A STOCHASTIC DYNAMIC PROGRAMMING ANALYSIS OF FARMLAND INVESTMENT AND FINANCIAL MANAGEMENT

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HEMAN DAS LOHANO

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ROBERT P. KING, ADVISER

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UMI Number: 3041944

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

UNIVERSITY OF MINNESOTA

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Heman Das Lohano

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Robert P. King

(Name of Faculty Adviser)

Robert P. King

(Signature of Faculty Adviser)

April 23, 2002

(Date)

GRADUATE SCHOOL

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To the Memory of
My Mother

Abstract

This dissertation investigates farm firm growth using a multiperiod investment portfolio problem that includes farmland, nonfarm assets, and debt financing on farmland. The investment portfolio problem is formulated as a stochastic continuous-state dynamic programming model.

Since this dynamic programming model lacks a closed-form solution, it is solved numerically with collocation methods. We develop a test for checking the accuracy of a stochastic continuous-state dynamic programming model. Using this accuracy test, we examine the accuracy of the solution for the investment portfolio problem. We compare the accuracy of collocation methods with Chebyshev and linear spline interpolations. We propose techniques for improving the accuracy and efficiency in solving a large-scale dynamic programming model.

We solve the investment portfolio problem that includes risky farmland and a riskless nonfarm asset or debt financing on farmland in the presence of transaction costs, credit constraints, stochastic land prices and farm returns. We explore how the optimal portfolio is adjusted in a dynamic and stochastic environment. Results show that the optimal portfolio depends on farm returns, farmland price, and liquid assets. We explore the effect of initial farm size, initial wealth levels, length of the planning horizon, interest rate, riskiness of returns, and risk aversion. Observed risk avoiding behavior in investment decisions is often attributed to risk aversion. We find that risk avoiding behavior in investment decisions can also be attributed to the length of

the decision maker's planning horizon. Also, unlike in a static model, changes in the riskiness of returns affect the optimal portfolio in the dynamic model, even when the decision maker is risk neutral.

The above portfolio problem is extended by adding a risky mutual fund investment. We find that it is optimal for farmers to include the mutual fund in the portfolio along with farmland investment. Furthermore, higher debt financing on farmland is optimal when mutual fund investment is included in the model. Finally, we find that the probability of exiting farming increases in the model with the mutual fund investment opportunity.

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Chapter 1

Introduction

In the process of economic development and growth, it is common for the number of farms to decline and the average farm size to increase (Upton and Haworth, p. 1). During the past four decades, for example, the agriculture sector of the United States has experienced a 50 percent decline in the number of farms, while the average farm size has doubled from about 200 acres to more than 400 acres per farm (Penson, Capps, and Rosson, p. 26).

This process of structural change in agriculture has motivated a number of farm firm growth[1] studies that focus on farmland purchase and sale decisions to explain changes in farm size from the individual farm standpoint (Minden, p. 38). In addition to concern about explaining the changing structure of agriculture, farm firm growth analysis addresses risk management issues of farmers' investment strategies (Baker)

[1]Farm firm growth theories are reviewed in Renborg.

and farm and lending policy issues relating to financial stress and farm firm survival (Boehlje and Eidman).

Farmland is a risky investment subject to numerous uncertainties. In addition to uncertainty of farm returns resulting from yield risks and output and input price risks, farmers also face farmland value risks. Farmland investments are usually financed with a combination of equity and debt capital. Through leverage, the use of debt capital makes it possible for farmers to finance additional farmland investment. However, using more debt capital also increases financial risk and so can have a long-term effect on the farm's financial well-being. Jensen and Langemeier (p. 85) note that during periods of financial stress, as in early 1980's, farms with high debt to asset ratios face a higher risk of failure. Since farmland is collateral for debt capital, low farmland values decrease the availability of debt capital. In periods of low farm income, with the associated decrease in land values, farmers are unable to meet planned cash flow commitments, which results in financial stress (Brake). Have the farmers who experience financial stress taken on too much financial risk? How does risk aversion affect investment and financial strategies? Collins and Karp note that optimal choice of leverage is also affected by farmer's age. What is the effect of the length of planning horizon on farmland investment strategies?

The Federal Agriculture Improvement Reform (FAIR) Act 1996 has changed farm policy, removing support prices and acreage restrictions. "Producers will ... bear greater income risk because payments are fixed and not related to the level of market prices" (Young and Shields, p. 97). How does this increase in riskiness affect farmers' investment and financing strategies?

Farmland investment and financing decisions are multiperiod in nature, and farmers make these decisions subject to relevant constraints in each time period. This suggests that consideration should be given to the relationship present and future decisions have with present and future outcomes. The time relationship of decisions and outcomes can be accommodated by a sequential decision model in a dynamic programming framework[2] (Minden, p. 39; Puterman, p. 1).

The literature on investment under uncertainty (e.g. Dixit and Pindyck) has emphasized that the firms must consider future decisions and uncertainty when making current decisions. The aspect of future decisions is viewed as an opportunity, and firms have option to wait before taking actions that may be difficult to reverse. Optimal decisions considering the option to wait and future decisions can be obtained using dynamic programming (Dixit and Pindyck, p. 93). In farmland investment decisions, the manager can also choose no purchase or sale besides the choice of buying or selling land.

Relatively few studies have analyzed the farm firm growth problem of farmland investment decision in a stochastic and dynamic framework. Minden formulated a conceptual farm firm growth model in a dynamic programming framework. He noted (p. 44) that dynamic programming does not require any particular functional form

[2]Dynamic programming is an approach for solving sequential decision models using Bellman's equation. Chapter 3 presents a brief overview on dynamic programming.

for the objective function and constraints, and thus is flexible for incorporating real world farm situations such as lumpiness of certain inputs.

Larson, Stauber, and Burt developed an empirical farm growth model for farmland investment and financial decisions in a stochastic dynamic programming framework. Their model incorporated transaction costs associated with selling land and uncertainty of crop yield. The values of farmland, buildings, and equipment were deterministic in their model. They solved the model numerically for wheat farms in Montana and found that the firm's survivability and failure depend on the initial debt to asset ratio, risk aversion, and the cost of investment including land price and interest rates.

The stochastic nature of farm returns and land values and their relationship have been characterized in many studies. Burt and other studies (e.g. Featherstone and Baker) have investigated the stochastic and dynamic nature of farmland returns and the linkages between farmland returns and land values. Motivated by these studies, Schnitkey, Taylor, and Barry extended the model of Larson, Stauber, and Burt by incorporating stochastic dynamic farm returns, and dynamic land prices that are related to farm returns. Schnitkey, Taylor, and Barry estimated the relationship between land price and farm returns but treated this as deterministic in their dynamic programming model. However, when land values are assumed to be deterministic, a large component of risk is not considered in evaluating the survivability of farms and the profitability of farmland purchase and sale decisions. Land value risk is particularly significant in investment decisions involving collateral risks and credit reserve risks, since land value serves as collateral in debt financing (Featherstone, Preckel, and Baker, p. 81). Incorporating land value risk is also important for full consideration of capital gain or loss risks.

Many studies (e.g. Pederson and Brake) have recognized that lenders allow a proportion, say 70 percent, of debt capital for financing farmland investment. Incorporating this constraint may have a significant impact on farmland investment and financial decisions. Also, for policy evaluation, it is important to ask how a change in interest rate affects farm survival and growth.

Farmers also have nonfarm investment opportunities, such as stocks, bonds, and mutual funds. Given numerous uncertainties in farm returns, it is important to ask what is the optimal mix of farm and non farm investment and what are the factors that influence investment portfolio adjustments.

Adding nonfarm investments to a farm growth dynamic programming model expands the size of the problem and poses technical challenges. In the case of continuous state and control variables, such as prices, returns, wealth, assets, the dynamic programming model may be solved by discretization methods, as used in the above studies[3]. Under this method, continuous state and control variables are discretized in finite sets, and the value function is solved and stored for each element in the state set. When solving the problem requires the value function at points other than state

[3]Other relevant studies of discrete state dynamic programming applications are Novak and Schnitkey (1993, 1994) and Schnitkey and Novak. They studied investment decisions applied to hog finishing barns with nonfarm assets.

set, it is approximated. This method is reliable if the discretization is sufficiently fine. In large-scale problems, however, discretization with many points is not possible due to limitations of computer storage capacity, and thus it may give poor approximation.

An alternative is a parametric approach, such as interpolation with polynomials or spline functions, for approximating the value function due to Bellman and Dreyfus and Bellman, Kalaba, and Kotkin. This approach is flexible, accurate, and numerically efficient (Miranda and Fackler). The model of the present study has four state variables of which two variables are stochastic. In the absence of a closed-form solution, it is essential to examine the accuracy of the numerical approximation of the solution to the model, especially in large-scale problems. Given the limitations of computer storage capacity, there is also a need for improving accuracy and efficiency in computing the solution for large-scale multidimensional dynamic programming models.

The present study explores a multiperiod portfolio problem in a dynamic programming framework, focusing on farmland investment decisions in the presence of transaction costs, credit constraints, stochastic farmland prices and farm returns, and risky and risk free nonfarm investment opportunities. Using this problem, this study also examines collocation methods for the reduction of computation effort in solving a large-scale dynamic programming model. The specific objectives of this study are:

1. To develop a dynamic programming model for farmland investment and financial planning.

2. To develop a test for accuracy of numerical solution of a continuous-state dynamic programming model.

3. To evaluate accuracy of alternative numerical methods, and to propose techniques for improving efficiency and accuracy in solving a large-scale dynamic programming model.

4. To identify optimal dynamic investment and financial strategies for corn and soybean producers of Southwestern Minnesota farmers.

5. To analyze the effect of initial farm size, available liquid assets, risk preferences, length of the planning horizon, interest rate and riskiness of farm returns on the farmland investment decisions.

6. To explore the effect of nonfarm investments on farmland investment decisions and farm growth.

Numerically solving a large-scale stochastic dynamic programming model involves technical challenges due to the stochastic nature of variables. The dynamic programming algorithm requires optimization and integration. State variables may go out of bounds. The utility function needs to be defined for all values of states. We address these issues for a continuous-state dynamic programming model using the investment problem developed for this study.

The remainder of this dissertation is organized as follows. Chapter 2 specifies the model for this study and defines the variables of the model. Chapter 3 provides a general overview of the theory of dynamic programming and numerical methods for solving dynamic programming problems. Econometric methods for estimating the stochastic state equations, data, estimation results, and other parameters of the model are presented in Chapter 4. Chapter 5 presents model implementation and accuracy analysis. Chapter 6 presents results for a base model that considers farmland investment in a context with limited nonfarm investment opportunities. Chapter 7 extends the model to include risky nonfarm investment and presents the results. Finally, Chapter 8 summarizes the results and draws conclusions.

Chapter 2

The Model

This chapter presents a model of farmland investment and financial planning with a single risk free opportunity for nonfarm investment. Chapter 7 presents an extended version of the model that also considers a risky alternative for nonfarm investment.

The model of farmland investment and financial planning is formulated in a stochastic dynamic programming framework and is an extension of models developed by Larson, Stauber, and Burt, and by Schnitkey, Taylor, and Barry. In the following sections, we first describe the economic environment and notation and then define the farmland investment problem. The choice of objective function and utility function is also discussed.

2.1 Economic Environment and Notation

At the beginning of each year t, a farm manager has farmland acreage L_t and liquid assets A_t. A negative value of A_t represents a net debt position. A positive value of A_t represents a riskless nonfarm asset. At the beginning of each year t, the manager can purchase or sell land, x_t:

$$L_{t+1} = L_t + x_t.$$

Farmland price per acre is denoted by P_t. Farming requires machinery and equipment. The price of machinery and equipment per acre is denoted by κ. There is a transaction cost per acre of tc_p on buying land, and no transaction cost on buying machinery and equipment. There is a transaction cost per acre on selling land and machinery and equipment, and the sum of these transaction costs per acre is denoted by tc_s. These expenses depend on x_t, and are equal to $(P_t + \kappa + tc) * x_t$, where $tc = tc_p$ if $x_t > 0$, and $tc = -tc_s$ if $x_t < 0$.

The purchase or sale decision for farmland is made in the beginning of each year t, so total land available for farming is $L_{t+1} = L_t + x_t$. The costs of production per acre are denoted by c, and the total costs of production are equal to $c * L_{t+1}$.

The above expenses are financed from liquid assets, A_t, and determine the net liquid assets after investment and production expenses. If the net liquid assets are negative, then the financing interest rate is r_b (borrowing rate). If the net liquid assets are positive, then these assets are invested at riskless interest rate r_l (lending rate). The farm manager faces a difference between the borrowing and lending rates such that $r_l < r_b$ (Hirshleifer, p. 196).

We define a composite crop representing a combination of crops produced for sale. Gross return per acre from the composite crop is denoted by R_t. Though total available land after making a decision about x_t, $L_{t+1} = L_t + x_t$, is used for producing the composite crop, the gross return is not realized until the beginning of the next year. Thus, gross revenue is equal to $R_{t+1} * L_{t+1}$, and is added into liquid assets in the beginning of the next year. Considering the above expenses and returns, the state transition equation of liquid assets is:

$$A_{t+1} = (1 + r)\,[A_t - (P_t + \kappa + tc) * x_t - c * L_{t+1}] + R_{t+1} * L_{t+1}, \tag{2.1}$$

where[1]

$$r = \begin{cases} r_b \text{ if } [A_t - (P_t + \kappa + tc) * x_t - c * L_{t+1}] > 0 \\ r_l \text{ otherwise,} \end{cases}$$

$$tc = \begin{cases} tc_p \text{ if } x_t > 0 \\ -tc_s \text{ otherwise.} \end{cases}$$

There are constraints on farmland purchase and sale decisions. The feasibility constraint is $\underline{L} \leq L_t \leq \overline{L}$ or $L_t = 0$, where $\underline{L}$ denotes minimum acres required for farming and $\overline{L}$ denotes maximum feasible acres the manager can own in the location.[2] For period $t + 1$, we have $\underline{L} \leq L_{t+1} \leq \overline{L}$ or $L_{t+1} = 0$, which defines the following

[1] The expression in brackets is net liquid assets after investment and production expenses at beginning of time t. As described above, the sign of net liquid assets determines borrowing or lending.

[2] The bounds on state variables are also required for numerically solving a dynamic programming problem and are specified in Chapter 5 for this model.

constraint on x_t:

$$\underline{L} - L_t \leq x_t \leq \overline{L} - L_t \text{ or } x_t = -L_t. \tag{2.2}$$

Another constraint is the bankruptcy condition. We define net wealth in period t, W_t, as the net value of all assets after they are sold:

$$W_t \equiv A_t + (P_t + \kappa - tc_s) * L_t$$

Under the bankruptcy condition, the farm is liquidated if net wealth is negative at any time:

$$x_t = -L_t \text{ if } W_t < 0 \tag{2.3}$$

Note that when $W_t \geq 0$, the manager can potentially choose to sell all land as given in equation (2.2). We assume that once all land is sold, however, the business cannot re-enter farming:

$$\text{if } x_t = -L_t, \text{ then } x_{t+1} = x_{t+2} = ... = x_T = 0 \tag{2.4}$$

Another constraint is on loans provided by the lender. This constraint allows purchase of land and the associated machinery and equipment as long as the debt to asset ratio is less than or equal to ρ, where $0 < \rho < 1$:

$$\text{if } x_t > 0, \text{ then } \frac{-A_t + (P_t + \kappa + tc_p) * x_t}{(P_t + \kappa - tc_s) * (L_t + x_t)} \leq \rho$$

This constraint can also be written as:

$$x_t \leq \max\left(0, \frac{A_t + \rho * (P_t + \kappa - tc_s) * L_t}{(P_t + \kappa + tc_p) - \rho * (P_t + \kappa - tc_s)}\right) \tag{2.5}$$

The above constraints, (2.2)-(2.5), limit the choice of control variable x_t. For notational convenience we denote X_t as the set of all levels of control variables that satisfy these constraints. Note that control space X_t is a function of variables P_t, L_t, A_t, and constants $\rho, \kappa, tc_p, tc_s, \underline{L}, \overline{L}$. Then, these constraints are represented by

$$x_t \in X_t \tag{2.6}$$

The farm manager faces uncertainty in the gross returns per acre, R_t, and farmland price per acre, P_t. R_t and P_t are assumed to follow Markov processes, which are given in the specification of the investment problem in the next section.

2.2 Farmland Investment Problem

The farm manager's objective is to choose an optimal policy $\{x_t^*\}_{t=0}^T$ that maximizes the expected utility of net terminal wealth subject to relevant constraints[3]:

$$\max_{\{x_t\}_{t=0}^T} E_0\left[U\left(W_{T+1}\right)\right] \tag{2.7}$$

subject to :

$$\begin{aligned} R_{t+1} &= g\left(R_t, \varepsilon_{1,t+1}\right), \\ P_{t+1} &= h\left(P_t, R_t, \varepsilon_{2,t+1}\right), \\ L_{t+1} &= L_t + x_t, \end{aligned}$$

[3]This is a preliminary specification of state equations of P_t and R_t. Their final specification, functional form, and estimation are described in Chapter 4. $t+1$ is time index used as a subscript in all variables including $\varepsilon_{1,t+1}$, where ε_1 is random shock in state equation of R_t.

$$W_{t+1} = (1+r)\left[\{W_t - (P_t+\kappa - tc_s)*L_t\} - (P_t+\kappa+tc)*x_t - c*L_{t+1}\right]$$
$$+R_{t+1}*L_{t+1} + (P_{t+1}+\kappa - tc_s)*L_{t+1},$$
$$x_t \in X_t, \text{ for } t=0,1,2,...,T, \ (R_0,P_0,L_0,W_0) \text{ are given},$$

where

$$r = \begin{cases} r_b \text{ if } \left[\{W_t - (P_t+\kappa - tc_s)*L_t\} - (P_t+\kappa+tc)*x_t - c*L_{t+1}\right] > 0 \\ r_l \text{ otherwise,} \end{cases}$$

$$tc = \begin{cases} tc_p \ \text{ if } x_t > 0 \\ -tc_s \text{ otherwise,} \end{cases}$$

and $0 < \rho < 1$.[4]

E_0 is expectation operator over the random shocks $\varepsilon_{1,t+1}, \varepsilon_{2,t+1}$, and U is the utility function.

In a dynamic programming context, the above investment problem has four state transition equations for state variables R_t, P_t, L_t, W_t, and one control variable x_t. In the economic environment described above, we note from the identity $W_t \equiv A_t + (P_t+\kappa - tc_s)*L_t$ that for a given state $(P_t+\kappa-tc_s)*L_t$, each level of A_t determines W_t, or each level of W_t determines A_t. Thus, the dynamic programming model can be formulated with either A_t or W_t as a state variable, in addition to the other state variables R_t, P_t, L_t. For period $t+1$, the identity of net wealth is $W_{t+1} \equiv A_{t+1} + (P_{t+1}+\kappa - tc_s)*L_{t+1}$, in which we substitute (2.1) for A_{t+1}. In the result we substitute $A_t \equiv W_t - (P_t+\kappa-tc_s)*L_t$, which gives the state equation for W_t

[4]The control space, X_t, is a function of state variables P_t, L_t, W_t and other constants including ρ, as defined in (2.6).

specified in the above problem. In addition, constraint (2.5) can be rewritten as:

$$x_t \leq \max\left(0, \frac{W_t - (1-\rho)*(P_t+\kappa-tc_s)*L_t}{(P_t+\kappa+tc_p)-\rho*(P_t+\kappa-tc_s)}\right)$$

R_t and P_t are stochastic state variables, because the next period's state is affected by random shocks. Net wealth, W_t, is also stochastic because the next period's state, W_{t+1}, depends on R_{t+1} and P_{t+1}. Riskiness of net wealth has important significance in the decision making process. It not only creates objective function risks, but it also affects bankruptcy risks, as given in (2.3).

The optimal policy $\{x_t^*\}_{t=0}^T$ is not a simple sequence of numbers because at least one variable is stochastic in the model. Rather, given (R_0, P_0, L_0, W_0), the farm manager chooses x_0^* and makes contingency plans x_t^* for periods $t = 1, 2, ..., T$. These contingency plans depend on (R_t, P_t, L_t, W_t), which will be known only after the realization of the shocks in period t, $\varepsilon_{1,t}, \varepsilon_{2,t}$.

In the above model, all variables, including farmland price, returns, costs, and interest rates, are in constant dollars. For numerically estimating this model, we adjust the data for inflation to convert them to constant dollars, as described in Chapter 4.

2.2.1 Choice of Objective Function

Various studies have discussed the choice of objective function for a sequential decision model from a theoretical and empirical standpoint. The most commonly used objective functions are the additive utility of consumption or profits and the utility of terminal wealth. The objective function of terminal wealth is suitable due its simplicity and the straightforward definition of risk aversion. Elton and Gruber (p. 86) state that, though this objective function ignores intermediate consumption, it is of interest because it represents a class of problems that actually exists and it provides insight into the nature of the multiperiod portfolio problem in a simple framework. Jeffrey and Eidman (p. 198) state that "additive utility ... is likely to be inappropriate" because the assumption of additive independence is unrealistic in many cases. They show that the objective function of the terminal wealth may be used to determine efficient sets for all forms of stochastic dominance and can be used to address the issue of long-run risk.[5]

In the above model, the objective function is the expected utility of terminal net wealth. In this framework, the withdrawal for consumption can be incorporated in the net wealth equation. To reduce the size of the problem, we do not include consumption as a control variable. We assume that consumption expenditures are met by the labor and management costs included in the production costs, $c * L_{t+1}$, which depend on the number of acres.[6]

[5]Featherstone, Preckel, and Baker (p. 85) also discuss the comparison of these objective functions.

[6]An alternative is a fixed amount of consumption for all farm sizes. However, this assumption implicitly puts more costs per acre. Furthermore, farmers with smaller farm size would not depend on only farm income.

2.2.2 Specification of Utility Function

The utility function used in the model should be able to represent risk neutrality or risk aversion. One utility function that offers flexibility in representing risk preferences and has been widely used in applications is:

$$U(W) = \frac{W^{1-\theta} - 1}{1 - \theta} \tag{2.8}$$

where $\theta \geq 0$. If $\theta = 0$, the utility function exhibits risk neutral preferences. If $\theta > 0$, it exhibits risk aversion. Note that $\lim_{\theta \to 1} U(W) = \ln(W)$.

Absolute risk aversion, $r(W)$, is defined as: $r(W) = -\frac{U''(W)}{U'(W)}$. Relative risk aversion, $rr(W)$, is defined as: $rr(W) = -W\frac{U''(W)}{U'(W)}$. For the above utility function, absolute risk aversion is $r(W) = \frac{\theta}{W}$, which decreases toward zero as wealth increases. Relative risk aversion is $rr(W) = \theta$, which is constant across wealth levels. One advantage of this utility function is that the relative risk aversion is unit free. Empirical evidence suggests that the preferences of most decision makers exhibit decreasing absolute risk aversion, especially with constant relative risk aversion of 1 (Arrow).

Chapter 3

Dynamic Programming and Numerical Methods

The farmland investment problem specified in Chapter 2 is a sequential decision model and can be solved by the dynamic programming approach[1]. Numerically solving the farmland investment problem requires estimates of the stochastic state equations and parameters of the model, which are presented in Chapter 4.

Dynamic programming "takes a sequential or multistage decision process containing many interdependent variables and converts it into a series of single-stage problems, each containing only a few variables" (Nemhauser, p. 6). The dynamic programming framework allows modelling sequential decision problems with nonlin-

[1]Since dynamic programming is an approach for solving sequential decision models, in the literature, these models are often referred to as dynamic programming models or dynamic programs. When outcomes are uncertain, these models are referred to as stochastic dynamic programs, or Markov decision processes. These models are also known as stochastic control problems in the mathematics and engineering literature.

earity and uncertainty. There is an extensive literature on dynamic programming and its application to economic problems. Stokey and Lucas with Prescott present a rigorous overview of dynamic programming and its properties with application to economic models. Rust surveys the literature on numerical methods and their properties for solving dynamic programming models in economics. Santos reviews some numerical techniques and their accuracy in solving economic models. Miranda and Fackler apply dynamic programming and numerical methods, implemented in MATLAB, to solve a wide range of dynamic decision problems.

This chapter presents a brief overview of the dynamic programming approach and numerical methods for solving sequential decision problems with discrete time, finite horizon, stochastic states, and continuous state and control variables. In Sections 3.1 and 3.2, we present this approach for a relatively general sequential decision problem. The farmland investment problem specified in Chapter 2 is a sequential decision problem with terminal optimization, which is a special case of the general problem. Based on Sections 3.1 and 3.2, the dynamic programming approach for the terminal optimization problem is presented in Section 3.3. Then we present numerical methods for solving the dynamic programming problem.

3.1 Dynamic Programming Approach

3.1.1 Sequential Decision Problem

A decision maker is faced with the problem of choosing a sequence of decision rules over a prespecified planning horizon to perform optimally with respect to a performance criterion. The decisions are made at the beginning of each period, $t = 0, 1, ..., T$, where T is a finite positive integer.

Consider a general problem with J continuous state variables and K continuous control variables. For $j = 1, 2, ..., J$, let S_j be a set of all levels of state variable j. The state space S is defined as $S = S_1 \times S_2 \times ... \times S_J$, which is the Cartesian product of all S_j's for $j = 1, 2, ..., J$. An arbitrary element of S is denoted by s, which is a vector of state variable levels. For each state variable $j = 1, 2, ..., J$, a random shock ε_j is distributed with a density function $f_j(\varepsilon_j)$. Let $\mathcal{E}_j$ be a set of all levels of random shocks. Then again, $\mathcal{E}$ is the Cartesian product of all $\mathcal{E}_j$'s for $j = 1, 2, ..., J$. An arbitrary element of $\mathcal{E}$ is denoted by ε. Since a problem may have both deterministic and stochastic state variables, this specification also characterizes deterministic state variables. For a state variable i, ε_i, $\mathcal{E}_i$, and $f_i(\varepsilon_i)$ can be defined to represent it as a deterministic state variable.[2] Let X_k be the nonempty set of all levels of control variable k. Then again, X is the Cartesian product of all X_k's for $k = 1, 2, ..., K$. An arbitrary element of X is denoted by x, which is a vector of control variable levels.

[2]For example, when the random shock is additive, we can define $\varepsilon_i = 0$ to be the only element in $\mathcal{E}_i$, and $f_i(\varepsilon_i) = 1$ to represent the deterministic variable.

Let u be a function from $S \times X$ into $\mathbf{R}$, and let G be a function from $S \times X \times \mathcal{E}$ into S. At any time t, if $x_t \in X$ is chosen in state $s_t \in S$, then $u(s_t, x_t)$ is the current period reward. The next period state, s_{t+1}, is given by $s_{t+1} = G(s_t, x_t, \varepsilon_{t+1})$,[3] where $\varepsilon_{t+1} \in \mathcal{E}$ is a random shock realized in the next period. We assume that the spaces S, X, $\mathcal{E}$, and the functions u, G, f_j, are for all t, that is, they are invariant to time. While the dynamic programming approach does not require this assumption, it is common in most applications, including the farmland investment problem of this study. Let $U(s_{T+1})$ be the salvage value function for time $T+1$. Let $\delta \in (0, 1]$ be the one-period discount factor.

The decision maker's objective function is:

$$\max_{\{x_t\}_{t=0}^{T}} E\left[\sum_{t=0}^{T} \delta^t u(s_t, x_t) + \delta^{T+1} U(s_{T+1})\right] \tag{3.1}$$

subject to:

$$s_{t+1} = G(s_t, x_t, \varepsilon_{t+1}), \; x_t \in X, \text{ for } t = 0, 1, 2, ..., T,$$

$$s_0 \in S \text{ given.}$$

In the case of all deterministic state variables, with the state s_0 known, the decision maker chooses a sequence of controls $\{x_t^*\}_{t=0}^{T}$ in period 0. In the case with at least one stochastic state variable, with the state s_0 known, the decision maker chooses x_0^*

[3]The control space, X, may be defined as $X = \{x : s \in S, \varepsilon \in E, \text{ and } G(s, x, \varepsilon) \in S\}$. However, especially with stochastic state variables, the condition of $G(s, x, \varepsilon) \in S$ with multiple state variables may yield stringent restrictions on the control space, or the condition may not satisfy for some state variables. In that case, extrapolation may be used. See implementation in Chapter 5 for details on the farmland investment problem.

and makes contingency plans x_t^* for periods $t = 1, 2, ..., T$. These contingency plans depend on s_t, which will be known after the realization of the shocks in period t, ε_t.

3.1.2 Bellman's Equation

The optimal policy for the sequence problem (3.1) is sought not only for period $t = 0$, but also for periods $t = 1, 2, ..., T$. Thus, problem (3.1) can be viewed as solving the following problems for each $t = 0, 1, ..., T$:

$$\max_{\{x_\tau\}_{\tau=t}^T} E\left[\sum_{\tau=t}^{T} \delta^{\tau-t} u\left(s_\tau, x_\tau\right) + \delta^{T+1-t} U\left(s_{T+1}\right)\right] \tag{3.2}$$

subject to:

$$s_{\tau+1} = G\left(s_\tau, x_\tau, \varepsilon_{\tau+1}\right),\ x_\tau \in X,\ \text{for } \tau = t, t+1, ..., T,$$

$$s_t \in S \text{ given.}$$

Note that s_t can take any value from S, so we seek the solution of the above problem for all $s_t \in S$. The formulation (3.2) is the basis for solving this problem by converting it into Bellman's Equation, and is explained by Bellman (p. 83) as the Principle of Optimality:

> "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

Define the value function for each $t = 0, 1, ..., T$, for all $s \in S$:

$$V_t(s) \equiv \max_{\{x_\tau\}_{\tau=t}^T} E\left[\sum_{\tau=t}^{T} \delta^{\tau-t} u\left(s_\tau, x_\tau\right) + \delta^{T+1-t} U\left(s_{T+1}\right) \mid s_t = s\right]$$

Note that we have omitted subscript t for the state s in the value function because the value function is defined for all s, not just the realized state in period t. From the above definition, it follows that the value function satisfies Bellman's equation:

$$V_t(s) = \max_x \{u(s,x) + \delta E[V_{t+1}(s_{t+1}) \mid s_t = s,\ x_t = x]\} \tag{3.3}$$

with the terminal (boundary) condition:

$$V_{T+1}(s) = U(s) \tag{3.4}$$

The value function is found by backward induction. Given $V_{T+1}(s)$ for all $s \in S$, as defined by (3.4), we find $V_T(s)$ for all $s \in S$ by using (3.3) for $t = T$ and attaining the maximum. Having $V_T(s)$ for all $s \in S$, we find $V_{T-1}(s)$ for all $s \in S$ by using (3.3) for $t = T - 1$ and attaining the maximum. Thus, by the backward induction, we find $V_t(s)$ for all $s \in S$ for each $t = 1, ..., T$.

