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Nonlinear Dynamics and Economic Instability: The Optimal Management of a Biological Population

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Assuming a competitive market, conditions are determined for when a steady-state equilibrium does not exist in the optimal dynamic management of a biological population. Irregular and unpredictable behavior (called "chaos") can arise from fully rational economic decision making. High interest rate, adjustment costs, and an inelastic demand can contribute to market instability.

Key words: chaos, limit cycle, optimal dynamics, steady state

Introduction

Much research has focused on the economic instability[1] of agricultural markets (e.g., Cochrane; Newbery and Stiglitz). Instability is a relevant topic given that most agricultural economists believe that a primary justification for government intervention in agriculture is to reduce instability (Pope and Hallam). Yet, agricultural instability remains poorly understood. Agricultural markets are clearly subject to exogenous shocks that affect food prices in unpredictable ways (e.g., weather effects). If farmers are risk averse and risk markets are incomplete, then competitive agricultural markets likely generate an inefficient allocation of resources (Newbery and Stiglitz). However, Newbery and Stiglitz's approach is basically static and neglects the role of dynamic adjustments. Cochrane's analysis of unstable agricultural markets relies on a cobweb analysis that reflects biological lags in the production process. Cochrane pointed out that supply and demand elasticities influence the dynamic path of market prices. He stressed that the inelasticity of food demand contributes to the instability of agricultural markets. Unfortunately, Cochrane's cobweb model assumes that farmers base their production decisions on lagged market prices. Such price expectations are not consistent with the rational expectations hypothesis. This raises the question: can market instability be generated dynamically by fully rational participants in competitive markets?

Most economists believe that market prices are regular and predictable. This regularity is often seen as a consequence of optimizing behavior by economic agents. For example, intertemporal arbitrage incentives tend to smooth the dynamic path of optimal resource use over time. Assuming the system being studied is stationary, this behavior can contribute to the convergence toward a steady-state equilibrium. Optimizing behavior, however, does not always lead to a steady-state equilibrium. The possibility of the optimal dynamic path

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[1]Throughout the article, the term "instability" is meant to characterize any situation that is not constant over time and not fully predictable ahead of time.

converging to a limit cycle is well documented (e.g., Benhabib and Nishimura; Clark; Boldrin and Woodford).[2] Irregular and unpredictable market equilibrium can arise without exogenous or stochastic shocks. Such irregular patterns are called "chaos" (e.g., May; Benhabib; Day and Chen; Hao; Li and Yorke; Grandmont). Chavas and Holt show that the management of the U.S. dairy herd can lead to a chaotic market equilibrium. However, Chavas and Holt used decision rules that were not derived under rational expectations nor optimizing behavior. Yet, chaotic dynamic paths are possible under optimal management (e.g., Deneckere and Pelikan; Boldrin and Montrucchio; Benhabib). Since chaotic paths correspond to situations of economic instability, this suggests a need to investigate further when chaotic behavior can be the outcome of the dynamic optimization of resource flow from biological populations.

This study determines when the optimal management of a biological population under competitive market results in a steady-state equilibrium, limit cycle, or chaos. Although much progress has been made on understanding economic dynamics in managed animal populations (e.g., Clark; Rosen), the dynamics of the associated market equilibrium are still not well understood. Rosen has presented a model of competitive market equilibrium in the dynamic management of a biological population. His research, however, is based on a simple model which assumes a linear-state equation and constant return to scale technology. These assumptions imply that market equilibrium necessarily leads to the existence of a steady state (Rosen, p. 550). In this article, we show that relaxing these stringent assumptions can have significant impacts on the nature of long-term equilibrium. We investigate optimal dynamics without imposing global concavity of the objective function, an assumption commonly made in previous research (e.g., Deneckere and Pelikan; Boldrin and Montrucchio). We provide empirical examples illustrating how limit cycles as well as chaos can arise as the outcome of optimizing behavior and market equilibrium. These results are obtained in completely deterministic models, implying that (stochastic) uncertainty is not necessary for the investigation of market instability. We show that such complexities in the nature of long-run equilibrium can arise even in a simple model involving a single state variable. We caution that our results do not mean that optimal behavior is necessarily chaotic; rather, they warn us against the danger of assuming that optimizing behavior always leads to a regular and predictable market equilibrium. Our analysis shows how irregular and unpredictable behavior can arise from rational economic decision making. It provides new insights into the possible causes of market instability.

The Model

Consider a population where the state variable x_t denotes the number of population members living at time t. We assume that $x_t \in X$, where $X = [0, H]$ is a convex and compact subset of $\Re^+$, H being a positive scalar. The natural evolution of the population over time is given by the state equation $x_{t+1} = f(x_t)$, where $f(x_t)$ is a continuous function from X to X. The function $f(x_t)$ is not necessarily concave everywhere. Indeed, in population growth, the function $f(x_t)$ is typically concave for large values of x but may be convex for small values of x (e.g., Clark, p. 17).

The population is assumed to be manageable. The management decision is to choose how many population members to harvest (i.e., to remove from the population) at each time

[2] In the absence of a steady-state equilibrium, a variable x_t exhibits a limit cycle if it eventually reaches a regular cycle such that $x_t = x_{t+k}$ as time t becomes large, k denoting the period of the cycle.

period. Let m_t denote the harvest rate at time t. Then, under management, the dynamics of the population is given by the following state equation:

(1) $$x_{t+1} = f(x_t) - m_t.$$

We focus our attention on the simple case where benefits from the animal population come only through the harvest variable m_t. At time t, the demand for m_t is specified in price-dependent form as:

$$p_t = p(m_t) \geq 0,$$

where p_t is the market price at time t, and $p(m_t)$ is a continuous, decreasing function of the harvest, m_t. The social benefits generated by the demand $p(m_t)$ at time t are given by:

$$D(m_t) = \int_0^{m_t} p(q)dq < \infty,$$

where the marginal benefit of consumption is equal to the market price: $\partial D(m_t) / \partial m_t = p(m_t) = p_t$.

