, M. "Returns to Scale in Electricity." In *Measurement in Economics*, ed., C. F. Christ, pp. 167–98. Stanford: Stanford University Press, 1963.
Panik, M. J. *Classical Optimization: Foundations and Extensions.* New York: North-Holland Publishing Company, 1976.
Paris, Q. "Testable Hypotheses of Technical Progress." Paper presented at the VII European Congress of Agricultural Economists, Stresa, Italy, 5–10 September, 1993.
———. "Keeping the Dream of Rigorous Hypothesis Testing Alive." *Amer. J. Agr. Econ.* 75th Anniversary Issue (1993):25–40.
Pope, R. D. "To Dual or Not to Dual." *West. J. Agr. Econ.* 7(1982):337–51.
Roy, R. "*De L'Utilité. Contribution a la Théorie des Choix.*" Paris: Hermann and Co., 1942.
———. "*La Distribution du Revenu Entre les Divers Biens.*" *Econometrica* 15(1947):205–25.
Samuelson, P. A. *Foundations of Economic Analysis.* Cambridge MA: Harvard University Press, 1947.
———."Prices of Factors and Goods in General Equilibrium." *Rev. Econ. Stud.* 21(1953):1–20.
Shephard, R. W. "Cost and Production Functions." Princeton: Princeton University Press, 1953.
Silberberg, E. "A Revision of Comparative Static Methodology in Economics, or, How to do Comparative Statics on the Back of an Envelope." *J. Econ. Theory* 7(1974):159–72.
———. *The Structure of Economics: A Mathematical Analysis*, 2nd. ed., New York: McGraw-Hill Publishing Company, 1990.
Taylor, C. R. "Stochastic Dynamic Duality: Theory and Empirical Applicability." *Amer. J. Agr. Econ.* 66(1984):351–57.
———. "Duality, Optimization, and Microeconomic Theory: Pitfalls for the Applied Researcher." *West. J. Agr. Econ.* 14(1989):200–12.
Taylor, T. G., and M. T. Monson. "Dynamic Factor Demands for Aggregate Southeastern United States Agriculture." *S. J. Agr. Econ.* 17(1985):1–9.
Treadway, A. B. "The Rational Multivariate Flexible Accelerator." *Econometrica* 39 (1971):845–55.
Uzawa, H. "Duality Principles in the Theory of Cost and Production." *Internat. Econ. Rev.* 5(1964):216–20.
Vasavada, U., and R. G. Chambers. "Investment in U.S. Agriculture." *Amer. J. Agr. Econ.* 68(1986):950–60.
Vasavada, U., and V. E. Ball. "A Dynamic Adjustment Model for U.S. Agriculture: 1948–79." *Agr. Econ.* 2(1988):123–37.