3.2 Optimal Policy

The approach of dynamic programming provides a method for finding the value functions $\{V_1(s), V_2(s), ..., V_T(s)\}$, and function $V_{T+1}(s)$ is given in (3.4), as described in the previous section. The result of this approach is that, instead of solving (3.1), at any time $t = 0, 1, ..., T$ the optimal policy for any given $s \in S$ can be obtained by:

$$x_t^*(s) = \arg\max_x \{u(s,x) + \delta E[V_{t+1}(s_{t+1}) \mid s_t = s,\ x_t = x]\}$$

subject to:

$$s_{t+1} = G\left(s_t, x_t, \varepsilon_{t+1}\right),\ x \in X,\ s \in S \text{ given.}$$

For finding optimal policy at $t = 0$ we use value function V_1, and at any t, value function V_{t+1}, in the above problem.

3.2.1 Comparative Dynamics

Comparative dynamic methods can be used to analyze the effect of changes in parameters of the model. For example, changes in the length of the planning horizon. risk aversion, and the interest rate can be explored using comparative dynamics. Denote γ as the set of parameters of the model to be varied. We take different levels of γ and find the value function V^{γ} for each level of γ as described in Section (4.1). Comparative dynamics for the representative agent can be done as follows.

Denote $\bar{s}$ as the state level. Then we solve for the optimal x_t for each level of γ at the fixed state level $\bar{s}$ by:

$$x_t^{*\gamma}\left(\bar{s}\right) = \arg\max_x \left\{u\left(\bar{s}, x\right) + \delta E\left[V_{t+1}^{\gamma}(s_{t+1}) \mid s_t = \bar{s},\ x_t = x\right]\right\} \tag{3.5}$$

subject to:

$$s_{t+1} = G\left(s_t, x_t, \varepsilon_{t+1}\right),\ x_t \in X,\ \bar{s} \in S \text{ given.}$$

3.3 Terminal Optimization

The farmland investment problem specified in Chapter 2 is a sequential decision problem with terminal optimization. The objective function under terminal optimization is a special case of problem (3.1). Let $u(.) = 0$ for $t = 0, 1, 2, ..., T$, and $\delta = 1$, then problem (3.1) can be written as follows. The decision maker's objective function is:

$$\max_{\{x_t\}_{t=0}^{T}} E\left[U\left(s_{T+1}\right)\right] \tag{3.6}$$

subject to:

$$s_{t+1} = G\left(s_t, x_t, \varepsilon_{t+1}\right), \; x_t \in X, \text{ for } t = 0, 1, 2, ..., T,$$

$$s_0 \in S \text{ given.}$$

The corresponding Bellman's equation can be written as:

$$V_t(s) = \max_x \left\{E\left[V_{t+1}(s_{t+1}) \mid s_t = s, \; x_t = x\right]\right\} \tag{3.7}$$

with the terminal (boundary) condition:

$$V_{T+1}\left(s\right) = U\left(s\right), \tag{3.8}$$

where the value function for each $t = 0, 1, ..., T$, for all $s \in S$, is defined as:

$$V_t(s) \equiv \max_{\{x_\tau\}_{\tau=t}^{T}} E\left[U\left(s_{T+1}\right) \mid s_t = s\right]$$

3.4 Numerical Solution Methods

Following Sections 3.1 and 3.2, we have the dynamic programming problem with discrete time, finite horizon, stochastic states, and continuous state and control variables. Finding the optimal policy by the dynamic programming approach requires the value functions $\{V_1(s), V_2(s), ..., V_T(s)\}$. Though we have a continuous-state dynamic programming problem, it lacks a closed-form solution in most applications, with the exception of some very simple deterministic dynamic programming models. Thus, the value functions must be numerically approximated by computational methods.

A variety of computational methods are available for numerically solving a dynamic programming model with continuous state and control variables. The choice of method depends on the assumptions for a given model. A special case of the dynamic programming model is the linear-quadratic problem, in which the state equations are assumed to be linear and the reward function to be quadratic. Optimal value and policy functions can be derived analytically in this case. The method for solving such problems, the linear-quadratic approach, is of limited value, however, since the necessary assumptions do not hold for many problems, including the farmland investment problem of this study.

Methods that do not make these assumptions include (1) the discretization method and (2) the parametric approach (Judd, 1998, p. 433). Under the discretization method, continuous state and control variables are discretized in finite sets, and the value function is solved and stored for the selected elements in the state space. Under the parametric approach, the value function is treated as a function of parameters, which are estimated using approximation methods. This approach yields a value function defined over all values in the range of the state space. Rust reviews different methods of parametrizing the value function and their properties.

In the discretization method, the value function is approximated with a step function (Judd, 1996, p. 562). This method poses problems in implementation since this method requires that, given the current state, the next period state must be an element of the set of points in state space defined for discretization. Furthermore, this method is reliable only if the discretization is made sufficiently fine. Despite advances in computer technology, however, this method becomes impractical with large-scale problems due to limitations of computer storage capacity (Judd, 1998, p. 433).

To overcome these limitations, Bellman and Dreyfus (p. 17) suggest using affine (linear) functions or a high degree polynomial to interpolate the value function between the selected points in state space. This approach is developed as interpolation by piecewise polynomials splines. Bellman and Dreyfus (p. 323) also introduced polynomial approximation for the whole domain of the state variables, formally presented in Bellman, Kalaba, and Kotkin. Interpolation by polynomials or piecewise polynomials is an efficient and common parametric approach used in dynamic programming models. Like the discretization method, this approach also requires solving the value function for a prespecified finite state set. Instead of storing the value function for each element in the state set, this approach fits a functional form. Miranda and

Fackler refer to the interpolation approach in dynamic programming as the collocation method, which is how it will be referred in this study. Since the investment portfolio has four state variables, we will use the collocation method for solving the dynamic programming model of the study.

3.4.1 Collocation Method

The collocation method approximates a functional equation in which the approximated function fits exactly at the prespecified points of the domain (Judd, 1998, p. 384). For solving dynamic models in economics and finance, the collocation method is often the most useful method. It is flexible, accurate, and numerically efficient in most applications (Miranda and Fackler).

The collocation method approximates a function with a linear combination of N basis functions using N prescribed points of the domain, called the collocation nodes, where N is a finite positive integer. For approximating the value functions $\{V_1(s), V_2(s), ..., V_T(s)\}$, the domain is the state space. We define a series of N basis functions $\{\phi_i(s)\}_{i=1}^{N}$ for $s \in S$. The value function $V_t(s)$ is approximated by:

$$V_t(s) \approx \sum_{i=1}^{N} c_{it}\phi_i(s)$$

where the basis coefficients $\{c_{it}\}_{i=1}^{N}$ are to be determined.

The value functions are approximated following backward induction using Bellman's equation. Now Bellman's equation (3.3) can be written as:

$$\sum_{i=1}^{N} c_{it}\phi_i(s) = \max_x \left\{ u(s,x) + \delta E\left[\sum_{i=1}^{N} c_{i,t+1}\phi_i(s_{t+1}) \mid s_t = s,\ x_t = x \right] \right\} \tag{3.9}$$

where $s_{t+1} = G(s_t, x_t, \varepsilon_{t+1})$. Note that the value function V_{T+1} is given (3.4) and is used for approximating V_T. For all other t, we use the approximated V_{t+1}, that is, $\sum_{i=1}^{N} c_{i,t+1}\phi_i(s)$. By the collocation method, (3.9) is solved for each of N nodes. Let $\{s_n\}_{n=1}^{N}$ be a series of N nodes selected from the state space such that $s_n \in S$.[4] Let v_{nt} be the maximum value in the above problem for each node. Then we have for each node $n = 1, 2, ..., N$:

$$\sum_{i=1}^{N} c_{it}\phi_i(s_n) = v_{nt} \tag{3.10}$$

This gives a system of N equations with the N unknown coefficients $\{c_i^t\}_{i=1}^{N}$. The N equations (3.10) may be expressed as:

$$\Phi * c_t = v_t$$

where Φ is an $N \times N$ matrix in which the nth row and ith column is $\phi_{it}(s_n)$, c_t is a column vector denoting the coefficients $\{c_{it}\}_{i=1}^{N}$, and v_t is a column vector of $\{v_{nt}\}_{n=1}^{N}$ for all nodes. Then we can find c_t by:

$$c_t = \Phi^{-1}V^t$$

Note that we have N nodes and basis functions for all state variables, $N = N_1 * N_2 * ... * N_J$, where N_j is number of nodes and basis functions for each state variable

[4]Note that we are using subscript n for the nodal index for explaining the collocation method.

$j = 1, 2, ..., J$. In this case, s and Φ are tensor products for all state variables. Miranda and Fackler show that matrix Φ^{-1} can be formed efficiently by inverting it for each state variable and then making the tensor product. This method saves storage and computational effort especially when the problem is large.

Implementation of the collocation method requires specification of (i) the type of collocation basis functions and nodes, and the number of nodes, (ii) an optimization method for finding the maximum value, and (iii) a method for finding expected value. We describe these specifications in Chapter 5, where we develop a test for accuracy analysis and evaluate alternative choices to find a good approximation for solving the model.

Chapter 4

Stochastic State Equations and Parameters

Numerically solving the farmland investment problem specified in Chapter 2 by dynamic programming requires estimates of the stochastic state equations and parameters of the model. This chapter describes estimation methods and data and presents estimates of the stochastic equation and parameters of the model.

In the farmland investment problem, there are two stochastic state equations for gross return per acre from crops and the farmland price: $R_{t+1} = g(R_t, \varepsilon_{1,t+1})$ and $P_{t+1} = h(P_t, R_t, \varepsilon_{2,t+1})$, respectively. The gross return equation is specified as an autoregressive model. The farmland price equation is a special autoregressive model conditional on gross returns, which is based on a model proposed by Burt. The Box-Jenkins approach to forecasting provides a method for finding the order of an autoregressive model. First we describe an autoregressive model and its properties. Then we describe the Box-Jenkins approach, estimation methods, data, and regression results.

4.1 Autoregressive Model

Consider a model with a pth-order autoregressive process, AR(p):

$$y_t = \beta_0 + \beta_1 y_{t-1} + \ldots + \beta_p y_{t-p} + \varepsilon_t \tag{4.1}$$

where y_t is random variable to forecast, $\beta_0, \beta_1, \ldots, \beta_p$ are parameters of the model, and ε_t is the error term (random shock) and is assumed to be a white noise process. Below we define white noise for the error term and the covariance stationarity for random variable y_t, a property of autoregressive process discussed later in this chapter. These definitions can be found in most time series econometric books such as Hamilton and Greene.

Definition 1 *A sequence $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ is described as white noise process if each element in the sequence satisfies:*

1. $E[\varepsilon_t] = 0$
2. $Var[\varepsilon_t] = \sigma_\varepsilon^2$
3. $Cov[\varepsilon_t, \varepsilon_\tau] = 0$ *for all* $\tau \neq t$.

Definition 2 *A stochastic process of variable y_t is covariance-stationary or weakly stationary if it satisfies the following requirements:*

1. $E[y_t]$ is a constant, independent of t.

2. $Var[y_t]$ is a constant, independent of t.

3. $Cov[y_t, y_s]$ is a function of t-s, but not of t or s.

The pth-order autoregressive process (4.1), $y_t = \beta_0 + \beta_1 y_{t-1} + ... + \beta_p y_{t-p} + \varepsilon_t$, is covariance stationary if (i) ε_t is a white noise process and (ii) the roots of the characteristic equation

$$1 - \beta_1 z + ... + \beta_p z^p = 0 \tag{4.2}$$

lie outside the unit circle. Condition (ii) puts restrictions on parameters $\beta_0, \beta_1, ..., \beta_p$. In this condition, z is used for finding the roots of the characteristics equation (4.2).[1]

We illustrate here some properties of an autoregressive process for $p = 1$, which is a first-order autoregressive process, AR(1): $y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t$. The characteristic equation for this process is $1 - \beta_1 z = 0$. The root of this equation is $\frac{1}{\beta_1}$, which lies outside the unit circle if $|\beta_1| < 1$. Hamilton (p. 53) shows that if the error term is a white noise process and $|\beta_1| < 1$, then

1. $E[y_t] = \mu = \frac{\beta_0}{1-\beta_1}$,
2. $Var[y_t] = E\left[(y_t - \mu)^2\right] = \frac{\sigma_\varepsilon^2}{1-\beta_1^2}$,
3. $Cov[y_t, y_{t-j}] = E[(y_t - \mu)(y_{t-j} - \mu)] = \frac{(\beta_1)^j . \sigma_\varepsilon^2}{1-\beta_1^2}$.

The first equation can be written as $\beta_0 = \mu(1 - \beta_1)$. As $E[y_{t+1}|y_t] = \beta_0 + \beta_1 y_t$, we have $E[y_{t+1}|y_t] = \mu(1-\beta) + \beta y_t$. Observe that if $\mu > y_t$, then we have $E[y_{t+1}|y_t] > y_t$. Since $0 < (1-\beta) < 2$ (or $|\beta_1| < 1$), multiplying by $(1-\beta)$ both sides of $\mu > y_t$ and

[1]The details on AR(1), AR(2), and AR(p) and the proof of covariance stationarity can be found in Hamilton (p. 53).

rearranging the terms yield $E[y_{t+1}|y_t] > y_t$. Similarly if $\mu < y_t$, then $E[y_{t+1}|y_t] < y_t$, and if $y_t = \mu$, then $E[y_{t+1}|y_t] = y_t$. This indicates that the series tends to wander around its mean value μ.

For an AR(2) model, $y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \varepsilon_t$, the roots of the characteristic equation (4.2) lie outside the unit circle if and only if $|\beta_2| < 1$, $(\beta_1 + \beta_2) < 1$, and $(\beta_2 - \beta_1) < 1$.

4.2 Box-Jenkins Approach

Before estimating the AR(p) model (4.1), we need to specify the order of autoregression, p. Box and Jenkins provide methods for modeling time series data for forecasting. The Box-Jenkins approach is an iterative method, which consists of identification of p, estimation, and a diagnostic test of the hypothesis that the error term is white noise. This method finds p so that the error term is white noise. We transform the data, if necessary, so that the assumption of a white noise error term is reasonable. For example, we begin with $p = 1$, estimate the model, and test the null hypothesis that the error term is white noise. We increase p until we find that error term is white noise (Hamilton, p. 110; Vandaele, p. 87). Estimation methods and diagnostic tests for white noise are described in next sections.

4.3 Estimation Method

For the pth-order autoregressive model (4.1), $y_t = \beta_0 + \beta_1 y_{t-1} + ... + \beta_p y_{t-p} + \varepsilon_t$, we have data to estimate the model. We have $(T + p)$ observations indexed as $(y_{-p+1}, y_{-p+1}, ..., y_0, y_1, ..., y_T)$. After adjusting for lagged variables, there are T observations for the regression model. Denote

$$Y \equiv \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}, X \equiv \begin{bmatrix} 1 & y_{1-1} & y_{1-2} & \cdots & y_{1-p} \\ 1 & y_{2-1} & y_{2-2} & \cdots & y_{2-p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & y_{T+p-1} & y_{T+p-2} & \cdots & y_T \end{bmatrix}, \beta \equiv \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}.$$

The ordinary least square (OLS) estimates of β and its variance are:

$$\widehat{\beta} = (X'X)^{-1} X'Y$$

$$Var\left[\widehat{\beta}\right] = \widehat{\sigma}_\varepsilon^2 (X'X)^{-1}$$

where $\widehat{\sigma}_\varepsilon^2 = \frac{e'e}{T-(p+1)}$ is variance estimate for error term, and $e = Y_1 - X\widehat{\beta}$ is a residual vector.

4.3.1 Properties of Estimators

Statistical inference about the OLS estimates requires assumptions about the error term, ε_t. The primary assumption is that the error term is a white noise process, which goes without loss of generality because it is required for the Box-Jenkins iterative method. We will test this assumption from the data.

Consider the AR(1) model $y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t$. If we assume that $\{\varepsilon_t\}$ is an independent and identically distributed (i.i.d.) sequence, then it implies that ε_t is independent of y_{t-1}. However from the autoregressive model, y_t is in part determined by ε_t. So, for period $(t-1)$, y_{t-1} is in part determined by ε_{t-1}, therefore y_{t-1} and ε_{t-1} are not independent. Also note that y_{t-1} depends on y_{t-2}, which is in part determined by ε_{t-2}. Thus y_{t-1} also depends on ε_{t-2}. This indicates that y_{t-1} is dependent on $(\varepsilon_{t-1}, \varepsilon_{t-2}, ...)$ and is independent of $(\varepsilon_t, \varepsilon_{t+1}, ...)$. Therefore, y_{t-1} is contemporaneously independent of error term $\{\varepsilon_t\}$. As the regressor is not independent of the entire vector of error terms, the OLS estimator $\widehat{\beta}$ is biased. The bias disappears as T becomes large. Therefore, the OLS estimator can be justified asymptotically (Hamilton, p. 215).

If we assume that the error term is normally distributed, Hamilton (p. 122) shows that the OLS estimator $\widehat{\beta}$ for the AR model is the same as obtained by conditional maximum likelihood estimation (MLE) method. Conditional MLE method provides consistent estimates, whereas exact MLE method does not.

4.4 Diagnostic Tests

The Box-Jenkins approach requires a white noise test for the error term. The autoregressive order is increased if the error term is not white noise. However, if the slope coefficient is not significantly different from zero, then a lower order autoregression may be tested. As mentioned in the previous section, the OLS method yields parameter estimates identical to those obtained from conditional MLE if the error term is normally distributed. Furthermore, information on the error term distribution is also needed in implementation of dynamic programming procedure. This section provides methods for testing the significance of estimators, and for testing white noise and normality of error terms.

4.4.1 Significance of Estimators

The OLS estimates of the slope parameters are tested by a t-test:

$$H_0 : \beta_i = 0, \; i = 1, 2, ..., p,$$

$$H_1 : \text{Not } H_0.$$

Under the null hypothesis, the t ratio, $t = \frac{\widehat{\beta}_i}{\sqrt{Var[\widehat{\beta}_i]}}$, has a t distribution with $(p+1)$ degrees of freedom.

4.4.2 Test for White Noise

The Ljung-Box (Ljung and Box) test is used to test that the error term is a white noise process. This test is particularly suitable for regressions with lagged dependent variables.

$$H_0 : \varepsilon_t \text{ is a white noise process,}$$

$$H_1 : \text{Not } H_0.$$

Under the null hypothesis, the Ljung-Box statistic

$$Q(K) = T(T+2)\sum_{k=1}^{K}\frac{r_k^2}{T-k}$$

is asymptotically distributed as chi-squared with K degrees of freedom, where $r_k = \frac{\sum_{t=k+1}^{T} e_t e_{t-k}}{\sum_{t=1}^{T} e_t^2}$, and $\{e_t\}_{t=1}^{T}$ are residuals from the OLS regression. This test is performed for each $K = 1, 2, ...,$ up to an appropriate number, say, 15. The Ljung-Box statistic has better finite-sample properties than other tests (Greene, p. 838).

4.4.3 Test for Normality

If we assume that the error term is normally distributed, then the OLS estimator is the same as obtained by conditional maximum likelihood estimation (MLE). The Bera-Jarque test (Bera and Jarque) is used to test for normality.

$$H_0 : \varepsilon_t \text{ is normally distributed,}$$

$$H_1 : \text{Not } H_0.$$

Under the null hypothesis, the Bera-Jarque statistic, BJ, is asymptotically distributed as chi-squared with two degrees of freedom:

$$BJ = T * \left(\frac{(\widehat{\mu}_3)^2}{6\left(\widehat{\sigma}^2\right)^3} + \frac{\left(\widehat{\mu}_4 - 3\left(\widehat{\sigma}^2\right)^2\right)^2}{24\left(\widehat{\sigma}^2\right)^4} \right),$$

where $\widehat{\sigma}_2 = \frac{1}{T}\sum_{t=1}^{T} e_t^2$, $\widehat{\mu}_3 = \frac{1}{T}\sum_{t=1}^{T} e_t^3$, and $\widehat{\mu}_4 = \frac{1}{T}\sum_{t=1}^{T} e_t^4$ are consistent estimators of the second, third, and fourth moments respectively.

For testing the null hypothesis in the above diagnostic tests, we will report t ratio, $Q(K)$, and BJ along with their p-values. The p-value for a test provides marginal significance level. For example, a p-value lower than 0.01 is taken as an evidence to reject the null hypothesis at 1 percent significance level.

4.5 Data

The stochastic state equations for the model introduced in Chapter 2 are estimated using time series data. Gross return from crops per acre, R_t, and farmland price, P_t, are adjusted for inflation using the Consumer Price Index. Gross return is calculated from data for farms that belong to the Southwestern Minnesota Farm Business Management Association (SWMFBMA), as published in their Annual Reports for the years 1967-99 (Olson et al.). The farm records of this region indicate that 44 percent of land is used for corn, 44 percent for soybeans, and 12 percent for other crops (Olson). Corn and soybeans are produced in the most common crop rotation. For simplicity, we assume that only corn and soybeans are produced, and gross return per acre is calculated for the 50-50 corn-soybean mix. Time series data for the price of land are obtained for Southwestern Minnesota for years 1966-1992 from the Minnesota Rural Real Estate Market (Schwab and Raup; Brekke, Tao, and Raup), and for years 1990-1999 from Minnesota Land Economics (Taff).[2]

[2]Appendix A provides data series with explanation on constructing the series for estimating the equations.

The investment portfolio model developed in Chapter 2 is at the level of the individual farm firm. However, time series data at the farm level are rare and often incomplete. Data described in the above paragraph are the averages of samples from the region. Thus, the estimation of the variance of error term may be underestimated by using this data set. Nevertheless, in the analysis of comparative dynamics (Section 3.2.1), we will explore the effect of increasing the variance on the model results (Chapter 6).

4.6 Returns from Crops

The state equation for gross return per acre from crops is modelled as an autoregression. The Box and Jenkins approach for forecasting suggests a first order autoregression model, AR(1). The suitable transformation of the data is natural logarithm. The gross returns are positive numbers because they are equal to price times crop yields, which is always positive. The regression results for the equation of the gross return per acre

$$\ln R_t = \beta_0 + \beta_1 \ln R_{t-1} + \varepsilon_{1t} \tag{4.3}$$

are presented in Table 4.1(a). The regression results show that the slope coefficient on $\ln R_{t-1}$ is significantly different from zero, as its p-value is less than 0.01. This specification gives a stable model by the diagnostic test of the error term. Following the Box-Jenkins approach, we test the null hypothesis that the error term is white

Table 4.1: Results for Gross Return Equation

(a) Regression Results

Variable	Coefficient Estimate	Standard Error	t-ratio	p-value
Constant	1.052028	0.679087	1.549180	0.1318
$\ln R_{t-1}$	0.821970	0.113723	7.227848	0.0000

Estimated $\text{Var}[\varepsilon_1] = 0.033155$, $R^2 = 0.635222$, adj $R^2 = 0.623063$.

(b) Ljung-Box Test for White Noise

K	$Q(K)$ statistic	p-value
1	0.0035	0.953
2	0.7275	0.695
3	0.7551	0.860
4	0.7747	0.942
5	1.6553	0.894
6	2.4892	0.870
7	3.8131	0.801
8	3.8452	0.871
9	6.5782	0.681
10	7.3832	0.689
11	8.3072	0.686
12	8.5760	0.739
13	10.474	0.655
14	10.684	0.711
15	10.743	0.771

(c) Bera-Jarque test for Normality

BJ statistic	p-value
3.160570	0.205916

noise. To test the null hypothesis, the Ljung-Box test results are presented in Table 4.1(b). This test does not reject the null hypothesis, as all p-values are much greater than 0.01. Thus, the hypothesis of white noise error term is maintained.

Normality of the error term is tested using the Bera-Jarque test. The Bera-Jarque test results are presented in Table 4.1(c) to test the null hypothesis that the error term is normally distributed. The test does not reject the null hypothesis, as the p-value is greater than 0.01.

4.7 Farmland Price

The specification of the farmland price state equation is based on Burt. The first order difference equation of the Burt model is $\ln P_t = \alpha_0 + \alpha_1 \ln P_{t-1} + \alpha_2 \ln R_{t-1} + \varepsilon_{2t}$. When the white noise test is performed on the residuals from the above regression, the test rejects the null hypothesis of the white noise error term. So, the second order difference equation of the Burt model is used and specified as[3]: $\ln P_t = \alpha_0 + \alpha_1 \ln P_{t-1} + \alpha_2 \ln P_{t-2} + \alpha_3 \ln R_{t-1} + \alpha_4 \ln R_{t-2} + \varepsilon_{2t}$. Since α_4 was not significant, the results suggest the following specification:

$$\ln P_t = \alpha_0 + \alpha_1 \ln P_{t-1} + \alpha_2 \ln P_{t-2} + \alpha_3 \ln R_{t-1} + \varepsilon_{2t} \tag{4.4}$$

Regression results of this model are presented in Table 4.2(a). We test the null hypothesis that the error term is white noise.The Ljung-Box test is presented in Table

[3]This specification is also estimated by Burt (p. 19).

Table 4.2: Results for Farmland Price Equation

(a) Regression Results

Variable	Coefficient Estimate	Standard Error	t-ratio	p-value
Constant	0.077732	0.399551	0.194548	0.8472
$\log P_{t-1}$	1.413025	0.132676	10.65019	0.0000
$\log P_{t-2}$	-0.597604	0.123220	-4.849899	0.0000
$\log R_{t-1}$	0.214150	0.074244	2.884407	0.0075

Estimated Var$[\varepsilon_2]$ = 0.009398, R^2= 0.946916, adj R^2= 0.941228.

(b) Ljung-Box Test for White Noise

K	$Q(K)$ statistic	p-value
1	0.0062	0.937
2	0.4677	0.791
3	0.8158	0.846
4	0.8198	0.936
5	2.8125	0.729
6	5.4445	0.488
7	5.6545	0.581
8	6.4914	0.592
9	6.4914	0.690
10	6.8153	0.743
11	8.8407	0.637
12	8.9801	0.705
13	9.0528	0.769
14	9.8600	0.772
15	11.029	0.751

(c) Bera-Jarque test for Normality

BJ statistic	p-value
0.030666	0.984784

4.2(b), and does not reject the null hypothesis. Thus, the hypothesis of white noise error term is maintained. Normality of the error term is tested using the Bera-Jarque test, and is presented in Table 4.2(c). The test does not reject the null hypothesis that the error term is normally distributed. Finally, we need to check for a statistical relationship between ε_{1t} and ε_{2t}, that is, between error term of return equation (4.3) and error term of farmland price equation (4.4). The covariance of their residuals is computed as 0.00007, and the correlation is equal to 0.003965. Since the covariance is close to zero, the OLS estimates are efficient.[4]

4.7.1 Reduction of State Variables for DP

The estimated equation of farmland price has two lagged variables, $\ln P_t = \alpha_0 + \alpha_1 \ln P_{t-1} + \alpha_2 \ln P_{t-2} + \alpha_3 \ln R_{t-1} + \varepsilon_{2t}$, which requires inclusion of two state variables. Burt and Taylor provide a statistical procedure to reduce a state variable for use in a dynamic programming problem. This procedure is based on the assumption that the stochastic variable follows a stationary process. This assumption holds when the error term is white noise and $|\alpha_2| < 1$, $(\alpha_1 + \alpha_2) < 1$, and $(\alpha_2 - \alpha_1) < 1$ in equation (4.4). The estimates of these parameters in Table 4.2(a) satisfy these inequalities, and the test of the white noise error term is maintained (Table 4.2(b)). Using the Burt and Taylor approach, we reduce a state variable in the land price equation. Table 4.3 presents the estimates obtained by their procedure. The reduced equation can be

[4] In the event of high correlation, these two equations may be specified as seemingly unrelated regressions (SUR) model, a topic found in most econometric books such as Greene.

Table 4.3: Reduction of a Lagged Variable for Land Price Equation

Variable	Coefficient Estimate
Constant	0.048655
$\ln P_{t-1}$	0.884465
$\ln R_{t-1}$	0.134044

$\text{Var}[\varepsilon_2] = 0.014619$.

represented by $\ln P_t = a_0 + a_1 \ln P_{t-1} + a_2 \ln R_{t-1} + \varepsilon_{2t}$, where $a_0 = \frac{\alpha_0}{1-\alpha_2}$, $a_1 = \frac{\alpha_1}{1-\alpha_2}$, $a_2 = \frac{\alpha_3}{1-\alpha_2}$, and the variance of error term is also adjusted by dividing by $(1 - (\alpha_2)^2)$.[5]

4.8 Estimation of Parameters

The farmland investment problem specified in Chapter 2 includes some parameters. Numerically solving the model requires their estimates. The interest rate on borrowing is assumed to be $r_b = 0.06$. This rate represents in constant dollars. It is based on reviewing last five-year interest rate on long-term loans in farm credit system (USDA), and is adjusted for inflation. On lending (riskless investment), the interest is assumed to be $r_l = 0.03$.

Estimation of the following parameters is based on farm data for Southwestern Minnesota and an interview with Dale Nordquist, who is familiar with the farm records. The price of machinery and equipment per acre is $\kappa = \$300$; the transaction

[5]Since the farmland price equation has lagged prices along with $\ln R_{t-1}$, following Burt and Taylor (p. 218), we consider the case of two interdependent processes: the gross return equation AR(1) and farmland price equations, as specified above. In this general case, the derivation yields reduction of lagged farmland price, while the gross return equation remains AR(1).

cost on farmland purchase per acre is $tc_p = 0.01 * P_t$; and the transaction cost on selling farmland and machinery and equipment per acre is $tc_s = 0.06 * P_t + 0.07 * \kappa$. The borrowing constraint has a limit on the maximum debt-to-asset ratio allowed for purchasing farmland: $\rho = 0.7$.

As described in Section 4.5, corn and soybeans are planted in equal proportion. so we calculate cost of production per acre for the 50-50 corn-soybean mix. From farm records of SWFBMA for year 1983-99 (Olson et al.), we find the average cost of production as \$217 per acre. For the farmer's own labor and management costs, we assume an opportunity cost of \$30 per acre. Thus, the total costs of production per acre is $c = \$247$.

Chapter 5

Model Implementation and Accuracy Analysis

Estimation of the parameters presented in Chapter 4 is required for numerically solving the farmland investment problem. This chapter describes the implementation of the collocation method and presents the results on an accuracy analysis of alternative choices in the collocation method.