The management of the animal population is assumed to be costly. At time t, the cost of choosing a harvest level m_t given an animal population (x_t, x_{t+1}) is denoted by $C(x_t, x_{t+1})$, where $C(x_t, x_{t+1})$ is a continuous, bounded function from $X \times X$ into $\Re$.[3]

Assuming that the cost and demand functions are stationary over time, consider the optimal management of the animal population. The net social benefit from the animal population at time t, B_t, can be measured as consumer surplus net of production cost:

(2) $$B_t = \int_0^{m_t} p(q)dq - C(x_t, x_{t+1}).$$

Given an infinite planning horizon, optimal behavior corresponds to:

(3) $$\begin{aligned} V(x_0) &= \max_{x,m}[\sum_{t=0}^{\infty} \beta^t B_t : \text{ equations (1) and (2), } x_t \in X] \\ &= \max_{x,m}[\sum_{t=0}^{\infty} \beta^t [\int_0^{m_t} p(q)dq - C(x_t,\, x_{t+1})]: \text{ equation (1), } x_t \in X] \\ &= \max_{x}[\sum_{t=0}^{\infty} \beta^t [\int_0^{f(x_t)-x_{t+1}} p(q)dq - C(x_t,\, x_{t+1})]: \; x_t \in X], \end{aligned}$$

[3]More generally, the cost function could be written as $C(x_t, x_{t+1}, m_t)$. However, this reduces to our specification after substituting (1) for m_t to give $C(x_t, x_{t+1}, f(x_t) - x_{t+1})$, which depends only on x_t and x_{t+1}. Alternatively, our specification can be rationalized if the cost function takes the form $C_1(x_t, x_{t+1}, m_t) = C(x_t, x_{t+1}) + C_0(m_t)$, where $C(x_t, x_{t+1})$ denotes the cost of managing the biological population, and $C_0(m_t)$ is the cost of marketing the product m_t. In this case, if $p_0(m_t)$ denotes the consumer demand function, then $p(m_t) = p_0(m_t) - \partial C_0/\partial m_t$ would be the derived demand function, i.e., the consumer demand function net of the marginal cost of marketing the product m_t to the consumer.

where $x = (x_1, x_2, \ldots)$; $m = (m_0, m_1, \ldots)$; and β is the discount factor, $0 < \beta < 1$. Under differentiability and assuming an interior solution, the first-order necessary condition for an optimum with respect to x_t in equation (3) is

(4) $$p_t \frac{\partial f_t}{\partial x_t} - \frac{\partial C_t}{\partial x_t} - \frac{1}{\beta}[\frac{\partial C_{t-1}}{\partial x_t} + p_{t-1}] = 0 \quad \text{for } x_t \in \text{int} X,$$

where $p_t = p(m_t); f_t = f(x_t); C_t = C(x_t, x_{t+1}); m_t = f(x_t) - x_{t+1}$; $t = 1, 2, 3, \ldots$; and "int X" denotes the interior of the set X. Equation (4) characterizes the dynamics of optimal management of the animal population x_t. It also represents the dynamics of competitive market equilibrium, where p_t is the competitive market price. Thus, model (3) provides a framework for analyzing the dynamic behavior of rational agents facing a competitive market under perfect information.

Using backward induction of dynamic programming, equation (3) can be alternatively expressed in terms of Bellman's equation as:

(5) $$V(x_t) = \max_{x_{t+1} \in X} [\int_0^{f(x_t)-x_{t+1}} p(q)\, dq - C(x_t, x_{t+1}) + \beta V(x_{t+1})]$$
$$= \max_{x_{t+1} \in X} [U(x_t, x_{t+1}) + \beta V(x_{t+1})],$$

where $V(x)$ is the optimal value function and

(6) $$U(x_t, x_{t+1}) = \int_0^{f(x_t)-x_{t+1}} p(q)\, dq - C(x_t, x_{t+1}),$$

for all t, $t = 1, 2, \ldots$. We assume throughout the article that $U(x_t, x_{t+1})$ in (6) is a bounded and continuous function of $(x_t, x_{t+1}) \in X \times X$. From Weierstrass theorem, the compactness of X then implies that the optimization problem (3) [or, alternatively, (5)] has an optimal solution.

Denote the solution of (5) by $x_{t+1}^* = h(x_t)$ for each $x_t \in X$. Thus, $x_{t+1}^* = h(x_t)$ is the optimal policy correspondence from optimally managing the animal population over time. In the case of an interior solution, this solution corresponds to the first-order condition:

(7a) $$\frac{\partial U(x_t, x_{t+1})}{\partial x_{t+1}} + \beta \frac{\partial V(x_{t+1})}{\partial x_{t+1}} = 0 \quad \text{for } x_{t+1} \in \text{int } X,$$

where

(7b) $$\frac{\partial U(x_t, x_{t+1})}{\partial x_{t+1}} = -p_t - \frac{\partial C_t}{\partial x_{t+1}}.$$

Also, note that the envelope theorem applied to (5) yields

(7c) $$\frac{\partial V(x_t)}{\partial x_t} = p_t \frac{\partial f_t}{\partial x_t} - \frac{\partial C_t}{\partial x_t} \quad \text{for } x_t \in \text{int } X.$$

Dynamic programming problem (5) is of course equivalent to problem (3). This can be seen simply by noting that the first-order condition (4) is equivalent to equations (7a), (7b), and (7c).

Dynamic Properties

Both state equation (1) and optimal policy correspondence $x^*_{t+1} = h(x_t)$ represent the dynamics of the population. Equation (1) gives population dynamics when the harvest rate m_t is exogenously fixed or given. Alternatively, $x^*_{t+1} = h(x_t)$ represents the dynamic path of the population under optimal management. In general, these two equations are different and, thus, generate different dynamic paths. This raises two questions: (*a*) What are the dynamic patterns associated with each equation?; and (*b*) What factors influence the differences between the two dynamic patterns? Before investigating these questions, we begin with a general review of some analytical tools available in the analysis of dynamic systems.

