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A & E NOTE

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THE EXPECTATION OF A SECOND DEGREE EXPRESSION IN A MATRIX QUADRATIC FORM CONNECTED WITH THE NONCENTRAL WISHART DISTRIBUTION

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ABSTRACT

Some years ago Giguère and Styan [1] considered the expression $S_A B S_A$, where B is a pxp nonrandom (not necessarily symmetric) matrix and $S_A := X'AX$, A being an nxn nonrandom symmetric idempotent matrix and vec X' having the distribution $\mathcal{N}_{np}(\text{vec } M', I_n \otimes V)$.
They presented (without proof) the expectation of $S_A B S_A$ under the centrality condition $M'A = 0$.
In this paper the expectation will be derived for arbitrary (not necessarily symmetric) A.
Earlier results by Magnus and Neudecker [2], Neudecker and Wansbeek [3] and Neudecker [4] will be invoked.

1. INTRODUCTION

Let x_i for $i=1,\dots,n$ be px1 random vectors that are jointly independent with (normal) distribution $\mathcal{N}_p(\mu_i, V)$.
If we then define $X := (x_1,\dots,x_n)'$ and $M := (\mu_1,\dots,\mu_n)'$, then vec X' will have the distribution $\mathcal{N}_{np}(\text{vec } M', I_n \otimes V)$.
It is well-known that the matrix quadratic form $S_A := X'AX$ has expectation

$$E(S_A) = M'AM + (\text{tr} A)V \qquad (1.1)$$

and dispersion

$$\begin{aligned} D(\text{vec } S_A) = {} & \text{tr}(A'A)(V\otimes V) + (\text{tr } A^2)K_{pp}(V\otimes V) \\ & + M'A'AM \otimes V + V \otimes M'AA'M \\ & + K_{pp}(M'A^2M\otimes V) + \{K_{pp}(M'A^2M\otimes V)\}' . \end{aligned} \qquad (1.2)$$

The first result can be found, inter alia, in Giguère and Styan [1]. The second result was presented by Neudecker [4], who extended an earlier result by Magnus and Neudecker [2].
It is intuitively clear that $E(S_A B S_A)$ can be obtained from (1.1) and (1.2).

This will now be achieved.
Two results by Neudecker and Wansbeek [3], viz.

$$\mathrm{vec}(A\otimes B) = (I_n\otimes K_{qm}\otimes I_p)(\mathrm{vec}\,A\otimes \mathrm{vec}\,B) \tag{1.3}$$

and

$$\mathrm{vec}\{(A'\otimes B)K_{mq}\} = (I_q\otimes K_{mn}\otimes I_p)\,\mathrm{vec}\{(A\otimes B)K_{nq}\}\ , \tag{1.4}$$

where A and B are arbitrary mxn and pxq matrices and K_{mn} is the mnxmn commutation matrix as studied by Magnus and Neudecker [2] will be used. Other properties concerning Kronecker multiplication and the commutation matrix that will be employed are

$$\mathrm{vec}\,ABC = (C'\otimes A)\mathrm{vec}\,B, \quad \text{for compatible matrices A, B and C} \tag{1.5}$$

$$K_{mn}\,\mathrm{vec}\,A = \mathrm{vec}\,A', \text{ where A is an mxn matrix} \tag{1.6}$$

$$K_{pm}(A\otimes B)K_{nq} = B\otimes A, \text{ where A and B are mxn and pxq matrices} \tag{1.7}$$

$$(\mathrm{vec}\,A)'\mathrm{vec}B = \mathrm{tr}\,A'B, \quad \text{for compatible matrices A and B.} \tag{1.8}$$

2. THE EXPECTATION OF $S_A BS_A$

The following theorem is going to be proved.

THEOREM. Let the px1 random vectors x_i be independently distributed each as $N_p(\mu_i,V)$ for i=1,...,n.
Let

$$X := (x_1,\ldots,x_n)' \quad \text{and} \quad M := (\mu_1,\ldots,\mu_n)'.$$

Consider the matrix quadratic form $S_A := X'AX$, where the nxn matrix A is nonrandom and not necessarily symmetric. Let further B be a pxp nonrandom (not necessarily symmetric) matrix. Then

$$\begin{aligned} E(S_A BS_A) &= (\mathrm{tr}\,A'A)VB'V + (\mathrm{tr}\,A^2)(\mathrm{tr}\,BV)V + VB'M'A'AM \\ &\quad + M'AA'MB'V + (\mathrm{tr}\,BV)M'A^2M + (\mathrm{tr}\,M'A^2MB)V \\ &\quad + \{M'AM + (\mathrm{tr}\,A)V\}B\{M'AM + (\mathrm{tr}\,A)V\}\ . \end{aligned}$$

Proof It is advisable to examine $\mathrm{vec}\,S_A BS_A = I_{p^2}(S_A'\otimes S_A)\mathrm{vec}\,B$

$$= \mathrm{vec}\{I_{p^2}(S_A'\otimes S_A)\mathrm{vec}\,B\} = (\mathrm{vec}\,B\otimes I_{p^2})'\mathrm{vec}(S_A'\otimes S_A)$$

$$= (\mathrm{vec}\,B\otimes I_{p^2})'(I_p\otimes K_{pp}\otimes I_p)(K_{pp}\otimes I_{p^2})(\mathrm{vec}\,S_A\otimes \mathrm{vec}\,S_A)\ . \tag{2.1}$$

This subresult follows from (1.5) and (1.6).