5.1 Model Implementation

The farmland investment problem specified in Chapter 2 is a sequential decision problem with terminal optimization (Section 3.3). In this problem, there are four state variables, gross return per acre from crops, R_t, farmland price per acre, P_t, farmland acreage, L_t, and net wealth, W_t. From Section 3.3 (page 25) for terminal optimization, Bellman's equation for any R, P, L, W, can be written as:

$$V_t(R,P,L,W) = \max_{x} \{E[V_{t+1}(R_{t+1}, P_{t+1}, L_{t+1}, W_{t+1}) \mid R_t = R, P_t = P, \\ L_t = L, W_t = W,\ x_t = x]\} \tag{5.1}$$

subject to the constraints of the farmland investment problem (2.2) with the terminal (boundary) condition:

$$V_{T+1}\left(R,P,L,W\right) = U\left(W\right)$$

The value function for each $t = 0, 1, ..., T$, is defined as:

$$V_t(R,P,L,W) \equiv \max_{\{x_\tau\}_{\tau=t}^{T}} E\left[U\left(W_{T+1}\right) \mid R_t = R, P_t = P, L_t = L, W_t = W\right] \tag{5.2}$$

Note that we have omitted the subscript t for the state variables in the value function because the value function is defined and solved for all states, not just the realized state in period t.

To solve the problem by dynamic programming method, the range of the four state variables, R_t, P_t, L_t, W_t, must be defined for estimation of the value function. The value function for this problem is solved for the following ranges of the states: $220 \leq R_t \leq 620$, $950 \leq P_t \leq 2,215$, $400 \leq L_t \leq 2,000$, and $0 \leq W_t \leq 6,000,000$. The ranges for R_t and P_t are based on a 90 percent region for their distributions, based on their estimated equations. The range for farmland is chosen for fully owned farms, where 400 is minimum acreage required for farming, and $2,000$ is maximum feasible number of acres the manager can own in the location. The lower bound of net wealth is due to the bankruptcy condition ($W_t < 0$), thus the lower bound $W_t \geq 0$

is the condition for staying in farming. The upper bound of net wealth is arbitrarily chosen to allow no debt at the upper bounds of farmland acreage and its price.

In solving the Bellman equation for each t, we obtain the value function for the ranges of the state specified above. However, given the state in period t from the above ranges, the Bellman equation contains the value function in period $t+1$, which needs to be computed for the states in period $t+1$. In backward recursion, we have the value function for the states in the above ranges. However, it needs to be computed when the state in period $t+1$ is not in the above the ranges.

The control variable, x_t, is restricted so that L_{t+1} satisfies the feasibility constraint: $400 \leq L_{t+1} \leq 2,000$ or $L_{t+1} = 0$. Given the range of farmland, the value function needs to be computed for $L_{t+1} = 0$, when all farmland is sold. As assumed in Section 2.1, once all farmland is sold, the business cannot re-enter farming, and the value function for state $L_{t+1} = 0$ can be computed as follows. When $L_{t+1} = 0$, the farm manager cannot use debt financing, since it is based on farmland as collateral. Thus, W_{t+1} is all invested in liquid assets. First, W_{T+1} is computed by compounding it at the liquid asset return rate, and then the utility function is evaluated as $U(W_{T+1})$. Since the liquid asset is a riskless investment, $U(W_{T+1})$ is the value function for $L_{t+1} = 0$, as defined in (5.2).

Net wealth in period $t+1$, W_{t+1}, can go out of the bounds from the above range due to the stochastic nature of gross return from crops and farmland price. Unlike the state variable L_{t+1}, it is generally not possible to use restrictions on the control variable to ensure that W_{t+1} will be in the allowable range.[1] When $W_{t+1} < 0$, the value function is computed for the bankruptcy condition. Under this condition, the farm is liquidated if net wealth is negative ($W_{t+1} < 0$) at any time, which makes $L_{t+1} = 0$. Thus, the value function can be computed for $L_{t+1} = 0$ as described in the above paragraph. However, the compounding is done at the debt rate because $W_{t+1} < 0$. When W_{t+1} is greater than the upper bound, the extra amount above the upper bound earns the liquid asset return rate and is compounded for period $T+1$ to compute the value function.[2] Note that compounding of net wealth for $W_t = 0$ yields $W_{T+1} = 0$. In order to maintain continuity of the value function, the value function for zero current net wealth, $W_t = 0$, is set as $U(W_{T+1}) = U(0)$.

For return from crops and farmland price, given the state R_t and P_t, the state equations give R_{t+1} and P_{t+1}. When the error term values for these state variables are on the tails of their distribution, R_{t+1} and P_{t+1} can go out of the bounds from the above ranges. To assure that the state in period $t+1$ is within the bounds, we assume $R_{t+1} = \max(\underline{R}, \min(g(R_t, \varepsilon_{1,t+1}), \overline{R}))$, where $\underline{R}$ is the lower bound, $\overline{R}$ is the upper bound from the range of gross return per acre given above, and $g(R_t, \varepsilon_{1,t+1})$ denotes the right hand side in the state equation of gross return per acre. This assumption assigns to the bounds the probability mass for the states beyond the

[1]For example, $W_t = 6000000$, $P_t = 1000$, $L_t = 1000$, and $x_t > -L_t$. If in the next period, $P_{t+1} = 1500$ and $R_{t+1} = 250$ with some probability, the net wealth will be above 6000000, and cannot be put into the bounds with the control variable.

[2]To compute the value function, which is utility of terminal wealth, first compounded liquid assets are added to the wealth from value function, where W_{t+1} is greater than the upper bound. The utility function of the sum was regarded as the value function.

bounds. Similarly, we make this assumption for the farmland price state variable to assure that P_{t+1} is within its bounds. This assumption is made because solving the Bellman equation requires the value function for the state in period $t + 1$.[3]

The objective function in the model is to maximize the expected utility of terminal net wealth. The utility function described in Chapter 2 in (2.8) is $U(W) \equiv \frac{W^{1-\theta}-1}{1-\theta}$, where $\theta \geq 0$ to represent risk neutrality or risk aversion. For the risk neutral case, when $\theta = 0$, utility function $U(W) \equiv \frac{W^{1-\theta}-1}{1-\theta} = W - 1$ is defined all W. Its strategically equivalent form is:

$$\tilde{U}(W) \equiv W,$$

and the objective function is to maximize terminal net wealth. For the risk averse case, $\theta > 0$, the utility function $U(W) \equiv \frac{W^{1-\theta}-1}{1-\theta}$ is specified only for $W \geq 0$. Furthermore, for $\theta = 1$, $U(W) \equiv \frac{W^{1-\theta}-1}{1-\theta} = \ln(W)$ is defined only for $W > 0$. As described above, given the state W_t from the nonnegative range, W_{t+1} can be negative or nonnegative due to the stochastic variables in the model. In order to define the utility function[4] for all W, we respecify it as:

$$\tilde{U}(W) = \begin{cases} U(W) & \text{if } W \geq b \\ \frac{U(b)}{b} * W & \text{if } W \leq b \end{cases}$$

[3]Another way to solve this problem, as proposed by Miranda and Fackler, is to widen the ranges of the states for given minimum and maximum error term from numerical integration. However, applying this method for these equations gives very wide bounds, which are well outside the range of value found in the data.

[4]From the bankruptcy condition, it may be considered that the decision maker is indifferent as to the amount of negative wealth. However, given positive current wealth, the decision maker's preferences are characterized in the range of negative wealth as an increasing function of wealth. Furthermore, indifference with the amount of negative wealth also gives a nonconcave utility function.

where $b > 0$ and $U(W) \equiv \frac{W^{1-\theta}-1}{1-\theta}$ for $\theta > 0$. This specification makes a linear curve for $W \leq b$ passing through the origin and thus makes a continuous and concave function.[5] This specification represents risk neutral preferences for $W \leq b$, while maintaining risk preferences by $U(W) \equiv \frac{W^{1-\theta}-1}{1-\theta}$ for $W \geq b$. We assume $b = 60000$ considering the range of the net wealth, [0,6000000].

As described in Section 2.2, due to the identity $W_t \equiv A_t + (P_t + \kappa - tc_s) * L_t$, the dynamic programming model can be formulated with either A_t or W_t as a state variable. In implementing the numerical methods of dynamic programming algorithm, the grid of all states is made to solve the problem. Using state A_t, there will be many states that imply negative net wealth, where the bankruptcy condition applies and the value function does not need to be estimated. Thus, formulating the model with A_t as state variable will be numerically inefficient.[6] Considering these issues, we formulated the model with W_t as a state variable.

[5]Featherstone, Preckel, and Baker (p. 87) also address the problem of defining the utility function for negative wealth. Their specification, however, gives a non continuous function. We also considered another specification by Keeney and Raifa (p. 173): $\widetilde{U}(W) = U(W + B) = \frac{(W+B)^{1-\theta}-1}{1-\theta}$, where B is chosen so that $(W + B) \geq 0$. However, this specification does not exhibit constant relative risk aversion. Furthermore, for the given ranges, B needs to be 2,000,000, which makes very low marginal utility $(W + B)^{-\theta}$ for given W.

[6]Other normalizations were also considered, such as defining state variable as $\frac{A_t}{(P_t+\kappa-tc_s)*L_t}$, whose range greater than or equal to -1 ensure nonnegative net wealth. However, this formulation increases the likelihood of next period state to go out of bounds.

5.2 Implementation of the Collocation Method

As mentioned on page 30 (Chapter 3, Section 3.4.1), implementing the collocation method to solve a dynamic programming problem requires specification of (i) the type of collocation basis functions and nodes, and the number of nodes, (ii) an optimization method for finding the maximum value, and (iii) a method for finding expected value. In this section, we describe these specifications. In Section 5.3, we develop a test for accuracy analysis and evaluate alternative choices to find a good approximation for solving the farmland investment problem. Based on the accuracy results, we find the best specification and present in the end of this chapter.

5.2.1 Basis Functions and Nodes

In implementing the collocation method, there are a number of choices available for collocation basis functions and nodes. The choice of basis functions and nodes depends on the characteristics of the function to be approximated. Besides choosing the type of basis functions and nodes, the number of nodes must also be chosen.

"Weierstrass' Approximation Theorem asserts that every continuous function on a closed interval can be approximated uniformly to any prescribed accuracy by a polynomial" (Schumaker, p. 91). However, he emphasizes that this theorem does not provide any guidance on the order of polynomials.

Polynomial basis functions have long been used in approximating a function due to their properties of smoothness, differentiability, and efficiency in numerically implementation. Bellman and Dreyfus and Bellman, Kalaba, and Kotkin suggest using polynomial basis functions with orthogonality properties, such as Chebyshev and Legendre polynomials. Though polynomials have attractive properties, they may not perform well due to their oscillating behavior. In general the accuracy of polynomials increases as the order of polynomials is increased. However, this does not hold for every function to be approximated, due to oscillatory behavior. For example, in Runge's function $\frac{1}{1+s^2}$, when approximated with Chebyshev polynomials and uniform nodes, the error grows as the order of polynomials is increased.[7] However, when Chebyshev nodes are used, instead of uniform nodes, the error decreases as the order of polynomials is increased.

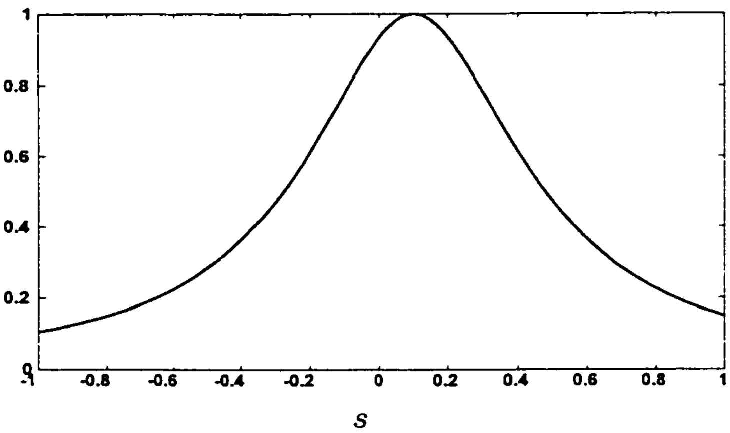
The choice of basis functions and nodes depends on the characteristics of the function to be approximated. Approximation theory suggests that use of Chebyshev polynomial basis functions coupled with Chebyshev nodes may be a superior choice for smooth functions. However, if the approximated function is not smooth, use of spline basis functions coupled with uniform nodes may be a better choice (Miranda and Fackler). Approximation with spline basis functions is made piecewise and has narrow supports. Thus, it avoids the typical oscillating behavior of polynomial interpolation (Santos, p. 338).

We explain here Chebyshev basis functions and nodes, and spline basis functions and uniform nodes. These basis functions and nodes are also explained by

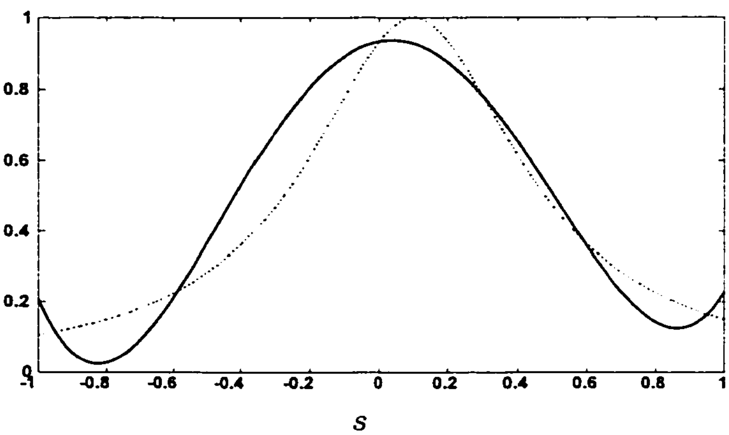
[7]The reader is referred to Schumaker (p. 101), Santos (p. 335), and Miranda and Fackler for examples and their discussion.

Miranda and Fackler, Gerald and Wheatly (Chapters 3 and 10), and Santos (p. 333). We illustrate these basis functions and nodes by using a type of Runge's function $F(s) = \frac{1}{1+7(s-0.1)^2}$, where $s \in [-1, 1]$. Figure 5.1(a) plots the original function. In Figure 5.1(b), this function is approximated by Chebyshev polynomial basis functions coupled with Chebyshev nodes. In Figure 5.1(c), this function is approximated by linear spline basis functions coupled with uniform nodes. In both cases, the number of nodes is 5. In Figures 5.2 (a) and (b), we use 9 nodes for approximation. For 9 nodes, Figures 5.2(c) shows locations of nodes for Chebyshev and uniform (i.e. equally spaced) nodes. Chebyshev nodes are more concentrated at the corners. Chebyshev nodes are appropriately located to avoid oscillation since Chebyshev interpolation is for the whole domain of the interval. However, in the spline interpolation, the function is approximated by dividing the domain in smaller intervals.

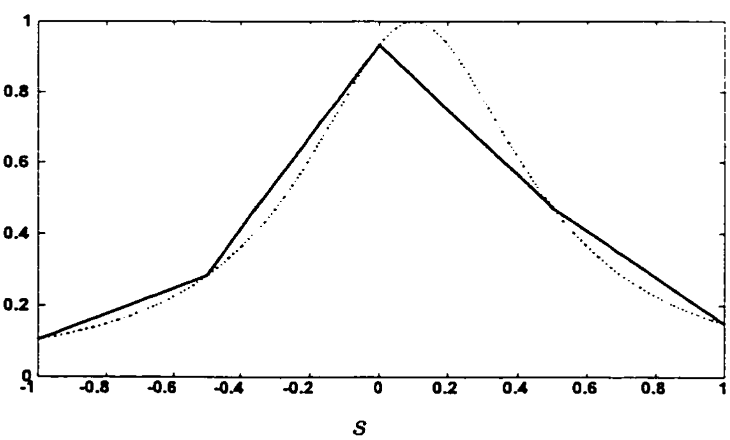
Note that, for both cases, the figures show that the approximated functions improve their accuracy as the number of nodes is increased from 5 to 9. When we compare 5.2(a) and 5.2(b), we find higher maximum absolute error in the spline interpolation than in the Chebyshev interpolation. This can be seen on these graphs at values of s between 0.1 and 0.2. However, the spline interpolation performs better than Chebyshev interpolation at other values in the domain.



(a) Original function: $F(s) = \frac{1}{1+7(s-0.1)^2}$

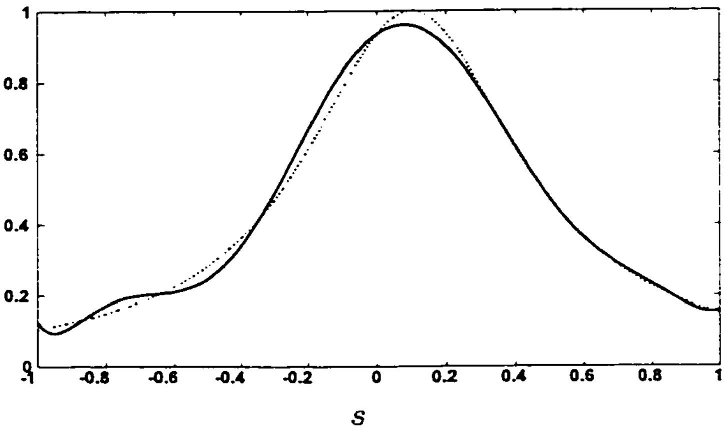


(b) Fitted with Chebyshev basis and nodes using 5 nodes

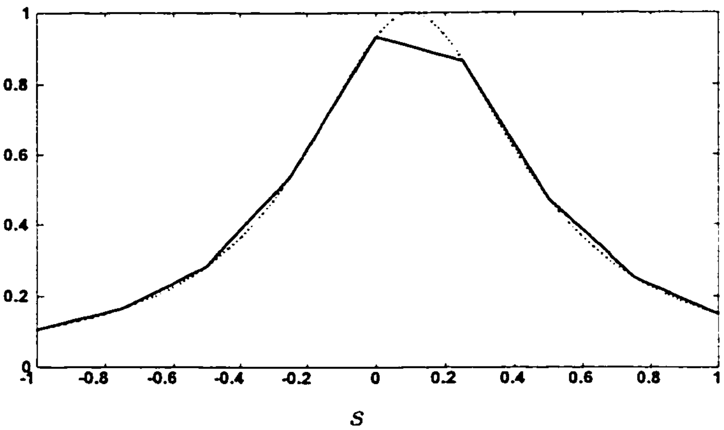


(c) Fitted with Linear spline basis and uniform nodes using 5 nodes

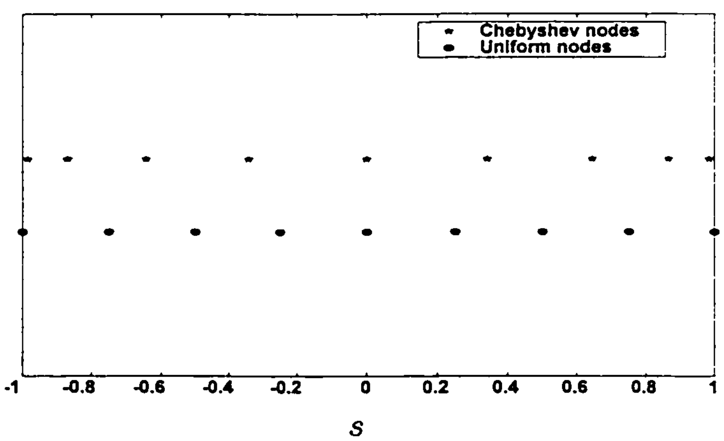
Figure 5.1: Approximation with 5 nodes



(a) Fitted with Chebyshev basis and nodes using 9 nodes



(b) Fitted with Linear spline basis and uniform nodes using 9 nodes



(c) Nodes

Figure 5.2: Approximation with 9 nodes

5.2.2 Optimization

When approximating the value function by the collocation methods, the optimal value function must be determined for each node using Bellman's equation. A variety of methods are available for numerical optimization, described in Miranda and Fackler, and Judd (1998, p. 93). The golden-section search and Nelder-Mead methods are derivative-free methods for finding a local optimum. These methods work for any continuous, bounded function defined on a finite interval. The Newton-Raphson method is a faster method for finding a local optimum. However, this method requires a twice differentiable function. Furthermore, this method only identifies a critical point in the function and does not make a distinction between a local maximum and a local minimum. This method requires second derivative information to determine whether the critical point is a local maximum or a local minimum. This method requires more search if we find, for example, a local minimum when we are searching a local maximum. Another problem with Newton-Raphson's method is that it may not converge (Judd, 1998, p. 96). It is important to note that the above methods provide a local optimum, instead of a global optimum required for most problems including the model of the study.

The grid search method specifies a grid of points in the interval and finds the optimum from those points (Judd, 1998, p. 100). Though this method is slow, it is likely to give a near global optimum, since the grid is specified for the entire range of the interval. Use of grid search method with the collocation method is referred to as hybrid method by Fackler and Miranda, because this method discretizes the control space while using continuous method for approximating the value function. The hybrid method is suitable for implementation in a matrix processing environment such as MATLAB or GAUSS (Fackler and Miranda). Discretization of the control space makes it possible to perform the optimization step for all elements of state set in a matrix. In the present study, we use this method for maximization in the dynamic programming model.

5.2.3 Integration

In Bellman's equation (3.9), the value function depends on the next period state, s_{t+1}. For each stochastic variable, the next period state is a function of a random shock, which has a continuous density function. In models with at least one stochastic state variable, the expected value function must be evaluated as part of dynamic programming procedure. Numerical integration is a practical approach for evaluating the expected value function.

Miranda and Fackler and Judd (1998, p. 251) describe a variety of methods available for numerical integration. The Gaussian quadrature method discretizes the continuous random variable to approximate the integral of a continuous density function. This method efficiently chooses these discrete points and their weights for approximating the integral (Judd, 1998, p. 257). A version of this method for normal random variables is called Gauss-Hermite quadrature, which is explained in Miranda and Fackler and Judd (1998, p. 261). From Chapter 4, the diagnostic tests indicate normal distribution of the error term for each stochastic variable equation: gross return per acre and farmland price. In the present study, we use Gauss-Hermite quadrature for evaluating the expected value function.

5.3 Accuracy Analysis

As noted in Section 3.4.1, solving a dynamic programming model requires the value function, which is a function of continuous-state variables. As in most applications the dynamic programming problem lacks a closed-form solution, so the value function must be numerically approximated by computational methods. It is important to know how accurate is the solution by numerical approximation, and from given alternatives, which scheme is more accurate for solving a problem. It is also important to recognize the limitations of computer storage capacity and of execution time. As illustrated in previous section, using more nodes may yield more accurate results, however, it requires more computer storage capacity and execution time in solving a dynamic programming problem. These limitations are even more serious with large-scale models and are referred to as the curse of dimensionality. Thus, there is a need to develop techniques for improving both accuracy and efficiency in solving large-scale dynamic programming problems.

The issue of accuracy in dynamic programming model is also addressed by Johnson et al., Santos, and Santos and Vigo-Aguiar. As noted in the literature, one method for checking accuracy is to compare the solution with an actual closed-form solution. However, this method cannot be applied in the absence of a closed-form solution. Another approach is to compare results from one method with results from a method assured to be more reliable (Santos, p. 346). However, this approach can be misleading since both numerical methods have error (Judd, 1998, p. 563). In this section, we develop a test for accuracy analysis by Monte Carlo simulation. Using this test, we examine the accuracy of collocation methods in solving the investment portfolio problem and propose techniques for improving their accuracy.

5.3.1 Accuracy Test

The accuracy test is explained by using the farmland investment problem and can be used for any finite planning horizon continuous-state dynamic programming model. In this study, this test is used for checking accuracy of the collocation method, however, it can also be used for checking the accuracy of other methods such as discretization.

As discussed in Section 5.2.1, suppose we have chosen a type of basis function and method for selecting nodes as required to solve the dynamic programming model by the collocation method. In the investment problem, there are four state variables, and suppose we have also chosen the number of nodes for each state variable. Let $N = \Pi_{j=1}^{4} N_j$ be the total nodes where N_j are nodes for state $j = 1, 2, 3, 4$.

For the accuracy analysis here, we assume a 20-year planning horizon. Following the dynamic programming procedure, we approximate the value function for each t by $V_t(R,P,L,W) \approx \sum_{i=1}^{N} c_{it}\phi_i(R,P,L,W)$. Having approximated the value function, the optimal policy x_t^* for any state, R,P,L,W, can be found by solving:

$$\max_x \{E[\sum_{i=1}^{N} c_{i,t+1}\phi_i(R_{t+1},P_{t+1},L_{t+1},W_{t+1}) \mid R_t = R, P_t = P, L_t = L, W_t = W]\} \tag{5.3}$$

subject to the constraints of the farmland investment problem (2.2).

Suppose we have a initial state R_0, P_0, L_0, W_0. For this initial state at $t = 0$ we solve the above problem (5.3) using the approximated value function. This gives the optimal policy x_0^* and the maximum objective function at that policy, $\tilde{V}$, for this initial state. We refer to $\tilde{V}$ as the estimated value function, since it is based on estimation by the collocation method.

The dynamic programing method, by using the value function, establishes that the optimal policy and its objective function take into account the future decisions. By definition of the value function of this problem, $\tilde{V}$ is the expected utility of net terminal wealth for this initial state. To test the accuracy of the solution, we simulate the objective function by the Monte Carlo method. For this initial state in $t = 0$, we apply the optimal policy x_0^* derived using (5.3). Given this optimal policy, the state in next period depends on the random shocks. In the model we have error terms in each of two stochastic state variable equations: gross return per acre, R_t, and farmland price per acre, P_t. These error terms have normal density functions and their parameters are estimated in Chapter 4. We draw a 20-year set of the error terms from these density functions. The optimal policy x_0^* and the error term in the next period determine the state in period $t = 1$. Then we find the optimal policy for period $t = 1$. Following this procedure to the end of the planning horizon, we find the states for each period. That is, this procedure gives a path of states from the initial state to the end of planning horizon state. Of our interest is the state in period $T+1$, $R_{T+1}, P_{T+1}, L_{T+1}, W_{T+1}$, and we compute $U(W_{T+1})$.

To compute $E[U(W_{T+1})]$, we draw 500 trails of error terms, and follow the above procedure for each of 500 trials. Since there are 20 years, these will be $500 * 20 = 10,000$ sets of error terms for each stochastic variable: R_t and P_t. From the above procedure, now we have 500 paths of states and 500 values of $U(W_{T+1})$. The average of these values is the expected utility of net terminal wealth, $E[U(W_{T+1})]$. We refer to this as the expected simulated value function, denoted by $E[\mathrm{V_{sim}}]$.

For the accuracy analysis here, we choose 81 different initial states in period $t = 0$ from the state space.[8] For each initial state, we follow the above procedure and compute $E[\mathrm{V_{sim}}]$ and $\tilde{V}$.

It is important to recognize that (5.3) has an approximated value function which is based on specification of the collocation method including the type of basis function, method for selecting nodes, and number of nodes. For accuracy analysis, we propose two tests. Our first test is for comparing alternative specifications (schemes) of the collocation method. Each scheme gives an optimal policy and the associated value

[8]Given the ranges of the state variables in Section 5.1, we choose $R_0 = [320, 420, 520]$, $P_0 = [1265, 1580, 1900]$, $L_0 = [800, 1200, 1600]$, $W_0 = [1500000, 3000000, 4500000]$. We make a grid of 81 states by their Cartesian product.

functions. We can compare alternative collocation schemes by determining which method gives a higher $E\left[\mathrm{V}_{\mathrm{sim}}\right]$, since the objective function is to be maximized. This comparison provides the gain in the objective function from one scheme to another.

The second test of accuracy is to measure the absolute error in approximating the value function. For each initial state level, the absolute error in the value function is calculated by dividing the difference between the expected simulated value function and the estimated value function by the expected simulated value function, then taking the absolute value:

$$\text{Absolute Error} = \left| \frac{E\left[\mathrm{V}_{\mathrm{sim}}\right] - \tilde{\mathrm{V}}}{E\left[\mathrm{V}_{\mathrm{sim}}\right]} \right|$$

The absolute error is calculated for each of 81 initial states. The maximum absolute error is calculated as the maximum of the absolute error from these 81 initial states. The average absolute error is calculated as the mean for 81 initial states. Computation of errors in this way provides information on the error in the value function associated with a collocation method. For comparing alternative collocation schemes, we can also compare the error associated with each scheme.

In our experiments, we will perform both tests. We expect that the smaller the absolute error in the value function, the higher the value of $E\left[\mathrm{V}_{\mathrm{sim}}\right]$ will be in the objective function. It is important to note that the second test provides the information on how well the value function is approximated. The first test provides the gain in the objective function, no matter how much error there is in the approximated value function.

5.3.2 Specification for Experiments

Chebyshev nodes are not evenly spaced, and their scheme does not assign the nodes in the corners of the range for interpolation. The function approximated by interpolation may give poor approximation for extrapolation. To avoid extrapolation, we increase the range of each state variable for Chebyshev nodes, so that the first and the last nodes are the lower and upper bound of the range respectively. Uniform nodes are evenly spaced and include the corners of the range for interpolation. Spline interpolation gives good approximation with an odd number of nodes (Schumaker). Thus, we choose odd numbers, 3,5,7,9,..., for the number of nodes.

In the present study, we use the Gaussian quadrature method of numerical integration. In the model, we have two stochastic state variables, R_t and P_t. For each state we use 5 nodes for the numerical integration. This gives 25 combinations with their probabilities.

All experiments for accuracy analysis are done for the risk neutral case, $\theta = 0$, and for the value function in period $t = 0$ with a 20-year planning horizon. All other parameters are as specified in Chapter 4.

The dynamic programming model is programmed in MATLAB. For evaluating the basis functions and nodes for the collocation methods, and for numerical integration, we use MATLAB code developed by Miranda and Fackler. The model was run using MATLAB on a Dell PC with 512 MB RAM and a 800 Mhz processor. The CPU time is measured in seconds.

5.3.3 Chebyshev and Linear Spline

In this section we compare accuracy between (i) Chebyshev basis function and nodes and (ii) linear spline basis function and uniform nodes. For all cases in this section we discretize the control space, X, in 81 levels to find the optimal policy.

Table 5.1 presents the CPU time and absolute error for the collocation with Chebyshev and linear spline methods. There are four state variables in the model, R_t, P_t, L_t, W_t, so the value function has four dimensions. The first column of Table 5.1 indicates the number of nodes in each dimension. As we increase the number of nodes, the average and maximum absolute errors decrease for both Chebyshev and linear spline collocations. Note that the marginal advantage of increasing the number of nodes declines, as shown in Figure 5.3(a). However, the CPU time increases exponentially with increases in the number of nodes, as shown in Figure 5.3(b). This is because the model is multidimensional. For example, with 3 nodes in each dimension, the total number of nodes is $3^4 = 81$, and with 9 nodes in each dimension, the total number of nodes is $9^4 = 6561$. Note that the value function estimation makes a good approximation with 5 nodes in each dimension, as the average error is around 2 percent.