The Nature of Dynamic Path

Consider a general first-order difference equation, $x_{t+1} = g(x_t)$, representing the dynamics of the state variable $x_t \in X$, X being a compact set. The state equation $x_{t+1} = g(x_t)$ can generate three kinds of dynamic patterns: (*a*) the state variable x_t can converge to a steady state; (*b*) it can converge to a limit cycle; or (*c*) it can exhibit chaos. While a steady $g(\cdot)$ state or a limit cycle can be generated by a linear function $g(\cdot)$, only a nonlinear function can generate chaos (Hao). Chaos is associated with "sensitive dependence to initial conditions," meaning that small changes in initial conditions give rise to divergent time paths. Such divergence implies that long-term predictions from a chaotic system are virtually impossible.[4]

Given a single state variable x_t, the nature of dynamics associated with the state equation $x_{t+1} = g(x_t)$ can be inferred using the Lyapunov exponent λ, defined as:

(8) $$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log|g'(g^j(x))|,$$

for $x \in X$, where $j(x_t) = x_{t+j}$, and $g^j(x) = \partial g / \partial x$ (Benettin, Galgani, and Strelcyn). The Lyapunov exponent λ in (8) measures the average (linearized) contraction (or expansion) rate of the forward dynamic path. A negative Lyapunov exponent ($\lambda < 0$) indicates either a limit cycle or a steady-state equilibrium. Alternatively, a positive Lyapunov exponent ($\lambda > 0$) identifies sensitivity to initial conditions and thus chaos (Benettin, Galgani, and Strelcyn; Wolf et al.).

Optimal Dynamics

The optimal policy correspondence $x^*_{t+1} = h(x_t)$ is used to determine the dynamic nature of optimal resource allocation. Of particular interest is the existence (or nonexistence) of a steady-state equilibrium in the optimally managed population. Optimal dynamics have been investigated by Benhabib and Nishimura, Deneckere and Pelikan, and Boldrin and Mon-

[4]This raises the issue of distinguishing between a random process and a chaotic deterministic process (see Brock and Sayers).

trucchio under the assumption that the function $U(x_t, x_{t+1})$ in (6) is concave in $(x_t, x_{t+1}) \in X \times X$. Here we relax this assumption and consider the more general case where $U(x_t, x_{t+1})$ is continuous but not necessarily concave in $(x_t, x_{t+1}) \in X \times X$. This allows us to consider a broader class of dynamic optimization problems. The following lemma is an extension of a rresult obtained by Benhabib and Nishimure (p. 293) but without assuming the concavity of $U(x_t, x_{t+1})$.

Lemma 1: Let (x_t, x_{t+1}) be a point on the graph of the optimal policy correspondence $x^*_{t+1} = h(x_t)$. If $\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1} > 0\ (<)\ 0$, then $h(x_t)$ is nondecreasing (nonincreasing) at x_t.

Proof: Choose any two points $x^a_t \in X$ and $x^b_t \in X$ such that x^b_t is in the neighborhood of x^a_t and $x^b_t \neq x^a_t$. Denote the corresponding optimal choices by $x^a_{t+1} = h(x^a_t)$ and $x^b_{t+1} = h(x^b_t)$. By definition of a maximum in (5), we have

$$V(x^a_t) = U(x^a_t, x^a_{t+1}) + \beta V(x^a_{t+1}) \geq U(x^a_t, x^b_{t+1}) + \beta V(x^b_{t+1}),$$

and

$$V(x^b_t) = U(x^b_t, x^b_{t+1}) + \beta V(x^b_{t+1}) \geq U(x^b_t, x^a_{t+1}) + \beta V(x^a_{t+1}).$$

Combining these two expressions yields

$$U(x^a_t, x^b_{t+1}) - U(x^a_t, x^a_{t+1}) \leq \beta V(x^a_{t+1}) - \beta V(x^b_{t+1}) \leq U(x^b_t, x^b_{t+1}) - U(x^b_t, x^a_{t+1})$$

or

$$D \equiv [U(x^b_t, x^b_{t+1}) - U(x^b_t, x^a_{t+1})] - [Ux^a_t, x^b_{t+1}) - U(x^a_t, x^a_{t+1})] \geq 0.$$

But, when evaluated at (x^a_t, x^a_{t+1}), note that sign $\{\partial^2 U / \partial x_t \partial x_{t+1}\}$ = sign $\{D / [(x^b_t - x^a_t)(x^b_{t+1} - x^a_{t+1})]\}$ if $[(x^b_t - x^a_t)(x^b_{t+1} - x^a_{t+1})] \neq 0$. Given $x^b_t \neq x^a_t$, there are two possibilities:

(i) $x^b_{t+1} = x^a_{t+1}$, or

(ii) $x^b_{t+1} \neq x^a_{t+1}$, in which case $\partial^2 U / \partial x_t \partial x_{t+1} > 0$ (<0) implies that $\{D / [(x^b_t - x^a_t)(x^b_{t+1} - x^a_{t+1})]\} > 0\ (<0)$. With $D \geq 0$, it follows that $D > 0$ and $[(x^b_t - x^a_t)(x^b_{t+1} - x^a_{t+1})] > 0\ (<0)$ if $\partial^2 U / \partial_{xt} \partial x_{t+1} > 0\ (<0)$.

In other words, when evaluated at $x^a_t, x_{t+1} = h(x_t)$ is a nondecreasing (nonincreasing) correspondence if $\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1} > (<)\ 0$.

Using lemma 1, a number of useful results can be obtained concerning the dynamic nature of optimal behavior. Extending the analysis presented by Deneckere and Pelikan, and Boldrin and Montrucchio, without assuming the concavity of $U(x_t, x_{t+1})$, we have the following proposition.

Proposition 1:

(i) If $\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1} > 0$ for all (x_t, x_{t+1}) on the graph of the policy correspondence h, then $x_t+j = h_j(x_t)$ converges monotonically to a steady state for all $x_t \in X$.

(ii) If $\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1} < 0$ for all (x_t, x_{t+1}) on the graph of the policy correspondence h, then $x_{t+j} = h^j(x_t)$ converges to a cycle of period one or two.

Proof: Proposition 1 can be shown as follows. Part (i) assumes that an increase in x_t raises the marginal benefit of x_{t+1} [i.e., $\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1} > 0$]. From lemma 1, $x_{t+1} = h(x_t)$ is then a nondecreasing correspondence. As the optimal choice of x in any period depends only on the value of x in the previous period, if x_{t+1} is at least (at most) equal to x_t, then x_{t+2} will necessarily be no less (no greater) than x_{t+1}, implying a monotonic (either increasing or decreasing) path. Given a bounded feasible region, this path must eventually converge to a steady state.

Similarly, consider part (ii) where an increase in x_t reduces the marginal benefit of x_{t+1} [i.e., where $\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1} < 0$]. From lemma 1, $x_{t+1} = h(x_t)$ is then a nonincreasing correspondence. This in turn implies that $x_{t+2} = h^2(x_t) = h(h(x_t))$ is nondecreasing in x_t. In this context, having x_{t+1} at least (at most) equal to x_t necessarily means that x_{t+2} will be no more (no less) than x_{t+1}, and that x_{t+2} will be no less (no more) than x_t. This implies oscillatory behavior of x_{t+j} as well as a monotonic path of x_{t+2j}, $j = 1, 2, \ldots$. Given a bounded feasible region, the optimal path of x_t must eventually converge to a cycle of period one or two.