Clearly $(I_p \otimes K_{pp} \otimes I_p)(K_{pp} \otimes I_{p^2})\ \text{vec}\ \mathcal{D}(\text{vec}\ S_A)$

$$= (I_p \otimes K_{pp} \otimes I_p)\ \text{vec}\ \mathcal{D}(\text{vec}\ S_A) K_{pp}$$

$$= (I_p \otimes K_{pp} \otimes I_p)\ \text{vec}[(\text{tr}\ A'A)(V \otimes V)K_{pp} + (\text{tr}\ A^2)(V \otimes V)$$

$$+ (M'A'AM \otimes V)K_{pp} + (V \otimes M'AA'M)K_{pp} + V \otimes M'A^2M$$

$$+ (M'A^2M \otimes V)']$$

$$= (\text{tr}\ A'A)\ \text{vec}(V \otimes V)K_{pp} + (\text{tr}\ A^2)(\text{vec}\ V \otimes \text{vec}\ V)$$

$$+ \text{vec}(M'A'AM \otimes V)K_{pp} + \text{vec}(V \otimes M'AA'M)K_{pp}$$

$$+ \text{vec}\ V \otimes \text{vec}\ M'A^2M + \text{vec}(M'A^2M)' \otimes \text{vec}\ V\ , \qquad (2.2)$$

by virtue of (1.2), (1.5), (1.7), (1.3) and (1.4).

Further $(I_p \otimes K_{pp} \otimes I_p)(K_{pp} \otimes I_{p^2})[\mathcal{E}(\text{vec}\ S_A) \otimes \mathcal{E}(\text{vec}\ S_A)]$

$$= (I_p \otimes K_{pp} \otimes I_p)(K_{pp} \otimes I_{p^2})[\text{vec}\{M'AM + (\text{tr}\ A)V\} \otimes \text{vec}\{M'AM + (\text{tr}\ A)V\}]$$

$$= (I_p \otimes K_{pp} \otimes I_p)[\text{vec}\{M'A'M + (\text{tr}\ A)V\} \otimes \text{vec}\{M'AM + (\text{tr}\ A)V\}]$$

$$= \text{vec}[\{M'A'M + (\text{tr}\ A)V\} \otimes \{M'AM + (\text{tr}\ A)V\}] \qquad (2.3)$$

by virtue of (1.1), (1.6) and (1.3).

Premultiplication by $(\text{vec}\ B \otimes I_{p^2})'$ and addition of the two expressions (2.2) and (2.3) leads to

$$(\text{tr}\ A'A)\ \text{vec}\ VB'V + (\text{tr}\ A^2)(\text{tr}\ BV)\ \text{vec}\ V$$

$$+ \text{vec}\ VB'M'A'AM + \text{vec}\ M'AA'MB'V$$

$$+ (\text{tr}\ BV)\ \text{vec}\ M'A^2M + (\text{tr}\ M'A^2MB)\ \text{vec}\ V$$

$$+ \text{vec}\{M'AM + (\text{tr}\ A)V\}B\{M'AM + (\text{tr}\ A)V\}\ , \qquad (2.4)$$

by virtue of (1.8), (1.5), (1.7), (1.6) and (1.3).

From (2.4) follows $\mathcal{E}(S_A B S_A)$ after deletion of vec operators.
If A is symmetric idempotent of rank k say, and the noncentrality condition $M'A = 0$ is imposed, the result is simplified to

$$\mathcal{E}(S_A B S_A) = k(\text{tr}\ BV)V + kVB'V + k^2 VBV\ , \qquad (2.5)$$

which is Giguère and Styan's (2.2.8).

[1] M.A. Giguère and G.P.H. Styan, Multivariate normal estimation with missing data on several variates. Trans. Seventh Prague Conference on Information Theory, Statistical Decision Functions, Random Processes, and of the Eighth European Meeting of Statisticians (Technical Univ. Prague, August 1974), pub. Academia, Prague, and D. Reidel, Dordrecht, volume B (1978), pp. 129-139.

[2] J.R. Magnus and H. Neudecker, The commutation matrix: some properties and applications. Ann. Statist. 7:381-394 (1979).

[3] H. Neudecker and T.J. Wansbeek, Some results on commutation matrices with statistical applications. Can. J. Statist. 11:221-231 (1983).

[4] H. Neudecker, The dispersion matrix of vec X'AX, A'=A, when X'X is a Wishart matrix. Note N5/84, Faculty of Actuarial Science and Econometrics, University of Amsterdam, The Netherlands (1984).