Table 5.2 presents the average (i.e. mean of 81 initial states) $E\left[\mathrm{V_{sim}}\right]$ for different numbers of nodes for Chebyshev and linear spline methods. We note that as the number of nodes increases, average $E\left[\mathrm{V_{sim}}\right]$ gets higher. Again note that the marginal advantage of increasing the number of nodes declines, as shown in Figure 5.3(c).

Table 5.1: CPU Time and Absolute Error in Value Function

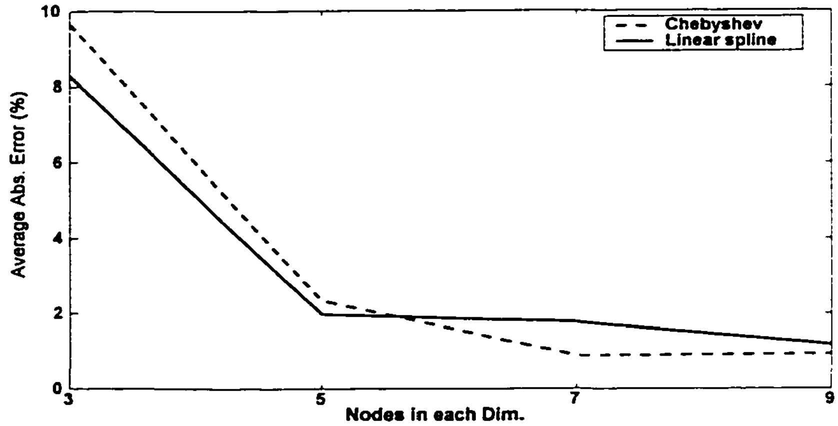
Nodes in each Dim.	Chebyshev CPU Time (Seconds)	Chebyshev Average Absolute Error (%)	Chebyshev Maximum Absolute Error (%)	Linear Spline CPU Time (Seconds)	Linear Spline Average Absolute Error (%)	Linear Spline Maximum Absolute Error (%)
3	105	9.65	18.68	219	8.27	26.50
5	500	2.34	4.20	766	1.98	7.38
7	5,959	0.88	2.32	3,858	1.77	5.25
9	66,845	0.93	2.27	24,665	1.18	3.42

Table 5.2: Expected Simulated Value Function

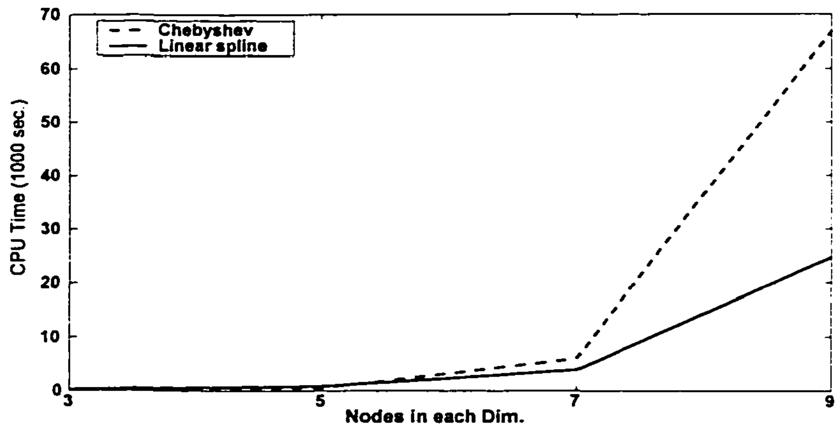
Nodes in each Dim.	Chebyshev Average $E[V_{sim}]$ ($10^6)	Chebyshev Comparison with 9 Nodes: % Decrease in Average $E[V_{sim}]$	Chebyshev Comparison with 9 Nodes: % Frequency $E[V_{sim}]$ decreased	Linear Spline Average $E[V_{sim}]$ ($10^6)	Linear Spline Comparison with 9 Nodes: % Decrease in Average $E[V_{sim}]$	Linear Spline Comparison with 9 Nodes: % Frequency $E[V_{sim}]$ decreased
3	10.038	0.26150	92.59	10.026	0.3823	98.77
5	10.054	0.11240	90.12	10.048	0.1609	82.72
7	10.065	0.00004	38.27	10.064	0.0058	53.09
9	10.065	- - -	- - -	10.064	- - -	- - -

Also note that, for all cases, the maximum gain in the objective function is less than 0.5 percent. For the Chebyshev approach, the average $E[V_{sim}]$ with 3 nodes is only 0.26150 percent less than the average $E[V_{sim}]$ with 9 nodes. However, the frequency, calculated from 81 initial states, of lower $E[V_{sim}]$ from 9 to 3 nodes is 92.59 percent. For the linear spline approach, this frequency is 98.77. This indicate that the probability of poor performance with 3 nodes, as compared to 9 nodes, is very high.

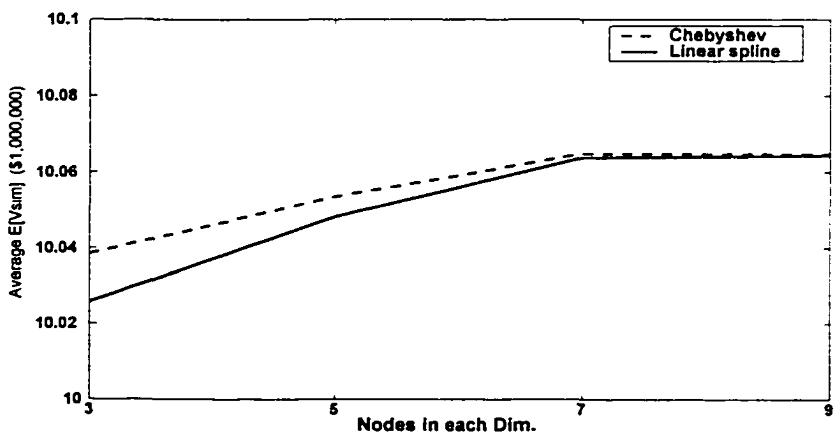
Table 5.3 compares the Chebyshev and linear spline collocation schemes. As noted above, with increase in the number of nodes, the CPU time increases exponentially.



(a)



(b)



(c)

Figure 5.3: (a) Average absolute error, (b) computation time, (c) average simulated value function

Table 5.3: Comparison of Chebychev and Linear Spline

Nodes in Each Dim.	Ratio of CPU Time Chebyshev/Spline	Ratio of Av. $E[V_{sim}]$ Chebyshev/Spline	Comparison of % Frequency $E[V_{sim}]$ increased with Cheb. vs. Spline	Spline vs. Cheb.
3	0.48	1.00126	62.96	37.04
5	0.65	1.00053	37.04	62.96
7	1.54	1.00011	43.21	56.79
9	2.71	1.00005	59.26	40.74

However, the linear spline collocation takes much less time than Chebyshev collocation with 7 and 9 nodes.[9]

As more nodes improve the accuracy, we compare the results with 9 nodes for Chebyshev and linear spline collocations. From Table 5.1, we note that 9 nodes gives 0.93 percent error with Chebyshev collocation and 1.18 percent with linear spline collocation. However, the CPU time in Chebyshev collocation is 2.7 times the CPU time in linear spline collocation (Table 5.3). Furthermore, the gain in $E\left[V_{sim}\right]$ with Chebyshev is only by the factor 1.00005, or 0.005 percent (Table 5.3). Thus for 9 nodes, the linear spline collocation may be a better choice than Chebyshev collocation for this problem. The fourth column of the table shows percent frequency, calculated from 81 initial states, $E\left[V_{sim}\right]$ with Chebyshev is greater than $E\left[V_{sim}\right]$ with linear spline. The last column of the table compares this frequency for linear spline versus Chebyshev. The frequency percent indicates no clear-cut superiority of either Chebyshev or linear spline.

[9]Miranda and Fackler indicate that this is due to the use of sparse function in MATLAB.

The approximation with spline basis functions is made piecewise and has narrow supports, thus it avoids the typical oscillating behavior of polynomial interpolation (Santos, p. 338). The solution of the model with Chebyshev collocation indicates some unexpected behavior of the policy function. Considering this and the above accuracy analysis, we choose the linear spline approach for our analysis.[10]

5.3.4 Optimization Procedure

For the accuracy analysis presented in the previous subsection, we discretized the control space, X, in 81 levels to find the optimal policy. In this section we introduce a two-stage grid search method. By this method, we first discretize the control space, X, in 41 levels to find the optimal policy. For the interval $[400, 2,000]$ of farmland acreage state variable, L_t, this discretizes the farmland purchase/sale control variable, x_t, with a 40-acre increment. Given the optimal policy from the first stage, we find a new control space for the second stage optimization For the second stage the lower bound is the optimal policy minus 40 acres, and the upper bound is the optimal policy plus 40 acres. This new control space is determined such that it satifies the constraints of the model. This control space is again discretized in 21 levels. The two-stage method of optimization is applied with linear spline basis functions and uniform nodes.

[10]As specified in Section 5.3.2, the accuracy results are presented for the risk neutral case ($\theta = 0$). The preliminary results of the risk averse case ($\theta = 1$) indicate that linear spline is much superior to Chebyshev by every criterion. As compared to Chebyshev, the linear spline collocation gives a higher average $E[\mathrm{V}_{\mathrm{sim}}]$, has less absolute error in the value function, and takes less CPU time.

Table 5.4 presents the CPU time, absolute error, and the average $E\left[\mathrm{V}_{\mathrm{sim}}\right]$ for the two stage optimization with the linear spline method. In this case also, as we increase the number of nodes, the average and maximum absolute errors decrease and $E\left[\mathrm{V}_{\mathrm{sim}}\right]$ gets higher.

Comparison between the one-stage and the two stage optimization methods is presented in Table 5.5. The table shows that the CPU time is reduced to about 77 percent with the two-stage method, as compared to the one-stage method. The average $E\left[\mathrm{V}_{\mathrm{sim}}\right]$ for the two-stage method is slightly higher than that for the one-stage method, as their ratio is greater than one, shown in Table 5.5. Furthermore. the frequency percent (defined in the previous subsection) also indicates a higher probability of better performance with the two-stage method than that with the one-stage method. The average absolute error for both methods is identical, which can be noted by comparing Table 5.4 and Table 5.1 for the linear spline case. These results show that the two-stage method is more efficient, as it takes less CPU time and performs slightly better than one-stage method. Thus we use two-stage with linear spline for our analysis.

5.3.5 Node Configuration

The accuracy analysis presented in the above subsections shows that accuracy increases as we increase the number of nodes, however this comes at a cost of increased CPU time. Despite advances in computer technology, solving a large-scale multidi-

Table 5.4: Two-stage Optimization with Linear Spline

Nodes in each Dim.	CPU Time (Seconds)	Average Absolute Error (%)	Maximum Absolute Error (%)	Average $E[V_{sim}]$ ($\$10^6$)
3	165	8.27	26.5	10.026
5	584	1.98	7.38	10.048
7	3,006	1.77	5.24	10.064
9	18,946	1.18	3.42	10.064

Table 5.5: Comparison of Two-stage and One-stage Optimization with Linear Spline

Nodes in Each Dim.	Ratio of CPU Time	Ratio of Av. $E[V_{sim}]$	Comparison of % Frequency $E[V_{sim}]$ increased	
	2-stage/1-stage	2-stage/1-stage	2-stage vs. 1-stage	1-stage vs. 2-stage
3	0.76	1.00001	70.07	29.93
5	0.76	1.00001	60.49	39.51
7	0.78	1.00001	71.60	28.40
9	0.77	1.00001	64.20	35.80

mensional dynamic programming model can only be done by limiting the number of nodes. The farmland investment problem has four state variables (R_t, P_t, L_t, W_t) requiring specification of the number of nodes for each state variable. We note that CPU time increases exponentially when we increase number of nodes in each dimension. Since this is a multidimensional approximation, it is possible that some state variables may need more nodes and other state variables may need less nodes. In this subsection we perform an accuracy analysis for alternative node configurations. We use two-stage optimization with linear spline collocation.

Previous results and results in Table 5.4 show that the model performs poorly with 3 nodes. Thus, for node configurations we start comparing 5 and 9 nodes. Rather than increasing nodes in every dimension, we first increase nodes in each dimension separately. Table 5.4 shows that the average $E[\mathrm{V_{sim}}]$ is $10.048 * 10^6$ when each state variable has 5 nodes (5,5,5,5). The following table shows the number of nodes for each state variable ordered by (R_t, P_t, L_t, W_t) with their average $E[\mathrm{V_{sim}}]$:

Nodes:	(9,5,5,5)	(5,9,5,5)	(5,5,9,5)	(5,5,5,9)
Average $E[\mathrm{V_{sim}}]$ (10^6) :	10.052	10.053	10.049	10.062

The results show that average $E[\mathrm{V_{sim}}]$ improves the most, from $10.048*10^6$ to $10.062*10^6$, when we increase the number of nodes for the state variable net wealth, W_t, from 5 to 9. The results also show that the average $E[\mathrm{V_{sim}}]$ improves the least when we increase number of nodes for the state variable farmland acreage, L_t, from 5 to 9. Thus this indicates the marginal benefit of increasing nodes is low for L_t and high for W_t.

Given these results, we start with 9 nodes in W_t and 5 nodes in the other state variables (5,5,5,9), where the average $E[\mathrm{V_{sim}}]$ is $10.062 * 10^6$, from the above table. Now we experiment of increasing number of nodes for R_t, P_t, L_t. When we increase the nodes of L_t from 5 to 9, (5,5,9,9), the average $E[\mathrm{V_{sim}}]$ remains unchanged, i.e. $10.062 * 10^6$. However, for each R_t, (9,5,5,9), and P_t, (5,9,5,9), the average $E[\mathrm{V_{sim}}]$ improves from $10.062 * 10^6$ to $10.065 * 10^6$. Also, the average $E[\mathrm{V_{sim}}]$ is $10.065 * 10^6$ for (7,7,5,9).

It is important to note that the total number of nodes is equal to the product of number of nodes for each state variable ($N = N_1 * N_2 * N_3 * N_4$). The CPU time increases as N increases. For node configuration (7,7,5,9), $N = 2,205$. If we increase two nodes of L_t, (7,7,7,9), $N = 3,087$. If we increase two nodes of W_t, (7,7,5,11), $N = 2,695$, which is much less than $3,087$.

Table 5.6 presents the CPU time, absolute error, and the average $E[\mathrm{V_{sim}}]$ with two stage optimization and the linear spline method for different number of nodes for W_t. In this case also, as we increase the number of nodes, the average and maximum absolute errors decrease and average $E[\mathrm{V_{sim}}]$ gets slightly higher. The results are compared with increasing number of nodes in every dimension in Figure 5.4. Figure 5.4(a) plots the average absolute error as a function of time. The figure shows that changing the node configuration gives less average absolute error for a given CPU time. Similarly, Figure 5.4(b) shows a higher average $E[\mathrm{V_{sim}}]$ with changed node configuration for a given CPU time.

From Table 5.6, we have results with nodes $(7, 7, 5, 21)$, where average $E[\mathrm{V_{sim}}]$ is $10.067 * 10^6$, the maximum absolute error is 1.7 percent, and the average absolute error is 0.6 percent. Without changing the node configuration, we have results for $(9, 9, 9, 9)$ in Table 5.4, where average $E[\mathrm{V_{sim}}]$ is $10.064 * 10^6$, the maximum absolute error is 3.42 percent, and the average absolute error is 1.18 percent. Thus, changing the node configuration yields more accurate approximation by each criterion. Furthermore, it is even more efficient, as the CPU time is $7,899$ seconds with nodes $(7, 7, 5, 21)$, which is much less than with nodes $(9, 9, 9, 9)$, where CPU time is $18,946$. This is due to the fact that the multiplication of number by increasing each number becomes a bigger number: $9 * 9 * 9 * 9 = 6,561$, whereas, $7 * 7 * 5 * 21 = 5,145$.

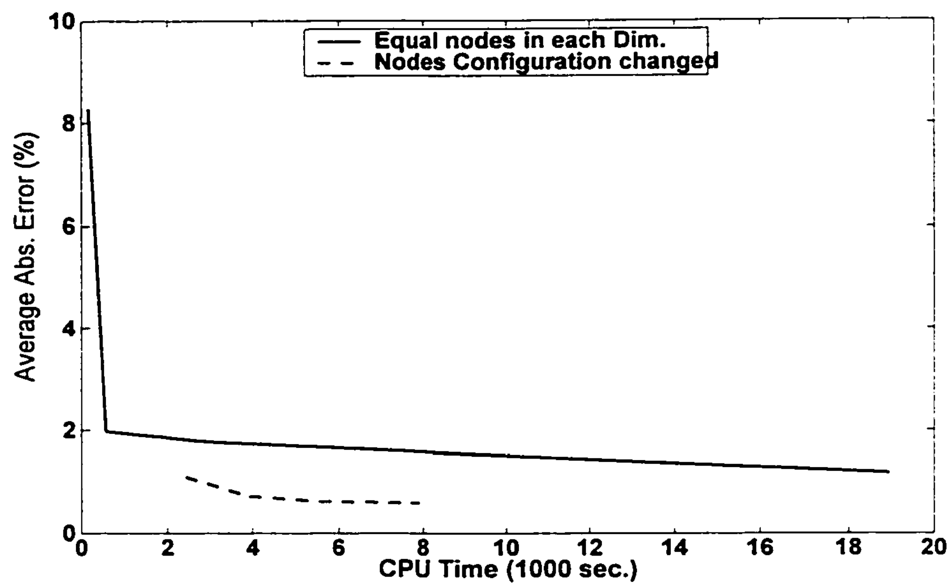
Table 5.6: Nodes Configuration with Two-stage Linear Spline

Nodes in each Dim.	Spline Two Stage: CPU Time (Seconds)	Average Abs. Error (%)	Maximum Abs. Error (%)	Average $E[V_{sim}]$ (\$10⁶)
7,7,5,9	2,435	1.10	3.28	10.065
7,7,5,13	3,877	0.73	2.28	10.067
7,7,5,17	5,568	0.63	1.84	10.067
7,7,5,21	7,899	0.60	1.70	10.067

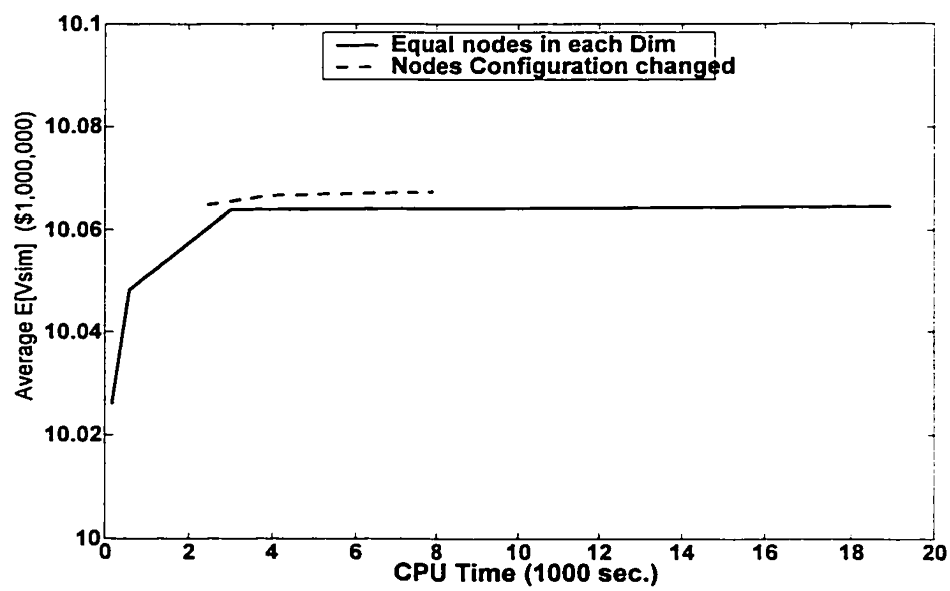
These results show that a change in node configuration can improve the performance of approximation with reduced error and less CPU time. This is because the multidimensional function may be related differently with its variables, some variables may require only few nodes and other variables may require more nodes to approximate it for a desired accuracy level.

From the accuracy analysis, we can search for the best specification for approximation of a problem. Furthermore, these accuracy results show the robustness of the approximated function.

For farmland investment model results in the next chapter, we use the two-stage with the linear spline and nodes $(7, 7, 5, 21)$. As in the accuracy analysis, for the model results we also use 41 control levels for the first stage and 21 control levels for the second stage of the optimization. Also for numerical integration, we use 5 nodes for each of the two stochastic state variables, which gives 25 combinations with their probabilities.



(a)



(b)

Figure 5.4: Node configuration

Chapter 6

Results

This chapter presents results from the farmland investment model specified in Chapter 2. First we present the model results for the risk neutral case. Next, for comparative dynamics, we present results after changes in the planning horizon, interest rate, variances of stochastic shocks, and relative risk aversion.

Before presenting the results, we rewrite the farmland investment problem with specification of the objective function and stochastic equations and with a summary of model parameters, based on Chapters 4 and 5.

The farmland investment model has four state variables: gross return per acre from crops, R_t, farmland price per acre, P_t, farmland acreage, L_t, and net wealth, W_t. The model has one control variable, x_t, which is number of acres of farmland purchased or sold. A positive value of x_t represents purchase, while a negative value represents sale of farmland.

The farmland investment problem specified in Chapter 2 can be written as:[1]

$$\max_{\{x_t\}_{t=0}^{T}} E_0\left[\tilde{U}\left(W_{T+1}\right)\right] \tag{6.1}$$

subject to :

$$
\begin{aligned}
\ln R_{t+1} &= \beta_0 + \beta_1 \ln R_t + \varepsilon_{1t+1}, \\
\ln P_{t+1} &= a_0 + a_1 \ln P_t + a_2 \ln R_t + \varepsilon_{2,t+1}, \\
L_{t+1} &= L_t + x_t, \\
W_{t+1} &= (1+r)\left[\{W_t - (P_t + \kappa - tc_s) * L_t\} - (P_t + \kappa + tc) * x_t - c * L_{t+1}\right] \\
&\quad + R_{t+1} * L_{t+1} + (P_{t+1} + \kappa - tc_s) * L_{t+1}, \\
x_t &\in X_t, \text{ for } t = 0, 1, 2, ..., T, \ (R_0, P_0, L_0, W_0) \text{ are given,}
\end{aligned}
$$

where

$$r = \begin{cases} r_b \text{ if } \left[\{W_t - (P_t + \kappa - tc_s) * L_t\} - (P_t + \kappa + tc) * x_t - c * L_{t+1}\right] > 0 \\ r_l \text{ otherwise,} \end{cases}$$

$$tc = \begin{cases} tc_p \ \text{ if } x_t > 0 \\ -tc_s \text{ otherwise.} \end{cases}$$

The objective in the model is to maximize the expected utility of terminal net wealth. For the risk neutral case, when $\theta = 0$, the utility function $U(W) \equiv \frac{W^{1-\theta}-1}{1-\theta} = W - 1$, which is defined for all W. Its strategically equivalent form is $\tilde{U}(W) \equiv W$, in which case the objective is to maximize expected terminal net wealth. For the risk averse case, $\theta > 0$, the utility function, $U(W) \equiv \frac{W^{1-\theta}-1}{1-\theta}$, is specified only for $W \geq 0$.

[1] $t+1$ is time index used as a subscript in all variables including $\varepsilon_{1,t+1}$, where ε_1 is random shock in state equation of R_t.

Furthermore, for $\theta = 1$, $U(W) \equiv \frac{W^{1-\theta}-1}{1-\theta} = \ln(W)$ is defined only for $W > 0$. To define preferences for all W including $W \leq 0$, as described in Chapter 5 on page 51, we have $\tilde{U}(W) = \frac{W^{1-\theta}-1}{1-\theta}$ if $W \geq b$, and $\tilde{U}(W) = \frac{b^{1-\theta}-1}{1-\theta} * \frac{W}{b}$if $W \leq b$, where $b > 0$.

The above specification of the state equations for R_t and P_t, their estimates, and other parameters are described in Chapter 4. Estimates for R_t equation are $\beta_0 = 1.052028$, $\beta_1 = 0.82197$, and error term $\varepsilon_1 \sim N(0, 0.033155)$, denoting a normal distribution with its mean and variance. Estimates for P_t equation are $a_0 = 0.048655$, $a_1 = 0.884465$, $a_2 = 0.134044$, and error term $\varepsilon_2 \sim N(0, 0.014619)$. E_0 is the expectation operator over the random shocks $\varepsilon_{1,t+1}, \varepsilon_{2,t+1}$. The interest rate on borrowing is $r_b = 0.06$. On lending (riskless investment), it is $r_l = 0.03$. The price of machinery and equipment per acre is $\kappa = \$300$; the transaction cost on farmland purchase per acre is $tc_p = 0.01 * P_t$; and the transaction cost on selling farmland and machinery and equipment per acre is $tc_s = 0.06 * P_t + 0.07 * \kappa$. The costs of production per acre is $c = \$247$. As described in Section 2.1, the control space X_t is defined by a set of four constraints including a borrowing constraint, where the maximum debt-to-asset ratio allowed for purchasing farmland is $\rho = 0.7$. To solve the above problem, Chapter 5 describes implementation of collocation method of dynamic programming.

Dynamics of the four state variables are given by the state equations of the model. Gross return and farmland price per acre, R_t and P_t, are stochastic and are unaffected by the control variable x_t because this is a firm level decision model where returns and prices cannot be affected by an individual firm's action.

Net wealth, W_t, is the sum of liquid assets and the net sale-value of farm assets, which is equal to the value of farmland and the associated farm machinery and equipment minus the transaction costs of selling them. The difference between net wealth and the net sale-value of farm assets determines liquid assets, which can be negative implying debt financing or positive implying riskless investment. Though the investment decision is the number of acres purchased or sold, this decision implicitly also determines the amount of either debt financing or investment in the riskless asset. The dynamics of farmland acreage, L_t, is determined through the purchase and sale decision x_t. This decision along with values of the two stochastic variables, R_t and P_t, determines the dynamics of net wealth, W_t.

The dynamics of gross return per acre from crops, R_t, and farmland price, P_t, are given by their respective state equations. Note that R_t and P_t follow mean reverting Markovian stochastic processes. At the beginning of each year, when the investment decision is made, these state equations are the basis for forecasts of the distribution of the gross return and farmland price in next period, given current gross return and farmland price.

There are two important properties of these processes, which are illustrated here for the gross return equation. First, the forecasted expected gross return is increasing in the current gross return, as the slope coefficient β_1 is positive. Second, in the long run, the expected crop return approaches the long-run mean, regardless of the current return, and is therefore referred to as a mean reverting process. The long-run mean of the gross return is \$390 per acre. As described on page 33, if the current gross return is \$390, the expected gross return in the next period is \$390. If the current gross return is greater than \$390, say $R_t = \$450$, the expected gross return in next period is greater than \$390 but less than \$450. Similarly, when the current gross return is less than \$390, the expected gross return in next period is less than \$390 but greater the current return.

The expected farmland price depends on both the current farmland price and the current gross returns. It is increasing in the current farmland price and the current gross return. Farmland price also follows a mean reverting process with a long-run mean of $\$1,500$ per acre.

As our analysis is at the level of the individual farm firm, the results are presented as a behavior of an individual firm solving the above problem in the given economic environment. The model results can also be analyzed to compare the optimal investment decision of farms with different wealth levels, planning horizon, riskiness, or risk preferences.

6.1 Results for Risk Neutral Case

This section presents results of the model for the risk neutral case, $\theta = 0$, and for a 20-year planning horizon, where decisions are made at the beginning of each year $t = 0, 1, ..., 19$. At any time t the farm manager can choose the number of acres to purchase or sell, subject to constraints based on current wealth and land ownership. In this section, we first present the optimal investment policy function at time $t = 0$. Next, we present the stochastic dynamic path, which examines how the process evolves over time starting from some initial state.

6.1.1 Optimal Investment Policy Function

The optimal investment policy is a function of current states: gross return, R_t, farmland price, P_t, farmland acres, L_t, and net wealth, W_t. Figure 6.1(a) presents a graph of the optimal farmland investment policy, x_t^*, as a function of farmland price and the gross return, (P_t, R_t), for $L_t = 600$ and $W_t = \$300,000$ at time $t = 0$. The optimal policy is also plotted in a two dimensional graph as a function of only farmland price for the gross returns of \$350 and \$450 in Figure 6.1(b). In this figure, when $R_t = 450$, the optimal investment policy is to buy 203 acres at the farmland price $P_t = \$950$. The optimal policy, x_t^*, decreases as P_t increases until P_t reaches \$1480, where $x_t^* = 0$. This remains optimal up to $P_t = \$2,075$. For $P_t > 2,075$, x_t^* is negative and nonincreasing. This shows that the optimal policy function is nonincreasing in P_t for given R_t. These graphs also show that the optimal policy function is nondecreasing in R_t for given P_t. For $P_t = \$950$, the optimal policy is to buy 203 acres for all R_t ranging from \$350 to \$450. For $P_t = \$1200$, the optimal policy is $x_t^* = 0$ for R_t from \$350 to \$380. For $R_t > 380$, x_t^* is positive and increases in R_t. The optimal policy is to buy 89 acres for R_t ranging from \$400 to \$450.