Proposition 1 implies that, whenever $[\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1}]$ is of uniform sign on $X \times X$, then the long-run optimal behavior of the population is either a steady state or a stable orbit of period two. In this case, any optimal trajectory is necessarily asymptotic to a periodic orbit of period one or two. Let us define "simple dynamics" as the situation where the optimal population exhibits periodic behavior with a period less than or equal to two. Then, having $[\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1}]$ of uniform sign on $X \times X$ is sufficient to guarantee that optimal management can only generate simple dynamics. Alternatively, we can define "complicated dynamics" as the situation where the optimal population exhibits long-run dynamic behavior that cannot be represented by cycles of period one or two. Then, the following Corollary to Proposition 1 is obtained.

Corollary 1: A necessary condition for an optimal population to exhibit complicated dynamics is that $[\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1}]$ is *not* of uniform sign on $X \times X$.

The question then is: What kind of complicated optimal dynamics can be generated in cases where $[\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1}]$ can take on both positive and negative values in $X \times X$? First, it should be noted that, in such situations, optimal dynamic behavior can be very complicated. This is illustrated in the following proposition. (For a proof, see Deneckere and Pelikan, p. 23).

Proposition 2: Assume that

$$\partial^2 U(x_t, x_{t+1}) / \partial x_t \partial x_{t+1} \begin{cases} > 0 & \text{for } x_{t+1} < \alpha \\ < 0 & \text{for } x_{t+1} > \alpha, \end{cases}$$

for some $\alpha \in \operatorname{int} X$, where $X = [0, H]$. Then, sufficient conditions for the optimal population to exhibit chaotic dynamics are

(i) the optimal policy correspondence $h(x_t)$ is continuous from X to X,

(ii) $h(0) = h(H) = 0$,

(iii) $\alpha < h(\alpha) \leq m$, $h^2(\alpha) \leq \{x \in (\alpha, H)$: $h(x) = \alpha$ and

(iv) $h(\alpha) \geq \{x \in (\alpha, H)$: $a = h(x)$, $\alpha = h(a)$, $a \in (0, \alpha)\}$.

Proposition 2 indicates that optimal chaotic behavior is possible under some conditions. Even in simple models, the dynamic properties of optimal behavior can be quite complex. Unfortunately, the conditions generating such complexities are still poorly understood. On the one hand, the necessary condition stated in Corollary 1 is not sufficient to obtain complicated dynamics. On the other hand, the sufficiency conditions stated in Proposition 2 are not necessary to generate chaotic dynamics. At this point of time, necessary and sufficient conditions for complicated dynamics are not known. This suggests that the analysis of complicated dynamics can best proceed by focusing on specific topics.

One topic has been the subject of considerable research: the role of the discount factor β in generating complicated optimal dynamics. Brock and Scheinkman have shown that a discount factor β close to 1 contributes to a convergent path toward a steady-state equilibrium. Writing the discount factor as $\beta = 1/(1+r)$ where r is the interest rate corresponds to a situation where the interest rate r is low. In this case, future benefits are weighted almost as much as current benefits, which provide incentive to smooth resource allocation over time and contributes to the convergence to a steady-state equilibrium.

Alternatively, Deneckere and Pelikan or Boldrin and Montrucchio have found that optimal chaos (where the optimal trajectory of the state variable is chaotic) can appear when the discount factor β in (3) is sufficiently small. This means the interest rate r must be sufficiently large, implying that future benefits are heavily discounted. The intuitive interpretation is that optimal chaotic dynamics can arise when current decisions are driven mostly by current benefits. Thus, a very high interest rate would provide little incentive to smooth resource allocation over time and could contribute to the nonexistence of a steady-state equilibrium. The question then is: How high must the interest rate r (or how low the discount rate β) be before chaotic dynamics are likely to appear? Deneckere and Pelikan's example (p. 24) indicates the presence of chaos when $\beta = 0.01$. Boldrin and Montrucchio (p. 37) propose a model where chaos appears when $\beta < 0.01263$. In either case, this corresponds to interest rates r around 100. Such extremely high interest rates seem very unlikely in most real world situations. This suggests that interest rates alone are probably not sufficient to motivate the existence of chaos in optimally controlled dynamic systems.

Thus, while we know that optimal paths can be quite complex, this does not imply that such complexities are necessarily present. Whether or not such complexities do arise depends on the dynamic system being considered. This stresses the need to evaluate further under what empirical conditions such complexities might arise.

Optimal Management of a Biological Population

This section illustrates the implications of our analysis for the nature of market dynamics under the optimal management of a biological population. From equation (6), we have

$$(9) \qquad \frac{\partial^2 U}{\partial x_t \partial x_{t+1}} = -\frac{\partial p_t}{\partial m_t}\frac{\partial f_t}{\partial x_t} - \frac{\partial^2 C_t}{\partial x_t \partial x_{t+1}},$$

where $\partial p_t / \partial m_t < 0$. Corollary 1 states that complicated dynamics can take place only if the second derivative $\partial^2 U / \partial x_t \partial x_{t+1}$ can change sign for $(x_t, x_{t+1}) \in X \times X$. From equation (9), obtaining complicated dynamics thus requires that $\partial f_t / \partial x_t$ can change sign and/or that $\partial^2 C_t / \partial x_t \partial x_{t+1}$ can change sign. This implies that complicated dynamics can never arise from a model with a linear state equation and a quadratic cost function C_t. Such specifications are commonly found in the literature. An example is given by Rosen who assumes a linear state equation and a linear cost function. In such a case, the model specification precludes uncovering complex forms of dynamic resource allocation. While this does not imply that such complex forms necessarily exist, it appears quite undesirable to eliminate them a priori.

Under what conditions are complicated dynamics likely to arise? Using equation (9), Corollary 1 and Proposition 2 suggest that complicated dynamics might arise when $\partial f_t / \partial x_t > 0 \,(< 0)$ for $x_t < \alpha_0 \,(> \alpha_0)$ for some $\alpha_0 \in \text{int } X$, and when $\partial^2 C_t / \partial x_t \partial x_{t+1} < (> 0)$ for $x_t < \alpha_1 (> \alpha_1)$ for some $\alpha_1 \in \text{int } X$. However, whether or not they do arise will depend on the problem investigated.