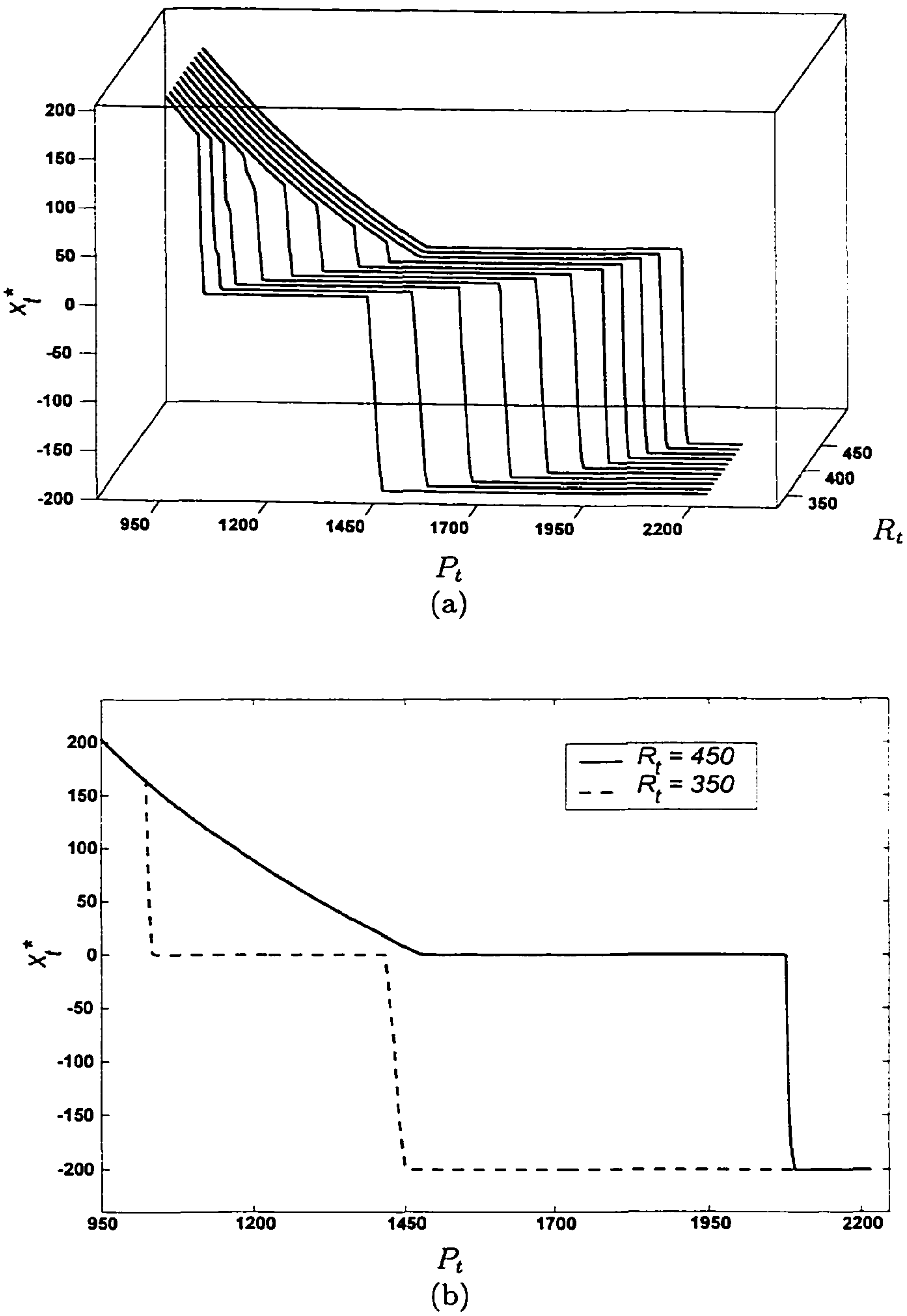


Figure 6.1: Optimal farmland investment policy, x_t^*, as a function of farmland price and the gross return, (P_t, R_t), for $L_t = 600$ and $W_t = 300,000$.

The optimal investment function can be divided into three categories: positive investment ($x_t^* > 0$), zero investment ($x_t^* = 0$), and negative investment ($x_t^* < 0$). A positive investment represents a decision to pay the partially irreversible investment cost and, in return, acquire an asset whose value can fluctuate. The rate of return on that asset also fluctuates over time. The optimal investment function is nonincreasing in farmland price because farmland price, along with farm machinery and equipment costs and transaction costs, determines the investment cost per acre. It is nondecreasing in the gross return because the forecasted expected gross return is increasing in the current gross return. The higher current gross return implies a higher stream of future expected gross returns, though both return streams converge to the long-run mean. Thus, the optimal investment policy depends on investment costs and the future stream of returns. The optimal policy also considers the sale of some or all of current farmland, and it depends on the current state.

For positive and negative investments, the investor chooses not only the purchase or sale decision, but also the optimal number of acres to purchase or sell. The optimal investment policy function takes into account future decisions and the flexibility of the option to wait. In addition to positive and negative investments, $x_t^* = 0$ is optimal over a range of state variable values, which is referred to as the range of inaction. Figure 6.1(b) shows a range of inaction from $P_t = \$1,480$ to $\$2,075$ when $R_t = \$450$. The range of inaction for $R_t = \$350$ is from $P_t = \$1,040$ to $\$1,415$. Figure 6.1(a) shows the ranges of inaction for additional values of R_t ranging from \$350 to \$450.

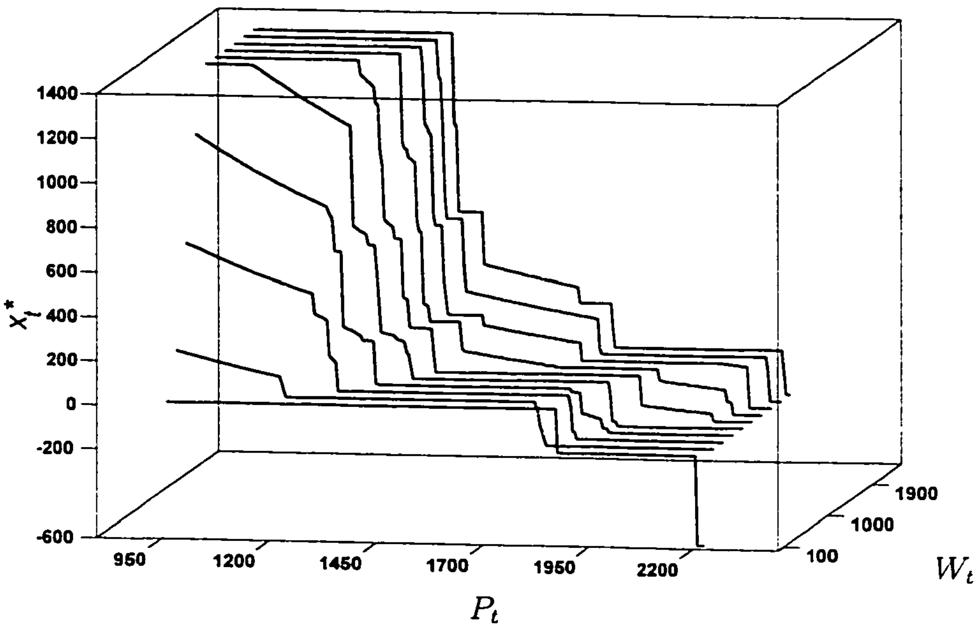
There are three causes for the range of inaction. First, there are transaction costs on buying or selling farmland. The range of inaction encompasses the states where it is not profitable either to purchase or to sell farmland. For a given value of P_t, the effective farmland price with the selling transaction cost is $0.94 * P_t$, which is lower than P_t. The effective price with buying transaction cost is $1.01 * P_t$, which is greater than P_t. Thus there is a gap between the effective purchasing and selling price. In the range of inaction, the effective price is too high to purchase and too low to sell farmland at this time period. Similar to these costs, the second cause of the range of inaction is the transaction cost on selling farm machinery and equipment, which is associated with the farmland investment decision. Third, there is a constraint on buying farmland through the debt-to-asset ratio borrowing constraint. When this constraint is binding, the investor is not allowed to purchase farmland due to financial constraints. However, the investor can choose to sell farmland or do nothing. Thus, the optimal decision can be to do nothing for a range of states when this constraint is binding.

In Figure 6.1(b), the range of inaction for R_t = \$350 is from P_t = \$1,040 to \$1,415, an interval of \$375. The range of inaction for R_t = \$450 is from P_t = \$1,480 to 2,075, an interval of \$595. Figure 6.1 shows that the size of the range of inaction increases as the gross return increases. This is due to the fact that the farm manager chooses not to sell when the gross returns are higher. From the buying side, actions are restricted by the availability of finances to buy farmland.

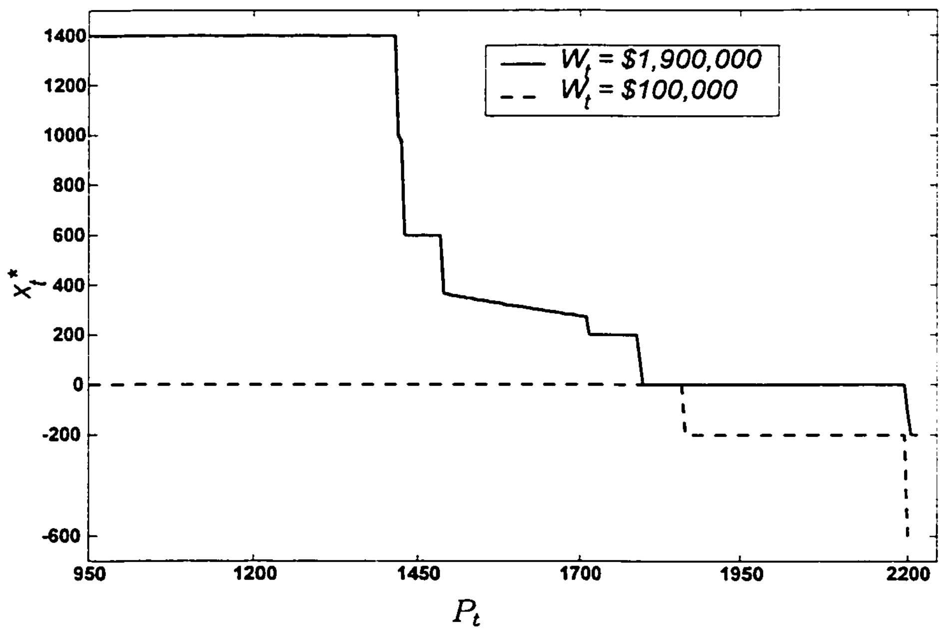
As compared to Figure 6.1, W_t replaces R_t in Figure 6.2. Figure 6.2(a) presents a graph of the optimal farmland investment policy, x_t^*, as a function of farmland price and net wealth, (P_t, W_t), for $L_t = 600$ and $R_t = 390$. The optimal policy is also plotted in a two dimensional graph as a function of only farmland price, P_t, for net wealth levels of \$100,000 and \$1,900,000 in Figure 6.2(b). These graphs show that the optimal investment function is nonincreasing in P_t and nondecreasing in W_t. As described above, the optimal investment policy depends on the investment costs and future returns. It is nonincreasing in farmland price because farmland price determines the investment cost per acre. The optimal investment policy is nondecreasing in W_t because of the lower opportunity cost of finances with a higher wealth level. This is due to the difference in interest rate on borrowing and lending. In addition, a wealthier farmer has access to more funds because the borrowing constraint due to the restriction on the debt to asset ratio is less binding.

As can be seen in Figure 6.2, for a given gross return, the length of the range of inaction decreases as W_t increases. This is because a wealthier farmer has access to more funds. Furthermore, the manager faces lower financial risk, such as risk of bankruptcy, with a higher wealth level.

In Figures 6.1 and 6.2, there is also a range of selling 200 acres. Given $L_t = 600$, the farmland acreage after selling 200 acres is 400 acres, which is the minimum acreage required to stay in farming. In Figure 6.1(b), $x_t^* = -200$ from $P_t = \$2,090$ to $\$2,215$ for $R_t = \$450$, and from $P_t = \$1,450$ to $\$2,215$ for $R_t = \$350$. Similarly, we have



(a)



(b)

Figure 6.2: Optimal farmland investment policy, x_t^*, as a function of farmland price and net wealth, (P_t, W_t), for $L_t = 600$ and $R_t = 390$.

ranges of staying in farming for levels of W_t in Figure 6.2. There are two reasons attributed to this behavior. First, the farm manager wants to keep the option open to remain in the farming. Second, the farm manager cannot choose farmland acreage between 0 and 400.

The dashed line of Figure 6.2(b) shows that when the farmland price is $2,200 or higher, the optimal investment policy is to sell all land for $W_t = \$100,000$ and $R_t = \$390$. The manager chooses to sell all land to avoid bankruptcy risk and considering the profitability of farmland investment.

Figures 6.1 and 6.2 show how the optimal policy depends on the current states. Thus the optimal portfolio of farmland and liquid assets or debt financing depends on investment costs and expectations about the future stream of returns based on information from current states. The optimal portfolio also takes into account the possibility of adjusting the portfolio in the future. To study the dynamic behavior, the next section presents transitional dynamics.

6.1.2 Transitional Dynamics

Analysis of transitional dynamics examines how variables in the model evolve along the transition path starting from some initial state. For a deterministic problem, it describes an exact path for a given initial state level. However, for a stochastic problem, it provides distributions of the variables along the path. The expected path only provides an incomplete summary information about the distributions of the state variable levels along the time path. However, the expected path can also be analyzed for comparative dynamics, such as comparing the overall difference in paths due to changes in the initial state or parameters of the model.

It is important to note that the optimal policy depends on the state in each period of time, as required under the Principle of Optimality stated on page 22. When at least one variable is stochastic, the optimal policy can be determined for the initial state, but for each of the remaining periods the optimal policy is a contingency plan. These contingency plans depend on the state, which will be known only after the realization of shocks in the stochastic variables.

Suppose the farm manager's initial state at time $t = 0$ is: gross return $R_0 = \$390$, farmland price $P_0 = \$1,500$, farmland acreage $L_0 = 600$, and net wealth $W_0 = \$300,000$. The optimal policy at this state is $x_0^* = 0$. This says that farmland at time $t = 1$, L_1, will also be 600 acres. But, R_1 and P_1 are stochastic and have distributions that are conditional on the current state. Since W_1 depends on R_1 and P_1, there is also a distribution of W_1. The optimal policy at time $t = 1$ depends on the state at time $t = 1$, which will only be known at time $t = 1$. Thus, there are infinitely many possible paths when stochastic variables have continuous distributions.

To explore the dynamic nature of the process over time, we use Monte Carlo simulation methods. We draw 500 trials for each year, which will be $500 * 20 = 10,000$ sets of error terms for each of the two stochastic variables R_t and P_t.[2] Using these

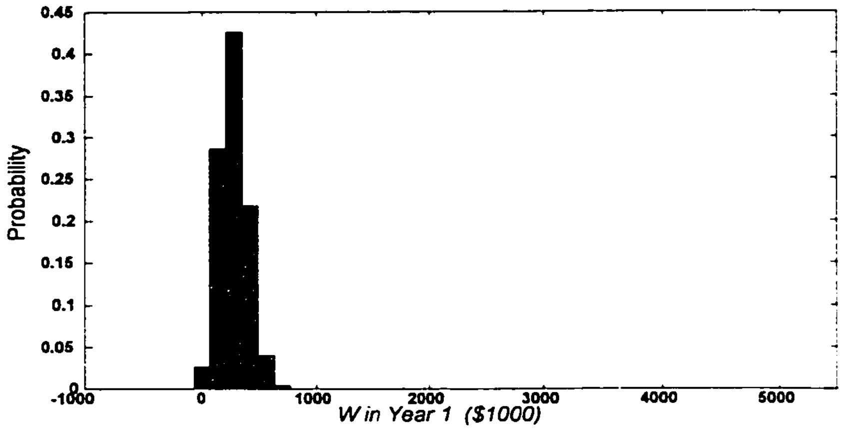
[2]These sets of error terms are the same as used for accuracy analysis in the last chapter.

error terms and following the optimal policy for each state in each time period yields 500 paths.

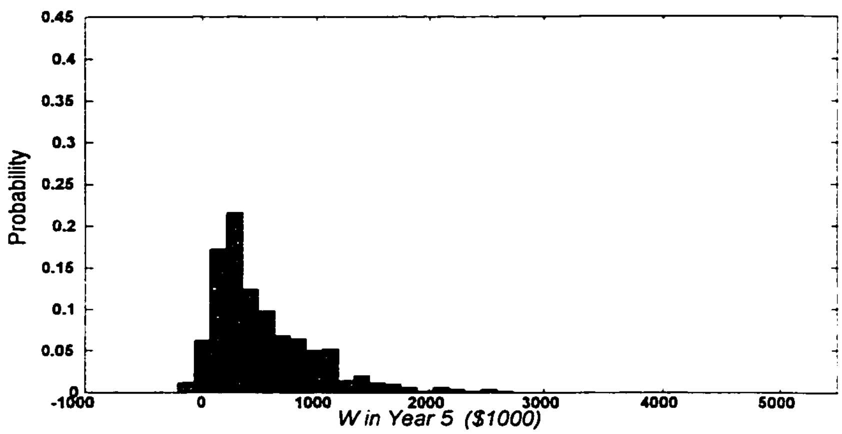
Figure 6.3 presents histograms of W_t for years t = 1, 5 and 10, given the initial state at time $t = 0$ of $R_0 = \$390$, $P_0 = \$1,500$, $L_0 = 600$, $W_0 = \$300,000$. These histograms describe the distributions of W_t, and describe the cumulative effect of uncertainty and the optimal policy. There is a zero probability of bankruptcy in year 1, as W_t in Figure 6.3(a) is always positive. However, Figures 6.3(b) and 6.3(c) show the possibility of negative net wealth in years 5 and 10, the result of bankruptcy. On positive side, it is also possible to accumulate considerable amount of net wealth over time.

Figure 6.4 presents histograms of L_t for years t = 1, 5 and 10. As noted above, the optimal policy in period 0 is $x_t^* = 0$, the farmland in year 1 is unchanged and is 600 acres. The optimal policy in the remaining periods depends on the state. Besides bankruptcy, the farm manager can also choose to sell all land. The probability of exiting farming, either by bankruptcy or choice, by year 10 is shown at $L = 0$ in the Figure 6.4(c). The figure shows that the distribution of farmland acreage is bimodal, with 0.43 probability of 400-499 acres and 0.18 probability of 2000 acres.

Table 6.1 presents summary statistics of simulation results for each of 10 years. The fourth column of the table shows that the probability of bankruptcy by year 10 is 0.036, which is also illustrated by the histogram with negative net wealth in 6.3(c). The sixth column of Table 6.1 shows that the probability of exiting farming, either



(a)



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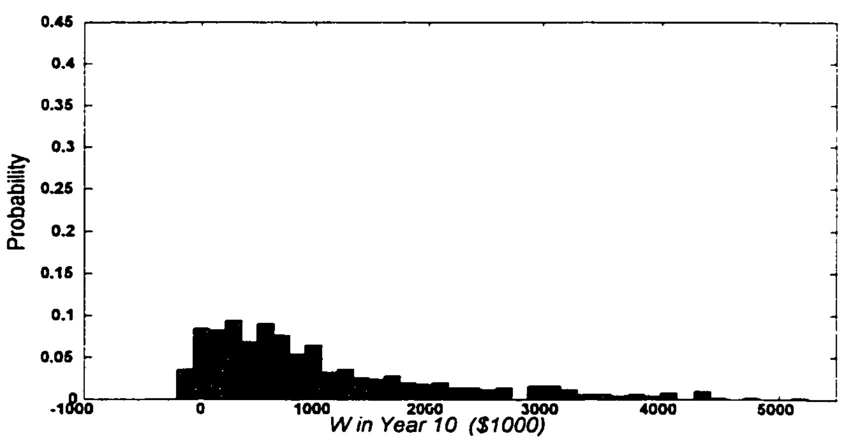


(c)

Figure 6.3: Distributions of net wealth (W) in years 1, 5, and 10 for the initial state $R_0 = 390$, $P_0 = 1500$, $L_0 = 600$, and $W_0 = 300,000$

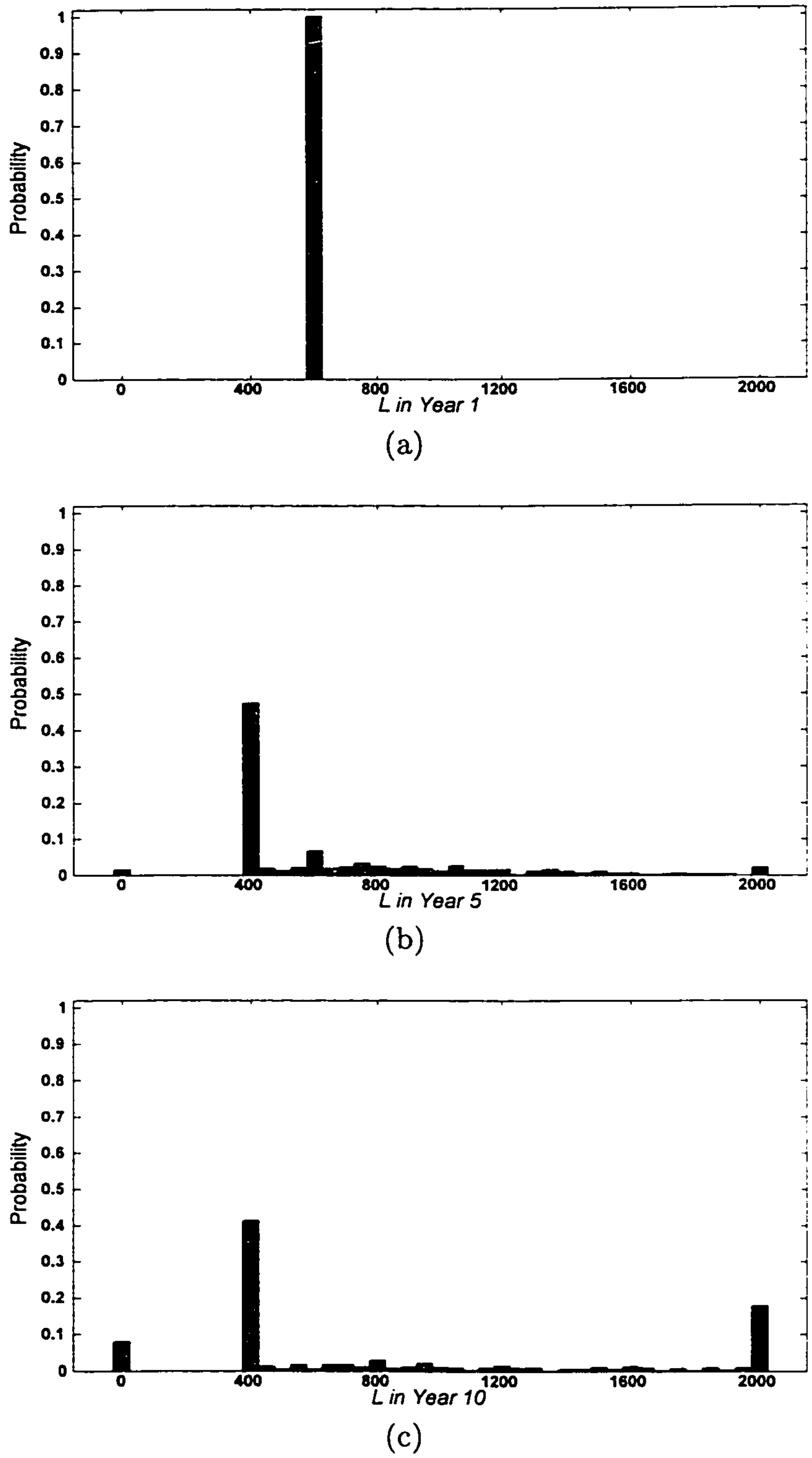


Figure 6.4: Distributions of farmland acreage (L) in years 1, 5, and 10 for the initial state $R_0 = 390$, $P_0 = 1500$, $L_0 = 600$, and $W_0 = 300,000$

Table 6.1: Simulation Results of Expected Path and Probabilities

	Expected Path of		Probability of Sell All by			If still farming		
Year t	Net Wealth W_t	Farmland acres L_t	Bank-ruptcy	Choice	Bankrupty + Choice	Farmland acres L_t	$E(\frac{LA}{FA})$	Prob. $-1< \frac{LA}{FA} \leq -0.7$
0	300,000	600	- - -	- - -	- - -	600	-0.70	- - -
1	346,724	600	0	0	0	600	-0.67	0.36
2	399,337	568	0	0	0	571	-0.60	0.20
3	449,372	588	0.004	0.002	0.006	594	-0.57	0.16
4	520,234	631	0.008	0.006	0.014	642	-0.53	0.13
5	603,958	697	0.012	0.008	0.020	715	-0.51	0.13
6	689,042	761	0.014	0.022	0.036	792	-0.48	0.15
7	785,165	811	0.022	0.028	0.050	854	-0.44	0.15
8	892,488	831	0.028	0.036	0.064	891	-0.38	0.12
9	998,480	851	0.032	0.048	0.080	927	-0.30	0.10
10	1123,020	880	0.036	0.048	0.084	963	-0.22	0.10

by bankruptcy or choice, by year 10 is 0.08, which is also illustrated by the histogram at 0 farmland acreage in Figure 6.4(c).

As simulation results provide 500 paths, the expected path can be computed by taking the average for each year. The second and third columns of Table 6.1 provide expected paths of W_t and L_t respectively. The results show that expected farmland acreage is 880 acres in year 10, given the initial farmland 600 acres. From the policy analysis, we found that a farmer with higher net wealth purchases more acres of farmland. The growth in expected farmland is due to growth in expected net wealth over time.

Net wealth, W_t, is the sum of liquid assets and the after-sale net value of farm assets, including farmland and associated farm machinery and equipment. Thus, liquid assets can be calculated for each state level as W_t minus the after-sale net value of farm assets. Positive liquid assets represent riskless investment in this model, while negative liquid assets imply debt financing. Table 6.1 also shows the liquid asset to farm asset ratio (LA/FA) in the eighth column. At the initial state at $t = 0$, LA/FA is -0.7, which implies a 0.7 debt-to-asset ratio. The results show that if the manager stays in farming, the mean debt-to-asset ratio is reduced from 0.7 to 0.22 in 10 years. The table also presents the expected farmland acreage and the probability of the debt to asset ratio being greater than 0.7 and less than 1 if the manager stays in farming. These exclude the farms that exit farming by bankruptcy or choice. As expected their expected farmland is higher than overall expected farmland presented in the third column. As net wealth grows over time, the results in the last column of the table show a reduction in the probability of a debt to asset ratio greater than 0.7 or less than 1.

Effect of Initial State

The above analysis of transitional dynamics was described by specifying an initial state at time $t = 0$ for each state variable. In this section, we investigate how the transition path is affected by changes in the initial state of net wealth and farmland acreage.

Figure 6.5 presents the expected paths of farmland acreage with three different initial net wealth levels: $W_0 = \$300,000$, $\$700,000$, and $\$1,000,000$, while keeping

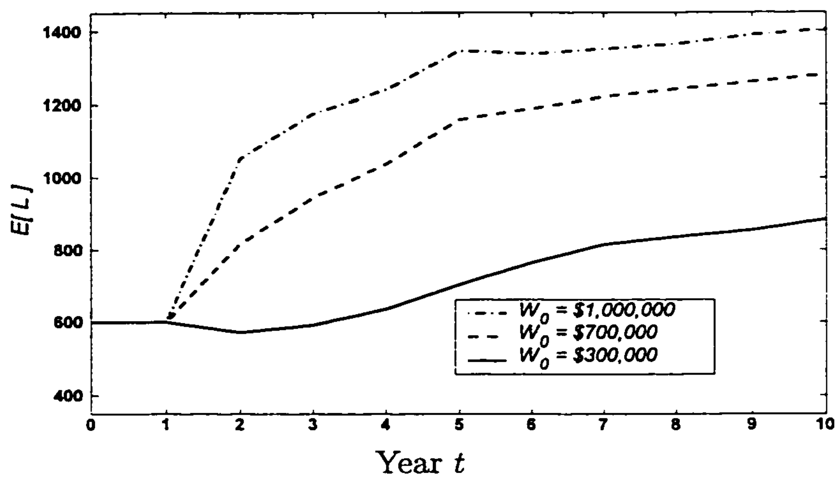


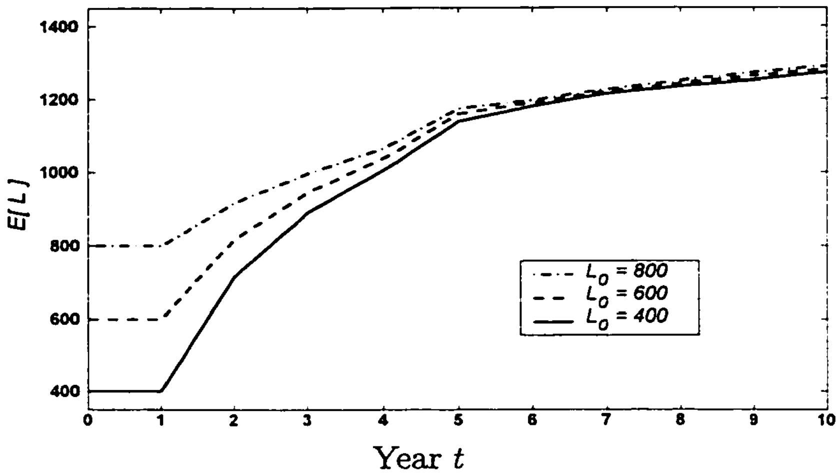
Figure 6.5: Expected paths of farmland acreage with different initial net wealth levels (W_0)

the initial state of the remaining state variables at $R_0 = \$390$, $P_0 = \$1,500$, $L_0 = 600$. The optimal investment policy at time $t = 0$ is $x_0^* = 0$ for all three initial states of net wealth, thus farmland at time $t = 1$ remains 600 acres. For given current state of R_0 and P_0, there are distributions of their states for $t \geq 1$. Thus optimal investment policy for $t \geq 1$ depends on the state at that time, and Figure 6.5 shows the expected value of farmland acreage. The figure shows that, with a higher initial net wealth level, expected farmland acreage starts growing faster at $t = 1$ and has higher expected level along the transition path. This is because a farmer with a higher net wealth has more farmland acreage in the portfolio. Since the initial farmland acreage is identical in all three cases, the farmer with higher initial net wealth grows faster to adjust the portfolio.

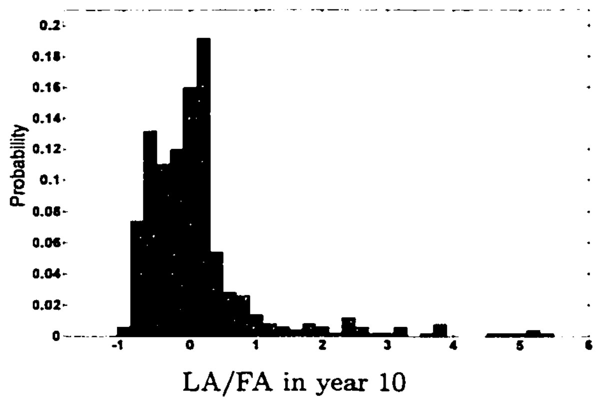
Next we analyze transitional dynamic paths with a fixed initial net wealth level but with different initial farmland acreage to explore adjustment in the portfolio by farmland purchase or sale decision. Figure 6.6(a) presents expected paths of farmland acreage with three different initial farmland acreage, $L_0 = 400$, 600, and 800, while keeping the initial state of the remaining state variables at $R_0 = \$390$, $P_0 = \$1,500$, $W_0 = \$700,000$. Figures 6.6(b), (c), and (d) present distributions of liquid asset to farm asset ratio (LA/FA) in year 10 for initial farmland acreage $L_0 = 400$, 600, and 800 respectively. Figure 6.6(a) shows that, with a lower initial farmland, expected farmland acreage starts growing faster to adjust the portfolio. Since the initial net wealth level is identical in all three cases, the figure shows that expected farmland acreage converges along the path.

It is important to note that the investment policy analysis presented in Section 6.1.1 represents short-term portfolio adjustments. For given gross return and farmland price, wealth is allocated between farmland investment and liquid assets. When farmland investment is more profitable, a higher farmland investment is possible with debt financing. When farmland investment is not profitable at current gross returns and farmland price, the investor would not purchase additional land or may even sell farmland.

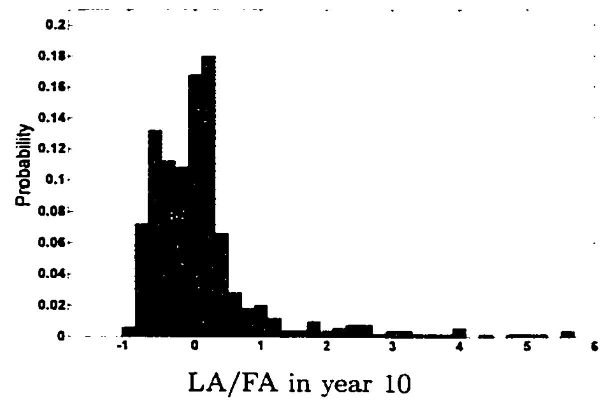
Following the optimal policy in each year gives the transition path. Figure 6.6(a) shows that adjustment in the expected farmland acreage occurs up to year 6, after which it converges to the level for the long-term portfolio. This can also be seen in distributions of LA/FA in year 10 presented in Figures 6.6(b), (c), and (d). These figures show a similar distribution for each case.



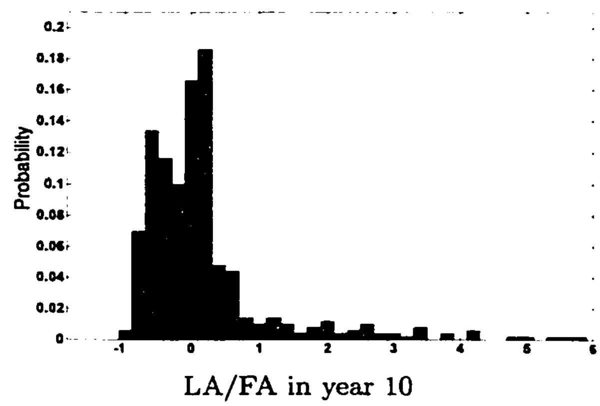
(a)



(b) $L_0 = 400$



(c) $L_0 = 600$



(d) $L_0 = 800$

Figure 6.6: (a) Expected paths of farmland acreage with different initial farmland, (b,c,d) Distribution of the liquid asset to farm asset ratio (LA/FA) in year 10 for each initial farmland

Table 6.2: Expected Paths of Farmland, Net Wealth, and Liquid Asset to Farm Asset Ratio for Different Initial Farmland Acreage

Year t	$L_0 = 400$			$L_0 = 600$			$L_0 = 800$		
	L_t	W_t	$\frac{\text{LA}}{\text{FA}}$	L_t	W_t	$\frac{\text{LA}}{\text{FA}}$	L_t	W_t	$\frac{\text{LA}}{\text{FA}}$
0	400	700,000	0.04	600	700,000	-0.31	800	700,000	-0.48
1	400	761,149	0.12	600	770,724	-0.25	800	780,299	-0.43
2	715	835,421	-0.12	817	856,523	-0.28	916	877,866	-0.36
3	891	932,802	-0.17	946	961,397	-0.24	997	990,357	-0.27
4	1,005	1,076,180	-0.17	1,037	1,108,337	-0.19	1,065	1,140,986	-0.20
5	1,138	1,230,954	-0.19	1160	1,267,104	-0.19	1,174	1,303,955	-0.19
6	1,180	1,389,300	-0.13	1189	1,429,520	-0.11	1,196	1,469,669	-0.10
7	1,214	1,558,151	-0.08	1220	1,600,281	-0.06	1,226	1,642,741	-0.04
8	1,233	1,737,664	-0.00	1241	1,782,041	0.02	1,249	1,826,326	0.03
9	1,250	1,924,549	0.10	1261	1,972,076	0.12	1,270	2,017,622	0.14
10	1,271	2,130,410	0.22	1280	2,180,861	0.25	1,288	2,228,640	0.27

Table 6.2 presents expected paths of farmland, net wealth, and the liquid asset to farm asset ratio for each case of initial farmland acreage. Though there is a long-term portfolio for a given wealth level, as seen in Figure 6.6, Table 6.2 shows a slightly higher expected farmland in year 10 with a higher initial farmland acreage. A farm with initial farmland of 400 acres has $1,271$ acres in year 10, and a farm with initial farmland of 600 acres has $1,280$ acres in year 10. Besides time as a factor of adjustment in the portfolio, the difference is also due to a difference in net wealth level. A farm with initial farmland of 400 acres has expected net wealth $\$2,130,410$ in year 10, and a farm with initial farmland of 600 acres has expected net wealth $\$2,180,861$ in year 10. The difference in the net wealth is due to transaction costs on the purchase of farmland and the rate of return in farmland investment. As noted above, expected farmland acreage grows faster with a lower initial farmland for the

long-term adjustment. A farm with 400 initial farmland acreage has to purchase more farmland to get to expected farmland of 1,271 in year 10, as compared to a farm that grows from initial farmland of 600 acres to 1,280 expected farmland acreage in year 10. As there is a transaction cost on farmland purchases, the farm with 400 initial farmland acreage would result in a lower expected wealth level.

In addition, the difference in net wealth is due to the rate of return on farmland investment. Though initial net wealth is identical for each case, a farm with a lower initial farmland level has a lower rate of return on wealth due to less investment in farmland during the adjustment to the long-term portfolio. This can be seen in Table 6.2 from the growth in net wealth from year 0 to year 1. When farmland is 400 acres in year 0, expected net wealth grows from 700,000 to 761,149, however, when farmland is 600 acres in year 0, the expected net wealth grows from 700,000 to 770,724.

Table 6.2 also presents expected values for the liquid asset to farm asset ratio (LA/FA) for each year. Its value in year 0 is 0.04, -0.31, and -0.48 for initial farmland of 400, 600, and 800 acres respectively. The expected value of LA/FA in 10 years converges to similar levels: 0.22, 0.25, and 0.27 for initial farmland of 400, 600, and 800 acres respectively. Like net wealth, we also find a slightly higher expected LA/FA with a higher initial farmland.

6.1.3 Comparative Dynamics

In this section we investigate how optimal portfolios are affected by changes in parameters of the model, such as the length of the planning horizon, interest rate, variances of stochastic shocks, and risk aversion.[3] The effect of changes in parameters of the model is analyzed for the optimal policy function and transitional dynamic path.

It is important to note that the optimal policy depends on the current state considering future optimal policy and states. For comparative dynamics, the optimal policy function is compared for a given state. Suppose, a change in some parameter leads to less investment in farmland investment under the optimal policy. However, the current policy also affects future state. Thus, future policy is affected not only by the change in the parameter, but also by the change in states resulting from adjustment of the current policy. When the transitional dynamic path is drawn, the change in current and future policy is also taken into account. We refer these effects as policy effects.

The transitional dynamic path is affected only by policy effects when comparative dynamics is done for changes in parameters which are not in state equations of the model. These parameters include length of planning horizon and risk aversion. When comparative dynamics is done for changes in parameters which are in state equations of the model, there can be a state effect as well as a policy effect on transitional

[3] In the previous section, the analysis of changes in initial state is also considered as comparative dynamics.

dynamic path. These parameters include interest rate and variances of stochastic shocks. Note that the optimal investment policy function is drawn for a given state, so comparing it captures the policy effect only. The transitional dynamic path captures both policy effect and state effect.

Effect of Planning Horizon

In previous sections, the optimal investment policy and transitional dynamics were analyzed for a 20-year planning horizon. In this section we compare the results of 20-year and 30-year planning horizons. Figure 6.7(a) presents the optimal investment policy, x_t^*, as a function of farmland price at time $t = 0$ for 20-year and 30-year planning horizons. This investment policy is for $R_t = \$390$, $L_t = 600$, and $W_t = \$700,000$. Figure 6.7(a) shows that, at price 1220, the optimal policy for the 20-year planning horizon is to purchase 820 acres. However, for the 30-year planning horizon, it is 814 acres. Purchase levels are lower for the 30-year planning horizon for prices ranging from \$1220 to \$1375. The optimal policy is identical for farmland prices ranging from \$950 to \$1219 because there is a constraint on the purchase imposed by the maximum allowable value for the debt to asset ratio. Similar differences in policy are seen at higher farmland price levels. Across the entire range of prices, the optimal investment policy for 30-year planning horizon is always less than or equal to that for the 20-year planning horizon. The difference in optimal investment policy can also be seen by the analysis of transitional dynamics. Figure 6.7(b) presents expected path

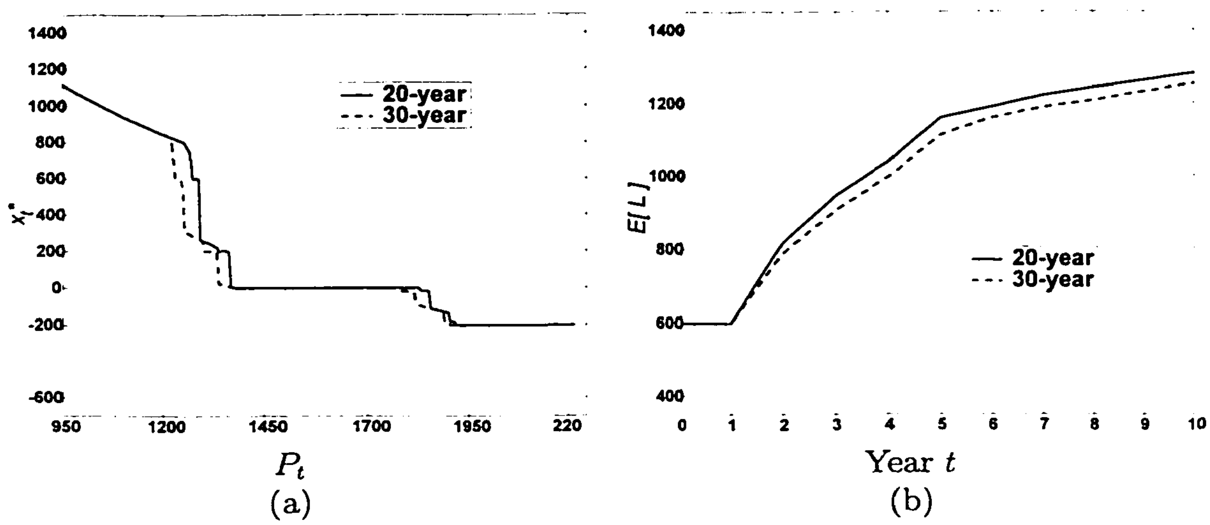


Figure 6.7: (a) Policy function, (b) expected path of farmland

of farmland acreage when the initial state is: $R_0 = \$390$, $P_0 = \$1500$, $L_0 = 600$, and $W_0 = \$700,000$. The figure shows the expected farmland acreage in year 10 for the 20-year planning horizon is 1280 acres, while it 1252 acres for the 30-year planning horizon. These results show that a farmer with a longer planning horizon tends to invest less in risky asset. Though the difference in farmland investment is small, this reflects risk avoiding behavior. Collins and Karp (p. 233) explain this behavior by stating "[i]f the horizon is sufficiently long, young farmers tend to be cautious because the value of farming is very large relative to the value of the nonfarm alternative."

There is also an end period effect on policy. The analysis of the optimal policy, not presented in graphs, indicates that farmland purchases are much lower during the last 5 years near the end of planning horizon. Furthermore, there is more probability of selling some or all land for earlier retirement near the end of planning horizon.

Effect of Borrowing Interest Rate

In previous sections the interest rate on borrowing funds was set at 0.06. To explore the effect of a change in the borrowing interest rate, the model was also solved with the rate of 0.07. In both cases, the interest rate is assumed to be deterministic and constant over time. Other parameters are unchanged – e.g., the interest rate on riskless investment remains at 0.03 and the planning horizon is 20 years. Figure 6.8(a) presents the optimal investment policy, x_t^*, as a function of farmland price at time $t = 0$ for borrowing interest rates 0.06 and 0.07. This investment policy is for $R_t = \$390$, $L_t = 600$, and $W_t = \$700,000$. Figure 6.8(b) presents expected path of farmland acreage when the initial state is: $R_0 = \$390$, $P_0 = \$1500$, $L_0 = 600$, and $W_0 = \$700,000$. Expected farmland acreage in year 10 is 1280 acres when the interest rate on borrowed funds is 0.06, while expected farmland is 1186 acres when the interest rate is 0.07. As expected, these results show that farmers tend to invest less in farmland when interest rates rise.

As described above, the interest rate is in the state equation of net wealth, so a higher borrowing interest rate has both policy and state effects. A higher rate increases interest costs on borrowed funds, leading to less purchase or more sale of farmland. Higher interest cost also reduces expected net wealth accumulation over time. Thus, less investment in farmland is also due to less net wealth.

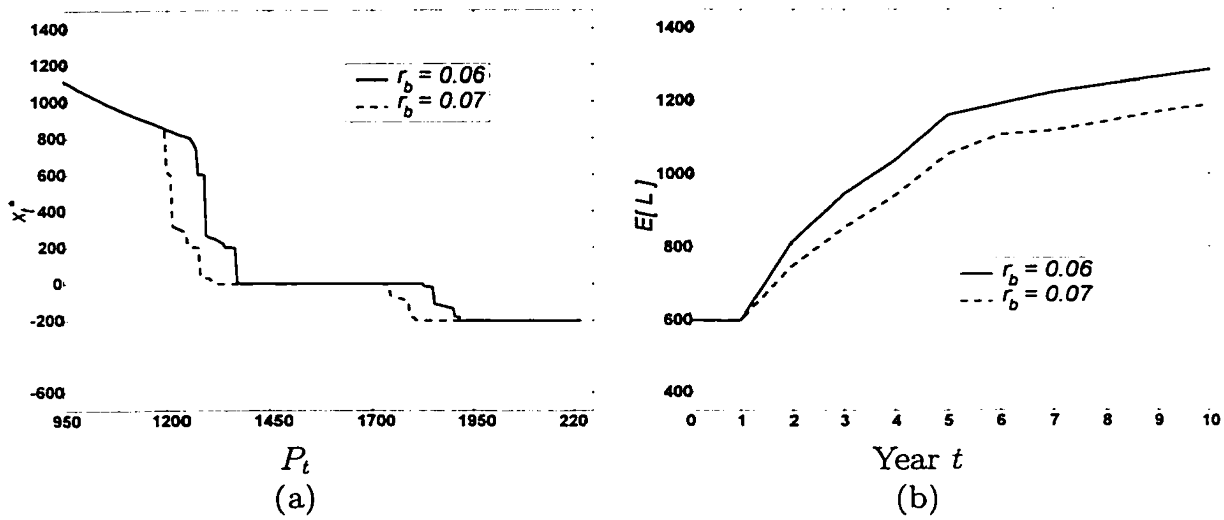


Figure 6.8: (a) Policy function, (b) expected path of farmland

Effect of Variances of Stochastic Shocks

In the investment model, there are two stochastic variables, gross return, R_t, and farmland price, P_t. The state equation of each of these state variables has an error term, which was estimated and presented with estimation of the equation in Chapter 4. In this section, we explore the effect of change in the variances of both error terms. The model is also solved after a 40 percent increase in the variances of error terms of both stochastic equations: R_t and P_t. Figure 6.9(a) presents the optimal investment policy, x_t^*, as a function of farmland price at time $t = 0$ for the base and higher variances. This investment policy is for $R_t = \$390$, $L_t = 600$, and $W_t = \$700,000$. Figure 6.9(b) presents the expected path of farmland acreage when the initial state is: $R_0 = \$390$, $P_0 = \$1500$, $L_0 = 600$, and $W_0 = \$700,000$.

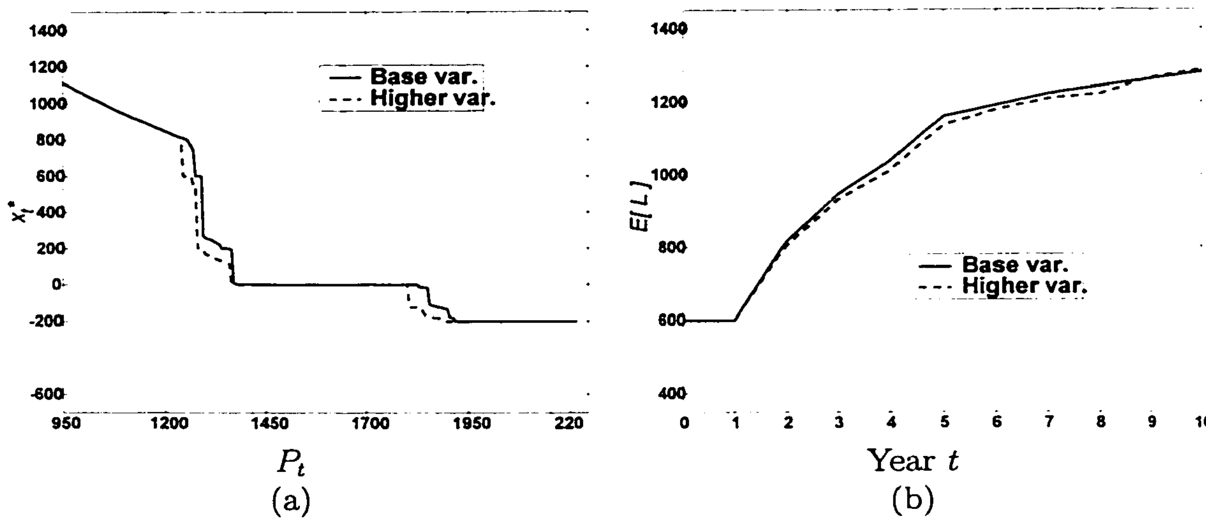


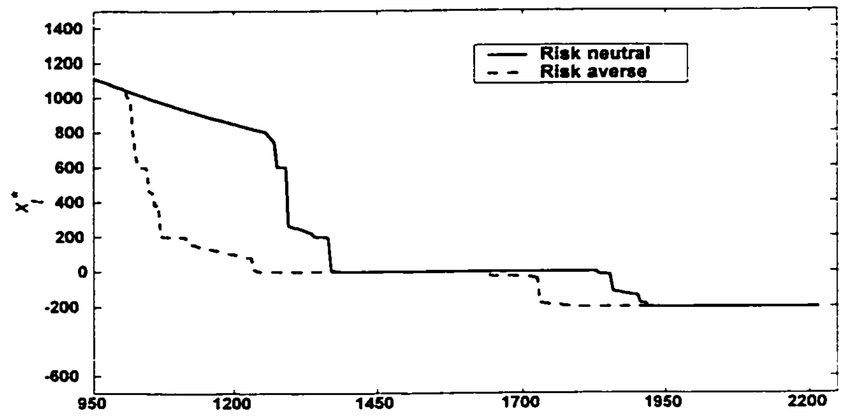
Figure 6.9: (a) Policy function, (b) expected path of farmland

In the two stochastic equations, the state variables R_t and P_t are transformed into natural logarithms. As the state equations have additive error terms, which are normally distributed, the state variables R_t and P_t have lognormal distributions. Thus, higher variances of error terms increase not only riskiness of R_t and P_t but also their expected values. Both factors affect the optimal investment policy. Figure 6.9(a) shows that optimal investment policy for higher variances is less than or equal to that for the base variances across the range of farmland prices. The difference in optimal policy suggests that the effect of increased risk outweighs the effect of the higher expected return from increasing variances. In addition to this policy effect, there are also state effects, as change in variances modifies the distributions and expected values of gross return and farmland price. Higher farmland price leads to less investment in farmland, while higher gross return leads to more investment. Figure 6.9(b) shows that the transitional dynamic path of expected farmland is only slightly changed due to both policy and state effects.

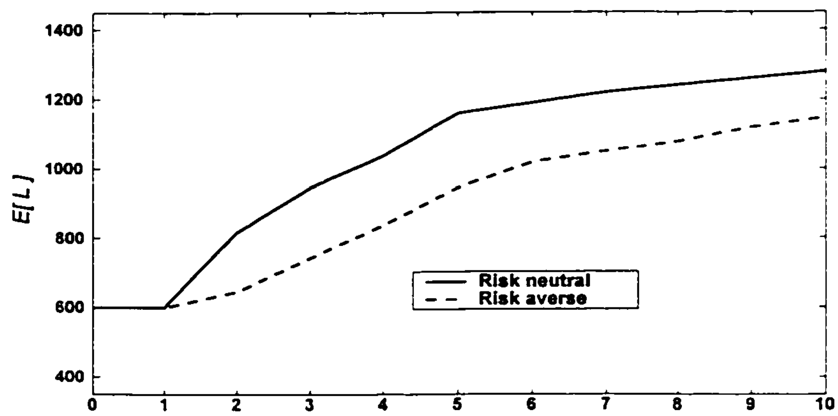
6.2 Risk Aversion

Previous sections presented results for the risk neutral case, with zero constant relative risk aversion, $\theta = 0$. In this section we investigate how optimal portfolios are affected by a change in relative risk aversion. Recall from Section 2.2.2 that $\lim\limits_{\theta \to 1} \frac{W^{1-\theta}-1}{1-\theta} = \ln(W)$ which has constant relative risk aversion equal to 1, $\theta = 1$. In this section we compare the results of risk neutral case to the risk averse case with relative risk aversion of 1.

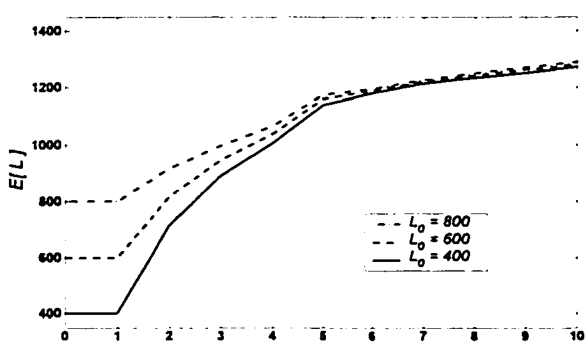
Figure 6.10(a) presents the optimal investment policy, x_t^*, as a function of farmland price at time $t = 0$ for risk neutral and risk averse cases. This investment policy is for $R_t = \$390$, $L_t = 600$, and $W_t = \$700,000$. Across the entire range of farmland prices, the optimal investment policy for the risk averse case is always less than or equal to that for the risk neutral case. Less investment in farmland reflects risk avoiding behavior of the risk averse farmer. This graph also shows that, at low farmland prices ranging from \$950 to \$1300, the risk neutral farmer chooses to purchase farmland where the debt to farm asset ratio constraint is binding. However, for the risk averse case, the constraint is biding only at very low price, only from \$950 to \$1000, and it is not binding from \$1000 to \$1300. This indicates that risk averse farmers have internal credit rationing, whereas there is external credit rationing for the risk neutral case.



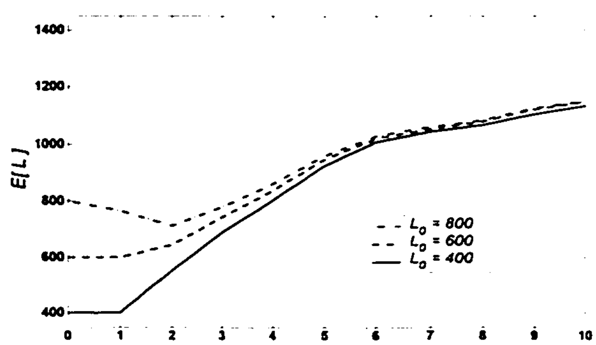
P_t
(a)



Year t
(b)



Year t
(c) Risk neutral



Year t
(d) Risk averse

Figure 6.10: Comparison of risk neutral and risk averse: (a) Policy function, (b) Expected path of farmland, (c,d) Expected paths of farmland with different initial farmland

To analyze transitional dynamics, suppose the farm manager's initial state at time $t = 0$ is : $R_0 = \$390$, $P_0 = \$1500$, $L_0 = 600$, and $W_0 = \$700,000$. At this state, the optimal policy at time $t = 0$ is $x_0^* = 0$ for both the risk averse and risk neutral cases. The policy is identical because the initial state is within the range of inaction for each case. Thus, the state at time $t = 1$ is identical for both cases. Though farmland at time $t = 1$, L_1, is still 600 acres, the gross return, R_1, and farmland price at time $t = 1$, R_1 and P_1, are stochastic and have their distributions. As net wealth at time $t = 1$, W_1, depends on R_1 and P_1, there is also distribution of W_1. For transitional dynamics we use Monte Carlo simulation method, as described in Section 6.1.2.

Figure 6.10(b) presents the expected paths of farmland acreage for risk averse and risk neutral cases when the initial state is: $R_0 = \$390$, $P_0 = \$1500$, $L_0 = 600$, and $W_0 = \$700,000$. Table 6.3 presents expected paths of farmland and net wealth and the path of standard deviation of net wealth for both cases. As described above that the initial optimal policy $x_0^* = 0$ is identical for both risk averse and risk neutral cases, expected farmland, expected net wealth, and standard deviation at time $t = 1$ are also identical for these cases.

For time $t \geq 2$, Figure 6.10(b) and Table 6.3 show that the risk averse farmer has less expected farmland along the path. Expected farmland in year 10 for the risk neutral case is 1280 acres, while for the risk averse case, expected farmland is 1147 acres. Like the policy function, the expected path of less investment in farmland reflects risk avoiding behavior of risk averse farmer. As the risky farmland investment

Table 6.3: Expected Paths of Farmland and Net Wealth and the Path of Standard Deviation of Net Wealth for Risk Nneutral and Risk Averse Cases

Year t	Risk neutral			Risk averse		
	$E(L_t)$ acres	$E(W_t)$ \$	$SD(W_t)$ \$	$E(L_t)$ acres	$E(W_t)$ \$	$SD(W_t)$ \$
0	600	700,000	0	600	700,000	0
1	600	770,724	118088	600	770724	118088
2	817	856,523	230416	644	854346	202338
3	946	961,397	351221	742	948875	303526
4	1,037	1,108,337	536938	836	1082241	474809
5	1160	1,267,104	705155	945	1227268	631849
6	1189	1,429,520	846159	1019	1374418	772468
7	1220	1,600,281	995691	1051	1537276	919471
8	1241	1,782,041	1149603	1077	1707447	1071887
9	1261	1,972,076	1267853	1119	1880548	1185280
10	1280	2,180,861	1407079	1147	2078243	1320962

has a higher expected rate of return and a higher variance than the riskless investment, less investment in farmland by the risk averse farmer reduces both the expected value and the variance of net wealth along the path, as presented in Table 6.3. The expected wealth in year 10 for the risk averse case is \$2,078,243, while it is \$2,180,861 for the risk neutral case. The standard deviation of wealth in year 10 for risk averse case is \$1,320,962, while it is \$1,407,079 for the risk neutral case.

To explore adjustment in the portfolio, we analyze transitional dynamic paths with different initial farmland acreage but with a fixed initial net wealth level, farmland price, and gross return. For the risk neutral and risk averse cases respectively Figures 6.10(c) and (d) present expected paths of farmland acreage with three different initial farmland acreage, L_0 = 400, 600, and 800 acres, while keeping the initial state of the remaining state variables at R_0 = \$390, P_0 = \$1,500, W_0 = \$700,000. The risk neutral case is discussed on page 96. As in the risk neutral case, we observe a similar portfolio adjustment in the risk averse case. Since the initial net wealth level is identical in all three cases of initial farmland, the figure shows that expected farmland acreage converges along the path. However, the risk averse case has less farmland acreage in the portfolio.

Results of comparative dynamics for a change in risk aversion show that a risk averse farmer makes a lower investment in risky farmland reflecting risk avoiding behavior. Differences in investment behavior are often attributed to the risk aversion. The literature in finance has raised the issue of big difference between returns on stocks and Treasury bills, and point out that unless a very high level of risk aversion is assumed, risk aversion alone cannot explain this difference (Chavas and Thomas; Kocherlakota). We find that, besides risk aversion, risk avoiding behavior of choosing the investment portfolio can also be attributed to other factors. In the previous section, with risk neutral preferences, the results show that a change in the planning horizon affects the investment decisions. Furthermore, unlike in a static model, a change in riskiness of returns and farmland prices also affects the investment decision in a dynamic model, even when the decision maker is risk neutral.

Chapter 7

Extended Model and Results

Results presented in Chapter 6 are for the base model specified in Chapter 2. The base model is a multiperiod investment portfolio problem with a risky farmland investment and with a riskless nonfarm investment or debt financing on farmland. This portfolio problem is solved in the presence of transaction costs, credit constraints, bankruptcy, and stochastic farmland prices and farm returns. In this chapter we extend the base model by adding a risky nonfarm investment, such as a mutual fund. The model implementation and results are also presented in this chapter.

7.1 The Extended Model

The extended model is a multiperiod investment portfolio problem with a risky farmland investment, risky nonfarm asset, and a riskless nonfarm asset or debt financing on farmland.

Risk in farmland investment is incorporated through both the farmland price and gross returns per acre (P_t, R_t), which follow Markov processes and are described by their state equations. Farmland investment is represented in units of acres. The state variable L_t is current farmland acreage and can be changed over time by control variable x_t. There are credit constraints, bankruptcy, and transaction costs on purchase and sale of farmland.

In this analysis, the risky nonfarm asset is an S&P (Standards and Poors) index mutual fund, which will be hereafter referred to as mutual fund. Investment in mutual fund is in units of dollars. Risk in the mutual fund investment is through its annual rate of return, which accounts for both the gain or loss in the value of the asset and the dividend in each year. The mutual fund rate of return plus 1 is denoted by M_t. We assume that there are no transaction costs on purchasing or selling of the mutual fund asset.[1]

Including the mutual fund in the model would generally require adding two state variables, one for its returns M_t and another for the dollar amount invested in mutual fund asset, and one new control variable representing the amount invested. However, given the data on its returns and assumption of no transaction costs in the mutual fund, the mutual fund investment opportunity can be added to the model by adding only a control variable x_{mt}, which denotes the dollar amount invested in mutual fund.

[1]We make this assumption for simplicity. However, there are many mutual fund assets available in the market that can be bought or sold with almost no or very little transaction costs.