For the purpose of illustration, consider the case where $f(x_t) = \gamma x_t(1 - x_t)$ and the state equation (1) takes the form:

$$x_{t+1} = \gamma x_t (1 - x_t) - m_t, \tag{10}$$

where $0 < \gamma < 4$ and $x_t \in X = (0, 1)$. With $m_t = 0$, equation (10) is the classical logistic difference equation commonly used to investigate population dynamics (e.g., Clark). As shown by May, this simple deterministic state equation can generate complex dynamics. For $m_t = 0$ and $\gamma \in (0, 3)$, x_t is stable and converges to a unique steady state, given any initial condition $x_0 \in X$. For $\gamma \in (3, 3.5699)$, x_t exhibits cyclical patterns. Finally, x_t produces chaotic patterns for selected values of $\gamma \in (3.5699, 4)$. In this case, the deterministic state variable follows a highly irregular trajectory (May; Baumol and Benhabib). Thus, the "uncontrolled" logistic equation (10) (with $m_t = 0$) can generate a steady state, a limit cycle, or chaos depending on the value of the tuning parameter γ .

The question of interest here is: What are the dynamic properties of the optimal policy correspondence $x^*_{t+1} = h(x_t)$ associated with the state equation (10)? More specifically, under what conditions does the optimal policy exhibit a steady state, a limit cycle, or chaos? From equation (10), a large value of γ contributes to having $\partial^2 U / \partial x_t \partial_{t+1} > 0 \; (< 0)$ for small values (large values) of x_t. From Corollary 1, this may be associated with complicated optimal dynamics. And, from Proposition 2, this might generate optimal chaos. However, knowing the nature of optimal dynamics is difficult in general: there is an infinite number of possible nonlinear functional forms for the benefit function B_t in (2), each functional form generating a different optimal path.

Our analysis focuses on a particular parametric specification of the demand function $p(q)$ and of the cost function $C(x_t, x_{t+1})$ in (2). More specifically, we assume that the (quantity dependent) demand function for m_t takes the form:[5]

$$m_t = A p_t^B, \tag{11}$$

where $A > 0$ and $B = \partial \ln m_t / \partial \ln p_t < 0$ denotes the (constant) price elasticity of demand. This simple specification allows evaluating the effect of the demand elasticity on optimal dynamics. The parameter A in (11) can be interpreted as the quantity m_t demanded at price $p_t = 1$.

[5]The corresponding price dependent demand function $p(m)$ used in equation (2) or (3) is $p = A^{-1/B} m^{1/B}$.

The cost function $C(x_t,\ x_{t+1})$ in (2) is assumed to take the form:

$$(12)\qquad C = c_0(x_t) + c_1(x_{t+1}) + c_2 x_t x_{t+1} + c_3 x_{t+1} x_t [c_4 - x_t],$$

where $c_0(x_t)$ and $c_1(x_{t+1})$ are parametric functions; and c_2, c_3, and $c_4 \in (0, 2)$ are parameters. Note that if c_3=0, then the cost function C in (12) can provide a second-order approximation to any differentiable cost function provided $c_0(x_t)$ and $c_1(x_{t+1})$ are quadratic functions. Since $(x_{t+1}x_t[c_4-x_t])$ is a third-order term, $c_3 \neq 0$ can then be interpreted as introducing a third-order term in a quadratic cost specification. Moreover, given $\partial^2 C / \partial x_t \partial x_{t+1} = c_3[c_4 - 2x_t]$ and $c_4 \in (0, 2)$, equation (9) implies that $c_3 < 0$ contributes to having $\partial^2 U / \partial x_t \partial x_{t+1} > 0$ (< 0) for small values (large values) of x_t. Corollary 1 and Proposition 2 then suggest that $c_3 < 0$ might be associated with optimal chaos.

Given equations (10), (11), and (12), the dynamic programming problem (5) was solved for selected values of the parameters β, A, B, and c. With $\beta = 1/(1+r)$, where r denotes the interest rate, analyzing the effects of the parameter β is done through parametric changes in the interest rate r. The solution to problem (5) was obtained using the successive approximation method (Bertsekas, p. 237). The method necessarily converges to a fixed point of the value function $V(x_t)$ (Bertsekas; Streufert). It was implemented by discretizing the state space X = (0, 1) in a grid of 200 equally spaced points. The associated optimal policy function was first examined for possible discontinuities. Except at the points of discontinuities, the optimal policy $h(x_t)$ was smoothed using a piecewise quadratic, continuous and first-differentiable spline function.[6] The smoothed optimal policy $x^*_{t+1} = h(x_t)$ was then simulated numerically.[7] This provided the basis for estimating the Lyapunov exponent [given in (8)] and for evaluating the nature of optimal dynamics.

Three sets of scenarios are evaluated. The first set assumes that $\gamma = 2.9$ in (10). This is a situation where the uncontrolled state equation (10) has a steady-state long-run equilibrium. The second set of scenarios assumes that $\gamma = 3.5$. This is a case where the uncontrolled state equation (10) exhibits a limit cycle (May). Finally, the third set of scenarios assumes that $\gamma = 3.9$. This is the case where the uncontrolled state equation (10) generates chaotic patterns (May). Thus we start from a state equation that can generate a wide variety of dynamic behavior. The question then is: How does optimal management influence the dynamics of the state variable?

For each set of scenarios, we evaluate seven different economic situations. These situations differ according to the elasticity of demand (B), the third-order term coefficient (c_3), and the interest rate [r, where $\beta = 1/(1+r)$]. In each case, the remaining parameters in (11) and (12) are assumed constant, taking the following values: $A = 0.1$, $c_0(x_1) = 0$, $c_1(x_{t+1}) = 0$, $c_2 = 0$, and $c_4 = 0.5$. Thus, we evaluate 21 scenarios: seven scenarios for each value of γ: $\gamma = 2.5, 3.5$, and 3.9. The 21 scenarios are presented in table 1, along with the corresponding estimate of the Lyapunov exponent λ. Each scenario is denoted by two indexes (i,j). The index i takes the value i = 1, 2, and 3 for $\gamma = 2.5, 3.5$, and 3.9, respectively. The index j takes the values j = 1, 2, . . . , 7, each corresponding to a particular economic situation represented by specific parameter values B, c_3 and r (see table 1).