As described in the next section (Section 7.2), the Box-Jenkins approach for forecasting yields a zero degree of autoregressive model for the equation of M_t. This result can be interpreted to mean that the forecasted mutual fund return in next period, M_{t+1}, is stochastic and has a distribution which is not conditional on the current year's mutual fund returns, M_t. Thus, the mutual fund return equation does not add a state variable in the model.

Since there are no transaction costs on purchasing or selling the mutual fund asset, the amount invested in the mutual fund can be decided every year and adjusted accordingly no matter how much investment is already made in the mutual fund. Each year the control variable $x_{mt} \geq 0$ determines mutual fund investment, which is financed from the current liquid assets through the net wealth equation. The invested dollar amount x_{mt} yields $M_{t+1} * x_{mt}$ in the next year, which is added to liquid assets in the next year. Thus, the mutual fund amount does not add a state variable in the model. Note that, without an additional state variable in the model, the control variable amount x_{mt} also represents the amount (state) in the mutual fund after the investment decision, and $M_{t+1} * x_{mt}$ represents end of year's amount (state) in the mutual fund.

Net wealth, W_t, is the sum of liquid assets and the net sale-value of farm assets including farmland and the associated farm machinery and equipment. Farm expenses and all investment costs are financed from the current liquid assets, and all returns are added to liquid assets in next period through the state equation for net wealth.

For a given net wealth level, investment decisions for farmland and the mutual fund, (x_t, x_{mt}), implicitly also determine debt financing or investment in riskless asset. Negative value of liquid assets after all expenses represents debt financing, and a positive value represents investment in the riskless investment.

The farm manager's objective is to choose an optimal policy $\{x_t^*, x_{mt}^*\}_{t=0}^T$ that maximizes the expected utility of net terminal wealth subject to relevant constraints:

$$\max_{\{(x_t, x_{mt})\}_{t=0}^T} E_0\left[\tilde{U}\left(W_{T+1}\right)\right]$$

$$\begin{aligned}
\text{subject to} \quad &: \\
\ln R_{t+1} &= \beta_0 + \beta_1 \ln R_t + \varepsilon_{1,t+1}, \\
\ln P_{t+1} &= a_0 + a_1 \ln P_t + a_2 \ln R_t + \varepsilon_{2,t+1}, \\
\ln M_{t+1} &= \gamma_0 + \varepsilon_{3,t+1} \\
L_{t+1} &= L_t + x_t, \\
W_{t+1} &= (1+r)[\{W_t - (P_t + \kappa - tc_s) * L_t\} - (P_t + \kappa + tc) * x_t - c * L_{t+1} \\
&\quad -x_{mt}] + R_{t+1} * L_{t+1} + (P_{t+1} + \kappa - tc_s) * L_{t+1} + M_{t+1} * x_{mt}, \\
x_t &\in X_t,\ x_{mt} \in X_{mt},\ \text{for } t = 0, 1, 2, ..., T, \\
&\quad (R_0, P_0, L_0, W_0) \ \text{are given},
\end{aligned}$$

where

$$r = \begin{cases} r_b \text{ if } [\{W_t - (P_t + \kappa - tc_s) * L_t\} - (P_t + \kappa + tc) * x_t - c * L_{t+1} - x_{mt}] > 0 \\ r_l \text{ otherwise,} \end{cases}$$

$$tc = \begin{cases} tc_p & \text{if } x_t > 0 \\ -tc_s & \text{otherwise,} \end{cases}$$

$0 < \rho < 1$,[2] E_0 is expectation operator and $\tilde{U}$ is the utility function.

In this model, we have four state variables: gross return per acre from crops, R_t, farmland price per acre, P_t, farmland acreage, L_t, and net wealth, W_t. The model has two control variables: x_t is number of acres of farmland purchased or sold, and x_{mt} is dollar value of the mutual fund. There are three stochastic variables, R_t, P_t, and M_t, and their random shocks are ε_{1t}, ε_{2t}, and ε_{3t} respectively. There are four constraints on farmland purchase or sale decisions, which are denoted by $x_t \in X_t$. These constraints include the feasibility constraint, bankruptcy condition, credit constraints, and the choice of exiting farming, which are described in Section 2.1.

The credit constraint allows purchase of land and the associated machinery and equipment as long as the debt to asset ratio is less than or equal to ρ, where $0 < \rho < 1$,

$$\frac{-\{W_t - (P_t + \kappa - tc_s) * L_t\} + (P_t + \kappa + tc_p) * x_t}{(P_t + \kappa - tc_s) * (L_t + x_t)} \leq \rho \tag{7.1}$$

This constraint can be written as:

$$x_t \leq \max\left(0, \frac{W_t - (1-\rho) * (P_t + \kappa - tc_s) * L_t}{(P_t + \kappa + tc_p) - \rho * (P_t + \kappa - tc_s)}\right)$$

The above constraint (7.1) describes the maximum availability of debt financing, which depends on W_t, P_t, L_t, x_t, and constants of the model. The farm manager can potentially use available debt finances based on the farmland investment decision,

[2] Control spaces X_t and X_{mt} depend on state variables P_t, L_t, W_t, and other constants including ρ, which is described below in this section.

and can also invest in mutual fund. Thus for given W_t, P_t, L_t, x_t, the constraint on mutual fund investment x_{mt} is given by:

$$\frac{-\{W_t - (P_t + \kappa - tc_s) * L_t\} + (P_t + \kappa + tc_p) * x_t + x_{mt}}{(P_t + \kappa - tc_s) * (L_t + x_t)} \leq \rho$$

This constraint can also be written as:

$$\begin{aligned} x_{mt} \leq & \max(0, [W_t - (P_t + \kappa - tc_s) * L_t - (P_t + \kappa + tc) * x_t \\ & + \rho * (P_t + \kappa - tc_s) * (L_t + x_t)]) \end{aligned}$$

In the model, X_{mt} denotes a set of levels of x_{mt} that satisfy the above constraint and $x_{mt} \geq 0$.

As described in Chapter 5 on page 51, the utility function needs to specified for all W. For the risk neutral case, when $\theta = 0$, we have the utility function $\widetilde{U}(W) \equiv W$, in which case the objective function is to maximize expected terminal net wealth. For the risk averse case, $\theta > 0$, we have $\widetilde{U}(W) = \frac{W^{1-\theta}-1}{1-\theta}$ if $W \geq b$, and $\widetilde{U}(W) = \frac{b^{1-\theta}-1}{1-\theta} * \frac{W}{b}$if $W \leq b$, where $b > 0$. Estimation of stochastic equations for state variables R_t and P_t and other parameters are used the same as for the base model, presented in Chapter 4 and summarized on page 79 (Chapter 6).

7.2 Estimation

In this section we estimate the equation for forecasting mutual fund return, M_t, using methods described in Chapter 4. For data on M_t, we use inflation-adjusted compound annual total return for the Standard and Poor's 500 Stock Composite Index (S&P 500) for year 1967-99 reported in Ibbotson Associates.[3]

The equation for mutual fund return, M_t, is modelled as an autoregressive model. M_t is one plus the rate of return per dollar invested. As the value of asset cannot be nonpositive, M_t is positive and the suitable transformation of data is natural logarithm. The first order autoregression model AR(1) is $\ln M_t = \gamma_0 + \gamma_1 \ln M_{t-1} + \varepsilon_{3t}$. When this model was estimated, and γ_1 was nonsignificant with p-value 0.65. Furthermore, the Box-Jenkins approach suggests this specification: $\ln M_t = \gamma_0 + \varepsilon_{3t}$. The estimation results are presented in Table 7.1(a). This specification gives a stable model by the diagnostic test of the error term following the Box-Jenkins approach, as described in Chapter 4. The Box-Jenkins approach requires a white noise error term. The Ljung-Box test is used to test the null hypothesis that the error term is a white noise process, and is presented in Table 7.1(b). The test fails to reject the null hypothesis, as all p-values are greater than 0.01. As the hypothesis of white noise error term is maintained, this model represents the result of Box-Jenkins approach.

Implementation in dynamic programming model requires distribution of error term. Normality of the error term is tested using the Bera-Jarque test, and is presented in Table 7.1(c). The null hypothesis is that error term is normally distributed. As the p-value is 0.020781, the test does not reject the null hypothesis at the significance of 0.01.

[3] Ibbotson Associates report the S&P returns as large company stocks: inflation adjusted total returns (compound annual return).

Table 7.1: Results for Mutual Fund Return Equation

(a) Regression Results

Variable	Coefficient Estimate	Standard Error	t-ratio	p-value
Constant	0.074476	0.028291	2.632527	0.0129
Estimated Var[ε_3] = 0.026412				

(b) Ljung-Box Test for White Noise

K	$Q(K)$ statistic	p-value
1	0.2512	0.616
2	1.4435	0.486
3	2.5108	0.473
4	7.2091	0.125
5	7.2998	0.199
6	7.4949	0.277
7	7.5139	0.377
8	7.6580	0.468
9	7.6924	0.565
10	7.7233	0.656
11	7.9295	0.720
12	8.0797	0.779
13	9.1235	0.764
14	9.5314	0.796
15	9.9251	0.824

(c) Bera-Jarque test for Normality

BJ statistic	p-value
7.747479	0.020781

7.3 Model Implementation

The extended model is a sequential decision problem with terminal optimization. In this problem, there are four state variables, gross return per acre from crops, R_t, farmland price per acre, P_t, farmland acreage, L_t, and net wealth, W_t. From Section 3.3 for terminal optimization, Bellman's equation for any R, P, L, W, can be written as:

$$V_t(R,P,L,W) = \max_{(x,x_m)} \{E[V_{t+1}(R_{t+1}, P_{t+1}, L_{t+1}, W_{t+1}) \mid R_t = R, P_t = P, L_t = L, W_t = W,\ x_t = x, x_{mt} = x_m]\}$$

subject to the constraints of the extended model with the terminal (boundary) condition: $V_{T+1}(R,P,L,W) = U(W)$. The value function for each $t = 0, 1, ..., T$, is defined as:

$$V_t(R,P,L,W) \equiv \max_{\{(x_\tau, x_{m\tau})\}_{\tau=t}^{T}} E\left[U(W_{T+1}) \mid R_t = R, P_t = P, L_t = L, W_t = W\right]$$

To solve the problem by the dynamic programming method, the range of the four state variables, R_t, P_t, L_t, W_t, must be defined for estimation of the value function. We use the identical ranges here as defined for the base model on page 5.1 in Chapter 5. The value function in dynamic programming is solved for the following ranges of the states: $220 \le R_t \le 620$, $950 \le P_t \le 2,215$, $400 \le L_t \le 2,000$, and $0 \le W_t \le 6,000,000$.

In solving the Bellman equation for each t, we obtain the value function for the ranges of the state specified above. However, given the state in period t from the above ranges, the Bellman equation contains the value function in period $t+1$, which needs to be computed for the states in period $t+1$.[4]

As assumed in the model, once all land is sold, the business cannot re-enter farming, and the value function for state $L_t = 0$ can be computed as follows. When $L_t = 0$, the farm manager cannot use debt financing, since it is based on farmland as collateral. In this case, the above investment problem is to choose optimal portfolio of nonfarm assets: mutual fund and a riskless asset. Thus, the value function for $L_t = 0$ is $E[U(W_{T+1})]$ from the nonfarm asset portfolio.

Net wealth in period $t+1$, W_{t+1}, can go out of the bounds from the above range due to the stochastic nature of gross return from crops and farmland price. When $W_{t+1} < 0$, the value function is computed for the bankruptcy condition. Under this condition, the farm is liquidated if net wealth is negative at any time, which makes $L_t = 0$. In this case also, the value function can be computed, however, $E[U(W_{T+1})]$ is calculated by compounding W_t at the debt rate. When W_{t+1} is greater than the upper bound, the extra amount above the upper bound earns return rate from the portfolio of nonfarm assets: a combination of the mutual fund and the riskless asset.[5]

In this extended model, we have three stochastic variables, R_t, P_t, and M_t. For each variable we use 3 nodes for the numerical integration. This gives 27 combinations with their probabilities. In this model there two control variables. For optimization,

[4]For details on implementation, the reader is referred to model implementation of the base model in Chapter 5.

[5]To compute the value function, which is utility of terminal wealth, first returns from nonfarm asset portfolio are added to the wealth from value function, where W_{t+1} is greater than the upper bound. The utility function of the sum was regarded as the value function.

we discretize the control spaces X_t in 41 uniform levels and X_{mt} in 25 uniform levels. This gives 1025 combinations. The optimal policy is found in two steps. First, for each element of X_t, we find optimal x_{mt}. Then, we find optimal $\{x_t, x_{mt}\}$.

7.4 Results

This section presents results for the extended model specified in Section 7.1. We present the model results for the risk averse case with relative risk aversion coefficient $\theta = 1$, and for 20 year planning horizon, where decisions are made at the beginning of each year $t = 0, 1, ..., 19$. At any time t the farm manager can choose the number of acres to purchase or sell, and dollar amount of mutual fund investment, subject to the constraints of the model. We first present the optimal investment policy function at time $t = 0$. Next, we present the stochastic dynamic path, which examines how the process evolves over time starting from some initial state. We also compare these results with the base model results of the risk averse case to investigate the effect of including mutual fund asset in the investment portfolio.

7.4.1 Optimal Investment Policy Function

There are two control variables in the extended model: number of acres to purchase or sell, x_t, and dollar amount of mutual fund investment, x_{mt}. The optimal policy for each control variable is a function of current states: gross return, R_t, farmland price,

P_t, farmland acres, L_t, and net wealth, W_t. We present the optimal policy at time $t = 0$.

Table 7.2 presents the optimal policy, (x_t^*, x_{mt}^*), as function of net wealth, W_t, when $R_t = 390$, $L_t = 600$, and $P_t = 1325$. The table shows the liquid asset to farm asset ratio (LA/FA) for each level of W_t. As expected, the optimal farmland policy, x_t^*, is nondecreasing function of W_t, ranging from -600 to 600 acres, which includes a policy of inaction, $x_t^* = 0$. The optimal mutual fund purchase policy, x_{mt}^*, is increasing function of W_t.

As described in the model, for each given wealth level and other states, the optimal decisions for farmland acreage and mutual fund investment implicitly also determine either debt financing or investment in a riskless asset. Table 7.2 shows LA/FA (*) which is LA/FA after the optimal decisions of (x_t^*, x_{mt}^*). Except for $W_t = 300,000$, the results show that the optimal choice of LA/FA (*) is -0.7, which implies a 0.7 debt to farm asset ratio, the maximum limit allowed through the credit constraint. This is because the interest rate on debt financing is 0.06 and the expected mutual fund return rate is 0.091641, based on the estimated equation for M_t. The table also shows (LA+x_{mt}^*)/FA (*), which adds mutual fund investment and the liquid assets. This ratio is realized by the farm manager, since mutual fund investment is also liquid that can be used for financing farm expenses. However, note that mutual fund returns are also stochastic. When $W_t = 300,000$, the optimal policy is to sell all farmland and invest in the mutual fund due to bankruptcy risks from the possibility of a drop

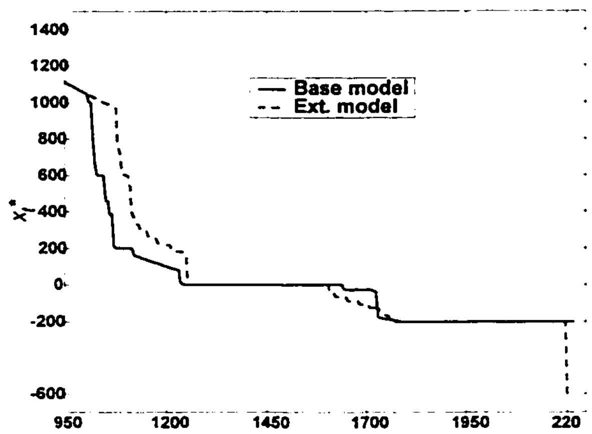
Table 7.2: Optimal Policy for Farmland and Mutual Fund

	$P_t = 1325$					$P_t = 1500$				
W_t $	$\frac{LA}{FA}$	x_t^* acres	x_{mt}^* $	$\frac{LA}{FA}$ (*)	$\frac{LA+x_{mt}^*}{FA^*}$ (*)	$\frac{LA}{FA}$	x_t^* acres	x_{mt}^* $	$\frac{LA}{FA}$ (*)	$\frac{LA+x_{mt}^*}{FA^*}$ (*)
300,000	-0.67	-600	300,000	- - -	- - -	-0.70	-600	300,000	- - -	- - -
500,000	-0.45	0	225,590	-0.70	-0.45	-0.51	0	146,985	-0.65	-0.51
700,000	-0.23	0	425,590	-0.70	-0.23	-0.31	0	395,980	-0.70	-0.31
900,000	-0.02	59	591,848	-0.70	-0.11	-0.11	0	595,980	-0.70	-0.11
1,100,000	0.20	200	711,370	-0.70	-0.12	0.09	0	795,980	-0.70	0.09
1,300,000	0.42	280	865,682	-0.70	-0.05	0.28	0	995,980	-0.70	0.28
1,500,000	0.64	600	882,930	-0.70	-0.22	0.48	0	1,195,980	-0.70	0.48

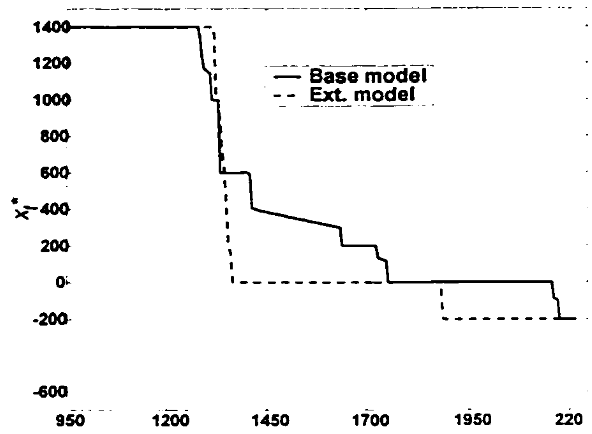
(*) indicates the ratio after investment decisions are made.

in farmland price and/or gross returns. When $P_t = 1500$, Table 7.2 also shows the similar results for the optimal policy. However, when the farmland price is higher, there is a less purchase of farmland as compared to that with $P_t = 1325$.

As shown in Table 7.2, the optimal choice of debt to asset ratio is 0.7 due to the mutual fund. It is important to note that, at a 0.7 debt to asset ratio, the amount of debt depends on farmland acreage along with farmland price. Now we will investigate the effect of including the mutual fund asset in the investment portfolio on the optimal policy for farmland purchase or sale. For this, we compare the farmland investment policy as a function of its price between the base and extended models. Figure 7.1(a) presents the optimal farmland investment policy, x_t^*, as a function of farmland price at time $t = 0$ for the base and extended models. This investment policy is for $R_t = \$390$, $L_t = 600$, and $W_t = \$700,000$. The figure shows that, when the farmland price is low, the optimal farmland policy for the extended model is to



(a) $W_t = \$700,000$



(b) $W_t = \$1,900,000$

Figure 7.1: Policy function

purchase more farmland as compared to that for the base model. When the farmland price is high, there is a slight difference in the policy. However, when the farmland price is very high, it is optimal to sell all farmland in the extended model. When all farmland is sold in the extended model, all wealth is invested in the mutual fund where the expected rate of return is 0.091641. In the base model, it is not optimal to sell all land because the rate of return on the riskless asset is only 0.03. Figure 7.1(b) presents the graph for a higher wealth level, $W_t = \$1,900,000$. Since the wealth level is high, the figure shows that in the base model it is optimal to purchase 300 acres at price 1450. However, in the extended model the farmer does not purchase or sell farmland, $x_t = 0$. Again, this is because the only alternative nonfarm investment opportunity in the base model is riskless asset with 0.03 rate of return. However, in the extended model, in addition to the riskless asset, there is also a mutual fund.

7.4.2 Transitional Dynamics

The above analysis shows how the optimal policy depends on the current states. This section presents transitional dynamics starting from some initial state, using the Monte Carlo simulation method, as described in Section 6.1.2 (Chapter 6). Tables 7.3(a) and (b) present the transition paths for the base model and extended model respectively. For both cases, the initial state at time $t = 0$ is: gross return per acre $R_0 = \$390$, farmland price $P_0 = \$1,500$, farmland acreage $L_0 = 600$, and net wealth $W_0 = \$700,000$. For both cases, the farmland optimal policy at this state is $x_0^* = 0$, thus, the farmland at time $t = 1$ is 600 acres. In the extended model however, the farmer manager can also invest in the mutual fund, and the optimal policy is to allocate $\$395,980$ to the mutual fund. The expected value of the mutual fund in year 1 is $\$427,275$. The expected net wealth in year 1 is $\$770,724$ in the base model, however, it is $778,260$ in the extended model due to the mutual fund.

Comparison of the results for the base model and the extended model can be viewed as comparative dynamics, as described in Section 6.1.3. In this case, comparison of the results is not just for a change in a parameter of the model, but also for a change in the economic environment of the model. In the extended model we have added the mutual fund as an alternative investment opportunity. As discussed in Section 6.1.3, there are policy and state effects on the results. In the last section, Figure 7.1 shows the policy effect of adding the mutual fund in the model – the change in optimal policy for a given state due to mutual fund. Along the dynamic path, adding

Table 7.3: Comparison of Base and Extended models with Risk Averse

(a) Base Model

Year t	$E(W_t)$ \$	$SD(W_t)$ \$	$E(L_t)$ acres	Prob exit	$E(L_t/\text{farming})$
0	700,000	0	600		600
1	770,724	118088	600	0	600
2	854,346	202338	644	0	644
3	948,875	303526	742	0	742
4	1,082,241	474809	836	0	836
5	1,227,268	631849	945	0	945
6	1,374,418	772468	1019	0.002	1,021
7	1,537,276	919471	1051	0.002	1,053
8	1,707,447	1071887	1077	0.002	1,080
9	1,880,548	1185280	1119	0.002	1,121
10	2,078,243	1320962	1147	0.002	1,150

(b) Extended Model

Year t	$E(W_t)$ \$	$SD(W_t)$ \$	$E(L_t)$ acres	Prob. of exit	$E(L_t/$ farming) acres	$E(x_{m,t-1}/$ farming) \$	$E(M_t * x_{m,t-1}/$ farming) \$
0	700,000	0	600	- - -	600	- - -	
1	778,260	133,766	600	0	600	395,980	427,275
2	875,229	234,942	700	0.002	701	400,134	434,991
3	995,978	355,849	789	0.032	815	440,477	482,109
4	1,160,160	537,164	877	0.050	923	502,625	550,728
5	1,343,970	689,667	957	0.080	1,040	626,955	689,807
6	1,532,417	849,276	988	0.100	1,098	803,562	877,576
7	1,745,440	1,045,669	979	0.114	1,105	1,018,577	1,115,410
8	1,984,503	1,254,115	937	0.132	1,079	1,275,737	1,400,663
9	2,234,284	1,456,560	930	0.148	1,092	1,556,755	1,698,547
10	2,559,297	1,735,698	922	0.162	1,100	1,856,309	2,074,514

the mutual fund also increases the expected net wealth and other states. Thus, the future optimal policy is also affected by change in the state.

Table 7.3 shows that the expected path of net wealth is higher for the extended model than that for the base model. Again, this is due to the high expected return on the mutual fund. However, the mutual fund is a risky asset. Thus, the standard deviation of net wealth along the path is also higher for the extended model than that for the base model.

Table 7.3 also presents the probability of exiting farming. In both cases, the probability of exiting farming is by choice, since the probability of bankruptcy is zero along the path for the initial state. The probability of exiting farming in 10 years is 0.002 for the base model, however, it is 0.162 for the extended model. The higher probability in the extended model is due to availability of the mutual fund in addition to the riskless asset. Table 7.3 also presents the expected path of farmland acreage if the farmer stays in farming. In the extended model, expected farmland acreage in year 10 is 1,100 acres, while it is 1,150 acres in the base model. Note that there are both policy and state effects. The expected farm land along the path is also affected by the expected net wealth along the path. The expected net wealth in year 10 for the base model is \$2,078,243, and its is \$2,559,297 in the extended model. We find a lower farmland acreage, 1,100 acres, in the extended model even with a higher net wealth levels due to the mutual fund. This result shows that the optimal portfolio has a lower overall farmland acreage to accommodate the mutual fund in the portfolio.

Chapter 8

Summary and Conclusions

Growth in average farm size and the decline in the number of farms have motivated farm firm growth analysis focusing on farmland purchase and sale decisions to explain the process of change from the individual farm standpoint. In addition, farm growth analysis investigates farmers' investment strategies under uncertainty and addresses the financial management and farm survival issues.

We develop a multiperiod investment portfolio problem with a risky farmland investment and with a riskless nonfarm investment or debt financing on farmland. Farmland prices and farm returns are stochastic, and there are credit constraints, bankruptcy, and transaction costs on purchase and sale of farmland. The investment portfolio problem is formulated as a stochastic dynamic programming model. We present an overview of the dynamic programming approach and numerical methods for solving the model. Solving the investment portfolio problem requires estimates of parameters of the model. Farmland price and farm returns are modelled using time series data for Southwestern Minnesota farmers. The econometric results yield Markov processes for farmland prices and farm returns.

The investment portfolio problem has four continuous state variables of which two variables are stochastic. Like in most applications, this dynamic programming model lacks a closed-form solution. We solve the model numerically using collocation methods. In the absence of a closed-form solution, it is essential to examine accuracy of the numerical approximation of the solution to the model, especially in a large-scale problem. In this dissertation, we develop a method for testing the accuracy of the numerical solution to a dynamic programming model. Using this method, we examine the accuracy of collocation methods in solving the investment portfolio problem and propose techniques for improving their accuracy.

Results of the investment portfolio problem show that the optimal investment policy depends on farm returns, farmland price, and liquid assets. This indicates that the policy is influenced by investment costs and the future stream of returns. There are also ranges of inaction – the states where the optimal policy in the current year is to wait, neither purchasing nor selling farmland.

The study explores the influence of planning horizon, interest rate, riskiness, and risk aversion on the farmland investment decisions. Results show that a risk averse farmer makes a lower investment in risky farmland reflecting risk avoiding behavior. We find that, besides risk aversion, risk avoiding behavior in choosing an investment portfolio can also be attributed to other personal characteristics, such as the planning horizon. Even when the decision maker is risk neutral, a change in riskiness of returns and farmland prices affects the investment decision in a dynamic model.

We extend the investment portfolio problem by adding a risky nonfarm asset: a mutual fund. The results of the model show that it is optimal for farmers to include the mutual fund in the portfolio with farmland investment. Furthermore, higher debt financing on farmland is optimal with the mutual fund. However, we find that optimal farmland investment levels are generally lower and that the probability of exiting farming increases due to the mutual fund investment opportunity.

This study focuses on the fact that farm firm growth is a dynamic process. Results of this study show that growth depends on many factors, including initial wealth, initial farm size, length of the planning horizon, interest rate, riskiness of returns, risk aversion, and nonfarm investment opportunities.

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Appendix A

Data Series

Time series data for the farmland price are obtained for years 1966-1992 and for year 1990-1999. Denote P^R as farmland price from the first series and P^T from the second series. These two series are for Southwestern Minnesota with slightly different sample of locations within the region. We found that $P^T = 0.86P^R$ in each year 1990-1992. These prices are averages of very large samples and may be representative. Furthermore, this transformation affects only the intercept in the farmland price equation. Since series of P^T are more recent data, we combined the two series in the scale of P^T.

As described in Section 4.5, gross return per acre is calculated as $0.5 * RC_t + 0.5 * RS_t$, where RC_t and RS_t denote gross return per acre from corn and soybeans respectively.

Data on the farmland price and gross return per acre are adjusted for inflation. Given the time series, we divide the value for each year by its CPI. We multiply all years of data by CPI of 1999 to represent data in dollars of 1999. The adjusted data are given in Table A.1.

Table A.1: Data for Estimating Gross Return and Farmland Price Equations

Year	R_t $	P_t $
1966	- - -	1101.373
1967	350.6082	1255.9
1968	377.9139	1295.552
1969	370.1516	1247.524
1970	453.8562	1200.841
1971	358.8482	1160.586
1972	538.5405	1198.911
1973	730.1508	1265.742
1974	673.6655	1751.463
1975	491.8571	2149.548
1976	425.4741	2687.036
1977	525.9859	3031.099
1978	582.0396	2775.843
1979	468.3158	3170.164
1980	640.6013	3106.952
1981	465.7194	3022.054
1982	380.9184	2871.816
1983	444.7517	2575.638
1984	411.3208	2186.923
1985	364.6233	1504.097
1986	310.0949	1037.281
1987	317.8306	910.2838
1988	327.6725	1055.175
1989	350.7132	1187.361
1990	334.1815	1148.055
1991	296.0147	1192.942
1992	260.9849	1272.665
1993	218.6841	1285.708
1994	284.9175	1206.306
1995	318.0595	1163.986
1996	334.0176	1198.979
1997	302.9265	1242.545
1998	277.8438	1349.33
1999	255.625	1264.341

Appendix B

MATLAB Programs

The MATLAB programs used in this study are presented in this Appendix. These programs use functions, which are also coded and presented here. Furthermore, these programs also use functions and utilities created by Miranda and Fackler.[1] Their book comes with a toolkit that contains these function and utilities. We used their toolkit that was downloaded in 1999.[2]

[1]Examples in Miranda and Fackler have been very helpful in implementing the procedure and in learning MATLAB programming.

[2]We notice a change in their toolkit. In toolkit of 1999, the tensor product of ϕ_j for $j = 1, 2, .., J$ is made by $\Phi = \phi_1 \otimes \phi_2 \otimes ... \otimes \phi_J$. In a toolkit of 2001, it is $\Phi = \phi_J \otimes \phi_{J-1} \otimes ... \otimes \phi_1$.