[6] When feasible, each segment of the spline approximation was chosen to include five points.

[7] The model was simulated using x_0 = 0.456 as initial value. In general, the results reported in table 1 were insensitive to this choice. However, we cannot rule out the possibility that complicated optimal dynamics might be associated with some initial conditions of (Lebesgue) measure zero. In this case, "topological chaos" might exist, although it would not be easily observable.

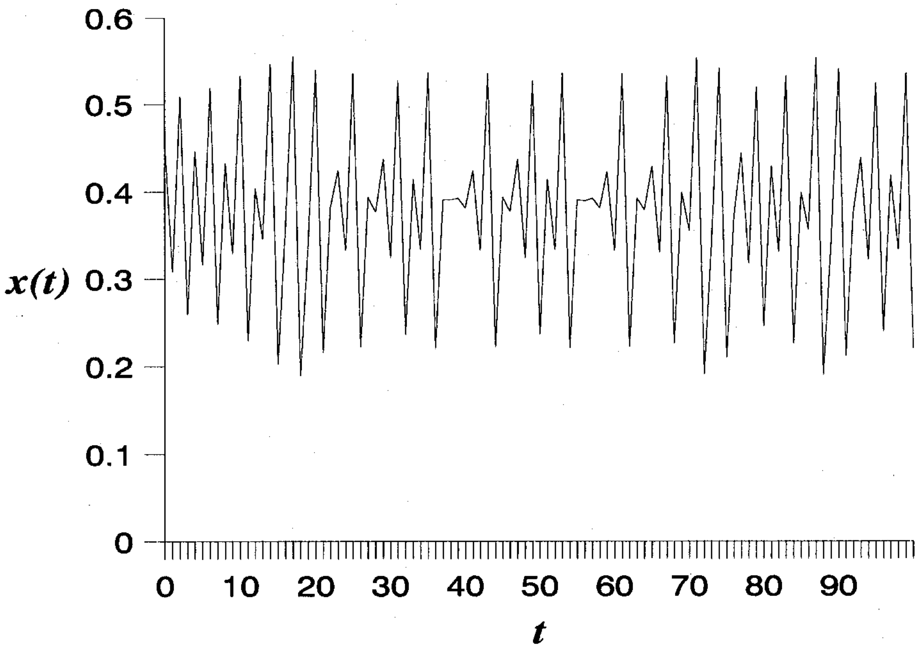

Figure 1. Optimal dynamic path (scenario (1.1))

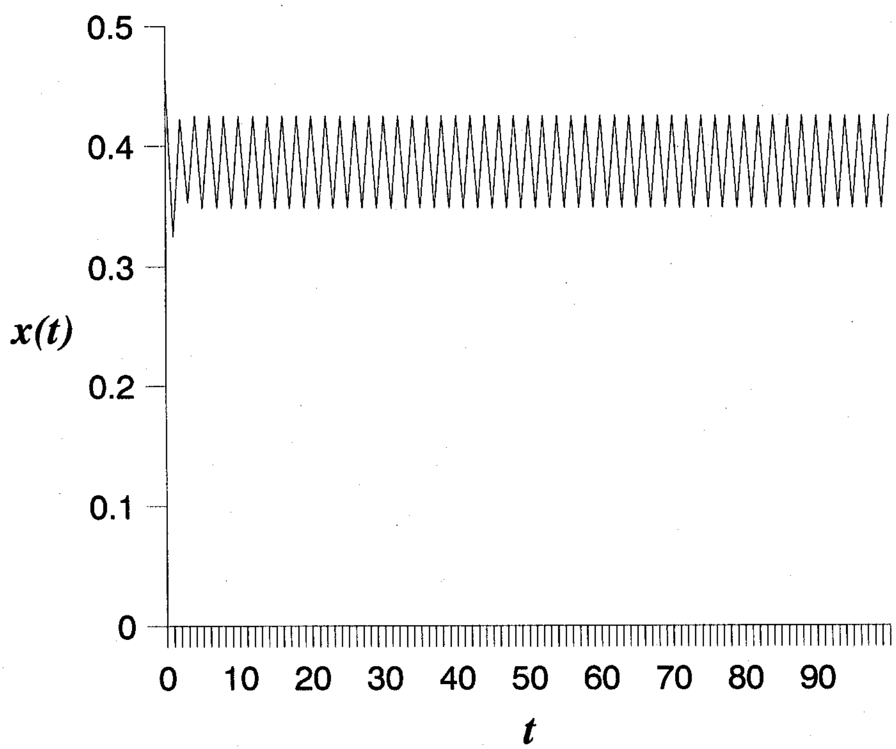

Figure 2. Optimal dynamic path (scenario (1.2))

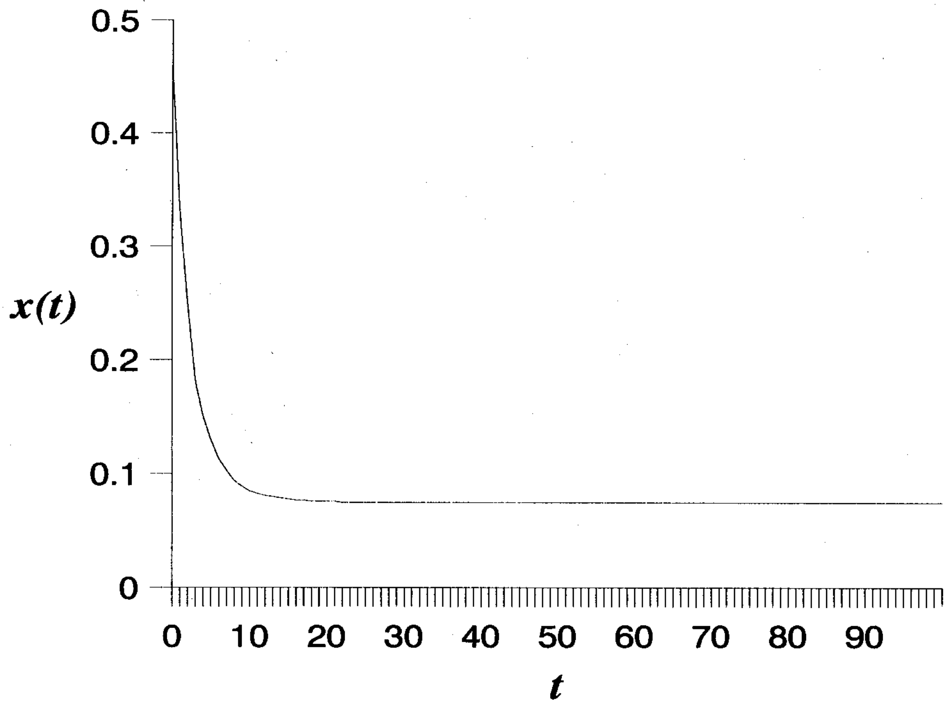

Figure 3. Optimal dynamic path (scenario (1.3))

Scenario (1.1) considers the following parameter values: $B = -0.01$, $c_3 = -0.05$, and $r = 1.6$. It represents a very inelastic demand. It also corresponds to a situation we associated with the possibility of chaos: a negative value for c_3 and a fairly large interest rate r. With a Lyapunov exponent equal to 0.2763 (see table 1), scenario (1.1) indeed generates an optimal chaotic market. This is illustrated in figure 1, which shows the chaotic path generated by the corresponding optimal policy correspondence $x_{t+1} = h(x_t)$. Scenario (1.1) is an example where, starting from a stable system, optimal behavior *does* generate market instability. The sensitivity of this result is evaluated in scenarios (1.2) through (1.7).

In scenarios (1.2) and (1.3), the elasticity of demand B is changed. With an elasticity of demand equal to -0.1, scenario (1.2) generates a limit cycle of period 2. Figure 2 shows the optimal path of the state variable x_t under scenario (1.2). With an elasticity of demand equal to -0.5, scenario (1.3) generates a steady-state equilibrium, as shown in figure 3. These results indicate that the inelasticity of demand contributes to market instability under optimal management. Alternatively, a more elastic demand contributes to the existence of a steady-state equilibrium, that is, to the convergence of optimal dynamics toward a unique long-run equilibrium.

Scenarios (1.4) and (1.5) consider the influence of changing the third-order term parameter c_3 in the cost function. They illustrate that, in the absence of third-order terms ($c_3 = 0$), optimal dynamics exhibit a steady-state long-run equilibrium. Finally, scenarios (1.6) and (1.7) evaluate the effects of the interest rate r on market dynamics. As expected, they indicate that a lower interest rate provides incentives to smooth the optimal path of x_t: market dynamics changes from chaos for $r = 1.6$, to a limit cycle for $r = 1.0$, and to a steady-state equilibrium for $r = 0.5$.

In general, similar results were obtained under scenarios (2.j) and (3.j), $j = 1, 2 \ldots, 7$. In particular, optimal paths exhibited chaos under a very inelastic demand (as measured by

Table 1. The Nature of Dynamic Market Equilibrium under Alternative Scenarios

Scenarios[a]	γ	B	c_3	r	Lyapunov Exponent[b]	Optimal Path
(1.1)	2.9	− 0.01	− 0.05	1.6	0.2763	chaos
(1.2)	2.9	− 0.1	− 0.05	1.6	− 1.3032	limit cycle (period = 2)
(1.3)	2.9	− 0.5	− 0.05	1.6	− 0.1132	steady state (x^e = 0.0667)[c]
(1.4)	2.9	− 0.01	0.00	1.6	− 0.0391	steady state (x^e = 0.2264)
(1.5)	2.9	− 0.5	0.00	1.6	− 0.1084	steady state (x^e = 0.0634)
(1.6)	2.9	− 0.01	− 0.05	1.0	− 0.8371	limit cycle (period = 2)
(1.7)	2.9	− 0.01	− 0.05	0.5	− 0.6275	steady state (x^e = 0.3574)
(2.1)	3.5	− 0.01	− 0.05	1.6	0.2371	chaos
(2.2)	3.5	− 0.1	− 0.05	1.6	0.2153	chaos
(2.3)	3.5	− 0.50	− 0.05	1.6	− 0.2074	steady state (x^e = 0.1351)
(2.4)	3.5	− 0.01	0.00	1.6	− 0.0319	steady state (x^e = 0.1499)
(2.5)	3.5	− 0.5	0.00	1.6	− 0.1960	steady state (x^e = 0.1267)
(2.6)	3.5	− 0.01	− 0.05	1.0	− 0.0152	limit cycle (period = 4)
(2.7)	3.5	− 0.01	− 0.05	0.5	− 0.6314	steady state (x^e = 0.3575)
(3.1)	3.9	− 0.01	− 0.05	1.6	0.2019	chaos
(3.2)	3.9	− 0.1	− 0.05	1.6	0.1365	chaos
(3.3)	3.9	− 0.5	− 0.05	1.6	− 0.2848	steady state (x^e = 0.1868)
(3.4)	3.9	− 0.01	0.00	1.6	− 0.1497	steady state (x^e = 0.2682)
(3.5)	3.9	− 0.5	0.00	1.6	− 0.2595	steady state (x^e = 0.1675)
(3.6)	3.9	− 0.01	− 0.05	1.0	− 0.3433	limit cycle (period = 4)
(3.7)	3.9	− 0.01	− 0.05	0.5	− 0.6312	steady state (x^e = 0.3575)

[a]The parameter γ is a parameter of the state equation (10). The parameter B is the price elasticity of demand in (11). The parameter c_3 characterizes the third-order term in the cost function (12), and r is the interest rate satisfying $\beta = 1/(1 + r)$. The other parameters held constant across all scenarios are: $A = 0.1$ in equation (11); and $c_0(x_t) = 0$, $c_1(x_{t+1}) = 0$, $c_2 = 0$, and $c_4 = 0.5$ in (12).

[b]The Lyapunov exponent (λ) was estimated from (8) by simulating the optimal policy correspondence $x^*_{t+1} = h(x_t)$ forward.

[c]The x^e denotes the long-run equilibrium value in the presence of a steady state.

the demand elasticity B), a negative c_3 coefficient, and a large interest rate r. The optimal paths converged to a steady-state equilibrium under a more elastic demand, a zero c_3 coefficient, and a smaller interest rate r. Some limit cycles existed in between the chaotic situations and the steady-state situations. The fact that these results were consistently obtained for different values of the parameter γ is worth emphasizing. First, it indicates a weak linkage between the dynamics of the uncontrolled state equation (10) and the dynamics of the optimally managed population. In our examples, knowing that equation (10) exhibits a steady state (or chaos) clearly does not imply that the optimal state equation $x^*_{t+1} = h(x_t)$ will also exhibit a steady state (or chaos). Second, our results suggests that the nature of optimal dynamics depends crucially on the particular economic situation (as reflected by the parameters B, c_3, and r).