For Base Model

```
%%%%%%%%%%%%%%%%%%%%%%file name: Parameters.m %%%%%%%%%%%%%%%%%%%%%%%%
% Base Model:
%Obj. func: Max: EU(W(T+1))
clear all;
tic;
%profile on -detail builtin
%____________________________

global basis beta0 beta1 alpha0 alpha1 alpha2 cost sminv smaxv smin...
smax n rp rn tcs tcb fme fds q qf e w T theta b valuef bound dau

% Parameters

file1 = 'ch77521th1.txt';           % Saving in text file
file2 = 'ch77521th1';               % Saving the work space
n1=9; n2=9; n3=9; n4=9;             % Number of nodes for each state
                                    % Number of x levels used to max
                                    % Bellman's eqn
%q = 81; qf = 0;                    % one-stage hybrid method
q = 41; qf = 21;                    % two-stage hybrid method

%basis = 'chebbas';
%node = 'chebnode';                 % Chebychev polynomial basis
                                    % functions and nodes
basis = 'splibas';
node = 'nodeunif';                  % Linear Spline basis functions
                                    % and uniform nodes
%____________________________
if qf==0
   vmaxh = 'vmaxh1';                % one-stage hybrid method
else
   vmaxh = 'vmaxh2';                % two-stage hybrid method
end

valuef = 'valuefw';                 % Value function
bound = 'boundi';                   % Constraints on control
%____________________________
m1=5; m2=5;                         % Number of nodes for
                                    % approximating integration
theta = 1;                          % Utility function parameters
b = 60000;
T = 19;                             % Planning horizon
t1 = 1;
beta0 = 1.052028;                   % Parameters for R state eqn
beta1 = 0.821970;
alpha0 = 0.048655;                  % Parameters for P state eqn
alpha1 = 0.884465;                  % (state reduced eqn)
alpha2 = 0.134044;
Ee = zeros(2,1);                    % Mean of error terms for R and P
eqns
VarCov = zeros(2,2);                % Variance-covariance matrix
```

```
VarCov(1,1) = 0.033155;
VarCov(2,2) = 0.014619;
cost = 247.0;                       % cost of production per acre
                                    % Ranges of state variables:
Rmin = 220.0; Rmax = 620.0;         % Gross Return per acre
Pmin = 950.0; Pmax = 2215.00;       % Price of land (dollars)
Lmin = 400.0; Lmax = 2000.0;        % Land (acres)
Wmin = 0; Wmax = 6000000;           % Net wealth (dollars)
sminv = [Rmin Pmin Lmin Wmin];      % All states in a vector
smaxv = [Rmax Pmax Lmax Wmax];
n = [n1 n2 n3 n4];
m = [m1 m2];
if node=='chebnode'
   smin = sminv - ((sminv-smaxv)./(2.*(cos((n-1+0.5).*pi./n)))+...
   (sminv-smaxv)./2);
   smax = smaxv + ((sminv-smaxv)./(2.*(cos((n-1+0.5).*pi./n)))+...
   (sminv-smaxv)./2);
else
   smin = sminv;
   smax = smaxv;
end

rp = 0.03;                          % Lending Interest rate
rn = rp + 0.03;                     % Borrowing Interest rate
tcs = 0.06;                         % Transac. cost on selling land (%)
tcb = 0.01;                         % on buying land (%)
fme = 300;                          % Farm mach. Equip. per acre (%)
fds = 0.07;                         % fme selling deduction (%)
dau = 0.7;                          % Debt-to-asset ratio (upper bound)



%%%%%%%%%%%%%%%%%%%%%%file name: Invest.m %%%%%%%%%%%%%%%%%%%%%%%

Parameters;
fid = fopen(file1,'w+t');     % Creating a file for output
                              % w+t for deleting old contents, and for
                              %reading and writing new output in text
fprintf(fid,'Max: E[U(Net Wealth(T+1))]\n');
fprintf(fid,'Number of nodes for states = %3i %3i %3i %3i\n',n);
fprintf(fid,'Number of x levels to find max = %3i %3i\n',q,qf);
fprintf(fid,'Theta in utility func = %g\n',theta);
fprintf(fid,'Time horizon T = %3i\n',T);
fprintf(fid,'Number of nodes for error terms = %3i %3i\n\n',m);
                              %g prints the number in compact notation
                        %3i is for printing integer which holds 3 digits
                              % \n is for starting new line in printing
yearnorm = ' Year  norm1ratio  norm2ratio  max&mean(abs(x-xold))
p(=<1)';
fprintf(yearnorm)                   % For displaying
fprintf(fid,yearnorm);              % For saving in the file
%_________________________
[e w]=qnwnorm(m,Ee,VarCov);         % Normal distri of error terms
```

```
D = length(n);                          % Computes number of states
phiinv = cell(1,D);
si = cell(1,D);                         % For nodes
for d=1:D
   si{d} =  feval(node,n(d),smin(d),smax(d));   % (n(d) by 1)
   if basis=='splibas'
      phiinv{d} = inv(feval(basis,n(d),smin(d),smax(d),si{d},0,1));
                                        % For linear spline
   else
      phiinv{d} = inv(feval(basis,n(d),smin(d),smax(d),si{d}));
   end
end
st = cgrid(si);
pn = size(st,1);                        % pn=prod(n)=n1*n2*n3*n4
copt = zeros(pn,T);                     % Matrix for c for 1:T
                           % First node w=0, others are big amounts>1
wi = find(st(:,4)>1);                   % To avoid rounding error for w>0
s = st(wi,:);
nn = size(s,1);

% Algorithm for 1:T (for finding value function)
c = [];
x = -s(:,3);                            % x in T+1
for t=T:-1:t1
   xold = x;                            % Store old value for comparing
   [x,v] = feval(vmaxh,s,c,t);          % Solve Bellman equation at nodes.
   vt = ones(pn,1).*utility(0);         % v=u(0) for w=0
   vt(wi) = v;                          % opt solution for w>0
   c=ckronx(phiinv,vt);                 % Coef. c for value funct in each t
   copt(:,t) = c;                       % Store them in the matrix (nn x T)
   change = (x-xold);
   mchange = max(abs(change));          % Compute maximum change
   achange = mean(abs(change));
   n1change = (norm(change,1))/(norm(xold,1));
   n2change = (norm(change))/(norm(xold));
   pchange = 100.*size(find(abs(change)<=1),1)/size(change,1);
   fprintf('\n%3i\t %4.6f\t %4.6f\t %6.4f\t %6.4f\t %6.4f\n',...
      t,n1change,n2change,mchange,achange,pchange)    % For displaying
   fprintf(fid,'\n%3i\t %4.6f\t %4.6f\t %6.4f\t %6.4f\t %6.4f\n',...
      t,n1change,n2change,mchange,achange,pchange);
end
clear phiinv s st change mchange achange pchange wi v vt c x xold d D
yearnorm
%____________________________
toc;                                          % Elapsed time since tic was
used
seconds = toc;
fprintf(fid,'\nCPU Time seconds= %15.2f\n',toc); % For saving in the
file
fclose(fid);       % Returns 0 if successful in closing output file
save(file2);
disp('For output, run result**.m');
%profile report invest;
```

```
%%%%%%%%%%%%%%%%%%%%%%file name: vmaxh1.m %%%%%%%%%%%%%%%%%%%%%%%%

function [x,v] = vmaxh1(s,c,t);
% Solves Bellman equation at state nodes: stochastic problem
% Uses 1-stage hybrid method:
% Calculates optimal v and x for each set of state nodes

global bound
                           % compute bounds of x for s: xl,xu are nn by 1
                                  % rows of s =rows of xl=rows of xu
[xl,xu] = feval(bound,s);
nn = size(s,1);
[vxq xq] = vx(s,c,t,xl,xu);    % vxq is (q+2 by nn), xq is (q+2 by nn)
[v ind] = max(vxq);            % v is (1 by nn), ind is (1 by nn)
x = zeros(1,nn);               % x is (1 by nn)
for j=1:nn
   x(j) = xq(ind(j),j);
end
v = v';                        % now v is (nn by 1)
x = x';                        % now x is (nn by 1)



%%%%%%%%%%%%%%%%%%%%%%file name: vmaxh2.m %%%%%%%%%%%%%%%%%%%%%%%%

function [x,v] = vmaxh2(s,c,t);
% Solves Bellman equation at state nodes: stochastic problem
% Uses 2-stage hybrid method:
% Calculates optimal v and x for each set of state nodes

global bound q
                           % compute bounds of x for s: xl,xu are nn by 1
                                  % rows of s =rows of xl=rows of xu
[xl,xu] = feval(bound,s);
nn = size(s,1);
[vxq xq] = vxi(s,c,t,xl,xu);   % vxq is (q+2 by nn), xq is (q+2 by nn)
[v ind] = max(vxq);            % v is (1 by nn), ind is (1 by nn)
xopt = zeros(1,nn);            % xopt is (1 by nn)
for j=1:nn
   xopt(j) = xq(ind(j),j);
end
x = xopt';                     % x is (nn by 1)

clear vxq xq ind
%**********************************

xas = 1600/(q-1);              % Lmax-Lmin = 1600
xlf = max(xl,(x - xas));
xuf = min(xu,(x + xas));
[vxq xq] = vxf(s,c,t,xlf,xuf);% vxq is (q+2 by nn), xq is (q+2 by nn)
[v ind] = max(vxq);            % v is (1 by nn), ind is (1 by nn)
xopt = zeros(1,nn);            % xopt is (1 by nn)
```

```
for j=1:nn
   xopt(j) = xq(ind(j),j);
end
v = v';                              % now v is (nn by 1)
x = xopt';                           % x is (nn by 1)



%%%%%%%%%%%%%%%%%%%%%%file name: vx.m %%%%%%%%%%%%%%%%%%%%%%%

function [vxq,xq] = vx(s,c,t,xl,xu);
% For each node, it computes v as a function of x
% when we plug state equation into value function and
% then plug a specific state s (the state node)
% vxq is (q+2 by nn), xq is (q+2 by nn)


global q valuef

nn= size(s,1);           % rows of s =rows of xl=rows of xu
xq = zeros(q+2,nn);
vxq = zeros(q+2,nn);
rxlu = xu-xl;                                                    %
Range from xl to xu
gap = rxlu ./(q-1);
xqi = zeros(nn,1);
vxqi = zeros(nn,1);
for qi=1:(q+2)
   if qi==1
      xqi = zeros(nn,1);
   elseif qi==2
      xqi = -s(:,3);
   else
      xqi = xl+(gap.*(qi-3));
   end
   vxqi = feval(valuef,s,c,t,xqi); % xqi is nn by 1, s is nn by 4
   xq(qi,:)=xqi';
   vxq(qi,:) = vxqi';
end


%%%%%%%%%%%%%%%%%%%%%%file name: vxi.m %%%%%%%%%%%%%%%%%%%%%%%



function [vxq,xq] = vxi(s,c,t,xl,xu);
% For each node, it computes v as a function of x when we plug state
equation
% into value function and then plug a specific state s (the state node)
% vxq is (q by nn), xq is (q by nn)


global q valuef bound
```

```
nn= size(s,1);                          % rows of s =rows of xl=rows of xu
xq = zeros(q,nn);
vxq = zeros(q,nn);
rxlu = xu-xl;                           % Range from xl to xu
gap = rxlu ./(q-1);
xqi = zeros(nn,1);
vxqi = zeros(nn,1);
for qi=1:q
   xqi = xl+(gap.*(qi-1));
   vxqi = feval(valuef,s,c,t,xqi); % xqi is nn by 1, s is nn by 4
   xq(qi,:)=xqi';
   vxq(qi,:) = vxqi';
end



%%%%%%%%%%%%%%%%%%%%%%file name: vxf.m %%%%%%%%%%%%%%%%%%%%%%%%

function [vxq,xq] = vxf(s,c,t,xl,xu);
% For each node, it computes v as a function of x when we plug state
equation
% into value function and then plug a specific state s (the state node)
% vxq is (qf+2 by nn), xq is (qf+2 by nn)


global qf valuef

nn= size(s,1);                                           % rows of s =rows
of xl=rows of xu
xq = zeros(qf+2,nn);
vxq = zeros(qf+2,nn);
rxlu = xu-xl;                                                          %
Range from xl to xu
gap = rxlu ./(qf-1);
xqi = zeros(nn,1);
vxqi = zeros(nn,1);
for qi=1:(qf+2)
   if qi==1
      xqi = zeros(nn,1);
   elseif qi==2
      xqi = -s(:,3);
   else
      xqi = xl+(gap.*(qi-3));
   end
   vxqi = feval(valuef,s,c,t,xqi); % xqi is nn by 1, s is nn by 4
   xq(qi,:)=xqi';
   vxq(qi,:) = vxqi';
end
```

```
%%%%%%%%%%%%%%%%%%%%%%file name: valuefw.m %%%%%%%%%%%%%%%%%%%%%%%%

function v = valuefw(s,c,t,x)
%VALFUNCW: value function for one control, 4 number of states
%s is set of all states
%x is (nn by 1), c is (nn by 1), s is (nn by 4), t is scalar

global basis n sminv smaxv smin smax e w T rp rn tcs tcb fme fds fdb si

nn = size(s,1);
v = zeros(nn,1);
vval = zeros(nn,1);
TWval = zeros(nn,1);
K = size(e,1);

if (nargin==3)
   K=1; wk=1;
elseif ((s(:,3)+x)<1)        % this is for g(:,3)==0 or <1
   K=1;
end

for k = 1:K
   if (nargin==3)
      g = s;
   else
      if ((s(:,3)+x)<1)
         wk = 1;
      else
         wk = w(k);
      end
      ek = e(k,:);
      g = feval('gstate',s,x,ek);
   end
            % g gives R, P, within bounds
            % Lt in bounds, but g3:L(t+1) is 0 or in bounds
            % W can go out of bounds % W is computed from bounded R, P.
   if g(:,3)<1                                  % if g(:,3)==0,sell all land
      gA = g(:,4);                              % g(:,4) is all liquid
                                                % asset since gL=0
      r = ones(nn,1).*rp;
      ngi = find(gA<0);                         %Negative gA index
      r(ngi) = rn;
      vval = utility((((1+r).^(T-t)).*gA));
   else                                         %(2)
      if isempty(c)            %(2.1)
         TW = g(:,4);                     % Terminal period net wealth
         vval = utility(TW);                    % Utility function
      else                                %(2.2)
         gf4 = g(:,4);
         nbri = find(gf4>0);                 % No bankruptcy index
         if isempty(nbri)==0    %(2.2.1)
         gm4 = min(gf4,smaxv(4));
         gd = gf4-gm4;                    % Out of bounds 4th state
         g(:,4) = gm4;
```

```
                                  % g(:,4) within upper bound
         D = length(n);
         phi = cell(1,D);
         for d=1:D
            if basis=='splibas'
                  phi{d} =
                  feval(basis,n(d),smin(d),smax(d),g(nbri,d),0,1);
            else
                  phi{d} = feval(basis,n(d),smin(d),smax(d),g(nbri,d));
            end
         end
         vval(nbri) = cdprodx(phi,c);
                                        % s(t+1)=g that computes v(s(t+1))
         TWval(nbri) = invutility(vval(nbri));

         ewi = find(gf4>smaxv(4));
                        % Positive land and more than Nmax state index
         if isempty(ewi)==0   %(2.2.1..)
            vval(ewi) = utility((TWval(ewi) + ...
            (((1+rp).^(T-t)).*gd(ewi))));
         end
      end

      bri = find(gf4<=0);            %Positive land and Bankruptcy index
      if isempty(bri)==0             %(2.2.2)
            vval(bri) = utility(((1+rn).^(T-t)).*gf4(bri));
      end
      end                  % end of: if isempty(c)
   end                            % end of: if g(:,3)==0
   v = v + vval.*wk;
end                               % end of: for k = 1:K


%%%%%%%%%%%%%%%%%%%%%%file name: boundi.m %%%%%%%%%%%%%%%%%%%%%%%%

function [xl,xu] = boundi(s);
% Bound function for one control

global sminv smaxv dau fme tcs tcb fds

xl = sminv(3)-s(:,3);                % Bounds from Land state

xu3 = smaxv(3)-s(:,3);
s2v = (1-tcs).*s(:,2) + (1-fds).*fme;     % selling price: for
normalization
s2b = ((1+tcb).*s(:,2)) + fme;         % buying price

xdm = max(0,((s(:,4) - (1-dau).*s2v.*s(:,3))./(s2b - dau.*s2v)));
xu = min(xu3,xdm);
```

```
%%%%%%%%%%%%%%%%%%%%%%file name: gstate.m %%%%%%%%%%%%%%%%%%%%%%%

function g = gstate(s,x,ek);
% State transition function for 4 states s,
% 1 control buy or sell land: x, error term for first 2 states

global beta0 beta1 alpha0 alpha1 alpha2 cost rn rp tcs tcb fme fds
sminv smaxv

[nn d]= size(s);
g = zeros(size(s));
gf1 = exp(beta0 + (beta1.*log(s(:,1))) + ek(:,1));
                    % R state eqn
gf2 = exp(alpha0 + (alpha1.*log(s(:,2))) + alpha2.*log(s(:,1)) +
ek(:,2));   % P state eqn
g(:,1) = max(sminv(1),min(gf1,smaxv(1)));
g(:,2) = max(sminv(2),min(gf2,smaxv(2)));
g(:,3) = s(:,3) + x;                              % Land state eqn

s2bs = ((1+tcb).*s(:,2)) + fme;
nxi = find(x<0);
s2bs(nxi) = ((1-tcs).*s(nxi,2)) + (1-fds).*fme;

s2v = (1-tcs).*s(:,2) + (1-fds).*fme;            % For normalization
g2v = (1-tcs).*g(:,2) + (1-fds).*fme;            %  "         "

At = s(:,4)-(s2v.*s(:,3));
r = ones(size(s,1),1).*rp;
if g(:,3)<1                         % this is for if g(:,3)==0, sell
all land
   ia = At - (s2bs.*x);             % selling: x is negative
   nai = find(ia<0);                                % Negative amount index
   r(nai) = rn;
   g(:,4) = (1+r).*ia;
else
   ia = At - (s2bs.*x) - (cost.*g(:,3));
   nai = find(ia<0);                                % Negative amount index
   r(nai) = rn;
   gAt = ((1+r).*ia) + (g(:,1).*g(:,3));
   g(:,4) = gAt + g2v.*g(:,3);
end
```

```
%%%%%%%%%%%%%%%%%%%%%%file name: utility.m %%%%%%%%%%%%%%%%%%%%%%%%

function u = utility(y)
% Utility function

global b theta

if theta==0
   u = y;
elseif theta==1
   u = (log(b)./b).*y;
      byi = find(y>=b);
   u(byi) = log(y(byi));
else
   disp('utility function for theta=0 or 1');
end


%%%%%%%%%%%%%%%%%%%%%%file name: utility.m %%%%%%%%%%%%%%%%%%%%%%%%

function y = invutility(u)
% Inverse of: utility function

global b theta

if theta==0
   y = u;
elseif theta==1
   y = (u.*b)./log(b);
   bui = find(u>=log(b));
   y(bui) = exp(u(bui));
else
   disp('utility function for theta=0 or 1');
end
```

Simulation for Accuracy:

```
%%%%%%%%%%%%%%%%%%%%%%file name: errors.m %%%%%%%%%%%%%%%%%%%%%%%%

% Error terms: generating randomly
clear all;

Ee = zeros(2,1);                                    % Mean of error
terms for R and P eqns
VarCov = zeros(2,2);                         % Variance-covariance
matrix
VarCov(1,1) = 0.033155;
VarCov(2,2) = 0.014619;
TRIALS = 500;
YEARS = 20;
TY = TRIALS*YEARS;
```

```
Em = MVNRND(Ee,VarCov,TY);
Em1 = Em(:,1);
Em2 = Em(:,2);
Er1 = reshape(Em1,TRIALS,YEARS);
Er2 = reshape(Em2,TRIALS,YEARS);
E = cell(1,YEARS);
for YR = 1:YEARS
   E{YR} = [Er1(:,YR) Er2(:,YR)];
end

%save errors
save errors E;


%%%%%%%%%%%%%%%%%%%%%%file name: simulatetr.m %%%%%%%%%%%%%%%%%%%%%%%%

clear all;
tic;
load errors
load su59521
sfile1 = 'simsu59521.txt';
sfile2 = 'simsu59521';

R = [320;420;520]; P=[1265;1580;1900];L=[800;1200;1600];
W=[1500000;3000000;4500000];

S = cell(1,4);
S{1} = R; S{2} = P; S{3} = L; S{4} = W;
s0 = cgrid(S);

J = size(s0,1);
gJ = cell(1,J);
Eutil = zeros(J,1);
years = length(E);
trials = size(E{1},1);                 % E{1} = [e1,e2,...,e(T+1)]'
yri = T + 1 - years;

%%%%%%%% Estimation of v0 from value function %%%%%%%%%%
if yri==T
   c1 = [];
else
   c1 = copt(:,yri+1);
end
[x0 v0] = feval(vmaxh,s0,c1,yri);

%%%%%%%% Estimation of v0 by simulation %%%%%%%%%%

for j = 1:size(s0,1)
   sj = s0(j,:);
   j
   util = zeros(trials,1);
   sjt = zeros(trials,4);
   gjt = zeros(size(sjt));
   sjt(:,1) = sj(1);
```

```
   sjt(:,2) = sj(2);
   sjt(:,3) = sj(3);
   sjt(:,4) = sj(4);
   for yr = yri:T
      if yr==T
         c1 = [];
      else
         c1 = copt(:,yr+1);
      end
      ini =  find(sjt(:,3)>=(sminv(3)-1) & sjt(:,4)>0);  %note
L=[0,400:2000],sminv(3)=400
      outi = find(sjt(:,3)<1 | sjt(:,4)<=0);
      sk = sjt(ini,:);
      xk = feval(vmaxh,sk,c1,yr);
      gk = zeros(size(sk));
      eyr = E{yr-yri+1};
      eyr1 = eyr(ini,:);
      gk = gstate(sk,xk,eyr1);
      gjt(ini,:) = gk;
      if isempty(outi)==0
         gjt(outi,3) = 0;
         s4o = sjt(outi,4);
         ro = ones(size(s4o,1),1).*rp;
         oi = find(s4o<0);
         ro(oi) = rn;
         gjt(outi,4) = (1+ro).*s4o;
      end
      sjt = gjt;
   end
   %gJ{j} = gjt;
   wealth = gjt(:,4);   % Wealth at T+1
   util = utility(wealth);
   Eutil(j) = mean(util);
end

Eutilv0s0 = [Eutil,v0,s0]';
fid = fopen(sfile1,'w+t');
fprintf(fid,'Number of nodes for states = %3i %3i %3i %3i\n\n',n);
fprintf(fid,'      E[U(W(T+1))]         v0      R0     P0       L0
W0\n');
fprintf(fid,'\n%15.2f %15.2f %7.0f %7.0f %7.0f %7.2f\n',Eutilv0s0);
toc;                               % Elapsed time since tic was used
fprintf(fid,'\nCPU Time seconds= %15.4f\n',toc); % For saving in the
file
fclose(fid);
save(sfile2);
```

For Some Result Output (graphs)

```
%%%%%%%%%%%%%%%%%%%%%%file name: figxprdata.m %%%%%%%%%%%%%%%%%%%%%%%%

% Policy Function
clear all;
load su77521                                              % Loading
workspace of run model

t = 0;
if t==T
   c = [];
else
   c = copt(:,t+1);
end

Lt = 600;
Wt = 300000;
P = [Pmin:5:Pmax];
R = [350:10:450]';
%W = [1000000;4000000];
%__________________________________________________
% Policy Function x(P) for different states gievn

S = cell(1,length(R));
XP = zeros(length(P),length(R));
      for ri = 1:length(R)
         Rt = R(ri);
         S{ri} = zeros(length(P),length(n));
         S{ri}(:,1) = Rt;
         S{ri}(:,2) = P;
         S{ri}(:,3) = Lt;
         S{ri}(:,4) = Wt;
         XP(:,ri) = feval(vmaxh,S{ri},c,t);
      end
save figxprdata


%%%%%%%%%%%%%%%%%%%%%%file name: figxpr.m %%%%%%%%%%%%%%%%%%%%%%%%


clear all;
load figxprdata
figure(2);
plot(P,XP(:,end),'k-',P,XP(:,1),'k--','linewidth',1.3);
set(gca,'fontsize',10,'xtick',[Pmin:250:2210],'ytick',[-200:50:200])
set(gcf,'papersize',[8.5,5.5]);
%xlabel('\itP','fontsize',12)
%ylabel('Net Wealth: W  ($1000)','fontsize',10)
ylabel('\itx_{t}*','fontsize',16)
legend('\itR_{t} = 450  ','\itR_{t} = 350  ',1)
%text(1620,265,'\itR = 450','fontsize',12);
%text(1450,-170,'\itR = 350','fontsize',12);
```

```
xlim([Pmin Pmax+30])
ylim([-240 240]);
%------------------------------------------------
[p,r]=meshgrid(P,R);
p = p'; r=r';

figure(1);
plot3(p,r,XP,'k-','linewidth',1.1);
view([6,14]);
set(gca,'fontsize',10,'xtick',[Pmin:250:2210],'ytick',[min(R):50:max(R
]);
  % 'ztick',[-Lt,[(400-Lt):200:Lmax-Lt]]);
box on; set(gcf,'papersize',[8.5,5.5]);
%xlabel('\itP_{\itt}','fontsize',10)
%ylabel('\itW_{\itt} ($1000)','fontsize',10)
zlabel('\itx_{t}*','fontsize',16)
ylim([325 525]);
zlim([-200,205]);


%%%%%%%%%%%%%%%%%%%%%%file name: pathdist.m %%%%%%%%%%%%%%%%%%%%%%%%

clear all;
load errors

load su77521


sfile2 = 'pathdistw700L800';

R = 390;
P = 1500;
L = 800;
W = 700000;
syear = 0;
%syear = [0,10,20,30];

years = 10;
%years = length(E);            % E = [e1,e2,...,e(T+1)]'
s0 = [R P L W];
gs = cell(1,3);                %storing for selected t distributions:
t=1,5,10
glafa = cell(1,3);
gT = cell(1,1);
gT{1} = zeros(years,4);        %for storing only means for all years
%gT{2} = zeros(years,4);
%gT{3} = zeros(years,4);
%gT{4} = zeros(years,4);
%gT{5} = zeros(years,4);
%gT{6} = zeros(years,4);
%gT{7} = zeros(years,4);
pout = cell(1,1);               %storing for selected t distributions:
t=1,5,10,20
pout{1} = zeros(years,6);       %for storing only means for all years
```

```
%pout{2} = zeros(years,6);
%pout{3} = zeros(years,6);
%pout{4} = zeros(years,6);
%pout{5} = zeros(years,6);
%pout{6} = zeros(years,6);
%pout{7} = zeros(years,6);

trials = size(E{1},1);        % each e is trials 500 by 2 for 2 states
sj = s0;

for tind = 1:length(syear)
yri = syear(tind);
%yri = 0;
sjt = zeros(trials,4);
gjt = zeros(size(sjt));
sjt(:,1) = sj(1);
sjt(:,2) = sj(2);
sjt(:,3) = sj(3);
sjt(:,4) = sj(4);

for yr = yri:(yri+years-1)              %0:T (0:19)
   if yr==T
      c1 = [];
   else
      c1 = copt(:,yr+1);
   end
   ini =  find(sjt(:,3)>=(sminv(3)-1) & sjt(:,4)>0);
                              %note L=[0,400:2000],sminv(3)=400
   outi = find(sjt(:,3)<1 | sjt(:,4)<=0);
                        %Besides bankruptcy, for W=0 also, V is known
   sk = sjt(ini,:);
   xk = feval(vmaxh,sk,c1,yr);                            %%%%
   gk = zeros(size(sk));
   eyr = E{yr-yri+1};
   eyr1 = eyr(ini,:);
   gk = gstate(sk,xk,eyr1);
   gjt(ini,:) = gk;
   if isempty(outi)==0
      gjt(outi,3) = 0;
      s4o = sjt(outi,4);
      ro = ones(size(s4o,1),1).*rp;
      oi = find(s4o<0);
      ro(oi) = rn;
      gjt(outi,4) = (1+ro).*s4o;
   end
   gT{tind}(yr+1-yri,3:4) = mean(gjt(:,3:4),1);
%%%%
   gT{tind}(yr+1-yri,1:2) = var(gjt(:,3:4),1);
   outbr0i = find(gjt(:,3)<1 | gjt(:,4)<=0);
   inibr0i =  find(gjt(:,3)>=(sminv(3)-1) & gjt(:,4)>0);
   outbri = find(gjt(:,4)<=0);
   pout{tind}(yr+1-yri,3) = length(outbr0i)./500;       %prob of out of
farming = BR+choosing out
   pout{tind}(yr+1-yri,1) = length(outbri)./500;     %prob of BR
```

```
   pout{tind}(yr+1-yri,2) = (length(outbr0i) - length(outbri))./500;
%prob of choosing out

   gjt2v = (1-tcs).*gjt(inibr0i,2) + (1-fds).*fme;
   if isempty(inibr0i)==0
   outdau = find((gjt(inibr0i,4) - (1-dau).*gjt2v.*gjt(inibr0i,3))<=0);
   pout{tind}(yr+1-yri,4) = length(outdau)./length(inibr0i);
   pout{tind}(yr+1-yri,5) =
mean((gjt(inibr0i,4)./(gjt2v.*gjt(inibr0i,3))))-1;
   pout{tind}(yr+1-yri,6) = mean(gjt(inibr0i,3));
   end
   sjt = gjt;
   if yr==0
      gs{1} = gjt;          %for years 1,5,10
      glafa{1} = (gjt(inibr0i,4)./(gjt2v.*gjt(inibr0i,3)))-1;
   elseif yr == 4
      gs{2} = gjt;
      glafa{2} = (gjt(inibr0i,4)./(gjt2v.*gjt(inibr0i,3)))-1;
   elseif yr == 9
      gs{3} = gjt;
      glafa{3} = (gjt(inibr0i,4)./(gjt2v.*gjt(inibr0i,3)))-1;
   end
end
end
save(sfile2);



%%%%%%%%%%%%%%%%%%%%%file name: pathdistfig.m %%%%%%%%%%%%%%%%%%%%%%%%

clear all;
load pathdistw;

syears = [1,5,10];

syear = 1;        %choose 1,2, or 3 (upto length(syears))

%minW1 = min(gs{syear}(:,2));
%maxW1 = max(gs{syear}(:,2));
minW1 = -140000;
maxW1 = 5460000;
range1 = maxW1 - minW1;
gap1 = range1/40;
edge1 = [minW1:gap1:maxW1];
fp1 = histc(gs{syear}(:,2),edge1);
figure;
bar(edge1/1000,fp1/500,1,'c');
xlim([-1000 5500])
ylim([0,0.45])
set(gcf,'papersize',[8.5,5.5]);
set(gca,'fontsize',12,'layer','top')%,'ytick',[0:0.02:0.23]);
xlabel(['\itW' ' in Year ' num2str(syears(syear)), '
($1000)'],'fontsize',16)
ylabel('Probability','fontsize',16)
```

```
edge2 = [0,Lmin:50:Lmax];
fp2 = histc(gs{syear}(:,1),edge2);
figure;
bar(edge2,fp2/500,1,'c');
xlim([-150,2150]);
ylim([0,1.02])
set(gcf,'papersize',[8.5,5.5]);
set(gca,'fontsize',12,'layer','top','xtick',[0,400,800,1200,1600,2000]);
xlabel(['\itL' ' in Year ' num2str(syears(syear))],'fontsize',16)
ylabel('Probability','fontsize',16)
```