Implications

The above results have implications for the research methods used in analyzing optimal dynamic behavior. They also provide new insights in the economics of market instability.

First, the arguments presented in the previous sections imply that quadratic cost functions and linear state equations preclude obtaining complicated optimal dynamics. Define a dynamically flexible functional form as a form that does not restrict a priori the qualitative nature of optimal dynamics. It follows that cost functions that can provide a quadratic local approximation to an arbitrary cost function are not dynamically flexible. This is in sharp contrast to static analysis (where such functions would be considered "flexible"). It stresses the need to consider third-order terms in the specification of cost (or profit) functions in dynamic problems (Epstein). Linear state equations are not dynamically flexible. Such linear equations are commonly used in dynamic models (e.g., Rosen) and often justified as "linear approximations" to some underlying (nonlinear) state equation. Clearly, such linear approximations can preclude finding complicated dynamics in optimally managed resources.

Second, our analysis has implications for the numerical methods used in solving dynamic optimization problems. Two broad classes of methods are available: optimal control methods which solve (3) for the optimal level of $x = (x_1, x_2, \ldots)$; and dynamic programming methods which solve (5) for the optimal policy correspondence $x^*_{t+1} = h(x_t)$. In the previous section, we chose a dynamic programming method. The reason is that, under optimal chaos, optimal control methods may not converge numerically. Solving the optimal state equation forward is a fairly standard way to obtain numerical solutions to optimal control problems (e.g., Bryson and Ho). But, under chaos, this leads to a locally unstable trajectory that is sensitive to initial conditions. This local instability will in general preclude the numerical convergence of the corresponding iterative optimal control algorithm. In other words, under optimal chaos, standard optimal control methods can fail to provide a numerical solution to a dynamic optimization problem. Dynamic programming methods are not subject to this limitation because they do not solve directly for the optimal path of the state variable. Rather they solve for the optimal policy correspondence $x^*_{t+1} = h(x_t)$. This optimal policy correspondence is stationary even under optimal chaos, as long as the planning horizon is sufficiently long (Bertsekas). In that sense, dynamic programming algorithms are superior to optimal control methods in obtaining numerical solutions to dynamic optimization problems.[8]

Third, our results illustrate that optimal chaos can arise even in a completely deterministic model of optimal resource allocation. Uncertainty or exogenous shocks are not necessary to obtain competitive market instability and unpredictability. This should not be interpreted to mean that uncertainty and exogenous shock do not play a role in population dynamics: they do. Rather, market instability can be endogenously generated (Boldrin and Montrucchio; Deneckere and Pelikan). The results presented in table 1 indicate that the inelasticity of demand (B) can contribute to optimal chaos. This is consistent with the findings of Chavas and Holt. Thus, markets that face very inelastic demands seem to be prone to be unstable. Agricultural markets involve biological populations and tend to exhibit inelastic demand for food. As a result, a number of agricultural markets could well be endogenously unstable, even in the presence of rational decision makers, perfect information and perfect competi-

[8]This is not to imply that dynamic programming algorithms are always simple to implement. First, they are subject to the "curse of dimensionality" when the number of state variables is large. Second, the numerical approximations used in obtaining the solution to Bellman's equation can influence the quality of the results. This should be kept in mind in the design and interpretation of any dynamic programming analysis.

tion. This suggests a need for further research on the economics of instability in agricultural markets.

Fourth, table 1 indicates that third-order terms in the cost function (with $c_3 < 0$) can contribute to market instability. These third-order terms could arise in many situations. For example, adjustment costs necessarily involve interaction effects between state variables at successive time periods. Thus the existence of adjustment costs may lead to terms such as $[c_3\, x_{t+1}\, x_t\, (c_4 - x_t)]$ in the cost function. It follows that adjustment costs can contribute to obtaining optimal chaos. This suggests that, in general, it would be inappropriate to assume that an optimal steady state exists in adjustment cost models. Yet, such an assumption is common in previous research (e.g., Epstein). This raises questions about the validity of a number of adjustment cost models found in the literature. Our analysis stresses the need for a more careful evaluation of the role of adjustment costs in dynamic resource allocation.

Finally, table 1 shows that a high (low) interest rate tends to contribute to complicated (simple) dynamics of the market equilibrium. This result is consistent with previous research (e.g., Brock and Scheinkman; Boldrin and Montrucchio; Deneckere and Pelikan). What is of interest here is that optimal chaos seems possible under an interest rate r much lower (about 60 times lower) than those discussed by Boldrin and Montrucchio or Deneckere and Pelikan. Thus, a high interest rate could contribute to market instability in situations somewhat more realistic than those identified in previous research. To the extent that high interest rates can be associated with imperfection in the capital and credit markets, this suggests that such imperfections would reduce the ability of rational decision makers to smooth resource allocation over time and could contribute to market instability. This stresses the need to investigate further the role of imperfect capital markets in the management of biological populations, in the functioning of agricultural markets, and in the evaluation of conservation policies.

Concluding Remarks

Our analysis strongly suggests that a steady-state equilibrium cannot be assumed in the analysis of competitive market allocation of biological resources, even with rational agents under perfect information. Limit cycles or chaotic patterns are possible in such competitive markets. The inelasticity of demand, the presence of adjustment costs, or imperfections in the credit market can contribute to endogenous market instability.

What are the policy implications of these results? Our analysis was done in the context of rational agents facing a competitive market under perfect information. This implies that the standard welfare theorem holds, implying a Pareto optimal resource allocation given the assumed institutional framework. However, the following question remains: Under market instability, does there exist some alternative institutional arrangement that would affect resource allocation in a Pareto improving way? The existence of such institutional arrangements cannot be ruled out. For example, if adjustment costs are generating chaotic markets, then any institution (e.g., a government program) that would help smooth intertemporal resource allocation could possibly reduce these adjustment costs sufficiently to generate a Pareto welfare improvement. This suggests a need to investigate further the policy implications of endogenous market instability in agriculture and natural resources. Finally, the role of market imperfections and/or imperfect information can clearly play a role in dynamic resource allocation. Examining such issues appears to be a good topic for future research.

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